

MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are “due”; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

— Dr. Steven Clontz (sclontz5@uncc.edu)

Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1). Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b - a) \leq L(f, P) \text{ and } U(f, P) \leq M(b - a).$$

Proof.

□

Lemma (6.2, The Refinement Lemma). Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If P^* is a refinement of P , then

$$L(f, P) \leq L(f, P^*) \text{ and } U(f, P^*) \leq U(f, P).$$

Proof.

□

Lemma (6.3). Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof.

□

Lemma (6.4). For a bounded function $f : [a, b] \rightarrow \mathbb{R}$,

$$\int_a^b f \leq \overline{\int_a^b f}.$$

Proof.

□

Exercise (2). For an interval $[a, b]$ and a positive number δ , show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ .

Solution. □

Exercise (3). Suppose that the bounded function $f : [a, b] \rightarrow \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\int_a^b f \leq 0 \leq \overline{\int_a^b f}.$$

Solution. □

Exercise (6). Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \rightarrow \mathbb{R}$ is constant.

Solution. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7). For a bounded function $f : [a, b] \rightarrow \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \int_a^b f \leq \overline{\int_a^b f} \leq U(f, P).$$

Proof. □

Theorem (6.8, The Archimedes-Riemann Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n).$$

Proof. □

Example (6.9). Show that a monotonically increasing function $f : [a, b] \rightarrow \mathbb{R}$ is integrable.

Solution. □

Example (6.11). Show that $\int_0^1 x^2 dx = \frac{1}{3}$.

Solution. □

Exercise (4). Prove that for a natural number n ,

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x \, dx = (b^2 - a^2)/2$.

Solution. □

Exercise (6b). Use the Archimedes-Riemann Theorem to show that for $0 \leq a < b$,

$$\int_a^b x^2 \, dx = \frac{b^3 - a^3}{3}.$$

Solution. □

Exercise (9). Suppose that the functions $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for f on $[a, b]$ and also an Archimedean sequence of partitions for g on $[a, b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity

Theorem (6.12, Additivity over Intervals). Let $f : [a, b] \rightarrow \mathbb{R}$ be integrable on $[a, b]$ and let $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$, and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. □

Theorem (6.13, Monotonicity of the Integral). Suppose $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. □

Lemma (6.14). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be bounded and let P partition $[a, b]$. Then

$$L(f, P) + L(g, P) \leq L(f + g, P) \quad \text{and} \quad U(f + g, P) \leq U(f, P) + U(g, P).$$

Moreover, for any number α ,

$$U(\alpha f, P) = \alpha U(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha L(f, P) \quad \text{if } \alpha \geq 0$$

$$U(\alpha f, P) = \alpha L(f, P) \quad \text{and} \quad L(\alpha f, P) = \alpha U(f, P) \quad \text{if } \alpha < 0.$$

Proof. □

Theorem (6.15, Linearity of the Integral). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable. Then for any two numbers α, β , the function $\alpha f + \beta g : [a, b] \rightarrow \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

Proof. □

Exercise (1). Suppose that the functions f, g, f^2, g^2, fg are integrable on $[a, b]$. Prove that $(f - g)^2$ is also integrable on $[a, b]$ and that $\int_a^b (f - g)^2 \geq 0$. Use this to prove that

$$\int_a^b fg \leq \frac{1}{2} \left[\int_a^b f^2 + \int_a^b g^2 \right].$$

Solution. □

Exercise (4). Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x : x \in S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

Solution. □

Exercise (6). Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on both $[a, c]$, $[c, b]$, then it is integrable on $[a, b]$.

Solution. □

6.4 Continuity and Integrability

Lemma (6.17). Let the function $f : [a, b] \rightarrow \mathbb{R}$ be continuous let P partition its domain. Then there is a partition interval of P that contains two points u, v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

Proof. □

Theorem (6.18). A continuous function on a closed bounded interval is integrable.

Proof. □

Theorem (6.19). Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b) . Then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of $[a, b]$.

Proof. □

Exercise (1). Determine whether each of the following statements is true or false, and justify your answer.

- (a) If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.
- (b) If $f : [a, b] \rightarrow \mathbb{R}$ is integrable, then f is continuous.
- (c) If $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.
- (d) A continuous function $f : (a, b) \rightarrow \mathbb{R}$ defined on an open interval (a, b) is bounded.
- (e) A continuous function $f : [a, b] \rightarrow \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

Solution. (a)

(b)

(c)

(d)

(e)

□

Exercise (5). Suppose that the continuous function $f : [a, b] \rightarrow \mathbb{R}$ has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all $x \in [a, b]$. Is this true if we only require integrability of the function?

Solution. □

Exercise (6). Suppose that $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all $x \in [0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point $x_0 \in [0, 1]$ at which $f(x_0) > 0$.

Solution. □

6.5 The First Fundamental Theorem: Integrating Derivatives

Lemma (6.21). Suppose $f : [a, b] \rightarrow \mathbb{R}$ is integrable and that the number A has the property that for every P partitioning $[a, b]$,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof. □

Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives). Let $F : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Moreover, suppose that its derivative $F' : (a, b) \rightarrow \mathbb{R}$ is both continuous and bounded. Then

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

Proof. □

Exercise (1). Let m, b be positive numbers. Find the value of $\int_0^1 mx + b \, dx$ in the following three ways:

- (a) Using elementary geometry, interpreting the integral as an area.
- (b) Using upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes-Riemann Theorem.
- (c) Using the First Fundamental Theorem (Integrating Derivatives).

Solution. □

Exercise (5). The monotonicity property of the integral implies that if the functions $g, h : [0, \infty) \rightarrow \mathbb{R}$ are continuous and $g(x) \leq h(x)$ for all $x \geq 0$, then

$$\int_0^x g \leq \int_0^x h \quad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \leq 1 \quad \text{if } x \geq 0.$$

$$\sin x \leq x \quad \text{if } x \geq 0.$$

$$1 - \cos x \leq \frac{x^2}{2} \quad \text{if } x \geq 0.$$

$$x - \sin x \leq \frac{x^3}{6} \quad \text{if } x \geq 0.$$

$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{if } x \geq 0.$$

(For this problem, you may assume that the sine and cosine functions are differentiable functions with the properties $\sin(0) = 0$, $\cos(0) = 1$, $\frac{d}{dx}[\sin(x)] = \cos(x)$, and $\frac{d}{dx}[\cos(x)] = -\sin(x)$.)

Solution. □

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem (6.26, The Mean Value Theorem for Integrals). Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a, b]$ at which

$$\frac{1}{b-a} \int_a^b f = f(x_0).$$

Proof. □

Proposition (6.27). Suppose that the function $f : [a, b] \rightarrow \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a, b].$$

Then the function $F : [a, b] \rightarrow \mathbb{R}$ is continuous.

Proof. □

Theorem (6.29, The Second Fundamental Theorem: Differentiating Integrals). Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then

$$\frac{d}{dx} \left[\int_a^x \right] = f(x) \quad \text{for all } x \in (a, b).$$

Proof. □

Exercise (2b). Suppose $f : [0, 2] \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t) \, dt \quad \text{for all } x \in [a, b]$$

and find a formula for $F(x)$ which does not involve integrals.

Solution.

□

Exercise (5). Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x-t)f(t) \, dt \quad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x .

Solution.

□

Exercise (12). Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous and that α, β are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a, b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all $x \in (a, b)$. Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

Solution.

□

Chapter 10

The Euclidean Space \mathbb{R}^n

10.1 The Linear Structure of \mathbb{R}^n and the Scalar Product

Proposition (10.2). Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then both of the following hold:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$\langle \alpha \mathbf{u} + \beta \mathbf{w}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle + \beta \langle \mathbf{w}, \mathbf{v} \rangle$$

Proof.

□

Lemma (10.4). For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, \mathbf{u}, \mathbf{v} are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Proof.

□

Lemma (10.5). For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} \neq \mathbf{0}$, define $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$ and $\mathbf{w} = \mathbf{u} - \lambda \mathbf{v}$. Then \mathbf{v}, \mathbf{w} are orthogonal and $\mathbf{u} = \mathbf{w} + \lambda \mathbf{v}$.

Proof.

□

Theorem (10.6, The Cauchy-Schwarz Inequality). For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|.$$

Proof.

□

Theorem (10.7, The Triangle Inequality). For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Exercise (3). Show that for $\mathbf{u} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(a) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(b) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

Proof. □

Exercise (4). For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ verify the identity

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (9). Let $\mathbf{u} \in \mathbb{R}^n$ and suppose $\|\mathbf{u}\| < 1$. Show that for $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v} - \mathbf{u}\| < 1 - \|\mathbf{u}\|$ implies $\|\mathbf{v}\| < 1$.

Solution. □

Exercise (10). Let $\mathbf{u} \in \mathbb{R}^n$ and $r > 0$. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are at a distance less than r from \mathbf{u} . Prove that if $0 \leq t \leq 1$, then the point $t\mathbf{v} + (1 - t)\mathbf{w}$ is also at a distance less than r from \mathbf{u} .

Solution. □

10.2 Convergence of Sequences in \mathbb{R}^n

Theorem (10.9, The Componentwise Convergence Criterion). Let $\{\mathbf{u}_k\}$ be a sequence in \mathbb{R}^n . Then $\{\mathbf{u}_k\}$ converges to \mathbf{u} if and only if $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$ for all $1 \leq i \leq n$.

Proof. □

Theorem (10.10). Let $\{\mathbf{u}_k\}, \{\mathbf{v}_k\}$ be sequences in \mathbb{R}^n such that $\{\mathbf{u}_k\}$ converges to \mathbf{u} and $\{\mathbf{v}_k\}$ converges to \mathbf{v} . Then for any $\alpha, \beta \in \mathbb{R}$,

$$\lim_{k \rightarrow \infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

Proof. □

Exercise (1). Let $\{\mathbf{u}_k\}$ be a sequence in \mathbb{R}^n that converges to \mathbf{u} . Prove the following for all $\mathbf{v} \in \mathbb{R}^n$:

$$\lim_{k \rightarrow \infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle.$$

Solution. □

Exercise (2). Let $\{\mathbf{u}_k\}$ be a sequence in \mathbb{R}^n and $\mathbf{u} \in \mathbb{R}^n$. Prove that if

$$\lim_{k \rightarrow \infty} \langle \mathbf{u}_k, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

holds for all $\mathbf{v} \in \mathbb{R}^n$, then $\{\mathbf{u}_k\}$ converges to \mathbf{u} .

Solution. □

Exercise (5). Let $\{\mathbf{u}_k\}$ be a sequence in \mathbb{R}^n that converges to \mathbf{u} where $\|\mathbf{u}\| = r > 0$. Prove that there is an index K where

$$\|\mathbf{u}_k\| > \frac{r}{2} \text{ if } k \geq K.$$

Solution. □

10.3 Open Sets and Closed Sets in \mathbb{R}^n

Example (10.11). Let $a < b$ be in \mathbb{R} . Then $\text{int}(a, b] = (a, b)$.

Proof. □

Example (10.12). Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational real numbers. Then $\text{int } \mathbb{Q} = \emptyset$.

Proof. □

Proposition (10.13). Every open ball $B_r(\mathbf{u})$ in \mathbb{R}^n is open.

Proof. □

Example (10.14). Let $a < b$ be in \mathbb{R} . Then $[a, b]$ is closed.

Proof. □

Example (10.15). The set

$$[-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

is closed in \mathbb{R}^2 .

Proof. □

Theorem (10.16, The Complementing Characterization). A subset $A \subseteq \mathbb{R}^n$ is open if and only if its complement $\mathbb{R}^n \setminus A$ is closed.

Proof. □

Proposition (10.17.i). The union of a collection of open subsets of \mathbb{R}^n is open.

Proof. □

Proposition (10.17.ii). The intersection of a collection of closed subsets of \mathbb{R}^n is closed.

Proof. □

Proposition (10.18.i). The intersection of a finite collection of open subsets of \mathbb{R}^n is open.

Proof.

□

Proposition (10.18.ii). The union of a finite collection of closed subsets of \mathbb{R}^n is closed.

Proof.

□

Proposition (10.19.i). $A \subseteq \mathbb{R}^n$ is open if and only if $A \cap \text{bd } A = \emptyset$.

Proof.

□

Proposition (10.19.ii). $A \subseteq \mathbb{R}^n$ is closed if and only if $\text{bd } A \subseteq A$.

Proof.

□

Exercise (2). Determine which of the following subsets of \mathbb{R}^2 are open, closed, neither, or both.

(a) $\{(x, y) : x^2 > y\}$

(b) $\{(x, y) : x^2 + y^2 = 1\}$

(c) $\{(x, y) : x \text{ is rational}\}$

(d) $\{(x, y) : x \geq 0, y \geq 0\}$

Solution. (a)

(b)

(c)

(d)

□

Exercise (3). Let $r > 0$ and $O = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| > r\}$. Prove that O is open.

Solution.

□

Exercise (7a). Show that $A \subseteq \mathbb{R}^n$ is open if and only if

$$\mathbf{w} + A = \{\mathbf{w} + \mathbf{u} : \mathbf{u} \in A\}$$

is open for all $\mathbf{w} \in \mathbb{R}^n$.

Solution.

□

Exercise (12). For $A \subseteq \mathbb{R}^n$, denote its closure by

$$\text{cl } A = \text{int } A \cup \text{bd } A.$$

Prove that $A \subseteq \text{cl } A$. Then prove that $A = \text{cl } A$ if and only if A is closed.

Solution.

□

Chapter 11

Continuity, Compactness, and Connectedness

11.1 Continuous Functions and Mappings

Proposition (11.1). For each $i \in \{1, \dots, n\}$, the i th projection map $p_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous.

Proof. □

Theorem (11.3). Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $h, g : A \rightarrow \mathbb{R}$ be continuous at \mathbf{u} . Then for $\alpha, \beta \in \mathbb{R}$, the following functions are continuous at \mathbf{u} :

$$\alpha h + \beta g : A \rightarrow \mathbb{R} \qquad h \cdot g : A \rightarrow \mathbb{R}.$$

Also if $g(\mathbf{v}) \neq 0$ for all $\mathbf{v} \in A$, then the following function is also continuous at \mathbf{u} :

$$\frac{h}{g} : A \rightarrow \mathbb{R}.$$

Proof. □

Theorem (11.5). Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $G : A \rightarrow \mathbb{R}^m$ be continuous at \mathbf{u} . Also let $G(A) \subseteq B \subseteq \mathbb{R}^m$ and $H : B \rightarrow \mathbb{R}^k$ be continuous at $G(\mathbf{u})$. Then the composition

$$H \circ G : A \rightarrow \mathbb{R}^k$$

is continuous at \mathbf{u} .

Proof. □

Theorem (11.9, The Componentwise Continuity Criterion). Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \rightarrow \mathbb{R}^m$. Then F is continuous at \mathbf{u} if and only if $F_i = p_i \circ F : A \rightarrow \mathbb{R}$ is continuous at \mathbf{u} for each $i \in \{1, \dots, m\}$.

Proof. □

Theorem (11.11, Exercise 12). Let $\mathbf{u} \in A \subseteq \mathbb{R}^n$ and $F : A \rightarrow \mathbb{R}^m$. Then F is continuous at \mathbf{u} if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{v} - \mathbf{u}\| < \delta$ implies $\|F(\mathbf{v}) - F(\mathbf{u})\| < \epsilon$.

Proof. □

Theorem (11.12). Let $U \subseteq \mathbb{R}^n$ be open and $F : U \rightarrow \mathbb{R}^m$. Then F is continuous if and only if $F^{-1}(V)$ is an open subset of \mathbb{R}^n for every open $V \subseteq \mathbb{R}^m$.

Proof. □

Example (11.15). Use corollary 11.13 and proposition 10.18.i to prove that $U = \{\mathbf{u} \in \mathbb{R}^n : a < \|\mathbf{u}\| < b\}$ is open. (You may assume $f(\mathbf{u}) = \|\mathbf{u}\|$ is continuous.)

Solution. □

Exercise (3). Fix a point $\mathbf{v} \in \mathbb{R}^n$. Prove that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by $f(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is continuous.

Solution. □

Exercise (6). Suppose $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous. Prove that $\{\mathbf{u} \in \mathbb{R}^n : f(\mathbf{u}) = g(\mathbf{u}) = 0\}$ is closed. (Hint: use corollary 11.13 and proposition 10.17.ii.)

Solution. □

Exercise (11). Let $A \subseteq \mathbb{R}^n$. The characteristic function $\phi_A : \mathbb{R}^n \rightarrow \mathbb{R}$ for A is defined to be

$$\phi_A(\mathbf{u}) = \begin{cases} 1 & \text{if } \mathbf{u} \in A \\ 0 & \text{if } \mathbf{u} \notin A \end{cases}.$$

Prove that ϕ_A is continuous at points in $\text{int } A$ and $\text{ext } A$, but not continuous at points in $\text{bd } A$.

11.2 Sequential Compactness, Extreme Values, and Uniform Continuity

Theorem (11.16). Every sequentially compact subset of \mathbb{R}^n is bounded and closed.

Proof. □

Theorem (11.17). Every bounded sequence in \mathbb{R}^n has a convergent subsequence.

Proof.

□

Theorem (11.18, The Sequential Compactness Theorem). A subset of \mathbb{R}^n is sequentially compact if and only if it is closed and bounded.

Proof.

□

Corollary (11.19). The generalized rectangle $\prod_{i=1}^n [a_i, b_i] = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is sequentially compact.

Proof.

□

Theorem (11.20). Let $A \subseteq \mathbb{R}^n$ and suppose $F : A \rightarrow \mathbb{R}^m$ is continuous. If A is sequentially compact then $F(A)$ is also sequentially compact.

Proof.

□

Lemma (11.21). Every nonempty sequentially compact subset of \mathbb{R} has a maximum and minimum element.

Proof.

□

Theorem (11.22, The Extreme Value Theorem, Bolzano-Weierstrass Theorem). Let $A \subseteq \mathbb{R}^n$ be nonempty sequentially compact and suppose $f : A \rightarrow \mathbb{R}^m$ is continuous. Then f attains a smallest and largest value.

Proof.

□

Theorem (11.24). Let $A \subseteq \mathbb{R}^n$ be nonempty. A is sequentially compact if and only if every continuous $f : A \rightarrow \mathbb{R}$ attains a smallest and largest value (i.e. it has the Extreme Value Property).

Proof.

□

Theorem (11.25, Exercise 5). Let $A \subseteq \mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}^m$ be continuous. If A is sequentially compact then f is uniformly continuous.

Proof.

□

Theorem (11.27, Exercise 11). Let $A \subseteq \mathbb{R}^n$ and $f : A \rightarrow \mathbb{R}^m$. f is uniformly continuous if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\mathbf{u} - \mathbf{v}\| < \delta$ implies $\|f(\mathbf{u}) - f(\mathbf{v})\| < \epsilon$.

Proof.

□

Exercise (2,3). Recall that $B_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| < r\}$. Is $B_r(\mathbf{u})$ bounded? Sequentially compact?

Solution.

□

Exercise (4). Let $D_r(\mathbf{u}) = \{\mathbf{v} : \|\mathbf{u} - \mathbf{v}\| \leq r\}$. Prove $D_r(\mathbf{u})$ is sequentially compact.

Solution.

□

Exercise (optional, Heine-Borel Theorem). Let $A \subseteq \mathbb{R}^n$. We say A is compact when for every collection \mathcal{U} such that A is open for all $U \in \mathcal{U}$ and $A \subseteq \bigcup \mathcal{U}$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{U}$ such that $A \subseteq \bigcup \mathcal{F}$. Prove that A is compact if and only if A is closed and bounded.

Solution.

□

Midterm Part 3

Choose two of the below problems (which you did not choose for Part 2) and typeset your solutions. Delete the other three. Each will be worth 20/100 points towards your midterm grade for a total of 40/100 points.

Exercise (1). Prove that if Q_n is a partition of $[a, b]$ refining the partition P_n of $[a, b]$ for each natural number n , and $\{P_n\}$ is an Archimedian sequence of partitions for f on $[a, b]$, then $\{Q_n\}$ is also Archimedian.

Solution. □

Exercise (2). Explain the error(s) in the following “proof”, and then give a counterexample showing that the theorem is false.

Theorem: If $f : [0, 1] \rightarrow \mathbb{R}$ is integrable, then f is also continuous.

Proof: Since f is integrable, we may define $F : [0, 1] \rightarrow \mathbb{R}$ by $F(x) = \int_0^x f$. It follows that $F(x)$ is a differentiable function, because it is an antiderivative of f . Thus $\frac{d}{dx}[F(x)] = f(x)$ by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude f is continuous.

Solution. □

Exercise (3). Recall that an **even** function satisfies the condition $f(x) = f(-x)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an even continuous function. Prove that

$$\frac{d}{dx} \left[\int_{-x}^x f \right] = 2f(x).$$

(Hint: Corollary 6.30 says that $\frac{d}{dx}[\int_x^0 f] = -f(x)$.)

Solution. □

Exercise (4). Prove the following theorem:

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{\mathbf{x}_k\}$ be a sequence of points in \mathbb{R}^n . If for every open set U containing \mathbf{x} , there is an index K such that $\mathbf{x}_k \in U$ for all $k \geq K$, then $\{\mathbf{x}_k\}$ converges to \mathbf{x} .

(Hint: $B_\epsilon(\mathbf{x})$ is open.)

Solution.

□

Exercise (5). Prove that any finite subset of \mathbb{R}^n is closed.

(Hint: First prove that any singleton subset of \mathbb{R}^n is closed.)

Solution.

□