MATH 3412 Notes — Spring 2016

Your Name Here

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

Proof. □

Lemma (6.2, The Refinement Lemma)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If $P^\star$ is a refinement of P, then

$$L(f, P) \leq L(f, P^\star) \text{ and } U(f, P^\star) \leq U(f, P).$$

Proof. □

Lemma (6.3)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. □

Lemma (6.4)**.** For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\overline{\int_a^b} f \leq \underline{\int_a^b} f.$$

Proof. □

Exercise (2)**.** For an interval $[a,b]$ and a positive number δ, show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a,b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ.

Solution. □

Exercise (3)**.** Suppose that the bounded function $f : [a,b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a,b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a,b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a,b]$ with $L(f,P) = U(f,P)$. Prove that $f : [a,b] \to \mathbb{R}$ is constant.

Solution. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7)**.** For a bounded function $f : [a,b] \to \mathbb{R}$ and a partition P of $[a,b]$,

$$L(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f,P).$$

Proof. □

Theorem (6.8, The Archimedes-Riemann Theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a,b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a,b]$ such that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f,P_n) = \int_a^b f = \lim_{n\to\infty} U(f,P_n).$$

Proof. □

Example (6.9)**.** Show that a monotonically increasing function $f : [a,b] \to \mathbb{R}$ is integrable.

Solution. □

Example (6.11)**.** Show that $\int_0^1 x^2\, dx = \frac{1}{3}$.

Solution. □

Exercise (4)**.** Prove that for a natural number n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x\,dx = (b^2 - a^2)/2$.

Solution. □

Exercise (6b)**.** Use the Archimedes-Riemann Theorem to show that for $0 \leq a < b$,

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Solution. □

Exercise (9)**.** Suppose that the functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ that is an Archimediean sequence of partitions for f on $[a,b]$ and also an Archimedean sequence of partitions for g on $[a,b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity