MATH 3412 Notes — Spring 2016

Your Name Here
UNC Charlotte

Updated: December 15, 2015

This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

— Dr. Steven Clontz (sclontz5@uncc.edu)

Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1). Suppose that the function $f:[a,b]\to\mathbb{R}$ is bounded and the numbers m,M have the property that

$$m \le f(x) \le M$$

for all x in [a, b]. Then, if P is a partition of the domain [a, b],

$$m(b-a) \le L(f,P)$$
 and $U(f,P) \le M(b-a)$.

Proof.

Lemma (6.2, The Refinement Lemma). Suppose that the function $f:[a,b] \to \mathbb{R}$ is bounded and that P is a partition of its domain [a,b]. If P^* is a refinement of P, then

$$L(f,P) \leq L(f,P^\star) \text{ and } U(f,P^\star) \leq U(f,P).$$

Proof. \Box

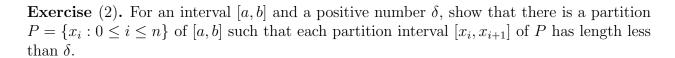
Lemma (6.3). Suppose that the function $f:[a,b]\to\mathbb{R}$ is bounded and that P_1,P_2 are partitions of its domain. Then $L(f,P_1)\leq U(f,P_2)$.

Proof.

Lemma (6.4). For a bounded function $f:[a,b] \to \mathbb{R}$,

$$\overline{\int_{a}^{b}} f \le \int_{a}^{b} f.$$

Proof.



 \square

Exercise (3). Suppose that the bounded function $f:[a,b] \to \mathbb{R}$ has the property that for each rational number x in the interval [a,b], f(x)=0. Prove that

$$\int_{\underline{a}}^{\underline{b}} f \le 0 \le \overline{\int_{\underline{a}}^{\underline{b}}} f.$$

 \Box

Exercise (6). Suppose that $f:[a,b] \to \mathbb{R}$ is a bounded function for which there is a partition P of [a,b] with L(f,P)=U(f,P). Prove that $f:[a,b]\to\mathbb{R}$ is constant.

Solution. \Box

6.2 The Archimedes-Riemann Theorem

Lemma (6.7).

Proof.

Theorem (6.8).

Proof. \Box