MATH 3142-001 — Spring 2016 — Dr. Clontz — Midterm Part 1

Name: Solutions

- Each problem is labeled with its worth toward the total grade of 100 points for this midterm.
- You do not need to show your work on these multiple-choice problems. No partial credit will be given.
- You may not use any notes/electronics on this portion of the exam.
- This part of the midterm is due after 20 minutes. Materials submitted late will be penalized by 50%.

For each of the following statements, choose if it is True or False.

1. (2 points) The lower integral $\underline{\int_a^b} f$ is defined to be $\cancel{\inf}\{L(f,P) : P \text{ partitions } [a,b]\}$.
   sup

   A. True

   B. False

2. (2 points) If $f : [a,b] \to \mathbb{R}$ is integrable, then there exists a sequence of partitions $\{P_n\}$ such that $\lim_{n\to\infty} L(f, P_n) = \int_a^b f$.

   A. True

   B. False

3. (2 points) The function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \in [0,2) \\ -1 & \text{if } x = 2 \end{cases}$$

   is integrable.

   A. True

   B. False

4. (2 points) All bounded functions $f : [a,b] \to \mathbb{R}$ are integrable.

   A. True

   B. False

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

5. (2 points) If $f : [a,b] \to \mathbb{R}$ is continuous, then $F : [a,b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ is differentiable.

   A. True

   B. False

Choose the most appropriate response for each.

6. (3 points) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Which of the following is false?

   A. $\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$

   B. $\|\mathbf{u}\|\|\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ $\qquad 6 = (2)(3) \not\leq 2+3 = 5$

   C. $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$

   D. $\|\mathbf{u} - \mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$

7. (3 points) If $\{\mathbf{u}_k\}, \{\mathbf{v}_k\}$ are sequences of vectors in $\mathbb{R}^n$ converging to $\mathbf{u}, \mathbf{v}$ respectively, then which of these must be true?

   A. $\lim_{k\to\infty} \mathbf{u}_k = \lim_{k\to\infty} \mathbf{v}_k$

   B. $\mathbf{v} + \lim_{k\to\infty} \mathbf{u}_k = \mathbf{u} + \lim_{k\to\infty} \mathbf{v}_k$ $= \underline{u} + \underline{v}$

   C. $\lim_{k\to\infty}(\mathbf{u}_k - \mathbf{v}_k) = \mathbf{0}$

   D. $\lim_{k\to\infty}\langle \mathbf{u}_k, \mathbf{v}_k \rangle = \|\mathbf{u}\| + \|\mathbf{v}\|$

8. (4 points) The set $\{\mathbf{x} \in \mathbb{R}^2 : p_1(\mathbf{x}) < p_2(\mathbf{x})\}$ is which of the following?

   A. Open, but not closed $\qquad \{(x,y) \in \mathbb{R}^2 : x < y\}$

   B. Closed, but not open

   C. Both open and closed

   D. Neither open nor closed

MATH 3142-001 — Spring 2016 — Dr. Clontz — Midterm Part 2

Name: ______________________________

- Each problem is labeled with its worth toward the total grade of 100 points for this midterm.
- You must choose two of the five problems to submit. These should be stapled to this cover sheet. Save the other three problems for your reference in Part 3.
- Each problem requires a rigorous proof. When in doubt, don't skip details. You may sketch pictures to help illustrate concepts, but the proof must still be valid without the use of illustrations.
- You may use any notes you wish once Part 1 has been submitted. Electronics are still disallowed.
- This part of the midterm is due after 70 minutes. Materials submitted late will be penalized by 50%.

MATH 3142-001 — Spring 2016 — Dr. Clontz — Midterm Part 3

Name: ______________________________

- Each problem is labeled with its worth toward the total grade of 100 points for this midterm.
- You must choose two of the three problems you didn't choose for Part 2 to submit online.
- Each problem requires a rigorous proof. When in doubt, don't skip details. You may sketch pictures to help illustrate concepts, but the proof must still be valid without the use of illustrations.
- You may use any notes you wish, and even collaborate with other students, as long as you submit your own work. **Plagiarism will be treated as a violation of academic honesty.**
- Solutions must be typeset in your Overleaf project using a template to be provided by the professor via email.
- This part of the midterm is due at 11:59pm on Friday, March 4. Materials submitted late (or Overleaf projects which will not compile) will not be graded.

1. (20 points) Prove that if Q_n is a partition of $[a, b]$ refining the partition P_n of $[a, b]$ for each natural number n, and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a, b]$, then $\{Q_n\}$ is also Archimedean.

Since Q_n refines P_n, it follows that

$$L(f, Q_n) \geq L(f, P_n), \quad -L(f, Q_n) \leq -L(f, P_n)$$

$$U(f, Q_n) \leq U(f, P_n), \text{ and } U(f, Q_n) - L(f, Q_n) \leq U(f, P_n) - L(f, P_n).$$

Since $\{P_n\}$ is Archimedean, we may note

$$\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0.$$

So we can show that

$$0 \leq \lim_{n \to \infty} U(f, Q_n) - L(f, Q_n)$$

$$\leq \lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0.$$

Therefore $\{Q_n\}$ is Archimedean.

2. (20 points) Explain the error(s) in the following "proof", and then give a counterexample showing that the theorem is false.

Theorem: If $f : [0,1] \to \mathbb{R}$ is integrable, then f is also continuous.

Proof: Since f is integrable, we may define $F : [0,1] \to \mathbb{R}$ by $F(x) = \int_0^x f$. It follows that $F(x)$ is a differentiable function, because it is an antiderivative of f. Thus $\frac{d}{dx}[F(x)] = f(x)$ by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude f is continuous.

→ Prop 6.27 only guarantees F is continuous when f is only integrable.

→ This is also untrue (at least, it was never given as a theorem).

Note that $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

is integrable by Thm 6.19, but not continuous at $x = 1$.

3. (20 points) Recall that an **even** function satisfies the condition $f(x) = f(-x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be an even continuous function. Prove that

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = 2f(x).$$

(Hint: Corollary 6.30 says that $\frac{d}{dx}[\int_x^0 f] = -f(x)$.) So by Chain Rule, $\frac{d}{dx}\left[\int_{-x}^{0} f\right] = -(-f(-x))$

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = \frac{d}{dx}\left[\int_{-x}^{0} f + \int_{0}^{x} f\right]$$

$$= -(-f(-x)) + f(x)$$

$$= f(-x) + f(x)$$

$$= f(x) + f(x)$$

$$= 2f(x). \quad \square$$

5. (20 points) Prove that any finite subset of $\mathbb{R}^n$ is closed.

(Hint: First prove that any singleton subset of $\mathbb{R}^n$ is closed.)

Let $C = \{\underline{x}\}$ for $\underline{x} \in \mathbb{R}^n$. If $\{\underline{u}_k\}$ is a sequence of points in C, then $\underline{u}_k = \underline{x}$ for all k. Thus $\{\underline{u}_k\}$ converges to $\underline{x} \in C$, so C is closed.

By Prop 10.18.ii, a finite union of closed sets is closed. Since every finite set $\{\underline{x}_1, \dots, \underline{x}_m\}$ is the union $\bigcup_{i=1}^{m} \{\underline{x}_i\}$, every finite set is closed.