

# MATH 3142 Notes — Spring 2016

Your Name Here

UNC Charlotte

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

— Dr. Steven Clontz (sclontz5@uncc.edu)

# Chapter 6

## Integration: Two Fundamental Theorems

### 6.1 Darboux Sums: Upper and Lower Integrals

**Lemma (6.1).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and the numbers  $m, M$  have the property that

$$m \leq f(x) \leq M$$

for all  $x$  in  $[a, b]$ . Then, if  $P$  is a partition of the domain  $[a, b]$ ,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

*Proof.*

□

**Lemma (6.2, The Refinement Lemma).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and that  $P$  is a partition of its domain  $[a, b]$ . If  $P^*$  is a refinement of  $P$ , then

$$L(f, P) \leq L(f, P^*) \text{ and } U(f, P^*) \leq U(f, P).$$

*Proof.*

□

**Lemma (6.3).** Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and that  $P_1, P_2$  are partitions of its domain. Then  $L(f, P_1) \leq U(f, P_2)$ .

*Proof.*

□

**Lemma (6.4).** For a bounded function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$\int_a^b f \leq \overline{\int_a^b f}.$$

*Proof.*

□

**Exercise (2).** For an interval  $[a, b]$  and a positive number  $\delta$ , show that there is a partition  $P = \{x_i : 0 \leq i \leq n\}$  of  $[a, b]$  such that each partition interval  $[x_i, x_{i+1}]$  of  $P$  has length less than  $\delta$ .

*Solution.* □

**Exercise (3).** Suppose that the bounded function  $f : [a, b] \rightarrow \mathbb{R}$  has the property that for each rational number  $x$  in the interval  $[a, b]$ ,  $f(x) = 0$ . Prove that

$$\int_a^b f \leq 0 \leq \int_a^b f.$$

*Solution.* □

**Exercise (6).** Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded function for which there is a partition  $P$  of  $[a, b]$  with  $L(f, P) = U(f, P)$ . Prove that  $f : [a, b] \rightarrow \mathbb{R}$  is constant.

*Solution.* □

## 6.2 The Archimedes-Riemann Theorem

**Lemma (6.7).** For a bounded function  $f : [a, b] \rightarrow \mathbb{R}$  and a partition  $P$  of  $[a, b]$ ,

$$L(f, P) \leq \int_a^b f \leq \int_a^b f \leq U(f, P).$$

*Proof.* □

**Theorem (6.8, The Archimedes-Riemann Theorem).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then  $f$  is integrable on  $[a, b]$  if and only if there is a sequence  $\{P_n\}$  of partitions of the interval  $[a, b]$  such that

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n).$$

*Proof.* □

**Example (6.9).** Show that a monotonically increasing function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable.

*Solution.* □

**Example (6.11).** Show that  $\int_0^1 x^2 dx = \frac{1}{3}$ .

*Solution.* □

**Exercise (4).** Prove that for a natural number  $n$ ,

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that  $\int_a^b x \, dx = (b^2 - a^2)/2$ .

*Solution.* □

**Exercise (6b).** Use the Archimedes-Riemann Theorem to show that for  $0 \leq a < b$ ,

$$\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3}.$$

*Solution.* □

**Exercise (9).** Suppose that the functions  $f : [a, b] \rightarrow \mathbb{R}$  and  $g : [a, b] \rightarrow \mathbb{R}$  are integrable. Show that there is a sequence  $\{P_n\}$  of partitions of  $[a, b]$  that is an Archimedean sequence of partitions for  $f$  on  $[a, b]$  and also an Archimedean sequence of partitions for  $g$  on  $[a, b]$ .

*Solution.* □

## 6.3 Additivity, Monotonicity, and Linearity

**Theorem (6.12, Additivity over Intervals).** Let  $f : [a, b] \rightarrow \mathbb{R}$  be integrable on  $[a, b]$  and let  $c \in (a, b)$ . Then  $f$  is integrable on  $[a, c]$  and  $[c, b]$ , and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

*Proof.* □

**Theorem (6.13, Monotonicity of the Integral).** Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable and that  $f(x) \leq g(x)$  for all  $x \in [a, b]$ . Then

$$\int_a^b f \leq \int_a^b g.$$

*Proof.* □

**Lemma (6.14).** Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be bounded and let  $P$  partition  $[a, b]$ . Then

$$L(f, P) + L(g, P) \leq L(f + g, P) \quad \text{and} \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Moreover, for any number  $\alpha$ ,

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P) \quad \text{if } \alpha \geq 0$$

$$U(\alpha f, P) = \alpha L(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha U(f, P) \quad \text{if } \alpha < 0.$$

*Proof.* □

**Theorem** (6.15, Linearity of the Integral). Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be integrable. Then for any two numbers  $\alpha, \beta$ , the function  $\alpha f + \beta g : [a, b] \rightarrow \mathbb{R}$  is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

*Proof.* □

**Exercise** (1). Suppose that the functions  $f, g, f^2, g^2, fg$  are integrable on  $[a, b]$ . Prove that  $(f - g)^2$  is also integrable on  $[a, b]$  and that  $\int_a^b (f - g)^2 \geq 0$ . Use this to prove that

$$\int_a^b fg \leq \frac{1}{2} \left[ \int_a^b f^2 + \int_a^b g^2 \right].$$

*Solution.* □

**Exercise** (4). Suppose that  $S$  is a nonempty bounded set of numbers and that  $\alpha$  is a number. Define  $\alpha S$  to be the set  $\{\alpha x : x \in S\}$ . Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

*Solution.* □

**Exercise** (6). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and let  $a < c < b$ . Prove that if  $f$  is integrable on both  $[a, c]$ ,  $[c, b]$ , then it is integrable on  $[a, b]$ .

*Solution.* □

## 6.4 Continuity and Integrability

**Lemma** (6.17). Let the function  $f : [a, b] \rightarrow \mathbb{R}$  be continuous let  $P$  partition its domain. Then there is a partition interval of  $P$  that contains two points  $u, v$  for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

*Proof.* □

**Theorem** (6.18). A continuous function on a closed bounded interval is integrable.

*Proof.* □

**Theorem (6.19).** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is bounded on  $[a, b]$  and continuous on  $(a, b)$ . Then  $f$  is integrable on  $[a, b]$  and the value of  $\int_a^b f$  does not depend on the values of  $f$  at the endpoints of  $[a, b]$ .

*Proof.* □

**Exercise (1).** Determine whether each of the following statements is true or false, and justify your answer.

- (a) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $\int_a^b f = 0$ , then  $f(x) = 0$  for all  $x \in [a, b]$ .
- (b) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable, then  $f$  is continuous.
- (c) If  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and  $f(x) \geq 0$  for all  $x \in [a, b]$ , then  $\int_a^b f \geq 0$ .
- (d) A continuous function  $f : (a, b) \rightarrow \mathbb{R}$  defined on an open interval  $(a, b)$  is bounded.
- (e) A continuous function  $f : [a, b] \rightarrow \mathbb{R}$  defined on a closed interval  $[a, b]$  is bounded.

*Solution.* (a)

(b)

(c)

(d)

(e)

□

**Exercise (5).** Suppose that the continuous function  $f : [a, b] \rightarrow \mathbb{R}$  has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that  $f(x) \leq 0$  for all  $x \in [a, b]$ . Is this true if we only require integrability of the function?

*Solution.* □

**Exercise (6).** Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and that  $f(x) \geq 0$  for all  $x \in [0, 1]$ . Prove that  $\int_0^1 f > 0$  if and only if there is a point  $x_0 \in [0, 1]$  at which  $f(x_0) > 0$ .

*Solution.* □

## 6.5 The First Fundamental Theorem: Integrating Derivatives

**Lemma (6.21).** Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and that the number  $A$  has the property that for every  $P$  partitioning  $[a, b]$ ,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

*Proof.* □

**Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives).** Let  $F : [a, b] \rightarrow \mathbb{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Moreover, suppose that its derivative  $F' : (a, b) \rightarrow \mathbb{R}$  is both continuous and bounded. Then

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

*Proof.* □

**Exercise (1).** Let  $m, b$  be positive numbers. Find the value of  $\int_0^1 mx + b \, dx$  in the following three ways:

- (a) Using elementary geometry, interpreting the integral as an area.
- (b) Using upper and lower Darboux sums based on regular partitions of the interval  $[0, 1]$  and using the Archimedes-Riemann Theorem.
- (c) Using the First Fundamental Theorem (Integrating Derivatives).

*Solution.* □

**Exercise (5).** The monotonicity property of the integral implies that if the functions  $g, h : [0, \infty) \rightarrow \mathbb{R}$  are continuous and  $g(x) \leq h(x)$  for all  $x \geq 0$ , then

$$\int_0^x g \leq \int_0^x h \quad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \leq 1 \quad \text{if } x \geq 0.$$

$$\sin x \leq x \quad \text{if } x \geq 0.$$

$$1 - \cos x \leq \frac{x^2}{2} \quad \text{if } x \geq 0.$$



$$x - \sin x \leq \frac{x^3}{6} \quad \text{if } x \geq 0.$$

$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{if } x \geq 0.$$

(For this problem, you may assume that the sine and cosine functions are differentiable functions with the properties  $\sin(0) = 0$ ,  $\cos(0) = 1$ ,  $\frac{d}{dx}[\sin(x)] = \cos(x)$ , and  $\frac{d}{dx}[\cos(x)] = -\sin(x)$ .)

*Solution.* □

## 6.6 The Second Fundamental Theorem: Differentiating Integrals

**Theorem** (6.26, The Mean Value Theorem for Integrals). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then there is a point  $x_0$  in the interval  $[a, b]$  at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

*Proof.* □

**Proposition** (6.27). Suppose that the function  $f : [a, b] \rightarrow \mathbb{R}$  is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function  $F : [a, b] \rightarrow \mathbb{R}$  is continuous.

*Proof.* □

**Theorem** (6.29, The Second Fundamental Theorem: Differentiating Integrals). Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then

$$\frac{d}{dx} \left[ \int_a^x f \right] = f(x) \quad \text{for all } x \in (a, b).$$

*Proof.* □

**Exercise** (2b). Suppose  $f : [0, 2] \rightarrow \mathbb{R}$  is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t) \, dt \quad \text{for all } x \in [a, b]$$

and find a formula for  $F(x)$  which does not involve integrals.

*Solution.*

□

**Exercise (5).** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Define

$$G(x) = \int_0^x (x-t)f(t) \, dt \quad \text{for all } x.$$

Prove that  $G''(x) = f(x)$  for all  $x$ .

*Solution.*

□

**Exercise (12).** Suppose that  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous and that  $\alpha, \beta$  are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a, b].$$

Prove that  $H(a) = 0$  and  $H'(x) = 0$  for all  $x \in (a, b)$ . Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

*Solution.*

□

# Chapter 10

## The Euclidean Space $\mathbb{R}^n$

### 10.1 The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

**Proposition (10.2).** Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and  $\alpha, \beta \in \mathbb{R}$ . Then both of the following hold:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$\langle \alpha \mathbf{u} + \beta \mathbf{w}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle$$

*Proof.*

□

**Lemma (10.4).** For  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,  $\mathbf{u}, \mathbf{v}$  are orthogonal if and only if  $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ .

*Proof.*

□

**Lemma (10.5).** For  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  where  $\mathbf{v} \neq \mathbf{0}$ , define  $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$  and  $\mathbf{w} = \mathbf{u} - \lambda \mathbf{v}$ . Then  $\mathbf{v}, \mathbf{w}$  are orthogonal and  $\mathbf{u} = \mathbf{w} + \lambda \mathbf{v}$ .

*Proof.*

□

**Theorem (10.6, The Cauchy-Schwarz Inequality).** For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

*Proof.*

□

**Theorem (10.7, The Triangle Inequality).** For any two vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ ,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

**Exercise (3).** Show that for  $\mathbf{u} \in \mathbb{R}^n$ ,  $\alpha \in \mathbb{R}$ :

(a)  $\|\mathbf{u}\| = 0$  if and only if  $\mathbf{u} = \mathbf{0}$ .

(b)  $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$ .

*Proof.* □

**Exercise (4).** For  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  verify the identity

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle.$$

*Solution.* □

**Exercise (9).** Let  $\mathbf{u} \in \mathbb{R}^n$  and suppose  $\|\mathbf{u}\| < 1$ . Show that for  $\mathbf{v} \in \mathbb{R}^n$ ,  $\|\mathbf{v} - \mathbf{u}\| < 1 - \|\mathbf{u}\|$  implies  $\|\mathbf{v}\| < 1$ .

*Solution.* □

**Exercise (10).** Let  $\mathbf{u} \in \mathbb{R}^n$  and  $r > 0$ . Suppose  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  are at a distance less than  $r$  from  $\mathbf{u}$ . Prove that if  $0 \leq t \leq 1$ , then the point  $t\mathbf{v} + (1 - t)\mathbf{w}$  is also at a distance less than  $r$  from  $\mathbf{u}$ .

*Solution.* □

## 10.2 Convergence of Sequences in $\mathbb{R}^n$

**Theorem (10.9, The Componentwise Convergence Criterion).** Let  $\{\mathbf{u}_k\}$  be a sequence in  $\mathbb{R}^n$ . Then  $\{\mathbf{u}_k\}$  converges to  $\mathbf{u}$  if and only if  $\{p_i(\mathbf{u}_k)\}$  converges to  $p_i(\mathbf{u})$  for all  $1 \leq i \leq n$ .

*Proof.* □

**Theorem (10.10).** Let  $\{\mathbf{u}_k\}, \{\mathbf{v}_k\}$  be sequences in  $\mathbb{R}^n$  such that  $\{\mathbf{u}_k\}$  converges to  $\mathbf{u}$  and  $\{\mathbf{v}_k\}$  converges to  $\mathbf{v}$ . Then for any  $\alpha, \beta \in \mathbb{R}$ ,

$$\lim_{k \rightarrow \infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

*Proof.* □

**Exercise (1).** Let  $\{\mathbf{u}_k\}$  be a sequence in  $\mathbb{R}^n$  that converges to  $\mathbf{u}$ . Prove the following for all  $\mathbf{v} \in \mathbb{R}^n$ :

$$\lim_{k \rightarrow \infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

*Solution.* □

**Exercise (2).** Let  $\{\mathbf{u}_k\}$  be a sequence in  $\mathbb{R}^n$  and  $\mathbf{u} \in \mathbb{R}^n$ . Prove that if

$$\lim_{k \rightarrow \infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

holds for all  $\mathbf{v} \in \mathbb{R}^n$ , then  $\{\mathbf{u}_k\}$  converges to  $\mathbf{u}$ .

*Solution.* □

**Exercise (5).** Let  $\{\mathbf{u}_k\}$  be a sequence in  $\mathbb{R}^n$  that converges to  $\mathbf{u}$  where  $\|\mathbf{u}\| = r > 0$ . Prove that there is an index  $K$  where

$$\|\mathbf{u}_k\| > \frac{r}{2} \text{ if } k \geq K.$$

*Solution.* □

## 10.3 Open Sets and Closed Sets in $\mathbb{R}^n$

**Example (10.11).** Let  $a < b$  be in  $\mathbb{R}$ . Then  $\text{int}(a, b] = (a, b)$ .

*Proof.* □

**Example (10.12).** Let  $\mathbb{Q} \subseteq \mathbb{R}$  be the set of rational real numbers. Then  $\text{int } \mathbb{Q} = \emptyset$ .

*Proof.* □

**Proposition (10.13).** Every open ball  $B_r(\mathbf{u})$  in  $\mathbb{R}^n$  is open.

*Proof.* □

**Example (10.14).** Let  $a < b$  be in  $\mathbb{R}$ . Then  $[a, b]$  is closed.

*Proof.* □

**Example (10.15).** The set

$$[-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

is closed in  $\mathbb{R}^2$ .

*Proof.* □

**Theorem (10.16, The Complementing Characterization).** A subset  $A \subseteq \mathbb{R}^n$  is open if and only if its complement  $\mathbb{R}^n \setminus A$  is closed.

*Proof.* □

**Proposition (10.17.i).** The union of a collection of open subsets of  $\mathbb{R}^n$  is open.

*Proof.* □

**Proposition (10.17.ii).** The intersection of a collection of closed subsets of  $\mathbb{R}^n$  is closed.

*Proof.* □

**Proposition** (10.18.i). The intersection of a finite collection of open subsets of  $\mathbb{R}^n$  is open.

*Proof.* □

**Proposition** (10.18.ii). The union of a finite collection of closed subsets of  $\mathbb{R}^n$  is closed.

*Proof.* □

**Proposition** (10.19.i).  $A \subseteq \mathbb{R}^n$  is open if and only if  $A \cap \text{bd } A = \emptyset$ .

*Proof.* □

**Proposition** (10.19.ii).  $A \subseteq \mathbb{R}^n$  is closed if and only if  $\text{bd } A \subseteq A$ .

*Proof.* □

**Exercise** (2). Determine which of the following subsets of  $\mathbb{R}^2$  are open, closed, neither, or both.

(a)  $\{(x, y) : x^2 > y\}$

(b)  $\{(x, y) : x^2 + y^2 = 1\}$

(c)  $\{(x, y) : x \text{ is rational}\}$

(d)  $\{(x, y) : x \geq 0, y \geq 0\}$

*Solution.* (a)

(b)

(c)

(d)

□

**Exercise** (3). Let  $r > 0$  and  $O = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| > r\}$ . Prove that  $O$  is open.

*Solution.* □

**Exercise** (7a). Show that  $A \subseteq \mathbb{R}^n$  is open if and only if

$$\mathbf{w} + A = \{\mathbf{w} + \mathbf{u} : \mathbf{u} \in A\}$$

is open for all  $\mathbf{w} \in \mathbb{R}^n$ .

*Solution.* □

**Exercise** (12). For  $A \subseteq \mathbb{R}^n$ , denote its closure by

$$\text{cl } A = \text{int } A \cup \text{bd } A.$$

Prove that  $A \subseteq \text{cl } A$ . Then prove that  $A = \text{cl } A$  if and only if  $A$  is closed.

*Solution.* □

# Chapter 11

## Continuity, Compactness, and Connectedness

### 11.1 Continuous Functions and Mappings

**Proposition** (11.1). For each  $i \in \{1, \dots, n\}$ , the  $i$ th projection map  $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous.

*Proof.* □

**Theorem** (11.3). Let  $\mathbf{u} \in A \subseteq \mathbb{R}^n$  and  $h, g : A \rightarrow \mathbb{R}$  be continuous at  $\mathbf{u}$ . Then for  $\alpha, \beta \in \mathbb{R}$ , the following functions are continuous at  $\mathbf{u}$ :

$$\alpha h + \beta g : A \rightarrow \mathbb{R} \qquad h \cdot g : A \rightarrow \mathbb{R}.$$

Also if  $g(\mathbf{v}) \neq 0$  for all  $\mathbf{v} \in A$ , then the following function is also continuous at  $\mathbf{u}$ :

$$\frac{h}{g} : A \rightarrow \mathbb{R}.$$

*Proof.* □

**Theorem** (11.5). Let  $\mathbf{u} \in A \subseteq \mathbb{R}^n$  and  $G : A \rightarrow \mathbb{R}^m$  be continuous at  $\mathbf{u}$ . Also let  $G(A) \subseteq B \subseteq \mathbb{R}^m$  and  $H : B \rightarrow \mathbb{R}^k$  be continuous at  $G(\mathbf{u})$ . Then the composition

$$H \circ G : A \rightarrow \mathbb{R}^k$$

is continuous at  $\mathbf{u}$ .

*Proof.* □

**Theorem** (11.9, The Componentwise Continuity Criterion). Let  $\mathbf{u} \in A \subseteq \mathbb{R}^n$  and  $F : A \rightarrow \mathbb{R}^m$ . Then  $F$  is continuous at  $\mathbf{u}$  if and only if  $F_i = p_i \circ F : A \rightarrow \mathbb{R}$  is continuous at  $\mathbf{u}$  for each  $i \in \{1, \dots, m\}$ .

*Proof.* □

**Theorem** (11.11, Exercise 12). Let  $\mathbf{u} \in A \subseteq \mathbb{R}^n$  and  $F : A \rightarrow \mathbb{R}^m$ . Then  $F$  is continuous at  $\mathbf{u}$  if and only if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|\mathbf{v} - \mathbf{u}\| < \delta$  implies  $\|F(\mathbf{v}) - F(\mathbf{u})\| < \epsilon$ .

*Proof.* □

**Theorem** (11.12). Let  $U \subseteq \mathbb{R}^n$  be open and  $F : U \rightarrow \mathbb{R}^m$ . Then  $F$  is continuous if and only if  $F^{-1}(V)$  is an open subset of  $\mathbb{R}^n$  for every open  $V \subseteq \mathbb{R}^m$ .

*Proof.* □

**Example** (11.15). Use corollary 11.13 and proposition 10.18.i to prove that  $U = \{\mathbf{u} \in \mathbb{R}^n : a < \|\mathbf{u}\| < b\}$  is open. (You may assume  $f(\mathbf{u}) = \|\mathbf{u}\|$  is continuous.)

*Solution.* □

**Exercise** (3). Fix a point  $\mathbf{v} \in \mathbb{R}^n$ . Prove that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$  is continuous.

*Solution.* □

**Exercise** (6). Suppose  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous. Prove that  $\{\mathbf{u} \in \mathbb{R}^n : f(\mathbf{u}) = g(\mathbf{u}) = 0\}$  is closed. (Hint: use corollary 11.13 and proposition 10.17.ii.)

*Solution.* □

**Exercise** (11). Let  $A \subseteq \mathbb{R}^n$ . The characteristic function  $\phi_A : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $A$  is defined to be

$$\phi_A(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in A \\ 0 & \text{if } \mathbf{u} \notin A \end{cases}.$$

Prove that  $\phi_A$  is continuous at points in  $\text{int } A$  and  $\text{ext } A$ , but not continuous at points in  $\text{bd } A$ .

*Solution.* □

## 11.2 Sequential Compactness, Extreme Values, and Uniform Continuity

**Theorem** (11.16). Every sequentially compact subset of  $\mathbb{R}^n$  is bounded and closed.

*Proof.* □



**Theorem** (11.17). Every bounded sequence in  $\mathbb{R}^n$  has a convergent subsequence.

*Proof.* □

**Theorem** (11.18, The Sequential Compactness Theorem). A subset of  $\mathbb{R}^n$  is sequentially compact if and only if it is closed and bounded.

*Proof.* □

**Corollary** (11.19). The generalized rectangle  $\prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$  is sequentially compact.

*Proof.* □

**Theorem** (11.20). Let  $A \subseteq \mathbb{R}^n$  and suppose  $F : A \rightarrow \mathbb{R}^m$  is continuous. If  $A$  is sequentially compact then  $F(A)$  is also sequentially compact.

*Proof.* □

**Lemma** (11.21). Every nonempty sequentially compact subset of  $\mathbb{R}$  has a maximum and minimum element.

*Proof.* □

**Theorem** (11.22, The Extreme Value Theorem, Bolzano-Weierstrass Theorem). Let  $A \subseteq \mathbb{R}^n$  be nonempty sequentially compact and suppose  $f : A \rightarrow \mathbb{R}^m$  is continuous. Then  $f$  attains a smallest and largest value.

*Proof.* □

**Theorem** (11.24). Let  $A \subseteq \mathbb{R}^n$  be nonempty.  $A$  is sequentially compact if and only if every continuous  $f : A \rightarrow \mathbb{R}$  attains a smallest and largest value (i.e. it has the Extreme Value Property).

*Proof.* □

**Theorem** (11.25, Exercise 5). Let  $A \subseteq \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}^m$  be continuous. If  $A$  is sequentially compact then  $f$  is uniformly continuous.

*Proof.* □

**Theorem** (11.27, Exercise 11). Let  $A \subseteq \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}^m$ .  $f$  is uniformly continuous if and only if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|\mathbf{u} - \mathbf{v}\| < \delta$  implies  $\|f(\mathbf{u}) - f(\mathbf{v})\| < \epsilon$ .

*Proof.* □

**Exercise** (3,4). Recall that  $B_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| < r\}$ . Is  $B_r(\mathbf{u})$  bounded? Sequentially compact?

*Solution.*

□

**Exercise (2).** Let  $D_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| \leq r\}$ . Prove  $D_r(\mathbf{u})$  is sequentially compact.

*Solution.*

□

**Exercise** (optional, Heine-Borel Theorem). Let  $A \subseteq \mathbb{R}^n$ . We say  $A$  is compact when for every collection  $\mathcal{U}$  such that  $A$  is open for all  $U \in \mathcal{U}$  and  $A \subseteq \bigcup \mathcal{U}$ , there exists a finite subset  $\mathcal{F} \subseteq \mathcal{U}$  such that  $A \subseteq \bigcup \mathcal{F}$ . Prove that  $A$  is compact if and only if  $A$  is closed and bounded.

*Solution.*

□

# Chapter 12

## Metric Spaces

**Definition.** A pair  $(X, d)$  is called a *metric space* if  $X$  is a set and  $d$  is a function  $d : X^2 \rightarrow [0, \infty)$  satisfying the following properties:

- Identity:  $d(p, q) = 0$  if and only if  $p = q$ .
- Symmetry:  $d(p, q) = d(q, p)$ .
- Triangle Inequality:  $d(p, q) \leq d(p, w) + d(w, q)$ .

**Theorem (12.2).**  $dist(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$  is a metric on  $\mathbb{R}^n$ .

*Proof.*

□

**Definition.** Let  $(X, d)$  be a metric space. For  $p \in X, r > 0$ ,

$$B_r(p) = \{q \in X : d(p, q) < r\}$$

is the open ball about  $p$  with radius  $r$ . For  $A \subseteq X$ ,

- $\text{int } A = \{q \in A : \exists r > 0 (B_r(q) \subseteq A)\}$
- $\text{ext } A = \{q \in A : \exists r > 0 (B_r(q) \subseteq X \setminus A)\}$
- $\text{bd } A = \{q \in A : \forall r > 0 (B_r(q) \cap A \neq \emptyset \text{ and } B_r(q) \setminus A \neq \emptyset)\}$

Call  $A$  open in  $(X, d)$  if  $A = \text{int } A$ . Note that these concepts match the definitions we gave for  $\mathbb{R}^n$  using the metric  $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$ .

**Theorem (12.8).** Let  $(X, d)$  be a metric space. Let  $p \in X, r > 0$ . Then  $B_r(p)$  is open.

*Proof.*

□

**Definition.** Let  $d$  be a metric on  $\mathbb{R}^n$ . We say  $d$  is *compatible with the usual topology on  $\mathbb{R}^n$*  if the open sets determined by  $d$  are exactly the open sets determined by  $dist$ .

**Example.**  $s : \mathbb{R}^n \rightarrow [0, \infty)$  defined by  $s(\mathbf{u}, \mathbf{v}) = \max\{|p_i(\mathbf{v}) - p_i(\mathbf{u})| : 1 \leq i \leq n\}$  is a metric on  $\mathbb{R}^n$ .

*Proof.* □

**Theorem.**  $s$  is compatible with the usual topology on  $\mathbb{R}^n$ .

*Proof.* □

**Example.**  $t : \mathbb{R}^n \rightarrow [0, \infty)$  defined by  $t(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n |p_i(\mathbf{v}) - p_i(\mathbf{u})|$  is a metric on  $\mathbb{R}^n$ .

*Proof.* □

**Theorem.**  $t$  is compatible with the usual topology on  $\mathbb{R}^n$ .

*Proof.* □

**Definition.**  $d : X \rightarrow [0, \infty)$  defined by  $d(p, q) = 1$  for  $p \neq q$  and  $d(p, p) = 0$  is called a *discrete metric* on  $X$ .

**Theorem** (12.4). The discrete metric on  $X$  is a metric.

*Proof.* □

**Theorem.** The discrete metric on  $\mathbb{R}^n$  is not compatible with the usual topology on  $\mathbb{R}^n$ . (Hint: show that every subset of a discrete metric space is open.)

*Proof.* □

**Definition.** Let  $C([a, b], \mathbb{R})$  be the set of all continuous functions  $f : [a, b] \rightarrow \mathbb{R}$ , and for  $f, g \in C([a, b], \mathbb{R})$  let  $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$ .

**Theorem** (12.3).  $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$  is a metric on  $C([a, b], \mathbb{R})$ .

*Proof.* □

**Definition.** Let  $\{p_k\}$  denote a *sequence* in a metric space  $(X, d)$ , i.e. a function from  $\mathbb{N}$  to  $X$ .

**Definition.** We say the sequence  $\{p_k\}$  *converges* to  $p \in X$  when

$$\lim_{k \rightarrow \infty} d(p_k, p) = 0.$$

**Definition.**  $C \subseteq X$  is said to be *closed* in the metric space  $(X, d)$  when for every sequence  $\{p_k\}$  of points in  $C$  converging to  $p \in X$ , it follows that  $p \in C$ .

**Example** (12.11). The set  $\{f \in C([a, b], \mathbb{R}) : f(x) \geq 0\}$  is closed.

*Proof.* □

**Theorem** (12.12, The Complementing Characterization). Let  $(X, d)$  be a metric space and  $A \subseteq X$ . Then  $A$  is open in  $(X, d)$  if and only if  $X \setminus A$  is closed in  $(X, d)$ .

*Proof.*

□

**Definition.** A sequence  $\{p_k\}$  in a metric space  $(X, d)$  is called a *Cauchy sequence* when for each  $\epsilon > 0$  there is an  $N \in \mathbb{N}$  such that  $k, l \geq N$  implies  $d(p_k, p_l) < \epsilon$ .

**Proposition** (12.15). Every convergent sequence in a metric space is Cauchy.

*Proof.*

□

**Lemma** (9.3). Every Cauchy sequence in  $(\mathbb{R}, dist)$  is bounded.

*Proof.*

□

**Theorem** (9.4). A sequence in  $(\mathbb{R}, dist)$  is Cauchy if and only if it is convergent.

*Proof.*

□

**Corollary** (Example 12.16). A sequence in  $(\mathbb{R}^n, dist)$  is Cauchy if and only if it is convergent.

*Proof.*

□

**Definition.** A *complete metric space* is a metric space where every Cauchy sequence is convergent.



# Chapter 13

## Differentiating Functions of Several Variables

### 13.1 Limits

**Definition.** Let  $A \subseteq \mathbb{R}^n$ . We call  $\mathbf{x}_* \in \mathbb{R}^n$  a *limit point* of  $A$  in the case that there exists a sequence in  $A \setminus \{\mathbf{x}_*\}$  which converges to  $\mathbf{x}_*$ .

**Definition.** Let  $A \subseteq \mathbb{R}^n$  have a limit point  $\mathbf{x}_* \in \mathbb{R}^n$ , and  $f : A \rightarrow \mathbb{R}$  be a function. Then we say the *limit of  $f$  as  $\mathbf{x}$  approaches  $\mathbf{x}_*$*  is  $L \in \mathbb{R}$ , or

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_*} f(\mathbf{x}) = L$$

in the case that whenever  $\{\mathbf{x}_k\}$  is a sequence of points in  $A \setminus \{\mathbf{x}_*\}$  converging to  $\mathbf{x}_*$ , then  $\{f(\mathbf{x}_k)\}$  is a sequence of real numbers which converges to  $L$ .

**Theorem (13.3).** Let  $A \subseteq \mathbb{R}^n$  and  $\mathbf{x}_*$  be a limit point of  $A$ . Suppose the functions  $f, g : A \rightarrow \mathbb{R}$  satisfy

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_*} f(\mathbf{x}) = L_1 \quad \text{and} \quad \lim_{\mathbf{x} \rightarrow \mathbf{x}_*} g(\mathbf{x}) = L_2.$$

Then

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_*} [f(\mathbf{x}) + g(\mathbf{x})] = L_1 + L_2$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_*} [f(\mathbf{x})g(\mathbf{x})] = L_1L_2.$$

And assuming  $g(\mathbf{x}) \neq 0$  for  $\mathbf{x} \in A$  and  $L_2 \neq 0$ ,

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_*} [f(\mathbf{x})/g(\mathbf{x})] = L_1/L_2.$$

*Proof.*

□

**Example (13.4).** The limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}$$

does not exist.

*Proof.* □

**Example (13.5).**

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3}{x^2 + y^2} = 0.$$

*Proof.* □

**Exercise (4).** Let  $m, n \in \mathbb{N}$ . Prove that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^n y^m}{x^2 + y^2}$$

exists if and only if  $m + n > 2$ .

*Solution.* □

**Exercise (5).** Give an example of a subset  $A \subseteq \mathbb{R}$  and point  $x \in A$  such that  $x$  is not a limit point of  $A$ .

*Solution.* □

**Exercise (12).** Show that  $A \subseteq \mathbb{R}^n$  is closed if and only if it contains all its limit points.

*Solution.* □

## 13.2 Partial Derivatives

**Definition.** For each  $1 \leq i \leq n$ , let  $\mathbf{e}_i \in \mathbb{R}^n$  satisfy  $p_i(\mathbf{e}_i) = 1$  and  $p_j(\mathbf{e}_i) = 0$  for  $j \neq i$ .

**Definition.** Let  $\mathbf{x} \in U \subseteq \mathbb{R}^n$  with  $U$  open. For a function  $f : U \rightarrow \mathbb{R}$ , define its *first-order partial derivative with respect to its  $i$ th component at  $\mathbf{x}$*  to be

$$\left[ \frac{\partial}{\partial x_i} f \right] (\mathbf{x}) = \frac{\partial f}{\partial x_i} (\mathbf{x}) = f_{x_i} (\mathbf{x}) = \lim_{t \rightarrow 0} \frac{f(\mathbf{x} + t\mathbf{e}_i) - f(\mathbf{x})}{t}$$

whenever the limit exists.

**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open. For a function  $f : U \rightarrow \mathbb{R}$  such that  $f_{x_i}(\mathbf{x})$  exists for all  $\mathbf{x} \in U$ , let  $f_{x_i} : U \rightarrow \mathbb{R}^n$  be defined as its *first-order partial derivative with respect to its  $i$ th component*.



**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open. A function  $f : U \rightarrow \mathbb{R}$  such that  $f_{x_i}$  exists for all  $1 \leq i \leq n$  is said to have *first-order partial derivatives*.

**Example** (13.8\*). If  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is defined by

$$f(x, y, z) = xyz - 3xy^2$$

then  $f_y : \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfies

$$f_y(x, y, z) = xz - 6xy.$$

*Proof.* □

**Example** (13.9). The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

has first-order partial derivatives, but is not continuous.

*Proof.* □

**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open. Then a function  $f : U \rightarrow \mathbb{R}$  is *continuously differentiable* provided that  $f_{x_i} : U \rightarrow \mathbb{R}$  exists and is continuous for  $1 \leq i \leq n$ .

**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open, and  $f : U \rightarrow \mathbb{R}$  have first-order partial derivatives. Then for  $1 \leq i, j \leq n$  let

$$\frac{\partial^2 f}{\partial x_j \partial x_i} : U \rightarrow \mathbb{R}$$

be the partial derivative of  $\partial f / \partial x_i : U \rightarrow \mathbb{R}$  with respect to its  $j$ th component. This is also denoted by  $f_{x_i x_j}$ . When  $i = j$ , this is also denoted by  $\frac{\partial^2 f}{\partial x_i^2}$ .

**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open. A function  $f : U \rightarrow \mathbb{R}$  such that  $f_{x_i x_j}$  exists for all  $1 \leq i, j \leq n$  is said to have *second-order partial derivatives*.

**Definition.** Let  $U \subseteq \mathbb{R}^n$  be open. A function  $f : U \rightarrow \mathbb{R}$  such that  $f_{x_i x_j}$  exists and is continuous for all  $1 \leq i, j \leq n$  is said to have *continuous second-order partial derivatives*.

**Lemma** (13.11). Let  $U \subseteq \mathbb{R}^2$  be open and nonempty, and suppose  $f : U \rightarrow \mathbb{R}$  has second-order partial derivatives. Then there are points  $(x_1, y_1), (x_2, y_2) \in U$  such that  $f_{xy}(x_1, y_1) = f_{yx}(x_2, y_2)$ .

*Proof.* □

**Theorem** (13.10). Let  $U \subseteq \mathbb{R}^n$  be open and nonempty, and suppose  $f : U \rightarrow \mathbb{R}$  has continuous second-order partial derivatives. Then for all  $1 \leq i, j \leq n$ , it follows that  $f_{x_i x_j} = f_{x_j x_i}$ .

*Proof for  $n=2$ .* □

**Example** (13.12, exercise 13). The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

has second-order partial derivatives, but

$$f_{xy}(0, 0) = -1 \quad \text{while} \quad f_{yx}(0, 0) = 1.$$

*Proof.* □

**Exercise** (4). Prove that  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfying  $|g(x, y)| \leq x^2 + y^2$  must have partial derivatives with respect to both  $x$  and  $y$  at the point  $(0, 0)$ .

*Solution.* □

### 13.3 The Mean Value Theorem and Directional Derivatives

**Lemma** (13.14, The Mean Value Lemma). Let  $U \subseteq \mathbb{R}^n$  be open and  $1 \leq i \leq n$ . Let  $f : U \rightarrow \mathbb{R}$  be a function with a partial derivative with respect to its  $i$ th component at each point in  $U$ . Let  $\mathbf{x} \in U$  and  $a \in \mathbb{R}$  such that  $\mathbf{x} + \theta a \mathbf{e}_i \in U$  for all  $\theta \in [0, 1]$ . Then there is some  $\theta \in (0, 1)$  such that

$$f(\mathbf{x} + a \mathbf{e}_i) - f(\mathbf{x}) = a \frac{\partial f}{\partial x_i}(\mathbf{x} + \theta a \mathbf{e}_i).$$

*Proof.* □

**Proposition** (13.15, The Mean Value Proposition). Let  $\mathbf{x} \in \mathbb{R}^n$  and  $r > 0$ . Let  $f : B_r(\mathbf{x}) \rightarrow \mathbb{R}$  be a function with first-order partial derivatives. Then if  $\mathbf{h} \in \mathbb{R}^n$  satisfies  $\|\mathbf{h}\| < r$ , then there are points  $\mathbf{z}_i \in B_{\|\mathbf{h}\|}(\mathbf{x})$  for  $1 \leq i \leq n$  satisfying

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^n p_i(\mathbf{h}) \frac{\partial f}{\partial x_i}(\mathbf{z}_i).$$

*Proof.* □

**Definition.** Let  $\mathbf{x} \in U \subseteq \mathbb{R}^n$  where  $U$  is open, let  $f : U \rightarrow \mathbb{R}$  be a function, and let  $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ . Define the *directional derivative* of  $f$  at  $\mathbf{x}$  in the direction  $\mathbf{p}$  by

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \lim_{t \rightarrow 0} \frac{f(\mathbf{x} + t\mathbf{p}) - f(\mathbf{x})}{t}$$

whenever that limit exists.

**Definition.** Let  $\mathbf{x} \in U \subseteq \mathbb{R}^n$  where  $U$  is open, and let  $f : U \rightarrow \mathbb{R}$  be a function with first-order partial derivatives at  $\mathbf{x}$ . Define its *gradient*  $\nabla f(\mathbf{x}) \in \mathbb{R}^n$  at  $\mathbf{x}$  to satisfy  $p_i(\nabla f(\mathbf{x})) = f_{x_i}(\mathbf{x})$  for all  $1 \leq i \leq n$ . If its gradient exists at every  $\mathbf{x} \in U$ , then let  $\nabla f : U \rightarrow \mathbb{R}^n$  be the *gradient function* defined by evaluating the gradient at each point.

**Theorem** (13.16, The Directional Derivative Theorem). Let  $U \subseteq \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$  be continuously differentiable. Then for each  $\mathbf{x} \in U$  and  $\mathbf{p} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ ,  $f$  has a directional derivative at  $\mathbf{x}$  in the direction of  $\mathbf{p}$  given by

$$\frac{\partial f}{\partial \mathbf{p}}(\mathbf{x}) = \langle \mathbf{p}, \nabla f(\mathbf{x}) \rangle.$$

*Proof.* □

**Theorem** (13.17, The Mean Value Theorem). Let  $U \subseteq \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$  be continuously differentiable. Let  $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$  such that  $\mathbf{x} + \theta \mathbf{h} \in U$  for all  $\theta \in [0, 1]$ . Then there is some  $\theta \in (0, 1)$  such that

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \langle \mathbf{h}, \nabla f(\mathbf{x} + \theta \mathbf{h}) \rangle.$$

*Proof.* □

**Corollary** (13.18). Let  $\mathbf{x} \in U \subseteq \mathbb{R}^n$  with  $U$  open and let  $f : U \rightarrow \mathbb{R}$  be continuously differentiable. Then the unit vector maximizing the value of the directional derivative of  $f$  at  $\mathbf{x}$  is

$$\mathbf{p}_0 = \frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|}.$$

*Proof.* □

**Theorem** (13.20). Let  $U \subseteq \mathbb{R}^n$  be open and  $f : U \rightarrow \mathbb{R}$  be continuously differentiable. Then  $f$  is continuous.

*Proof.* □

**Exercise** (4). Suppose that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has first-order partial derivatives and that  $\mathbf{x}$  is a local minimizer for  $f$ , that is, there exists some  $\epsilon > 0$  such that for all  $\mathbf{y} \in B_\epsilon(\mathbf{x})$ ,  $f(\mathbf{y}) \geq f(\mathbf{x})$ . Prove that  $\nabla f(\mathbf{x}) = \mathbf{0}$ .

*Solution.* □



# Midterm Part 3

Choose two of the below problems (which you did not choose for Part 2) and typeset your solutions. Delete the other three. Each will be worth 20/100 points towards your midterm grade for a total of 40/100 points.

**Exercise (1).** Prove that if  $Q_n$  is a partition of  $[a, b]$  refining the partition  $P_n$  of  $[a, b]$  for each natural number  $n$ , and  $\{P_n\}$  is an Archimedian sequence of partitions for  $f$  on  $[a, b]$ , then  $\{Q_n\}$  is also Archimedian.

*Solution.* □

**Exercise (2).** Explain the error(s) in the following “proof”, and then give a counterexample showing that the theorem is false.

**Theorem:** If  $f : [0, 1] \rightarrow \mathbb{R}$  is integrable, then  $f$  is also continuous.

**Proof:** Since  $f$  is integrable, we may define  $F : [0, 1] \rightarrow \mathbb{R}$  by  $F(x) = \int_0^x f$ . It follows that  $F(x)$  is a differentiable function, because it is an antiderivative of  $f$ . Thus  $\frac{d}{dx}[F(x)] = f(x)$  by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude  $f$  is continuous.

*Solution.* □

**Exercise (3).** Recall that an **even** function satisfies the condition  $f(x) = f(-x)$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an even continuous function. Prove that

$$\frac{d}{dx} \left[ \int_{-x}^x f \right] = 2f(x).$$

(Hint: Corollary 6.30 says that  $\frac{d}{dx}[\int_x^0 f] = -f(x)$ .)

*Solution.* □

**Exercise (4).** Prove the following theorem:

Let  $\mathbf{x} \in \mathbb{R}^n$  and let  $\{\mathbf{x}_k\}$  be a sequence of points in  $\mathbb{R}^n$ . If for every open set  $U$  containing  $\mathbf{x}$ , there is an index  $K$  such that  $\mathbf{x}_k \in U$  for all  $k \geq K$ , then  $\{\mathbf{x}_k\}$  converges to  $\mathbf{x}$ .

(Hint:  $B_\epsilon(\mathbf{x})$  is open.)

*Solution.* □

**Exercise (5).** Prove that any finite subset of  $\mathbb{R}^n$  is closed.

(Hint: First prove that any singleton subset of  $\mathbb{R}^n$  is closed.)

*Solution.* □