

# MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are “due”; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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# Chapter 6

## Integration: Two Fundamental Theorems

### 6.1 Darboux Sums: Upper and Lower Integrals

**Lemma (6.1).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and the numbers  $m, M$  have the property that

$$m \leq f(x) \leq M$$

for all  $x$  in  $[a, b]$ . Then, if  $P$  is a partition of the domain  $[a, b]$ ,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

*Proof.*

□

**Lemma (6.2, The Refinement Lemma).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and that  $P$  is a partition of its domain  $[a, b]$ . If  $P^*$  is a refinement of  $P$ , then

$$L(f, P) \leq L(f, P^*) \text{ and } U(f, P^*) \leq U(f, P).$$

*Proof.*

□

**Lemma (6.3).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and that  $P_1, P_2$  are partitions of its domain. Then  $L(f, P_1) \leq U(f, P_2)$ .

*Proof.*

□

**Lemma (6.4).** For a bounded function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$\int_a^b f \leq \overline{\int_a^b f}.$$

*Proof.*

□

**Exercise (2).** For an interval  $[a, b]$  and a positive number  $\delta$ , show that there is a partition  $P = \{x_i : 0 \leq i \leq n\}$  of  $[a, b]$  such that each partition interval  $[x_i, x_{i+1}]$  of  $P$  has length less than  $\delta$ .

*Solution.* □

**Exercise (3).** Suppose that the bounded function  $f : [a, b] \rightarrow \mathbb{R}$  has the property that for each rational number  $x$  in the interval  $[a, b]$ ,  $f(x) = 0$ . Prove that

$$\int_a^b f \leq 0 \leq \overline{\int_a^b f}.$$

*Solution.* □

**Exercise (6).** Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function for which there is a partition  $P$  of  $[a, b]$  with  $L(f, P) = U(f, P)$ . Prove that  $f : [a, b] \rightarrow \mathbb{R}$  is constant.

*Solution.* □

## 6.2 The Archimedes-Riemann Theorem

**Lemma (6.7).** For a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $P$  of  $[a, b]$ ,

$$L(f, P) \leq \int_a^b f \leq \overline{\int_a^b f} \leq U(f, P).$$

*Proof.* □

**Theorem (6.8, The Archimedes-Riemann Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then  $f$  is integrable on  $[a, b]$  if and only if there is a sequence  $\{P_n\}$  of partitions of the interval  $[a, b]$  such that

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n).$$

*Proof.* □

**Example (6.9).** Show that a monotonically increasing function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable.

*Solution.* □

**Example (6.11).** Show that  $\int_0^1 x^2 dx = \frac{1}{3}$ .

*Solution.* □

**Exercise (4).** Prove that for a natural number  $n$ ,

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that  $\int_a^b x \, dx = (b^2 - a^2)/2$ .

*Solution.* □

**Exercise (6b).** Use the Archimedes-Riemann Theorem to show that for  $0 \leq a < b$ ,

$$\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3}.$$

*Solution.* □

**Exercise (9).** Suppose that the functions  $f : [a, b] \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}$  are integrable. Show that there is a sequence  $\{P_n\}$  of partitions of  $[a, b]$  that is an Archimedean sequence of partitions for  $f$  on  $[a, b]$  and also an Archimedean sequence of partitions for  $g$  on  $[a, b]$ .

*Solution.* □

## 6.3 Additivity, Monotonicity, and Linearity

**Theorem (6.12, Additivity over Intervals).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable on  $[a, b]$  and let  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, c]$  and  $[c, b]$ , and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

*Proof.* □

**Theorem (6.13, Monotonicity of the Integral).** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable and that  $f(x) \leq g(x)$  for all  $x \in [a, b]$ . Then

$$\int_a^b f \leq \int_a^b g.$$

*Proof.* □

**Lemma (6.14).** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded and let  $P$  partition  $[a, b]$ . Then

$$L(f, P) + L(g, P) \leq L(f + g, P) \quad \text{and} \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Moreover, for any number  $\alpha$ ,

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P) \quad \text{if } \alpha \geq 0$$

$$U(\alpha f, P) = \alpha L(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha U(f, P) \quad \text{if } \alpha < 0.$$

*Proof.* □

**Theorem** (6.15, Linearity of the Integral). Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be integrable. Then for any two numbers  $\alpha, \beta$ , the function  $\alpha f + \beta g : [a, b] \rightarrow \mathbb{R}$  is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

*Proof.* □

**Exercise** (1). Suppose that the functions  $f, g, f^2, g^2, fg$  are integrable on  $[a, b]$ . Prove that  $(f - g)^2$  is also integrable on  $[a, b]$  and that  $\int_a^b (f - g)^2 \geq 0$ . Use this to prove that

$$\int_a^b fg \leq \frac{1}{2} \left[ \int_a^b f^2 + \int_a^b g^2 \right].$$

*Solution.* □

**Exercise** (4). Suppose that  $S$  is a nonempty bounded set of numbers and that  $\alpha$  is a number. Define  $\alpha S$  to be the set  $\{\alpha x : x \in S\}$ . Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

*Solution.* □

**Exercise** (6). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and let  $a < c < b$ . Prove that if  $f$  is integrable on both  $[a, c]$ ,  $[c, b]$ , then it is integrable on  $[a, b]$ .

*Solution.* □

## 6.4 Continuity and Integrability

**Lemma** (6.17). Let the function  $f : [a, b] \rightarrow \mathbb{R}$  be continuous let  $P$  partition its domain. Then there is a partition interval of  $P$  that contains two points  $u, v$  for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

*Proof.* □

**Theorem** (6.18). A continuous function on a closed bounded interval is integrable.

*Proof.* □

**Theorem (6.19).** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded on  $[a, b]$  and continuous on  $(a, b)$ . Then  $f$  is integrable on  $[a, b]$  and the value of  $\int_a^b f$  does not depend on the values of  $f$  at the endpoints of  $[a, b]$ .

*Proof.* □

**Exercise (1).** Determine whether each of the following statements is true or false, and justify your answer.

- (a) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $\int_a^b f = 0$ , then  $f(x) = 0$  for all  $x \in [a, b]$ .
- (b) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, then  $f$  is continuous.
- (c) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $f(x) \geq 0$  for all  $x \in [a, b]$ , then  $\int_a^b f \geq 0$ .
- (d) A continuous function  $f : (a, b) \rightarrow \mathbb{R}$  defined on an open interval  $(a, b)$  is bounded.
- (e) A continuous function  $f : [a, b] \rightarrow \mathbb{R}$  defined on a closed interval  $[a, b]$  is bounded.

*Solution.* (a)

(b)

(c)

(d)

(e)

□

**Exercise (5).** Suppose that the continuous function  $f : [a, b] \rightarrow \mathbb{R}$  has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that  $f(x) \leq 0$  for all  $x \in [a, b]$ . Is this true if we only require integrability of the function?

*Solution.* □

**Exercise (6).** Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and that  $f(x) \geq 0$  for all  $x \in [0, 1]$ . Prove that  $\int_0^1 f > 0$  if and only if there is a point  $x_0 \in [0, 1]$  at which  $f(x_0) > 0$ .

*Solution.* □

## 6.5 The First Fundamental Theorem: Integrating Derivatives

**Lemma (6.21).** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and that the number  $A$  has the property that for every  $P$  partitioning  $[a, b]$ ,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

*Proof.* □

**Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives).** Let  $F : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Moreover, suppose that its derivative  $F' : (a, b) \rightarrow \mathbb{R}$  is both continuous and bounded. Then

$$\int_a^b F'(x) dx = F(b) - F(a).$$

*Proof.* □

**Exercise (1).** Let  $m, b$  be positive numbers. Find the value of  $\int_0^1 mx + b dx$  in the following three ways:

- (a) Using elementary geometry, interpreting the integral as an area.
- (b) Using upper and lower Darboux sums based on regular partitions of the interval  $[0, 1]$  and using the Archimedes-Riemann Theorem.
- (c) Using the First Fundamental Theorem (Integrating Derivatives).

*Solution.* □

**Exercise (5).** The monotonicity property of the integral implies that if the functions  $g, h : [0, \infty) \rightarrow \mathbb{R}$  are continuous and  $g(x) \leq h(x)$  for all  $x \geq 0$ , then

$$\int_0^x g \leq \int_0^x h \quad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\begin{aligned} \cos x &\leq 1 && \text{if } x \geq 0. \\ \sin x &\leq x && \text{if } x \geq 0. \\ 1 - \cos x &\leq \frac{x^2}{2} && \text{if } x \geq 0. \\ x - \sin x &\leq \frac{x^3}{6} && \text{if } x \geq 0. \\ x - \frac{x^3}{6} &\leq \sin x \leq x && \text{if } x \geq 0. \end{aligned}$$



## 6.6 The Second Fundamental Theorem: Differentiating Integrals

**Theorem** (6.26, The Mean Value Theorem for Integrals). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then there is a point  $x_0$  in the interval  $[a, b]$  at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

*Proof.*

□

**Proposition** (6.27). Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function  $F : [a, b] \rightarrow \mathbb{R}$  is continuous.

*Proof.*

□

**Theorem** (6.29, The Second Fundamental Theorem: Differentiating Integrals). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then

$$\frac{d}{dx} \left[ \int_a^x \right] = f(x) \quad \text{for all } x \in (a, b).$$

*Proof.*

□

**Exercise** (2b). Suppose  $f : [0, 2] \rightarrow \mathbb{R}$  is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t) \, dt \quad \text{for all } x \in [a, b]$$

and find a formula for  $F(x)$  which does not involve integrals.

*Solution.*

□

**Exercise** (5). Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Define

$$G(x) = \int_0^x (x-t)f(t) \, dt \quad \text{for all } x.$$

Prove that  $G'''(x) = f(x)$  for all  $x$ .

*Solution.*

□

**Exercise (12).** Suppose that  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous and that  $\alpha, \beta$  are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a, b].$$

Prove that  $H(a) = 0$  and  $H'(x) = 0$  for all  $x \in (a, b)$ . Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

*Solution.*

□

# Chapter 10

## The Euclidean Space $\mathbb{R}^n$

### 10.1 The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

**Proposition (10.2).** Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and  $\alpha, \beta \in \mathbb{R}$ . Then both of the following hold:

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{v}, \mathbf{u} \rangle \\ \langle \alpha \mathbf{u} + \beta \mathbf{w}, \mathbf{v} \rangle &= \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle\end{aligned}$$

*Proof.* □

**Lemma (10.4).** For  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,  $\mathbf{u}, \mathbf{v}$  are orthogonal if and only if  $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ .

*Proof.* □

**Lemma (10.5).** For  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  where  $\mathbf{v} \neq \mathbf{0}$ , define  $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$  and  $\mathbf{w} = \mathbf{u} - \lambda \mathbf{v}$ . Then  $\mathbf{v}, \mathbf{w}$  are orthogonal and  $\mathbf{u} = \mathbf{w} + \lambda \mathbf{v}$ .

*Proof.* □

**Theorem (10.6, The Cauchy-Schwarz Inequality).** For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

*Proof.* □

**Theorem (10.7, The Triangle Inequality).** For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

**Exercise (3).** Show that for  $\mathbf{u} \in \mathbb{R}^n$ ,  $\alpha \in \mathbb{R}$ :

(a)  $\|\mathbf{u}\| = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

(b)  $\|\alpha \mathbf{u}\| = |\alpha| \|\mathbf{u}\|$ .