MATH 3142 Streaming Lecture — 2016-02-26

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Updated: February 26, 2016

Exercise (1)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$. Prove the following for all $\mathbf{v} \in \mathbb{R}^n$:

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle.$$

Solution. **[F]:** Let $\mathbf{u}_k = (\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, ..., \mathbf{u}_{k_n})$ and $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n)$. Recall the proposition "$\mathbf{u}_k = (\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, ..., \mathbf{u}_{k_n})$ and $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n)$ then $\lim_{k\to\infty} \mathbf{u}_k = \mathbf{u}$ iff$\lim_{k\to\infty} \mathbf{u}_{k_i} = \mathbf{u}_i$ for each i". Since $\mathbf{u}_k$ in $\mathbb{R}^n$ converges to the point $\mathbf{v}$. Then $\lim_{k\to\infty} \mathbf{u}_{k_i} = \mathbf{u}_i$. Therefore,

$$\begin{aligned}
\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle &= \langle \lim_{k\to\infty} \mathbf{u}_k, \mathbf{v}\rangle \\
&= \langle \lim_{k\to\infty} (\mathbf{u}_{k_1}, \mathbf{u}_{k_2}, ..., \mathbf{u}_{k_n}), \rangle \\
&= \langle (\lim_{k\to\infty} \mathbf{u}_{k_1}, \lim_{k\to\infty} \mathbf{u}_{k_2}, ..., \lim_{k\to\infty} \mathbf{u}_{k_n}), \mathbf{v}\rangle \\
&= \langle (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n), \mathbf{v}\rangle \\
&= \langle \mathbf{u}, \mathbf{v}\rangle
\end{aligned}$$

$\square$

Solution. **[C]:** Let $\mathbf{v}$ be a point in R^n. By the definition of the scalar product,

$$\langle \mathbf{u}_k, \mathbf{v}\rangle = (u_{1k}v_1 + ... + u_{nk}v_n).$$

Note that $\{\mathbf{u}_k\} = (u_{1k}, ..., u_{nk})$ converges to $\mathbf{u} = (u_1, ..., u_n)$. It follows that $\langle \mathbf{u}_k, \mathbf{v}\rangle$ converges to $\langle \mathbf{u}, \mathbf{v}\rangle = (u_1v_1 + ... + u_nv_n)$. Therefore,

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle.$$

$\square$

Solution. **[A]:** By the Componentwise Convergence Theorem, for any index $0 \le i \le n$, $\lim_{k\to\infty} p_i(\mathbf{u}_k) = p_i(\mathbf{u})$. Then, using the properties of limits, we have

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \lim_{k\to\infty} \sum_{i=1}^{n} p_i(\mathbf{v}) p_i(\mathbf{u}_k) = \sum_{i=1}^{n} p_i(\mathbf{v}) \lim_{k\to\infty} p_i(\mathbf{u}_k) = \sum_{i=1}^{n} p_i(\mathbf{v}) p_i(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v}\rangle$$

$\square$

Exercise (Section 6.1 #5)**.** Suppose that $f, g : [a, b] \to \mathbb{R}$ are bounded, and have the property that $g(x) \leq f(x)$ for all $x \in [a, b]$.

(a) For P partitioning $[a, b]$, show that $L(g, P) \leq L(f, P)$.

(b) Use part (a) to show that $\underline{\int_a^b} g \leq \underline{\int_a^b} f$.

Solution. (a) Let $m(f, i) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $m(g, i) = \inf\{g(x) : x \in [x_{i-1}, x_i]\}$. Applying the defintions of

$$L(f, P) = \sum_{i=1}^{n} m(f, i)(x_i - x_{i-1})$$

and

$$L(g, P) = \sum_{i=1}^{n} m(g, i)(x_i - x_{i-1}),$$

and noting that $m(g, i) \leq m(f, i)$, we have that:

$$L(g, P) = \sum_{i=1}^{n} m(g, i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} m(f, i)(x_i - x_{i-1}) = L(f, P).$$

(b) Again, applying definitions:

$$\underline{\int_a^b} g = \sup\{L(g, P) : P \text{ partitions } [a, b]\} \leq \sup\{L(f, P) : P \text{ partitions } [a, b]\} = \underline{\int_a^b} f.$$

□

Exercise (Section 6.4 #8)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and continuous except at one point $x_0 \in (a, b)$. Prove that f is integrable. (Hint: let $f_1 : [a, x_0] \to \mathbb{R}$ and $f_2 : [x_0, b] \to \mathbb{R}$ be defined by $f_1(x) = f(x)$ and $f_2(x) = f(x)$, then apply Theorem 6.19.)

Solution. Let $f_1 : [a, x_0] \to \mathbb{R}$ and $f_2 : [x_0, b] \to \mathbb{R}$ be defined by $f_1(x) = f(x)$ and $f_2(x) = f(x)$. Since f is continuous everywhere in $[a, b]$ except for x_0, it follows that f_1 is continuous on (a, x_0) and f_2 is continuous on (x_0, b). It follows by Thm 6.19 that f_1, f_2 are both integrable.

Let $\{P_n^1\}$ be an Archemedian sequence for f_1 on $[a, x_0]$ and $\{P_n^2\}$ be an Archemedian sequence for f_2 on $[x_0, b]$. Then $P_n = P_n^1 \cup P_n^2$ is a partition of $[a, b]$.

We now show that f is integrable on $[a, b]$ by showing that $\{P_n\}$ is Archemedian for f on $[a, b]$:

$$\begin{aligned}
\lim_{n\to\infty} U(f, P_n) - L(f, P_n) &= \lim_{n\to\infty} (U(f, P_n^1) + U(f, P_n^2)) - (L(f, P_n^1) + L(f, P_n^2)) \\
&= \left(\lim_{n\to\infty} U(f, P_n^1) - L(f, P_n^1)\right) + \left(\lim_{n\to\infty} U(f, P_n^2) - L(f, P_n^2)\right) \\
&= \left(\lim_{n\to\infty} U(f_1, P_n^1) - L(f_1, P_n^1)\right) + \left(\lim_{n\to\infty} U(f_2, P_n^2) - L(f_2, P_n^2)\right) = 0 + 0 = 0.
\end{aligned}$$

□

Exercise (Not in book.)**.** Let $A \subseteq \mathbb{R}^n$. We say A is **clopen** if it is both an open and closed subset of $\mathbb{R}^n$. Prove that the only nonempty clopen subset of $\mathbb{R}$ is $\mathbb{R}$ itself. (Hint: let $x \in A$ and $R = \{r > 0 : B_r(x) \subseteq A\}$, and then show that R has no least upper bound.)

Solution. Let $x \in A$ and $R = \{r > 0 : B_r(x) \subseteq A\}$, where A is a clopen subset of $\mathbb{R}$. Suppose that q is a least upper bound for R. Since for $0 < r < q$ we have that $B_r(x) \subseteq A$, it follows that $\bigcup_{0<r<q} B_r(x) = B_q(x) \subseteq A$. Note that $B_q(x) = (x - q, x + q)$.

Note that $\{x + q(1 - \frac{1}{k})\}$ is a sequence of points in $(x - q, x + q) \subseteq A$, and

$$\lim_{k\to\infty} x + q(1 - \frac{1}{k}) = x + q(1 - 0) = x + q.$$

It follows that $x + q \in A$. Similarly, we may show that $x - q \in A$.

Let $p > 0$ satisify both $B_p(x - q) = (x - q - p, x - q + p) \subseteq A$ and $B_p(x + q) = (x + q - p, x + q + p) \subseteq A$. Note that

$$B_{p+q}(x) = (x-q-p, x+q+p) = (x-q-p, x-q+p) \cup (x-q, x+q) \cup (x+q-p, x+q+p)$$

$$= B_p(x - q) \cup B_q(x) \cup B_p(x + q) \subseteq A.$$

So since $p + q \in R$ and $p + q > q$, q is not an upper bound for R. Contradiction.

Let $y \in \mathbb{R}$ be distinct from x. Let $r = |x - y| > 0$. Since R has no upper bound, $2r \in R$. Since $y \in B_{2r}(x) \subseteq A$, so $y \in A$. This shows that $\mathbb{R} \subseteq A \subseteq \mathbb{R}$, so $A = \mathbb{R}$. $\square$