

# CSC279 HW3

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## Question 10

### (PART A)

#### Parametrizing the Segment $pq$ :

Let  $p = (x_0, y_0)$  and  $q = (x_1, y_1)$  be the endpoints of the segment  $pq$ . Any point  $x(t)$  on  $pq$  can be expressed as:

$$x(t) = (1-t)p + tq = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \quad t \in [0, 1]$$

#### Dual Lines Corresponding to Points on $pq$ :

The dual line  $\hat{x}(t)$  Corresponding to  $x(t)$  is:

$$y = A(t)x - B(t)$$

where:

$$A(t) = (1-t)x_0 + tx_1$$

$$B(t) = (1-t)y_0 + ty_1$$

Now we can express rewrite the equation of  $\hat{x}(t)$  as:

$$\begin{aligned} y &= [x_0 + t(x_1 - x_0)]x - [y_0 + t(y_1 - y_0)] \\ &= x_0x - y_0 + t[(x_1 - x_0)x - (y_1 - y_0)] \end{aligned}$$

Let:

$$D = (x_1 - x_0)x - (y_1 - y_0)$$

Then:

$$y = x_0x - y_0 + tD$$

For a fixed  $x$ ,  $y$  varies linearly with  $t$  from  $y = x_0x - y_0$  (when  $t = 0$ ) to  $y = x_1x - y_1$  (when  $t = 1$ ). The Union of all dual lines  $\hat{x}(t)$  for  $t \in [0, 1]$  is the set of all points  $(x, y)$  in the plane satisfying:

$$\min(y_0, y_1) \leq y - x \cdot \min(x_0, x_1) \leq \max(y_0, y_1)$$

Equivalently, by eliminating parameter  $t$ , we can get similar inequality:  
 We aim to eliminate the parameter  $t$  from the parametric equations to describe the union of dual lines  $\hat{x}(t)$ .  
 Solving for  $t$ :

$$t = \frac{y - x_0x + y_0}{D}$$

**Note:** The sign of  $D$  affects the inequality direction.  
 If  $D > 0$ , then

$$0 \leq \frac{y - x_0x + y_0}{D} \leq 1 \Rightarrow 0 \leq y - x_0x + y_0 \leq D$$

If  $D < 0$ , then

$$0 \geq \frac{y - x_0x + y_0}{D} \geq 1 \Rightarrow D \leq y - x_0x + y_0 \leq 0$$

Both cases can be unified by the product inequality:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes the region between the lines  $y = x_0x - y_0$  and  $y = x_1x - y_1$ , where the expressions  $y - x_0x + y_0$  and  $y - x_1x + y_1$  have opposite signs or are zero.

Hence, the union of all dual lines is:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes all points  $(x, y)$  that lie between  $y = x_0x - y_0$ ,  $y = x_1x - y_1$ . which forms a region called a "double wedge".

## (PART B)

Input:

$S = \{ s_i \mid i = 1 \text{ to } n \}$

output:

$l_{\max}$  # Line has max. intersections point with other sections

Algorithm FindMaxIntersectingLine(S):

Initialize an empty list  $L_{\text{dual\_lines}}$ .

For each segment  $s_i$  in  $S$  do:

Let  $p_i = (x_{\{0i\}}, y_{\{0i\}})$  and  $q_i = (x_{\{1i\}}, y_{\{1i\}})$

Compute the dual lines:

$L_{\{p_i\}}: y = x_{\{0i\}} x - y_{\{0i\}}$

$L_{\{q_i\}}: y = x_{\{1i\}} x - y_{\{01\}}$

Add  $L_{\{p_i\}}$  and  $L_{\{q_i\}}$  to  $L_{\text{dual\_lines}}$ .

Construct the arrangement  $A$  of the lines in  $L_{\text{dual\_lines}}$ :

# TIME COMPLEXITY:  $O(n^2 \log n)$

Use the line sweep algorithm to compute the  $A$ .

Store the faces, edges, and vertices of the  $A$ .

Initialize count  $c_f = 0$  for a starting PLANE  $f_0$  at INFINITY

Traverse arrangement  $A$  to label each face with the number

of double wedges covering it:

# TIME COMPLEXITY:  $O(n^2)$

For each edge  $e$  in  $A$ :

Determine which double wedges have boundaries along  $e$

For each face  $f$  adjacent to  $e$ :

\* $c_{\{f'\}}$  is the count of the adjacent face before crossing  $e$

If crossing  $e$  enters a double wedge  $W_i$ , then  $c_f = c_{\{f'\}} + 1$

If crossing  $e$  exits a double wedge  $W_i$ , then  $c_f = c_{\{f'\}} - 1$

Keep track of the face  $f_{\max}$  with the maximum count  $c_{\max}$  during traversal

Let  $(a_{\max}, b_{\max})$  be a point inside face  $f_{\max}$ .

Compute the line  $l_{\max}$  in the primal plane corresponding to  $(a_{\max}, b_{\max})$ :

$l_{\max}: y = a_{\max} x - b_{\max}$

Return  $l_{\max}$

## Question 11

Input:

Simple polygon  $P$  given as a list of vertices  $[v_1, v_2, \dots, v_n]$  in order

Output:

Whether there exists a line  $l$  such that  $P$  is monotone w.r.t.  $l$

Algorithm DetermineMonotonicity( $P$ ):

BadIntervals = empty\_list()

# TIME COMPLEXITY:  $O(n)$

For  $i$  from 1 to  $n$ :

$v_{\text{prev}} = P[(i - 2) \bmod n]$

$v = P[i - 1]$

$v_{\text{next}} = P[i \bmod n]$

    Compute vectors  $e_1 = v - v_{\text{prev}}$

    Compute vectors  $e_2 = v_{\text{next}} - v$

    Compute cross product  $cp = e_1.x * e_2.y - e_1.y * e_2.x$

    If  $cp < 0$  (vertex is concave):

        Compute angles  $a_1 = \text{atan2}(e_1.y, e_1.x) \bmod 2\pi$

        Compute angles  $a_2 = \text{atan2}(e_2.y, e_2.x) \bmod 2\pi$

        # Compute perpendicular angles

$\text{Theta}_1 = (a_1 + \pi/2) \bmod 2\pi$

$\text{Theta}_2 = (a_2 + \pi/2) \bmod 2\pi$

$\text{delta\_theta} = (\text{Theta}_2 - \text{Theta}_1) \bmod 2\pi$

        If  $\text{delta\_theta} > \pi$ :

            Swap  $\text{Theta}_1$  and  $\text{Theta}_2$

        # Create Interval

        If  $\text{Theta}_1 \leq \text{Theta}_2$ :

            Interval = [ $\text{Theta}_1$ ,  $\text{Theta}_2$ ]

        Else:

            # NOTE: 'v' denotes union

            Interval = [ $\text{Theta}_1$ ,  $2\pi$ ) v  $[0, \text{Theta}_2]$

        Add Interval to BadIntervals

# TIME COMPLEXITY:  $O(n \log n)$

Sort BadIntervals by their start angles

Merge overlapping intervals in BadIntervals to get a list of disjoint intervals

If the merged intervals cover  $[0, 2\pi)$ :

    return FALSE

```
return TRUE
```

## Question 12

Input:

```
RedPoints = [ (x1, y1), (x2, y2), ..., (xn, yn) ]  
BluePoints = [ (x1', y1'), (x2', y2'), ..., (xn', yn') ]
```

Output:

Coefficients (a, b, c) defining the parabola  $y = a x^2 + b x + c$   
or report "No solution exists" if impossible

Algorithm FindSeparatingParabola(RedPoints, BluePoints):

Initialize an empty list Constraints = []

# NOTE: e is a small positive number

# TIME COMPLEXITY:  $O(n)$

For each red point  $(x_i, y_i)$  in RedPoints do:

Make inequality:

$$a x_i^2 + b x_i + c - y_i < 0$$

Convert to standard LP form (inequalities with  $\leq$ ):

$$a x_i^2 + b x_i + c - y_i \leq -e$$

Add this to Constraints

# TIME COMPLEXITY:  $O(n)$

For each blue point  $(x_j, y_j)$  in RedPoints do:

Make inequality:

$$a x_j^2 + b x_j + c - y_j > 0$$

Convert to standard LP form (inequalities with  $\leq$ ):

$$-a x_j^2 - b x_j - c + y_j \leq -e$$

Add this to Constraints

ObjectiveFunction: Minimize 0

# TIME COMPLEXITY:  $O(n)$

Solution = LinearProgramming(Constraints, ObjectiveFunction)

If Solution is feasible then:

Output "Parabola found with coefficients:"

Output "a =", Solution.a

Output "b =", Solution.b

Output "c =", Solution.c

Output "No solution exists"

## Question 13

For this question, we use an approximation to replace  $L_2$  norm in order to use LP.

$$\begin{cases} r \geq 0 \\ w \geq 0 \\ \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, k\} : \\ \quad (p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \geq r \cos \left( \frac{\pi}{k} \right) \\ \quad (p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \leq (r + w) \sec \left( \frac{\pi}{k} \right) \end{cases}$$

### Variables

- $c_x, c_y$ : Coordinates of the center  $c$  of the annulus
- $r \geq 0$ : Inner Radius
- $w \geq 0$ : Width of the annulus ( $w = R - r$ )

### Objective Function:

$$\min w$$

### Constraint:

1.  $r \geq 0, w \geq 0$
2. To approximate  $L_1$  norm, we pick  $k$  directions. For a regular  $k$ -gon inscribed in the unit circle, define angles:

$$\theta_j = \frac{2\pi j}{k}, \quad j = 1, 2, \dots, k$$

3. Compute unit vectors in these directions:

$$u_j = (\cos \theta_j, \sin \theta_j)$$

4. For each point  $p_i = (p_{i,x}, p_{i,y})$  and each direction  $u_j$ :

- Inner Boundary Constraint:

$$(p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \geq r \cos \left( \frac{\pi}{k} \right)$$

- Outer Boundary Constraint:

$$(p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \leq (r + w) \sec \left( \frac{\pi}{k} \right)$$

The factors  $\cos \left( \frac{\pi}{k} \right)$  and  $\sec \left( \frac{\pi}{k} \right)$  adjust for the approximation error due to replacing the circle with a regular  $k$ -gon.