MATH 3142 Midterm Part 1

Clote - Spring 2016

For each of the following statements, choose if it is True or False.

1. (2 points) The lower integral $\underline{\int_a^b} f$ is defined to be $\cancel{\inf}$ sup $\{L(f,P) : P \text{ partitions } [a,b]\}$.

   A. True

   B. False

2. (2 points) If $f : [a,b] \to \mathbb{R}$ is integrable, then there exists a sequence of partitions $\{P_n\}$ such that $\lim_{n\to\infty} L(f,P_n) = \int_a^b f$.

   A. True

   B. False

3. (2 points) The function $f : [0,2] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x \in [0,2) \\ -1 & \text{if } x = 2 \end{cases}$$

   is integrable.

   A. True

   B. False

4. (2 points) All bounded functions $f : [a,b] \to \mathbb{R}$ are integrable.

   A. True

   B. False

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

5. (2 points) If $f : [a,b] \to \mathbb{R}$ is continuous, then $F : [a,b] \to \mathbb{R}$ defined by $F(x) = \int_a^x f$ is differentiable.

   A. True

   B. False

Choose the most appropriate response for each.

6. (3 points) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Which of the following is false?

    A. $\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$

    B. $\|\mathbf{u}\|\|\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ $\qquad 6 = (2)(3) \nleq 2+3 = 5$

    C. $\|\mathbf{u}\| + \|\mathbf{v}\| \geq \|\mathbf{u} + \mathbf{v}\|$

    D. $\|\mathbf{u} - \mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$

7. (3 points) If $\{\mathbf{u}_k\}, \{\mathbf{v}_k\}$ are sequences of vectors in $\mathbb{R}^n$ converging to $\mathbf{u}, \mathbf{v}$ respectively, then which of these must be true?

    A. $\lim_{k\to\infty} \mathbf{u}_k = \lim_{k\to\infty} \mathbf{v}_k$

    B. $\mathbf{v} + \lim_{k\to\infty} \mathbf{u}_k = \mathbf{u} + \lim_{k\to\infty} \mathbf{v}_k$ $= \underline{u} + \underline{v}$

    C. $\lim_{k\to\infty}(\mathbf{u}_k - \mathbf{v}_k) = \mathbf{0}$

    D. $\lim_{k\to\infty}\langle \mathbf{u}_k, \mathbf{v}_k \rangle = \|\mathbf{u}\| + \|\mathbf{v}\|$

8. (4 points) The set $\{\mathbf{x} \in \mathbb{R}^2 : p_1(\mathbf{x}) < p_2(\mathbf{x})\}$ is which of the following?

    A. Open, but not closed $\qquad \Rightarrow \{(x,y) \in \mathbb{R}^2 : x < y\}$

    B. Closed, but not open

    C. Both open and closed

    D. Neither open nor closed

MATH 3142 Midterm Parts 2 & 3

1. (20 points) Prove that if Q_n is a partition of $[a,b]$ refining the partition P_n of $[a,b]$ for each natural number n, and $\{P_n\}$ is an Archimedean sequence of partitions for f on $[a,b]$, then $\{Q_n\}$ is also Archimedean.

Since Q_n refines P_n, it follows that

$$L(f,Q_n) \geq L(f,P_n), \quad -L(f,Q_n) \leq -L(f,P_n)$$
$$U(f,Q_n) \leq U(f,P_n), \text{ and } U(f,Q_n) - L(f,P_n) \leq U(f,P_n) - L(f,P_n).$$

Since $\{P_n\}$ is Archimedean, we may note

$$\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0.$$

So we can show that

$$0 \leq \lim_{n\to\infty} U(f,Q_n) - L(f,Q_n)$$
$$\leq \lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0.$$

Therefore $\{Q_n\}$ is Archimedean.

2. (20 points) Explain the error(s) in the following "proof", and then give a counterexample showing that the theorem is false.

Theorem: If $f : [0,1] \to \mathbb{R}$ is integrable, then f is also continuous.

Proof: Since f is integrable, we may define $F : [0,1] \to \mathbb{R}$ by $F(x) = \int_0^x f$. It follows that $F(x)$ is a differentiable function, because it is an antiderivative of f. Thus $\frac{d}{dx}[F(x)] = f(x)$ by the Second Fundamental Theorem of Calculus. Since the derivative of any differentiable function is continuous, we conclude f is continuous.

Prop 6.27 only guarantees F is continuous when f is only integrable.

This is also untrue (at least, it was never given as a theorem).

Note that $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

is integrable by Thm 6.19, but not continuous at $x = 1$.

3. (20 points) Recall that an **even** function satisfies the condition $f(x) = f(-x)$. Let $f : \mathbb{R} \to \mathbb{R}$ be an even continuous function. Prove that

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = 2f(x).$$

(Hint: Corollary 6.30 says that $\frac{d}{dx}[\int_x^0 f] = -f(x)$.) So by Chain Rule, $\frac{d}{dx}\left[\int_{-x}^{0} f\right] = -(-f(-x))$

$$\frac{d}{dx}\left[\int_{-x}^{x} f\right] = \frac{d}{dx}\left[\int_{-x}^{0} f + \int_{0}^{x} f\right]$$

$$= -(-f(-x)) + f(x)$$

$$= f(-x) + f(x)$$

$$= f(x) + f(x)$$

$$= 2f(x). \quad \square$$

4. (20 points) Prove the following theorem:

Let $\mathbf{x} \in \mathbb{R}^n$ and let $\{\mathbf{x}_k\}$ be a sequence of points in $\mathbb{R}^n$. If for every open set U containing $\mathbf{x}$, there is an index K such that $\mathbf{x}_k \in U$ for all $k \geq K$, then $\{\mathbf{x}_k\}$ converges to $\mathbf{x}$.

(Hint: $B_\epsilon(\mathbf{x})$ is open.)

Let $\varepsilon > 0$, Since $B_\varepsilon(\underline{x})$ is open, there exists $K \in \mathbb{N}$ where $\underline{x}_k \in B_\varepsilon(\underline{x})$ for all $k \geq K$. It follows that $\text{dist}(\underline{x}_k, \underline{x}) < \varepsilon$ for all $k \geq K$. Therefore $\{\underline{u}_k\}$ converges to $\underline{u}$.

5. (20 points) Prove that any finite subset of $\mathbb{R}^n$ is closed.
(Hint: First prove that any singleton subset of $\mathbb{R}^n$ is closed.)

Let $C = \{\underline{x}\}$ for $\underline{x} \in \mathbb{R}^n$. If $\{\underline{u}_k\}$ is a sequence of points in C, then $\underline{u}_k = x$ for all k. Thus $\{\underline{u}_k\}$ converges to $\underline{x} \in C$, so C is closed.

By Prop 10.18. ii, a finite union of closed sets is closed. Since every finite set $\{\underline{x}_1, \ldots, \underline{x}_m\}$ is the union $\bigcup_{i=1}^{m} \{\underline{x}_i\}$, every finite set is closed.