MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

Proof. □

Lemma (6.2, The Refinement Lemma)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If $P^\star$ is a refinement of P, then

$$L(f, P) \leq L(f, P^\star) \text{ and } U(f, P^\star) \leq U(f, P).$$

Proof. □

Lemma (6.3)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. □

Lemma (6.4)**.** For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. □

Exercise (2)**.** For an interval $[a, b]$ and a positive number δ, show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ.

Solution. □

Exercise (3)**.** Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

Solution. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7)**.** For a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P).$$

Proof. □

Theorem (6.8, The Archimedes-Riemann Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f = \lim_{n\to\infty} U(f, P_n).$$

Proof. □

Example (6.9)**.** Show that a monotonically increasing function $f : [a, b] \to \mathbb{R}$ is integrable.

Solution. □

Example (6.11)**.** Show that $\int_0^1 x^2\, dx = \frac{1}{3}$.

Solution. □

Exercise (4)**.** Prove that for a natural number n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x\,dx = (b^2 - a^2)/2$.

Solution. □

Exercise (6b)**.** Use the Archimedes-Riemann Theorem to show that for $0 \le a < b$,

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Solution. □

Exercise (9)**.** Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimediean sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity

Theorem (6.12, Additivity over Intervals)**.** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$, and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. □

Theorem (6.13, Monotonicity of the Integral)**.** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and that $f(x) \le g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \le \int_a^b g.$$

Proof. □

Lemma (6.14)**.** Let $f, g : [a, b] \to \mathbb{R}$ be bounded and let P partition $[a, b]$. Then

$$L(f, P) + L(g, P) \le L(f + g, P) \ \text{ and } \ U(f + g, P) \le U(f, P) + U(g, P).$$

Moreover, for any number α,

$$U(\alpha f, P) = \alpha U(f, P) \ \text{ and } \ L(\alpha f, P) = \alpha L(f, P) \ \text{ if } \alpha \ge 0$$
$$U(\alpha f, P) = \alpha L(f, P) \ \text{ and } \ L(\alpha f, P) = \alpha U(f, P) \ \text{ if } \alpha < 0.$$

Proof. □

Theorem (6.15, Linearity of the Integral)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then for any two numbers α, β, the function $\alpha f + \beta g : [a, b] \to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

Proof. □

Exercise (1)**.** Suppose that the functions f, g, f^2, g^2, fg are integrable on $[a, b]$. Prove that $(f - g)^2$ is also integrable on $[a, b]$ and that $\int_a^b (f - g)^2 \geq 0$. Use this to prove that

$$\int_a^b fg \leq \frac{1}{2}\left[\int_a^b f^2 + \int_a^b g^2\right].$$

Solution. □

Exercise (4)**.** Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x : x \in S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on both $[a, c], [c, b]$, then it is integrable on $[a, b]$.

Solution. □

6.4 Continuity and Integrability

Lemma (6.17)**.** Let the function $f : [a, b] \to \mathbb{R}$ be continuous let P partition its domain. Then there is a partition interval of P that contains two points u, v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

Proof. □

Theorem (6.18)**.** A continuous function on a closed bounded interval is integrable.

Proof. □

Theorem (6.19)**.** Supose $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b). Then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of $[a, b]$.

Proof. □

Exercise (1)**.** Determine whether each of the following statements is true or false, and justify your answer.

(a) If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is integrable, then f is continuous.

(c) If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.

(e) A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

Solution. (a)

(b)

(c)

(d)

(e)

□

Exercise (5)**.** Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all $x \in [a, b]$. Is this true if we only require integrability of the function?

Solution. □

Exercise (6)**.** Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all $x \in [0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point $x_0 \in [0, 1]$ at which $f(x_0) > 0$.

Solution. □

6.5 The First Fundamental Theorem: Integrating Derivatives

Lemma (6.21)**.** Suppose $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every P partitioning $[a, b]$,

$$L(f, P) \le A \le U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof. □

Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives)**.** Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Moreover, suppose that its derivative $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Proof. □

Exercise (1)**.** Let m, b be positive numbers. Find the value of $\int_0^1 mx + b\, dx$ in the following three ways:

(a) Using elementary geometry, interpreting the integral as an area.

(b) Using upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes-Riemann Theorem.

(c) Using the First Fundamental Theorem (Integrating Derivatives).

Solution. □

Exercise (5)**.** The monotonicity property of the integral implies that if the functions $g, h : [0, \infty) \to \mathbb{R}$ are continuous and $g(x) \le h(x)$ for all $x \ge 0$, then

$$\int_0^x g \le \int_0^x h \quad \text{for all } x \ge 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \le 1 \quad \text{if } x \ge 0.$$

$$\sin x \le x \quad \text{if } x \ge 0.$$

$$1 - \cos x \le \frac{x^2}{2} \quad \text{if } x \ge 0.$$

$$x - \sin x \leq \frac{x^3}{6} \quad \text{if } x \geq 0.$$
$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{if } x \geq 0.$$

(For this problem, you may assume that the sine and cosine functions are differentiable functions with the properties $\sin(0) = 0$, $\cos(0) = 1$, $\frac{d}{dx}[\sin(x)] = \cos(x)$, and $\frac{d}{dx}[\cos(x)] = -\sin(x)$.)

Solution. □

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem (6.26, The Mean Value Theorem for Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

Proof. □

Proposition (6.27)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function $F : [a, b] \to \mathbb{R}$ is continuous.

Proof. □

Theorem (6.29, The Second Fundamental Theorem: Differentiating Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x\right] = f(x) \quad \text{for all } x \in (a, b).$$

Proof. □

Exercise (2b)**.** Suppose $f : [0, 2] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t)\, dt \quad \text{for all} x \in [a, b]$$

and find a formula for $F(x)$ which does not involve integrals.

Solution. □

Exercise (5)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x-t)f(t)\,dt \quad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x.

Solution. □

Exercise (12)**.** Suppose that $f, g : [a,b] \to \mathbb{R}$ are continuous and that α, β are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a,b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all $x \in (a,b)$. Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

Solution. □

Chapter 10

The Euclidean Space $\mathbb{R}^n$

10.1 The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

Proposition (10.2)**.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then both of the following hold:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$\langle \alpha\mathbf{u} + \beta\mathbf{w}, v \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle$$

Proof. □

Lemma (10.4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Proof. □

Lemma (10.5)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} \neq \mathbf{0}$, define $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$ and $\mathbf{w} = \mathbf{u} - \lambda\mathbf{v}$. Then $\mathbf{v}, \mathbf{w}$ are orthogonal and $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$.

Proof. □

Theorem (10.6, The Cauchy-Schwarz Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Proof. □

Theorem (10.7, The Triangle Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Exercise (3)**.** Show that for $\mathbf{u} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(a) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(b) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

Proof. □

Exercise (4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ verify the identity

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (9)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and suppose $\|\mathbf{u}\| < 1$. Show that for $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v} - \mathbf{u}\| < 1 - \|\mathbf{u}\|$ implies $\|\mathbf{v}\| < 1$.

Solution. □

Exercise (10)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and $r > 0$. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are at a distance less than r from $\mathbf{u}$. Prove that if $0 \leq t \leq 1$, then the point $t\mathbf{v} + (1 - t)]\mathbf{w}$ is also at a distance less than r from $\mathbf{u}$.

Solution. □

10.2 Convergence of Sequences in $\mathbb{R}^n$

Theorem (10.9, The Componentwise Convergence Criterion)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$. Then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ if and only if $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$ for all $1 \leq i \leq n$.

Proof. □

Theorem (10.10)**.** Let $\{\mathbf{u}_k\}$, $\{\mathbf{v}_k\}$ be sequences in $\mathbb{R}^n$ such that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ and $\{\mathbf{v}_k\}$ converges to $\mathbf{v}$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\lim_{k\to\infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

Proof. □

Exercise (1)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$. Prove the following for all $\mathbf{v} \in \mathbb{R}^n$:

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (2)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^n$. Prove that if

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

holds for all $\mathbf{v} \in \mathbb{R}^n$, then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$.

Solution. □

Exercise (5)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$ where $\|\mathbf{u}\| = r > 0$. Prove that there is an index K where

$$\|\mathbf{u}_k\| > \frac{r}{2} \text{ if } k \geq K.$$

Solution. □

10.3 Open Sets and Closed Sets in $\mathbb{R}^n$

Example (10.11)**.** Let $a < b$ be in $\mathbb{R}$. Then $\operatorname{int}(a, b] = (a, b)$.

Proof. □

Example (10.12)**.** Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational real numbers. Then $\operatorname{int} \mathbb{Q} = \emptyset$.

Proof. □

Proposition (10.13)**.** Every open ball $B_r(\mathbf{u})$ in $\mathbb{R}^n$ is open.

Proof. □

Example (10.14)**.** Let $a < b$ be in $\mathbb{R}$. Then $[a, b]$ is closed.

Proof. □

Example (10.15)**.** The set

$$[-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

is closed in $\mathbb{R}^2$.

Proof. □

Theorem (10.16, The Complementing Characterization)**.** A subset $A \subseteq \mathbb{R}^n$ is open if and only if its complement $\mathbb{R}^n \setminus A$ is closed.

Proof. □

Proposition (10.17.i)**.** The union of a collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.17.ii)**.** The intersection of a collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.18.i). The intersection of a finite collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.18.ii). The union of a finite collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.19.i). $A \subseteq \mathbb{R}^n$ is open if and only if $A \cap \operatorname{bd} A = \emptyset$.

Proof. □

Proposition (10.19.ii). $A \subseteq \mathbb{R}^n$ is closed if and only if $\operatorname{bd} A \subseteq A$.

Proof. □

Exercise (2). Determine which of the following subsets of $\mathbb{R}^2$ are open, closed, neither, or both.

(a) $\{(x, y) : x^2 > y\}$

(b) $\{(x, y) : x^2 + y^2 = 1\}$

(c) $\{(x, y) : x \text{ is rational}\}$

(d) $\{(x, y) : x \geq 0,\ y \geq 0\}$

Solution. (a)

(b)

(c)

(d)

□

Exercise (3). Let $r > 0$ and $O = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| > r\}$. Prove that O is open.

Solution. □

Exercise (7a). Show that $A \subseteq \mathbb{R}^n$ is open if and only if

$$\mathbf{w} + A = \{\mathbf{w} + \mathbf{u} : \mathbf{u} \in A\}$$

is open for all $\mathbf{w} \in \mathbb{R}^n$.

Solution. □

Exercise (12). For $A \subseteq \mathbb{R}^n$, denote its closure by

$$\operatorname{cl} A = \operatorname{int} A \cup \operatorname{bd} A.$$

Prove that $A \subseteq \operatorname{cl} A$. Then prove that $A = \operatorname{cl} A$ if and only if A is closed.

Solution. □

Chapter 11

Continuity, Compactness, and Connectedness

11.1 Continuous Functions and Mappings

Proposition (11.1)**.** For each $i \in \{1, \dots, n\}$, the ith projection map $p_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof. $\square$

Theorem (11.3)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $h, g : A \to \mathbb{R}$ be continuous at $\mathbf{u}$. Then for $\alpha, \beta \in \mathbb{R}$, the following functions are continuous at $\mathbf{u}$:

$$\alpha h + \beta g : A \to \mathbb{R} \qquad\qquad h \cdot g : A \to \mathbb{R}.$$

Also if $g(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in A$, then the following function is also continuous at $\mathbf{u}$:

$$\frac{h}{g} : A \to \mathbb{R}.$$

Proof. $\square$

Theorem (11.5)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $G : A \to \mathbb{R}^m$ be continuous at $\mathbf{u}$. Also let $G(A) \subseteq B \subseteq \mathbb{R}^m$ and $H : B \to \mathbb{R}^k$ be continuous at $G(\mathbf{u})$. Then the composition

$$H \circ G : A \to \mathbb{R}^k$$

is continuous at $\mathbf{u}$.

Proof. $\square$

Theorem (11.9, The Componentwise Continuity Criterion)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F is continuous at $\mathbf{u}$ if and only if $F_i = p_i \circ F : A \to \mathbb{R}$ is continuous at $\mathbf{u}$ for each $i \in \{1, \dots, n\}$.

Proof. □

Theorem (11.11, Exercise 12)**.** Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \to \mathbb{R}^m$. Then F is continuous at $\mathbf{u}$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{v} - \mathbf{u}\| < \delta$ implies $\|F(\mathbf{v}) - F(\mathbf{u})\| < \epsilon$.

Proof. □

Theorem (11.12)**.** Let $U \subseteq \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$. Then F is continuous if and only if $F^{-1}(V)$ is an open subset of $\mathbb{R}^n$ for every open $V \subseteq \mathbb{R}^m$.

Proof. □

Example (11.15)**.** Use corollary 11.13 and proposition 10.18.i to prove that $U = \{\mathbf{u} \in \mathbb{R}^n : a < \|\mathbf{u}\| < b\}$ is open. (You may assume $f(\mathbf{u}) = \|\mathbf{u}\|$ is continuous.)

Solution. □

Exercise (3)**.** Fix a point $\mathbf{v} \in \mathbb{R}^n$. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is continuous.

Solution. □

Exercise (6)**.** Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous. Prove that $\{\mathbf{u} \in \mathbb{R}^n : f(\mathbf{u}) = g(\mathbf{u}) = 0\}$ is closed. (Hint: use corollary 11.13 and proposition 10.17.ii.)

Solution. □

Exercise (11)**.** Let $A \subseteq \mathbb{R}^n$. The characteristic function $\phi_A : \mathbb{R}^n \to \mathbb{R}$ for A is defined to be

$$\phi_A(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in A \\ 0 & \text{if } \mathbf{u} \notin A \end{cases}.$$

Prove that ϕ_A is continuous at points in $\operatorname{int} A$ and $\operatorname{ext} A$, but not continuous at points in $\operatorname{bd} A$.

11.2 Sequential Compactness, Extreme Values, and Uniform Continuity

Theorem (11.16)**.** Every sequentially compact subset of $\mathbb{R}^n$ is bounded and closed.

Proof. □

Theorem (11.17)**.** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Proof. □

Theorem (11.18, The Sequential Compactness Theorem)**.** A subset of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded.

Proof. □

Corollary (11.19)**.** The generalized rectangle $\prod_{i=1}^{n}[a_i, b_i] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is sequentially compact.

Proof. □

Theorem (11.20)**.** Let $A \subseteq \mathbb{R}^n$ and suppose $F : A \to \mathbb{R}^m$ is continuous. If A is sequentially compact then $F(A)$ is also sequentially compact.

Proof. □

Lemma (11.21)**.** Every nonempty seqeuntially compact subset of $\mathbb{R}$ has a maximum and minimum element.

Proof. □

Theorem (11.22, The Extreme Value Theorem, Bolzano-Weierstrass Theorem)**.** Let $A \subseteq \mathbb{R}^n$ be nonempty sequentially compact and suppose $f : A \to \mathbb{R}^m$ is continuous. Then f attains a smallest and largest value.

Proof. □

Theorem (11.24)**.** Let $A \subseteq \mathbb{R}^n$ be nonempty. A is sequentially compact if and only if every continuous $f : A \to \mathbb{R}$ attains a smallest and largest value (i.e. it has the Extreme Value Property).

Proof. □

Theorem (11.25, Exercise 5)**.** Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be continuous. If A is sequentially compact then f is uniformly continuous.

Proof. □

Theorem (11.27, Exercise 11)**.** Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. f is uniformly continuous if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{u} - \mathbf{v}\| < \delta$ implies $\|f(\mathbf{u}) - f(\mathbf{v})\| < \epsilon$.

Proof. □

Exercise (3,4)**.** Recall that $B_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| < r\}$. Is $B_r(\mathbf{u})$ bounded? Sequentially compact?

Solution. □

Exercise (2)**.** Let $D_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| \leq r\}$. Prove $D_r(\mathbf{u})$ is sequentially compact.

Solution. $\square$

Exercise (optional, Heine-Borel Theorem)**.** Let $A \subseteq \mathbb{R}^n$. We say A is compact when for every collection $\mathcal{U}$ such that U is open for all $U \in \mathcal{U}$ and $A \subseteq \bigcup \mathcal{U}$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup \mathcal{F}$. Prove that A is compact if and only if A is closed and bounded.

Solution. $\square$

Chapter 12

Metric Spaces

Definition. A pair (X, d) is called a *metric space* if X is a set and d is a function $d : X^2 \to [0, \infty)$ satisfying the following properties:

- Identity: $d(p, q) = 0$ if and only if $p = q$.
- Symmetry: $d(p, q) = d(q, p)$.
- Triangle Inequality: $d(p, q) \leq d(p, w) + d(w, q)$.

Theorem (12.2)**.** $dist(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$ is a metric on $\mathbb{R}^n$.

Proof. □

Definition. Let (X, d) be a metric space. For $p \in X, r > 0$,

$$B_r(p) = \{q \in X : d(p, q) < r\}$$

is the open ball about p with radius r. For $A \subseteq X$,

- $\operatorname{int} A = \{q \in A : \exists r > 0(B_r(q) \subseteq A)\}$
- $\operatorname{ext} A = \{q \in A : \exists r > 0(B_r(q) \subseteq X \setminus A)\}$
- $\operatorname{bd} A = \{q \in A : \forall r > 0(B_r(q) \cap A \neq \emptyset \text{ and } B_r(q) \setminus A \neq \emptyset)\}$

Call A open in (X, d) if $A = \operatorname{int} A$. Note that these concepts match the definitions we gave for $\mathbb{R}^n$ using the metric $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{q} - \mathbf{p}\|$.

Theorem (12.8)**.** Let (X, d) be a metric space. Let $p \in X, r > 0$. Then $B_r(p)$ is open.

Proof. □

Definition. Let d be a metric on $\mathbb{R}^n$. We say d is *compatible with the usual topology on* $\mathbb{R}^n$ if the open sets determined by d are exactly the open sets determined by $dist$.

Example. $s : \mathbb{R}^n \to [0, \infty)$ defined by $s(\mathbf{u}, \mathbf{v}) = \max\{|p_i(\mathbf{v}) - p_i(\mathbf{u})| : 1 \leq i \leq n\}$ is a metric on $\mathbb{R}^n$.

Proof. □

Theorem. s is compatible with the usual topology on $\mathbb{R}^n$.

Proof. □

Example. $t : \mathbb{R}^n \to [0, \infty)$ defined by $t(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n |p_i(\mathbf{v}) - p_i(\mathbf{u})|$ is a metric on $\mathbb{R}^n$.

Proof. □

Theorem. t is compatible with the usual topology on $\mathbb{R}^n$.

Proof. □

Definition. $d : X \to [0, \infty)$ defined by $d(p, q) = 1$ for $p \neq q$ and $d(p, p) = 0$ is called a *discrete metric* on X.

Theorem (12.4)**.** The discrete metric on X is a metric.

Proof. □

Theorem. The discrete metric on $\mathbb{R}^n$ is not compatible with the usual topology on $\mathbb{R}^n$. (Hint: show that every subset of a discrete metric space is open.)

Proof. □

Definition. Let $C([a, b], \mathbb{R})$ be the set of all continuous functions $f : [a, b] \to \mathbb{R}$, and for $f, g \in C([a, b], \mathbb{R})$ let $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$.

Theorem (12.3)**.** $d(f, g) = \max\{|f(x) - g(x)| : x \in [a, b]\}$ is a metric on $C([a, b], \mathbb{R})$.

Proof. □

Definition. Let $\{p_k\}$ denote a *sequence* in a metric space (X, d), i.e. a function from $\mathbb{N}$ to X.

Definition. We say the sequence $\{p_k\}$ *converges* to $p \in X$ when

$$\lim_{k \to \infty} d(p_k, p) = 0.$$

Definition. $C \subseteq X$ is said to be *closed* in the metric space (X, d) when for every sequence $\{p_k\}$ of points in C converging to $p \in X$, it follows that $p \in C$.

Example (12.11)**.** The set $\{f \in C([a, b], \mathbb{R}) : f(x) \geq 0\}$ is closed.

Proof. □

Theorem (12.12, The Complementing Characterization)**.** Let (X, d) be a metric space and $A \subseteq X$. Then A is open in (X, d) if and only if $X \setminus A$ is closed in (X, d).

Proof. $\square$

Definition. A sequence $\{p_k\}$ in a metric space (X, d) is called a *Cauchy sequence* when for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $k, l \geq N$ implies $d(p_k, p_l) < \epsilon$.

Proposition (12.15)**.** Every convergent sequence in a metric space is Cauchy.

Proof. $\square$

Lemma (9.3)**.** Every Cauchy sequence in $(\mathbb{R}, dist)$ is bounded.

Proof. $\square$

Theorem (9.4)**.** A sequence in $(\mathbb{R}, dist)$ is Cauchy if and only if it is convergent.

Proof. $\square$

Corollary (Example 12.16)**.** A sequence in $(\mathbb{R}^n, dist)$ is Cauchy if and only if it is convergent.

Proof. $\square$

Definition. A *complete metric space* is a metric space where every Cauchy sequence is convergent.

Chapter 13

Differentiating Functions of Several Variables

13.1 Limits

Definition. Let $A \subseteq \mathbb{R}^n$. We call $\mathbf{x}_* \in \mathbb{R}^n$ a *limit point* of A in the case that there exists a sequence in $A \setminus \{\mathbf{x}_*\}$ which converges to $\mathbf{x}_*$.

Definition. Let $A \subseteq \mathbb{R}^n$ have a limit point $x_* \in \mathbb{R}$, and $f : A \to \mathbb{R}$ be a function. Then we say the *limit of f as $\mathbf{x}$ approaches $\mathbf{x}_*$* is $L \in \mathbb{R}$, or

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = L$$

in the case that whenever $\{\mathbf{x}_k\}$ is a sequence of points in $A \setminus \{\mathbf{x}_*\}$ converging to $\mathbf{x}_*$, then $\{f(\mathbf{x}_k)\}$ is a sequence of real numbers which converges to L.

Theorem (13.3)**.** Let$A \subseteq \mathbb{R}^n$ and $\mathbf{x}_*$ be a limit point of A. Suppose the functions $f, g : A \to \mathbb{R}$ satsify

$$\lim_{\mathbf{x} \to \mathbf{x}_*} f(\mathbf{x}) = L_1 \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{x}_*} g(\mathbf{x}) = L_2.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x}) + g(\mathbf{x})] = L_1 + L_2$$

and

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x})g(\mathbf{x})] = L_1 L_2.$$

And assuming $g(\mathbf{x}) \neq 0$ for $x \in A$ and $L_2 \neq 0$,

$$\lim_{\mathbf{x} \to \mathbf{x}_*} [f(\mathbf{x})/g(\mathbf{x})] = L_1/L_2.$$

Proof. □

Example (13.4)**.** The limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not exist.

Proof. □

Example (13.5)**.**

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0.$$

Proof. □

Exercise (4)**.** Let $m, n \in \mathbb{N}$. Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^n y^m}{x^2+y^2}$$

exists if and only if $m + n > 2$.

Solution. □

Exercise (5)**.** Give an example of a subset $A \subseteq \mathbb{R}$ and point $x \in A$ such that x is not a limit point of A.

Solution. □

Exercise (12)**.** Show that $A \subseteq \mathbb{R}^n$ is closed if and only if it contains all its limit points.

Solution. □

13.2 Partial Derivatives

Definition. For each $1 \leq i \leq n$, let $\mathbf{e}_i \in \mathbb{R}^n$ satisfy $p_i(\mathbf{e}_i) = 1$ and $p_j(\mathbf{e}_i) = 0$ for $j \neq i$.

Definition. Let $\mathbf{x} \in U \subseteq \mathbb{R}^n$ with U open. For a function $f : U \to \mathbb{R}$, define its *first-order partial derivative with respect to its ith component at* $\mathbf{x}$ to be

$$\left[\frac{\partial}{\partial x_i} f\right](\mathbf{x}) = \frac{\partial f}{\partial x_i}(\mathbf{x}) = f_{x_i}(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{e}_i) - f(\mathbf{x})}{t}$$

whenever the limit exists.

Definition. Let $U \subseteq \mathbb{R}^n$ be open. For a function $f : U \to \mathbb{R}$ such that $f_{x_i}(\mathbf{x})$ exists for all $\mathbf{x} \in U$, let $f_{x_i} : U \to \mathbb{R}^n$ be defined as its *first-order partial derivative with respect to its ith component.*

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that f_{x_i} exists for all $1 \leq i \leq n$ is said to have *first-order partial derivatives.*

Example (13.8*)**.** If $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$f(x, y, z) = xyz - 3xy^2$$

then $f_y : \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$f_y(x, y, z) = xz - 6xy.$$

Proof. □

Example (13.9)**.** The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

has first-order partial derivatives, but is not continuous.

Proof. □

Definition. Let $U \subseteq \mathbb{R}^n$ be open. Then a function $f : U \to \mathbb{R}$ is *continuously differentiable* provided that $f_{x_i} : U \to \mathbb{R}$ exists and is continuous for $1 \leq i \leq n$.

Definition. Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ have first-order partial derivatives. Then for $1 \leq i, j \leq n$ let

$$\frac{\partial^2 f}{\partial x_j \partial x_i} : U \to \mathbb{R}$$

be the partial derivative of $\partial f / \partial x_i : U \to \mathbb{R}$ with respect to its jth component. This is also denoted by $f_{x_i x_j}$. When $i = j$, this is also denoted by $\frac{\partial^2 f}{\partial x_i^2}$.

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that $f_{x_i x_j}$ exists for all $1 \leq i, j \leq n$ is said to have *second-order partial derivatives.*

Definition. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}$ such that $f_{x_i x_j}$ exists and is continuous for all $1 \leq i, j \leq n$ is said to have *continuous second-order partial derivatives.*

Lemma (13.11)**.** Let $U \subseteq \mathbb{R}^2$ be open and nonempty, and suppose $f : U \to \mathbb{R}$ has second-order partial derivatives. Then there are points $(x_1, y_1), (x_2, y_2) \in U$ such that $f_{xy}(x_1, y_1) = f_{yx}(x_2, y_2)$.

Proof. □

Theorem (13.10)**.** Let $U \subseteq \mathbb{R}^n$ be open and nonempty, and suppose $f : U \to \mathbb{R}$ has continuous second-order partial derivatives. Then for all $1 \leq i, j \leq n$, it follows that $f_{x_i x_j} = f_{x_j x_i}$.

Proof for n=2. □

Example (13.12, exercise 13)**.** The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

has second-order partial derivatives, but

$$f_{xy}(0,0) = -1 \quad \text{while} \quad f_{yx}(0,0) = 1.$$

Proof. □

Exercise (4)**.** Prove that $g : \mathbb{R}^2 \to \mathbb{R}$ satisfying $|g(x,y)| \leq x^2 + y^2$ must have partial derivatives with respect to both x and y at the point $(0,0)$.

Solution. □

Midterm Part 3

Choose two of the below problems (which you did not choose for Part 2) and typeset your solutions. Delete the other three. Each will be worth 20/100 points towards your midterm grade for a total of 40/100 points.

Exercise (1)**.** Prove that if Q_n is a partition of $[a, b]$ refining the partition P_n of $[a, b]$ for each natural number n, and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$, then $\{Q_n\}$ is also Archimedean.

Solution. □

Exercise (2)**.** Explain the error(s) in the following "proof", and then give a counterexample showing that the theorem is false.

Theorem: If $f : [0, 1] \to \mathbb{R}$ is integrable, then f is also continuous.

Proof: Since f is integrable, we may define $F : [0, 1] \to \mathbb{R}$ by $F(x) = \int_0^x f$. It follows that $F(x)$ is a differentiable function, because it is an antiderivative of f. Thus $\frac{d}{dx}[F(x)] = f(x)$ by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude f is continuous.

Solution. □

Exercise (3)**.** Recall that an **even** function satisfies the condition $f(x) = f(-x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be an even continuous function. Prove that

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = 2f(x).$$

(Hint: Corollary 6.30 says that $\frac{d}{dx}[\int_x^0 f] = -f(x)$.)

Solution. □

Exercise (4)**.** Prove the following theorem:

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{\mathbf{x}_k\}$ be a sequence of points in $\mathbb{R}^n$. If for every open set U containing $\mathbf{x}$, there is an index K such that $\mathbf{x}_k \in U$ for all $k \geq K$, then $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$.

(Hint: $B_\epsilon(\mathbf{x})$ is open.)

Solution. □

Exercise (5)**.** Prove that any finite subset of $\mathbb{R}^n$ is closed.

(Hint: First prove that any singleton subset of $\mathbb{R}^n$ is closed.)

Solution. □