MATH 3142 Notes — Spring 2016

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This document is a template for you to take notes in my MATH 3142 course. For your note check grade, you are required to complete all proofs/solutions for the problems specified. This template will be updated periodically throughout the course; you are responsible for updating your copy as the template is updated. See the syllabus for more details.

You should maintain your notes on Overleaf.com and provide me with a link so I can check on them. I'll give you notice before notes are "due"; when they are due I will download a copy myself from Overleaf.

This is not a replacement for the textbook for this course, *Advanced Calculus* by Patrick M. Fitzpatrick. Many proofs are outlined in that text, as well as all the relevant definitions and other results not included in these notes.

A proof is valid if and only if it uses concepts proven previously in the book. For example, you cannot prove a lemma in Chapter 6 using a theorem from Chapter 10, but using a proposition from Chapter 4 is allowed.

I hope you enjoy working through these results. Please email me with any questions.

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Chapter 6

Integration: Two Fundamental Theorems

6.1 Darboux Sums: Upper and Lower Integrals

Lemma (6.1)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and the numbers m, M have the property that

$$m \leq f(x) \leq M$$

for all x in $[a, b]$. Then, if P is a partition of the domain $[a, b]$,

$$m(b-a) \leq L(f, P) \text{ and } U(f, P) \leq M(b-a).$$

Proof. □

Lemma (6.2, The Refinement Lemma)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P is a partition of its domain $[a, b]$. If $P^\star$ is a refinement of P, then

$$L(f, P) \leq L(f, P^\star) \text{ and } U(f, P^\star) \leq U(f, P).$$

Proof. □

Lemma (6.3)**.** Suppose that the function $f : [a, b] \to \mathbb{R}$ is bounded and that P_1, P_2 are partitions of its domain. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. □

Lemma (6.4)**.** For a bounded function $f : [a, b] \to \mathbb{R}$,

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof. □

Exercise (2)**.** For an interval $[a, b]$ and a positive number δ, show that there is a partition $P = \{x_i : 0 \leq i \leq n\}$ of $[a, b]$ such that each partition interval $[x_i, x_{i+1}]$ of P has length less than δ.

Solution. □

Exercise (3)**.** Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number x in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition P of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

Solution. □

6.2 The Archimedes-Riemann Theorem

Lemma (6.7)**.** For a bounded function $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$,

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P).$$

Proof. □

Theorem (6.8, The Archimedes-Riemann Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}$ of partitions of the interval $[a, b]$ such that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Moreover, for any such sequence of partitions,

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f = \lim_{n\to\infty} U(f, P_n).$$

Proof. □

Example (6.9)**.** Show that a monotonically increasing function $f : [a, b] \to \mathbb{R}$ is integrable.

Solution. □

Example (6.11)**.** Show that $\int_0^1 x^2\, dx = \frac{1}{3}$.

Solution. □

Exercise (4)**.** Prove that for a natural number n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Then use this fact and the Archimedes-Riemann Theorem to show that $\int_a^b x\,dx = (b^2-a^2)/2$.

Solution. □

Exercise (6b)**.** Use the Archimedes-Riemann Theorem to show that for $0 \le a < b$,

$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3}.$$

Solution. □

Exercise (9)**.** Suppose that the functions $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ that is an Archimedian sequence of partitions for f on $[a,b]$ and also an Archimedean sequence of partitions for g on $[a,b]$.

Solution. □

6.3 Additivity, Monotonicity, and Linearity

Theorem (6.12, Additivity over Intervals)**.** Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$ and let $c \in (a,b)$. Then f is integrable on $[a,c]$ and $[c,b]$, and furthermore

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. □

Theorem (6.13, Monotonicity of the Integral)**.** Suppose $f, g : [a,b] \to \mathbb{R}$ are integrable and that $f(x) \le g(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f \le \int_a^b g.$$

Proof. □

Lemma (6.14)**.** Let $f, g : [a,b] \to \mathbb{R}$ be bounded and let P partition $[a,b]$. Then

$$L(f,P) + L(g,P) \le L(f+g,P) \quad \text{and} \quad U(f+g,P) \le U(f,P) + U(g,P).$$

Moreover, for any number α,

$$U(\alpha f,P) = \alpha U(f,P) \quad \text{and} \quad L(\alpha f,P) = \alpha L(f,P) \quad \text{if } \alpha \ge 0$$
$$U(\alpha f,P) = \alpha L(f,P) \quad \text{and} \quad L(\alpha f,P) = \alpha U(f,P) \quad \text{if } \alpha < 0.$$

Proof. □

Theorem (6.15, Linearity of the Integral)**.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then for any two numbers α, β, the function $\alpha f + \beta g : [a, b] \to \mathbb{R}$ is integrable and

$$\int_a^b [\alpha f + \beta g] = \alpha \int_a^b f + \beta \int_a^b g.$$

Proof. □

Exercise (1)**.** Suppose that the functions f, g, f^2, g^2, fg are integrable on $[a, b]$. Prove that $(f - g)^2$ is also integrable on $[a, b]$ and that $\int_a^b (f - g)^2 \geq 0$. Use this to prove that

$$\int_a^b fg \leq \frac{1}{2}\left[\int_a^b f^2 + \int_a^b g^2\right].$$

Solution. □

Exercise (4)**.** Suppose that S is a nonempty bounded set of numbers and that α is a number. Define αS to be the set $\{\alpha x : x \in S\}$. Prove that

$$\sup \alpha S = \alpha \sup S \quad \text{and} \quad \inf \alpha S = \alpha \inf S \quad \text{if } \alpha \geq 0$$

while

$$\sup \alpha S = \alpha \inf S \quad \text{and} \quad \inf \alpha S = \alpha \sup S \quad \text{if } \alpha < 0.$$

Solution. □

Exercise (6)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and let $a < c < b$. Prove that if f is integrable on both $[a, c], [c, b]$, then it is integrable on $[a, b]$.

Solution. □

6.4 Continuity and Integrability

Lemma (6.17)**.** Let the function $f : [a, b] \to \mathbb{R}$ be continuous let P partition its domain. Then there is a partition interval of P that contains two points u, v for which the following estimate holds:

$$0 \leq U(f, P) - L(f, P) \leq [f(v) - f(u)][b - a].$$

Proof. □

Theorem (6.18)**.** A continuous function on a closed bounded interval is integrable.

Proof. □

Theorem (6.19)**.** Supose $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on (a, b). Then f is integrable on $[a, b]$ and the value of $\int_a^b f$ does not depend on the values of f at the endpoints of $[a, b]$.

Proof. □

Exercise (1)**.** Determine whether each of the following statements is true or false, and justify your answer.

(a) If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) If $f : [a, b] \to \mathbb{R}$ is integrable, then f is continuous.

(c) If $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f \geq 0$.

(d) A continuous function $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) is bounded.

(e) A continuous function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$ is bounded.

Solution. (a)

(b)

(c)

(d)

(e)

□

Exercise (5)**.** Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has the property

$$\int_c^d f \leq 0 \quad \text{whenever } a \leq c < d \leq b.$$

Prove that $f(x) \leq 0$ for all $x \in [a, b]$. Is this true if we only require integrability of the function?

Solution. □

Exercise (6)**.** Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that $f(x) \geq 0$ for all $x \in [0, 1]$. Prove that $\int_0^1 f > 0$ if and only if there is a point $x_0 \in [0, 1]$ at which $f(x_0) > 0$.

Solution. □

6.5 The First Fundamental Theorem: Integrating Derivatives

Lemma (6.21)**.** Suppose $f : [a, b] \to \mathbb{R}$ is integrable and that the number A has the property that for every P partitioning $[a, b]$,

$$L(f, P) \leq A \leq U(f, P).$$

Then

$$\int_a^b f = A.$$

Proof. □

Theorem (6.22, The First Fundamental Theorem: Integrating Derivatives)**.** Let $F : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Moreover, suppose that its derivative $F' : (a, b) \to \mathbb{R}$ is both continuous and bounded. Then

$$\int_a^b F'(x)\ dx = F(b) - F(a).$$

Proof. □

Exercise (1)**.** Let m, b be positive numbers. Find the value of $\int_0^1 mx + b\ dx$ in the following three ways:

(a) Using elementary geometry, interpreting the integral as an area.

(b) Using upper and lower Darboux sums based on regular partitions of the interval $[0, 1]$ and using the Archimedes-Riemann Theorem.

(c) Using the First Fundamental Theorem (Integrating Derivatives).

Solution. □

Exercise (5)**.** The monotonicity property of the integral implies that if the functions $g, h : [0, \infty) \to \mathbb{R}$ are continuous and $g(x) \leq h(x)$ for all $x \geq 0$, then

$$\int_0^x g \leq \int_0^x h \quad \text{for all } x \geq 0.$$

Use this and the First Fundamental Theorem to show that each of the following inequalities implies the next:

$$\cos x \leq 1 \quad \text{if } x \geq 0.$$
$$\sin x \leq x \quad \text{if } x \geq 0.$$
$$1 - \cos x \leq \frac{x^2}{2} \quad \text{if } x \geq 0.$$
$$x - \sin x \leq \frac{x^3}{6} \quad \text{if } x \geq 0.$$
$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{if } x \geq 0.$$

6.6 The Second Fundamental Theorem: Differentiating Integrals

Theorem (6.26, The Mean Value Theorem for Integrals)**.** Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then there is a point x_0 in the interval $[a,b]$ at which

$$\frac{1}{b-a}\int_a^b f = f(x_0).$$

Proof. □

Proposition (6.27)**.** Suppose that the function $f : [a,b] \to \mathbb{R}$ is integrable. Define

$$F(x) = \int_a^x f \quad \text{for all } x \in [a,b].$$

Then the function $F : [a,b] \to \mathbb{R}$ is continuous.

Proof. □

Theorem (6.29, The Second Fundamental Theorem: Differentiating Integrals)**.** Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then

$$\frac{d}{dx}\left[\int_a^x\right] = f(x) \quad \text{for all } x \in (a,b).$$

Proof. □

Exercise (2b)**.** Suppose $f : [0,2] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2 \end{cases}.$$

Define

$$F(x) = \int_a^x f(t)\ dt \quad \text{for all} x \in [a,b]$$

and find a formula for $F(x)$ which does not involve integrals.

Solution. □

Exercise (5)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define

$$G(x) = \int_0^x (x-t)f(t)\ dt \quad \text{for all } x.$$

Prove that $G''(x) = f(x)$ for all x.

Solution. $\square$

Exercise (12)**.** Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous and that α, β are real numbers. Define

$$H(x) = \int_a^x [\alpha f + \beta g] - \alpha \int_a^x [f] - \beta \int_a^x [g] \quad \text{for all } x \in [a, b].$$

Prove that $H(a) = 0$ and $H'(x) = 0$ for all $x \in (a, b)$. Use this fact and the Identity Criterion to give an alternate proof of Theorem 6.15 for continuous functions.

Solution. $\square$

Chapter 10

The Euclidean Space $\mathbb{R}^n$

10.1 The Linear Structure of $\mathbb{R}^n$ and the Scalar Product

Proposition (10.2)**.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Then both of the following hold:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$$

$$\langle \alpha\mathbf{u} + \beta\mathbf{w}, v \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle + \beta\langle \mathbf{w}, \mathbf{v} \rangle$$

Proof. □

Lemma (10.4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{v}$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Proof. □

Lemma (10.5)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ where $\mathbf{v} \neq \mathbf{0}$, define $\lambda = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$ and $\mathbf{w} = \mathbf{u} - \lambda\mathbf{v}$. Then $\mathbf{v}, \mathbf{w}$ are orthogonal and $\mathbf{u} = \mathbf{w} + \lambda\mathbf{v}$.

Proof. □

Theorem (10.6, The Cauchy-Schwarz Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Proof. □

Theorem (10.7, The Triangle Inequality)**.** For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Exercise (3)**.** Show that for $\mathbf{u} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$:

(a) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(b) $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.

Proof. □

Exercise (4)**.** For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ verify the identity

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle.$$

Solution. □

Exercise (9)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and suppose $\|\mathbf{u}\| < 1$. Show that for $\mathbf{v} \in \mathbb{R}^n$, $\|\mathbf{v} - \mathbf{u}\| < 1 - \|\mathbf{u}\|$ implies $\|\mathbf{v}\| < 1$.

Solution. □

Exercise (10)**.** Let $\mathbf{u} \in \mathbb{R}^n$ and $r > 0$. Suppose $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are at a distance less than r from $\mathbf{u}$. Prove that if $0 \leq t \leq 1$, then the point $t\mathbf{v} + (1 - t)]\mathbf{w}$ is also at a distance less than r from $\mathbf{u}$.

Solution. □

10.2 Convergence of Sequences in $\mathbb{R}^n$

Theorem (10.9, The Componentwise Convergence Criterion)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$. Then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ if and only if $\{p_i(\mathbf{u}_k)\}$ converges to $p_i(\mathbf{u})$ for all $1 \leq i \leq n$.

Proof. □

Theorem (10.10)**.** Let $\{\mathbf{u}_k\}$, $\{\mathbf{v}_k\}$ be sequences in $\mathbb{R}^n$ such that $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$ and $\{\mathbf{v}_k\}$ converges to $\mathbf{v}$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\lim_{k\to\infty} [\alpha\mathbf{u}_k + \beta\mathbf{v}_k] = \alpha\mathbf{u} + \beta\mathbf{v}.$$

Proof. □

Exercise (1)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$. Prove the following for all $\mathbf{v} \in \mathbb{R}^n$:

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle.$$

Solution. □

Exercise (2)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^n$. Prove that if

$$\lim_{k\to\infty} \langle \mathbf{u}_k, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$$

holds for all $\mathbf{v} \in \mathbb{R}^n$, then $\{\mathbf{u}_k\}$ converges to $\mathbf{u}$.

Solution. □

Exercise (5)**.** Let $\{\mathbf{u}_k\}$ be a sequence in $\mathbb{R}^n$ that converges to $\mathbf{u}$ where $\|\mathbf{u}\| => 0$. Prove that there is an index K where

$$\|\mathbf{u}_k\| > \frac{r}{2} \text{ if } k \geq K.$$

Solution. □

10.3 Open Sets and Closed Sets in $\mathbb{R}^n$

Example (10.11)**.** Let $a < b$ be in $\mathbb{R}$. Then $\operatorname{int}(a, b] = (a, b)$.

Proof. □

Example (10.12)**.** Let $\mathbb{Q} \subseteq \mathbb{R}$ be the set of rational real numbers. Then $\operatorname{int}\mathbb{Q} = \emptyset$.

Proof. □

Proposition (10.13)**.** Every open ball $B_r(\mathbf{u})$ in $\mathbb{R}^n$ is open.

Proof. □

Example (10.14)**.** Let $a < b$ be in $\mathbb{R}$. Then $[a, b]$ is closed.

Proof. □

Example (10.15)**.** The set

$$[-1, 1] \times [-1, 1] = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$$

is closed in $\mathbb{R}^2$.

Proof. □

Theorem (10.16, The Complementing Characterization)**.** A subset $A \subseteq \mathbb{R}^n$ is open if and only if its complement $\mathbb{R}^n \setminus A$ is closed.

Proof. □

Proposition (10.17.i)**.** The union of a collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.17.ii)**.** The intersection of a collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.18.i)**.** The intersection of a finite collection of open subsets of $\mathbb{R}^n$ is open.

Proof. □

Proposition (10.18.ii)**.** The union of a finite collection of closed subsets of $\mathbb{R}^n$ is closed.

Proof. □

Proposition (10.19.i)**.** $A \subseteq \mathbb{R}^n$ is open if and only if $A \cap \operatorname{bd} A = \emptyset$.

Proof. □

Proposition (10.19.ii)**.** $A \subseteq \mathbb{R}^n$ is closed if and only if $\operatorname{bd} A \subseteq A$.

Proof. □

Exercise (2)**.** Determine which of the following subsets of $\mathbb{R}^2$ are open, closed, neither, or both.

(a) $\{(x, y) : x^2 > y\}$

(b) $\{(x, y) : x^2 + y^2 = 1\}$

(c) $\{(x, y) : x \text{ is rational}\}$

(d) $\{(x, y) : x \geq 0,\ y \geq 0\}$

Solution. (a)

(b)

(c)

(d)

□

Exercise (3)**.** Let $r > 0$ and $O = \{\mathbf{u} \in \mathbb{R}^n : \|\mathbf{u}\| > r\}$. Prove that O is open.

Solution. □

Exercise (7a)**.** Show that $A \subseteq \mathbb{R}^n$ is open if and only if

$$\mathbf{w} + A = \{\mathbf{w} + \mathbf{u} : \mathbf{u} \in A\}$$

is open for all $\mathbf{w} \in \mathbb{R}^n$.

Solution. □

Exercise (12)**.** For $A \subseteq \mathbb{R}^n$, denote its closure by

$$\operatorname{cl} A = \operatorname{int} A \cup \operatorname{bd} A.$$

Prove that $A \subseteq \operatorname{cl} A$. Then prove that $A = \operatorname{cl} A$ if and only if A is closed.

Solution. □

Chapter 11

Continuity, Compactness, and Connectedness

11.1 Continuous Functions and Mappings