CSC279 HW5

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Collaborator

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PROBLEM 15

Answer:

1. Assume q is inside P. We want to find the closest point to q in $\{p_1, \ldots, p_n\}$.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Arrange all points on the circumference of a circle like regular convex n-polygon enclosing point q as its center. If the algorithm is deterministic and assumes an easy case, some points remain unvisited. For those unvisited point, we one of them closer. Therefore, the algorithm can not find the correct closest point and outputting incorrect results.

2. Assume q is outside P. We want to find the closest point to q in $\{p_1, \ldots, p_n\}$.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Place all points on a quarter-circle like regular convex n-polygon enclosing point q while ensuring P is convex and non-enclosing. Assume easy, then there will be some point that the algorithm (deterministic) will not visit. For an unvisited point, we move it closer. Therefore, the algorithm can not find the correct closest point and outputting incorrect results.

3. Assume q is inside P. We want to find the farthest point to q in $\{p_1, \ldots, p_n\}$.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Similar to Q1, but this time we move an unvisited point further.

4. Assume q is outside P. We want to find the farthest point to q in $\{p_1, \ldots, p_n\}$.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Similar to Q2, but this time we move an unvisited point further.

5. Assume q is inside P. We want to find the closest point to q on P.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Similar to Q1 again, The number of edges equals the number of points, so we still need at least $O(n)$ runtime.

6. Assume q is outside P. We want to find the closest point to q on P.
   Reasoning:
   Easy and can be solved within $O(logn)$.
   The algorithm finds the closest point to q on a convex polygon P in $O(\log n)$ time using the unimodal nature of the distance function from q to P. Using ternary search on the edges of P, it narrows the search interval until the closest edge is identified, and then computes the closest point on that edge. The convexity of P ensures the unimodal property, guaranteeing the correctness of the ternary search.

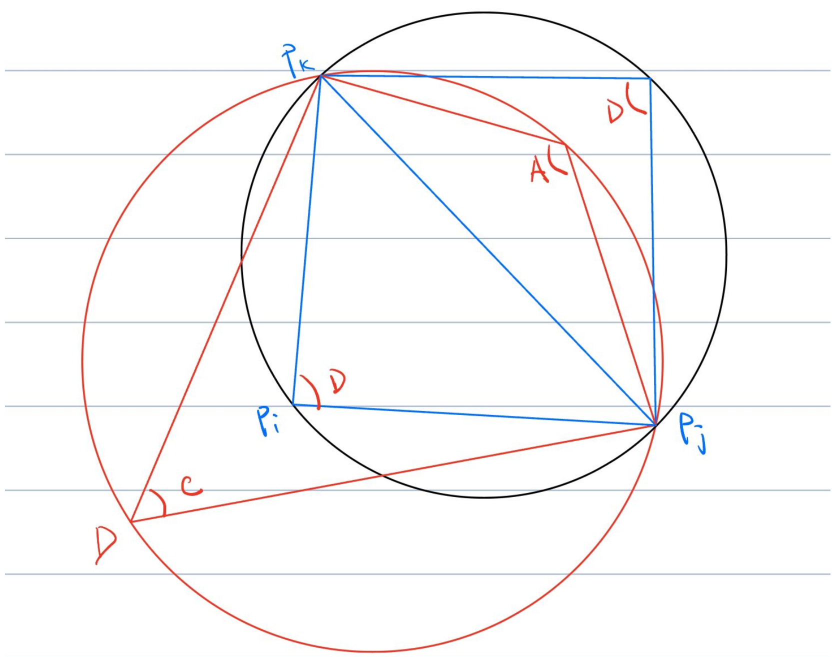
```
def closest_point_on_convex_polygon(q, P):
    n = len(P)
    low = 0
    high = n - 1

while high - low > 3:
    m1 = low + (high - low) // 3
    m2 = high - (high - low) // 3

    D_m1 = distance_to_edge(q, P[m1], P[(m1 + 1) % n])
    D_m2 = distance_to_edge(q, P[m2], P[(m2 + 1) % n])

    if D_m1 < D_m2:
        high = m2
    else:
        low = m1

min_dist = float('inf')
closest_point = None

for i in range(low, high + 1):
    p1 = P[i]
    p2 = P[(i + 1) % n]
    c = closest_point_on_segment(q, p1, p2)
    D = distance(q, c)
    if D < min_dist:
        min_dist = D
        closest_point = c

return closest_point
```

7. Assume q is inside P. We want to find the farthest point to q on P.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Similar to Q3, so similar argument can be made.

8. Assume q is outside P. We want to find the farthest point to q on P.
   Reasoning:
   Hard and required $\Omega(n)$ runtime. Similar to Q4. The farthest point must lie on the circumcircle as all other points are on an n-gon, so similar argument can be made.

PROBLEM 16

Proof. **Theorem:**
A convex polygon is fully contained within the largest circumcircle formed by three of its consecutive vertices.
Lemma:
Let $P = \{p_1, p_2, \ldots, p_n\}$ represent a convex polygon. Suppose a triangle T is formed by three vertices of P, and the circumcircle of T contains P. For any edge $\overline{p_a p_b}$ of T, there exists a vertex p_c between p_a and p_b such that the circumcircle of $T_{ab,c}$ contains P.
Proof
Let p_i, p_j, p_k be three vertices defining the triangle $T_{ij,k}$. Without loss of generality, consider the edge $\overline{p_j p_k}$ and its corresponding arc on the circumcircle. There exists $c \in (i, j)$ such that for every $a \in (i, j)$, the circumcircle of $T_{ij,c}$ contains p_a. Now, we examine two cases for p_c:
Case 1: p_c **lies on the circumcircle of** $T_{ij,k}$ If p_c is on the circumcircle of $T_{ij,k}$, then the circumcircle of $T_{j,k,c}$ is the same as that of $T_{ij,k}$. Thus, the lemma holds.
Case 2: p_c **lies inside the circumcircle of** $T_{ij,k}$ Construct a quadrilateral



with vertices p_j, p_c, p_k, D that forms a cyclic quadrilateral. By the properties of cyclic quadrilaterals:

$$\angle A + \angle D = \pi \quad \text{and} \quad \angle B + \angle C = \pi.$$

Using these properties:

$$\angle A + \angle B = \pi - \angle C < \pi - \angle D \implies \angle A > \angle D.$$

For any point z inside or on the circumcircle, it cannot satisfy $\angle p_j z p_k > \angle p_j p_c p_k$. Thus, z must lie outside the circumcircle of $T_{ij,k}$, ensuring that the circumcircle of $T_{ij,c}$ contains all points between the arc $p_j p_c p_k$. Hence, the lemma is proved.

Triangulation Construction

1. Start with three consecutive vertices p_i, p_{i+1}, p_{i+2} such that the circumcircle of $T_{p_i p_{i+1} p_{i+2}}$ contains P.

2. For each new triangle T, select any edge $\overline{p_a p_b}$. If there are no points of P within the range of vertices $\overline{p_a p_b}$, skip this edge. Otherwise, find a point p_c such that the circumcircle of $T_{p_a p_b p_c}$ contains P.

3. Repeat this process iteratively, ensuring that every newly constructed triangle satisfies the condition that its circumcircle contains P.

This method guarantees that the entire polygon P is contained within the circumcircle of the final triangulation. $\square$

PROBLEM 17

General Algorithm Thoughts:

1. **Construct the Voronoi diagram** for all sites.
2. **For each Voronoi cell**, examine its corners (vertices).
3. **Check if any corner is at a distance $\geq l + r$ from its associated site**.
   - **Reasoning**: Corners are the farthest points within a cell from the site.
   - They are **equidistant to the site and neighboring sites**.
   - If a corner is at distance $\geq l + r$ from the site, it is also that far from neighboring sites.
4. **Conclusion**: If such a corner exists, the site is "good" because all points at that corner are sufficiently distant from all relevant sites.

Potentially A More Refined and Rigorous Version

The algorithm identifies all "good" points by first constructing the Voronoi diagram of the given points, which efficiently captures proximity relationships in $O(n \log n)$ time. For each point p_i, it examines only its neighboring points in the Voronoi diagram, as these are the only ones that could potentially interfere with placing a new circle. By computing the angular intervals where a circle of radius ℓ touching C_i would intersect any neighboring C_j, the algorithm determines the directions that are blocked. If there exists at least one direction where such interference does not occur, the point p_i is therefore "good".

```
# Helper Functions
def compute_interfering_angles(p_i, p_j, r, l, d_ij):
    # Calculate the angle between p_i and p_j
    delta_x = p_j.x - p_i.x
    delta_y = p_j.y - p_i.y
    alpha = atan2(delta_y, delta_x)

    # Law of Cosines to find the angular width
    cos_theta = (d_ij**2 + (r + l)**2 - (r + l)**2) / (2 * d_ij * (r + l))
    if abs(cos_theta) <= 1:
        theta = acos(cos_theta)
        # The interfering interval is [alpha - theta, alpha + theta]
        interval = [(alpha - theta) % (2 * pi), (alpha + theta) % (2 * pi)]
        # Handle interval wrapping around 2pi
        if interval[0] > interval[1]:
            return [(interval[0], 2 * pi), (0, interval[1])]
        else:
```

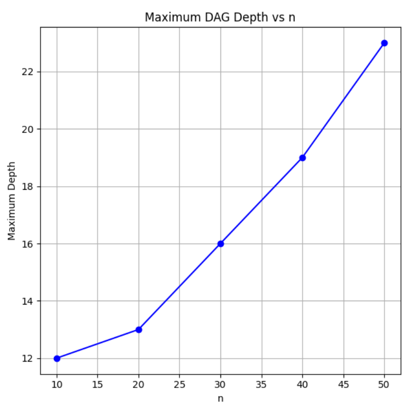
```
            return [interval]
    else:
        # Circles do not intersect; no interfering angles
        return []

# Main Function
def find_good_points(P, r, l):
    # Construct the Voronoi diagram
    # Need O(nlogn)
    V = voronoi_diagram(P)

    good_points = []

    # For each point p_i
    # Need O(n)
    for p_i in P:
        interfering_angles = []  # List to store interfering angular intervals

        # Get neighboring points in the Voronoi diagram
        neighbors = V.get_neighbors(p_i)

        # For each neighbor p_j
        for p_j in neighbors:
            d_ij = distance(p_i, p_j)

            # Only consider neighbors that may interfere
            if d_ij < 2 * (r + l):
                # Compute the angular intervals of interference
                angles = compute_interfering_angles(p_i, p_j, r, l, d_ij)
                interfering_angles.extend(angles)

        # Compute the union of interfering intervals
        interfering_union = union_of_intervals(interfering_angles)

        # Determine the complement of the union over [0, 2pi)
        non_interfering_angles = complement_of_intervals(interfering_union, 0, 2 * pi)

        # If there is at least one non-interfering angle, p_i is good
        if non_interfering_angles:
            good_points.append(p_i)

    return good_points
```

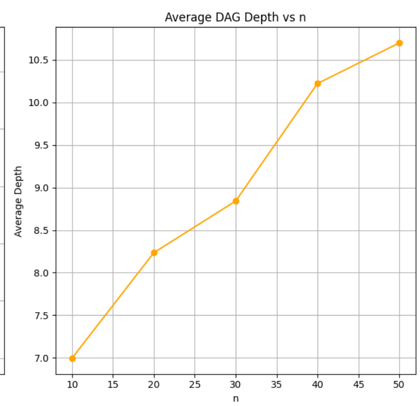
PROBLEM 18

Result:

Processing n	Max Depth	Avg Depth
10	8	5.45273631840796
20	12	7.750312109862672
30	18	8.939478067740144
40	17	9.610121836925961
50	21	9.93361327734453

Table 1: Depth statistics for different values of n





Short Analysis:
The result I got is reasonable, with the average depths increasing in $log n$ as expected, indicating that the algorithm effectively maintains a balanced DAG structure. Although the maximum depths are somewhat higher than theoretical predictions, they remain within an acceptable range considering the algorithm's randomness and the potential for local depth increases during edge legalization.
Implementation:
The following code of randomized Delaunay triangulation algorithm with history DAG was implemented in Python:

```
# Data Structures Definitions

class Point:
    def __init__(self, x, y):
        self.x = x
        self.y = y
```

```
class SegmentWrapper:
    """
    Helper class to uniquely identify a segment irrespective of point order.
    """
    def __init__(self, p1, p2):
        # Ensure consistent ordering
        if (p1.x, p1.y) < (p2.x, p2.y):
            self.p1, self.p2 = p1, p2
        else:
            self.p1, self.p2 = p2, p1

    def __eq__(self, other):
        return (self.p1.x, self.p1.y) == (other.p1.x, other.p1.y) and \
               (self.p2.x, self.p2.y) == (other.p2.x, other.p2.y)

    def __hash__(self):
        return hash(((self.p1.x, self.p1.y), (self.p2.x, self.p2.y)))

    def __repr__(self):
        return f"Segment(({self.p1.x}, {self.p1.y}) - ({self.p2.x}, {self.p2.y}))"

class Triangle:
    def __init__(self, p1, p2, p3):
        self.vertices = [p1, p2, p3]  # Points
        self.edges = [
            SegmentWrapper(p1, p2),
            SegmentWrapper(p2, p3),
            SegmentWrapper(p3, p1)
        ]
        self.neighbors = {}  # edge -> adjacent triangle
        self.children = []  # For history DAG

    def contains_point(self, point):
        return point in self.vertices

    def __repr__(self):
        verts = ', '.join([f"({p.x}, {p.y})" for p in self.vertices])
        return f"Triangle({verts})"

class HistoryDAGNode:
    def __init__(self, triangle):
        self.triangle = triangle  # Triangle object
        self.children = []        # List of HistoryDAGNode

    def add_child(self, child_node):
```

```
        self.children.append(child_node)

# Delaunay Triangulation Class

class DelaunayTriangulation:
    def __init__(self, points):
        self.points = points  # List of Point objects
        self.segments = {}    # SegmentWrapper -> list of adjacent Triangles
        self.triangles = []   # List of Triangle objects
        self.history_root = None

    def create_super_triangle(self):
        """
        Create a super-triangle that encompasses all the points.
        """
        min_x = min(p.x for p in self.points)
        max_x = max(p.x for p in self.points)
        min_y = min(p.y for p in self.points)
        max_y = max(p.y for p in self.points)

        dx = max_x - min_x
        dy = max_y - min_y
        delta_max = max(dx, dy) * 100  # Make it large enough

        # Create three points that form a super-triangle
        p1 = Point(min_x - delta_max, min_y - delta_max)
        p2 = Point(min_x + 2 * delta_max, min_y - delta_max)
        p3 = Point(min_x - delta_max, max_y + 2 * delta_max)

        super_triangle = Triangle(p1, p2, p3)
        self.triangles.append(super_triangle)
        self.history_root = HistoryDAGNode(super_triangle)

        # Add segments of the super-triangle
        for edge in super_triangle.edges:
            self.segments.setdefault(edge, []).append(super_triangle)

    def locate_containing_triangle(self, point):
        """
        Traverse the history DAG to locate the triangle containing the point.
        """
        node = self.history_root
        while node.children:
            found = False
            for child in node.children:
                if self.point_in_triangle(point, child.triangle):
```

```
                node = child
                found = True
                break
        if not found:
            break  # Point not found in any child; fallback
    # Now, node.triangle should contain the point
    return node.triangle, node

@staticmethod
def point_in_triangle(p, triangle):
    """
    Check if point p is inside the given triangle using barycentric coordinates.
    """
    def sign(p1, p2, p3):
        return (p1.x - p3.x) * (p2.y - p3.y) - \
               (p2.x - p3.x) * (p1.y - p3.y)

    b1 = sign(p, triangle.vertices[0], triangle.vertices[1]) < 0.0
    b2 = sign(p, triangle.vertices[1], triangle.vertices[2]) < 0.0
    b3 = sign(p, triangle.vertices[2], triangle.vertices[0]) < 0.0

    return ((b1 == b2) and (b2 == b3))

def insert_point(self, point):
    """
    Insert a single point into the triangulation.
    """
    containing_triangle, containing_node = self.locate_containing_triangle(point)

    # Remove the containing triangle
    self.triangles.remove(containing_triangle)

    # Create new triangles by connecting the point
    # to the vertices of the containing triangle
    t1 = Triangle(point, containing_triangle.vertices[0],
        containing_triangle.vertices[1])
    t2 = Triangle(point, containing_triangle.vertices[1],
        containing_triangle.vertices[2])
    t3 = Triangle(point, containing_triangle.vertices[2],
        containing_triangle.vertices[0])

    # Set neighbors
    self.set_neighbors(t1, containing_triangle)
    self.set_neighbors(t2, containing_triangle)
    self.set_neighbors(t3, containing_triangle)
```

```python
        # Add new triangles
        self.triangles.extend([t1, t2, t3])

        # Update segments
        for t in [t1, t2, t3]:
            for edge in t.edges:
                self.segments.setdefault(edge, []).append(t)

        # Update history DAG
        child_nodes = [
            HistoryDAGNode(t1),
            HistoryDAGNode(t2),
            HistoryDAGNode(t3)
        ]
        for child in child_nodes:
            containing_node.add_child(child)

        # Edge Legalization
        for t in [t1, t2, t3]:
            for edge in t.edges:
                if point in [edge.p1, edge.p2]:
                    self.legalize_edge(edge, t, point)

    def set_neighbors(self, new_triangle, old_triangle):
        """
        Set the neighboring triangles for the new triangle.
        """
        for edge in new_triangle.edges:
            if edge in self.segments:
                for neighbor in self.segments[edge]:
                    if neighbor != old_triangle and neighbor != new_triangle:
                        new_triangle.neighbors[edge] = neighbor
                        neighbor.neighbors[edge] = new_triangle

    def legalize_edge(self, edge, triangle, point):
        """
        Legalize an edge to restore the Delaunay condition.
        """
        if edge not in triangle.neighbors:
            return  # Boundary edge

        neighbor = triangle.neighbors[edge]
        if neighbor is None:
            return

        # Find the opposite point in the neighbor triangle
```

```
        opposite_point = [v for v in neighbor.vertices
                          if v not in [edge.p1, edge.p2]][0]

        if self.in_circumcircle(opposite_point, triangle):
            # Perform edge flip
            new_triangles = self.edge_flip(triangle, neighbor, edge,
                                            point, opposite_point)
            for new_t in new_triangles:
                for e in new_t.edges:
                    if point in [e.p1, e.p2]:
                        self.legalize_edge(e, new_t, point)

    def edge_flip(self, t1, t2, edge, point, opposite_point):
        """
        Flip the shared edge between two triangles.
        """
        # Remove old triangles
        self.triangles.remove(t1)
        self.triangles.remove(t2)

        # Remove edge from segments
        self.segments[edge].remove(t1)
        self.segments[edge].remove(t2)
        if not self.segments[edge]:
            del self.segments[edge]

        # Create new edge
        new_edge = SegmentWrapper(point, opposite_point)

        # Create new triangles
        new_t1 = Triangle(point, edge.p1, opposite_point)
        new_t2 = Triangle(point, opposite_point, edge.p2)

        # Update segments
        for t in [new_t1, new_t2]:
            for e in t.edges:
                self.segments.setdefault(e, []).append(t)

        # Update neighbors
        self.update_neighbors_after_flip(t1, t2, new_t1, new_t2, edge, new_edge)

        # Add new triangles
        self.triangles.extend([new_t1, new_t2])

        # Update history DAG
        parent_node = HistoryDAGNode(None)
```

```
        t1_node = self.find_history_node(self.history_root, t1)
        t2_node = self.find_history_node(self.history_root, t2)
        parent_node.add_child(t1_node)
        parent_node.add_child(t2_node)
        new_t1_node = HistoryDAGNode(new_t1)
        new_t2_node = HistoryDAGNode(new_t2)
        parent_node.add_child(new_t1_node)
        parent_node.add_child(new_t2_node)

        return [new_t1, new_t2]

    def update_neighbors_after_flip(self, t1, t2, new_t1, new_t2, old_edge, new_edge):
        """
        Update neighbor relationships after an edge flip.
        """
        # Set neighbors for new_t1
        new_t1.neighbors[new_edge] = new_t2
        new_t2.neighbors[new_edge] = new_t1

        # Update other neighbors
        for e in new_t1.edges:
            if e != new_edge:
                for neighbor in self.segments[e]:
                    if neighbor != new_t1:
                        new_t1.neighbors[e] = neighbor
                        neighbor.neighbors[e] = new_t1

        for e in new_t2.edges:
            if e != new_edge:
                for neighbor in self.segments[e]:
                    if neighbor != new_t2:
                        new_t2.neighbors[e] = neighbor
                        neighbor.neighbors[e] = new_t2

    def find_history_node(self, node, triangle):
        """
        Find the history DAG node corresponding to the given triangle.
        """
        if node.triangle == triangle:
            return node
        for child in node.children:
            result = self.find_history_node(child, triangle)
            if result is not None:
                return result
        return None
```

```
def in_circumcircle(self, point, triangle):
    """
    Check if a point is inside the circumcircle of a triangle.
    """
    ax, ay = triangle.vertices[0].x - point.x, triangle.vertices[0].y - point.y
    bx, by = triangle.vertices[1].x - point.x, triangle.vertices[1].y - point.y
    cx, cy = triangle.vertices[2].x - point.x, triangle.vertices[2].y - point.y

    det = (ax * (by * (cx**2 + cy**2) - cy * (bx**2 + by**2)) -
           ay * (bx * (cx**2 + cy**2) - cx * (bx**2 + by**2)) +
           (ax**2 + ay**2) * (bx * cy - cx * by))

    return det > 0

def build_triangulation(self):
    """
    Build the triangulation by inserting all points.
    """
    self.create_super_triangle()
    for point in self.points:
        self.insert_point(point)
    self.remove_super_triangle()

def remove_super_triangle(self):
    """
    Remove any triangles that share a vertex with the super-triangle.
    """
    # Super-triangle vertices
    super_vertices = set(self.history_root.triangle.vertices)
    self.triangles = [t for t in self.triangles
                      if not any(v in super_vertices for v in t.vertices)]

def calculate_dag_depths(self):
    """
    Calculate maximum and average depths of the history DAG.
    """
    depths = []
    queue = deque([(self.history_root, 0)])

    while queue:
        node, depth = queue.popleft()
        if not node.children:
            depths.append(depth)
        else:
            for child in node.children:
                queue.append((child, depth + 1))
```

```
        max_depth = max(depths) if depths else 0
        avg_depth = sum(depths) / len(depths) if depths else 0
        return max_depth, avg_depth

    def plot_triangulation(self):
        """
        Plot the triangulation.
        """
        plt.figure(figsize=(8, 8))
        for triangle in self.triangles:
            x_coords = [vertex.x for vertex in triangle.vertices + [triangle.vertices[0]]]
            y_coords = [vertex.y for vertex in triangle.vertices + [triangle.vertices[0]]]
            plt.plot(x_coords, y_coords, 'k-')
        plt.scatter([p.x for p in self.points],
                    [p.y for p in self.points], color='red', s=10)
        plt.axis('equal')
        plt.title('Delaunay Triangulation')
        plt.show()
```