MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Definition. Let n be a natural number. We define $[n] = \{1, 2, ..., n\}$. If i is an index, we write "for $i \in [n]$" in place of the usual "for $1 \leq i \leq n$."

Lemma (6.1)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition on $[a, b]$. By assumption, m is a lower bound of $f([a, b])$. Restricting f to $[x_{i-1}, x_i]$, we have $m \leq m_i$ for all $i \in [n]$ since m_i is the infimum of $f([x_{i-1}, x_i])$. Then, by definition,

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{n} m(x_i - x_{i-1}) \\
&= m\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= m(b - a).
\end{aligned}$$

Similarly, M is an upper bound of $f([a,b])$ and so, when restricting f to $[x_{i-1}, x_i]$, we have $M \geq M_i$ since M_i is the supremum of $f([x_{i-1}, x_i])$ (for all $i \in [n]$). Hence, we have

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} M(x_i - x_{i-1}) \\ &= M\sum_{i=1}^{n}(x_i - x_{i-1}) \\ &= M(b-a). \end{aligned}$$

Therefore, $m(b-a) \leq L(f,P)$ and $U(f,P) \leq M(b-a)$. □

Lemma (6.2, The Refinement Lemma)**.** Suppose that the function $f : [a,b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a,b]$. If $P^\star$ is a refinement of P, then

$$L(f,P) \leq L(f,P^\star) \text{ and } U(f,P^\star) \leq U(f,P).$$

Proof. Let $P = \{x_0, x_1, ..., x_n\}$ be a partition on $[a,b]$, and let P^* be its refinement. For $i \in [n]$, define P_i to be the partition on $[x_{i-1}, x_i]$ by the points of P^* inside this interval. Since $m_i \leq f(x)$ for $x \in [x_{i-1}, x_i]$, applying the previous lemma to the restriction of f on $[x_{i-1}, x_i]$, we have $m_i(x_i - x_{i-1}) \leq L(f, P_i)$. It follows that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} L(f,P_i) \\ &= L(f,P^*). \end{aligned}$$

Likewise, the previous lemma gives us $M_i(x_i - x_{i-1}) \geq U(f,P_i)$. Hence,

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{n} U(f,P_i) \\ &= U(f,P^*). \end{aligned}$$

□

Lemma (6.3)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. Let $P = P_1 \cup P_2$ be the common refinement of partitions P_1 and P_2. By the Refinement Lemma, $L(f, P_1) \leq L(f, P)$ and $U(f, P) \leq U(f, P_2)$. Then, since $L(f, P) \leq U(f, P)$, the transitivity of $\leq$ implies that $L(f, P_1) \leq U(f, P_2)$. $\square$

Lemma (6.4)**.** For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. Let P be any partition on $[a, b]$. By the previous lemma, $U(f, P) \geq L(f, P')$ for all partitions P' on $[a, b]$. It follows that

$$\underline{\int_a^b} f \leq U(f, P).$$

Since P was arbitrary, the above shows that $\underline{\int_a^b} f$ is a lower bound for all such $U(f, P)$. Therefore,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

$\square$

Exercise (2)**.** For an interval $[a, b]$ and a positive number δ, show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ.

Solution. Let $[a, b]$ be an interval ($b > a$) and $\delta > 0$. By the Archimedean property, there exists a natural number n such that $\frac{\delta}{b-a} > \frac{1}{n}$. It follows that we can form partition intervals of equal length $\frac{b-a}{n}$:

$$\begin{aligned} \delta &> \frac{b-a}{n} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \\ &= \frac{1}{n} [n(x_{i+1} - x_i)] \\ &= x_{i+1} - x_i \end{aligned}$$

$\square$

Exercise (3)**.** Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

Solution. Let $P = \{x_0, x_1, ..., x_n\}$ be an arbitrary partition on $[a, b]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $m_i \leq 0$ and $M_i \geq 0$ for all $i \in [n]$. This implies $L(f, P) \leq 0$ and $U(f, P) \geq 0$. Consequently,

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

□

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

Solution. Let P be the partition where $L(f, P) = U(f, P)$. Then

$$0 = U(f, P) - L(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}).$$

Since $x_i > x_{i-1}$, $(x_i - x_{i-1}) > 0$. Similarly, $(M_i - m_i) \geq 0$. This implies that the term $(M_i - m_i)(x_i - x_{i-1})$ is nonnegative, but since the entire sum is zero, we must have $M_i = m_i$ for all $i \in [n]$. It follows that f takes the same value within each partition interval, and since $[x_{i-1}, x_i] \cap [x_i, x_{i+1}] = \{x_i\}$, f takes the same value for all of $[a, b]$. Therefore, f is constant. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7)**.** For a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P).$$

Proof. By definition, $\overline{\int_a^b} f \leq U(f, P)$ and $L(f, P) \leq \underline{\int_a^b} f$. Then, by Lemma 6.4 we have $\underline{\int_a^b} f \leq \overline{\int_a^b} f$. The result follows. □

Theorem (6.8, The Archimedes-Riemann Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f = \lim_{n\to\infty} U(f, P_n).$$

Proof. □

Example (6.9)**.** Show that a monotonically increasing function $f : [a, b] \to \mathbb{R}$ is integrable.

Solution. Let P_n be the regular partition on $[a, b]$. Since f is monotonically increasing, on a partition interval $[x_{i-1}, x_i]$, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Then

$$\begin{aligned}
\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{i=1}^{n} m_i(x_i - x_{i-1})\right] \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})\right] \\
&= \lim_{n\to\infty}\left[\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\frac{b-a}{n}\right] \\
&= \lim_{n\to\infty}\frac{b-a}{n}\left[\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))\right] \\
&= \lim_{n\to\infty}\frac{b-a}{n}(f(b) - f(a)) \\
&= 0.
\end{aligned}$$

Therefore, by Theorem 6.8, f is integrable on $[a, b]$. □

Example (6.11)**.** Show that $\int_0^1 x^2\, dx = \frac{1}{3}$.

Solution. Since $f(x) = x^2$ is monotonically increasing on $[0, 1]$, f is integrable by the above example. Let $P_n = \{x_0, x_1, ..., x_n\}$ be the regular partition on $[0, 1]$. Then $x_i = \frac{i}{n}$ and using the fact that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$, we get

$$\begin{aligned}
\int_0^1 x^2, dx &= \lim_{n\to\infty} U(f, P_n) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\frac{i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \frac{2n^2+3n+1}{6n^2} \\
&= \frac{1}{3}.
\end{aligned}$$

□

Exercise (4)**.** Prove that for a natural number n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x\,dx = (b^2 - a^2)/2$.

Solution. First, we prove the summation holds by induction on n. If $n = 1$, $\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$. Assume this identity holds for all natural numbers $k \leq n$ and now consider $n+1$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n^2+3n+2}{2} \\
&= \frac{(n+1)((n+1)+1)}{2}
\end{aligned}$$

and hence the induction is complete.

We note that $f(x) = x$ is monotonically increasing on $\mathbb{R}$ and consequently integrable. Thus, for a regular partition P_n on $[a, b]$, we have

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty} U(f, P) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} x_i \frac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \\
&= \lim_{n\to\infty} \frac{b-a}{n}\left[\sum_{i=1}^{n} a + \frac{b-a}{n}\sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \frac{b-a}{n}\left[na + \frac{b-a}{n} \cdot \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} [(ab - a^2) + \frac{(b-a)^2(n+1)}{2n}] \\
&= ab - a^2 + \frac{(b-a)^2}{2} \\
&= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2} \\
&= \frac{b^2 - a^2}{2}.
\end{aligned}$$

□

Exercise (6b)**.** Use the Archimedes-Riemann Theorem to show that for $0 \le a < b$,

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Solution. □

Exercise (9)**.** Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedian sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity

Theorem (6.12, Additivity over Intervals)**.** Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$, and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. □

Theorem (6.13, Monotonicity of the Integral)**.** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. □

Lemma (6.14)**.** Let $f, g : [a, b] \to \mathbb{R}$ be bounded and let P partition $[a, b]$. Then

$$L(f, P) + L(g, P) \leq L(f + g, P) \quad \text{and} \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Moreover, for any number α,

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P) \quad \text{if } \alpha \geq 0$$

$$U(\alpha f, P) = \alpha L(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha U(f, P) \quad \text{if } \alpha < 0.$$

Proof. □

Theorem (6.15, Linearity of the Integral)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then for any two numbers α, β, the function $\alpha f + \beta g : [a, b] \to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

Proof. □

Exercise (1)**.** Suppose that the functions f, g, f^2, g^2, fg are integrable on $[a, b]$. Prove that $(f - g)^2$ is also integrable on $[a, b]$ and that $\int_a^b (f - g)^2 \geq 0$. Use this to prove that

$$\int_a^b fg \leq \frac{1}{2} \left[\int_a^b f^2 + \int_a^b g^2 \right].$$

Solution. □

Exercise (4)**.** Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x : x \in S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \sup \alpha S = \alpha \inf S \quad \text{if } \alpha < 0.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on both $[a, c], [c, b]$, then it is integrable on $[a, b]$.

Solution. □

6.4 Continuity and Integrability

Lemma (6.17)**.** Let the function $f : [a, b] \to \mathbb{R}$ be continuous let P partition its domain. Then there is a partition interval of P that contains two points u, v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

Proof. □

Theorem (6.18)**.** A continuous function on a closed bounded interval is integrable.

Proof. □

Theorem (6.19)**.** Supose $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b). Then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of $[a, b]$.

Proof. □

Exercise (1)**.** Determine whether each of the following statements is true or false, and justify your answer.

(a) If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is integrable, then f is continuous.

(c) If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.

(e) A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

Solution. (a)

(b)

(c)

(d)

(e)

□

Exercise (5)**.** Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all $x \in [a, b]$. Is this true if we only require integrability of the function?

Solution. □

Exercise (6)**.** Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all $x \in [0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point $x_0 \in [0, 1]$ at which $f(x_0) > 0$.

Solution. □

6.5 The First Fundamental Theorem: Integrating Derivatives

Lemma (6.21)**.** Suppose $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every P partitioning $[a, b]$,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof. □

Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives)**.** Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Moreover, suppose that its derivative $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Proof. □

Exercise (1)**.** Let m, b be positive numbers. Find the value of $\int_0^1 mx + b\, dx$ in the following three ways:

(a) Using elementary geometry, interpreting the integral as an area.

(b) Using upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes-Riemann Theorem.

(c) Using the First Fundamental Theorem (Integrating Derivatives).

Solution. □

Exercise (5)**.** The monotonicity property of the integral implies that if the functions $g, h : [0, \infty) \to \mathbb{R}$ are continuous and $g(x) \le h(x)$ for all $x \ge 0$, then

$$\int_0^x g \le \int_0^x h \quad \text{for all } x \ge 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \le 1 \quad \text{if } x \ge 0.$$

$$\sin x \le x \quad \text{if } x \ge 0.$$

$$1 - \cos x \le \frac{x^2}{2} \quad \text{if } x \ge 0.$$

$$x - \sin x \le \frac{x^3}{6} \quad \text{if } x \ge 0.$$

$$x - \frac{x^3}{6} \le \sin x \le x \quad \text{if } x \ge 0.$$

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem (6.26, The Mean Value Theorem for Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

Proof. □

Proposition (6.27)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function $F : [a, b] \to \mathbb{R}$ is continuous.

Proof. □

Theorem (6.29, The Second Fundamental Theorem: Differentiating Integrals)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x\right] = f(x) \quad \text{for all } x \in (a, b).$$

Proof. □

Exercise (2b)**.** Suppose $f : [0, 2] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t)\ dt \quad \text{for all} x \in [a, b]$$

and find a formula for $F(x)$ which does not involve integrals.

Solution. □

Exercise (5)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x - t) f(t)\ dt \quad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x.

Solution. □

Exercise (12)**.** Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and that α, β are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a, b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all $x \in (a, b)$. Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

Solution. □

Chapter 10

The Euclidean Space $\mathbb{R}^n$