CSC279 HW5

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Collaborator

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Question 15

PROBLEM 15—Closest point, farthest point.

1. Assume q is inside P. We want to find the closest point to q in $\{p_1, \ldots, p_n\}$.
2. Assume q is outside P. We want to find the closest point to q in $\{p_1, \ldots, p_n\}$.
3. Assume q is inside P. We want to find the farthest point to q in $\{p_1, \ldots, p_n\}$.
4. Assume q is outside P. We want to find the farthest point to q in $\{p_1, \ldots, p_n\}$.
5. Assume q is inside P. We want to find the closest point to q on P.
6. Assume q is outside P. We want to find the closest point to q on P.
7. Assume q is inside P. We want to find the farthest point to q on P.
8. Assume q is outside P. We want to find the farthest point to q on P.

Answer:
Question 1 - 4 Reasoning: The distance from q to the vertices p_i is unimodal function along the ordered sequence. This is true whether q is inside or outside P. Along the vertices, we can use ternary (binary) search on sequence of vertices to find MIN or MAX distance.

1. **Problem 1:** q inside P; find the closest point to q in $\{p_1, \ldots, p_n\}$.
   - **Solution:** $O(\log n)$ time.
   - **Reasoning:** The distance function is unimodal along the vertices, allowing ternary search.
2. **Problem 2:** q outside P; find the closest point to q in $\{p_1, \ldots, p_n\}$.

- **Solution:** $O(\log n)$ time.
- **Reasoning:** Similar to Problem 1, use ternary search due to the unimodal distance function.

3. **Problem 3:** q inside P; find the farthest point from q in $\{p_1, \dots, p_n\}$.
   - **Solution:** $O(\log n)$ time.
   - **Reasoning:** Unimodal distance function allows ternary search for the maximum.

4. **Problem 4:** q outside P; find the farthest point from q in $\{p_1, \dots, p_n\}$.
   - **Solution:** $O(\log n)$ time.
   - **Reasoning:** Same as Problem 3, apply ternary search on the unimodal distance function.

Question 5 - 6 Reasoning:
The distance from q to the boundary of the convex polygon P is a convex function along the perimeter, regardless of whether q is inside or outside P. (i.e. The distance decreases to a minimum point and then increases, forming a single through)

1. **Problem 5:** q inside P; find the closest point to q on P.
   - **Solution:** $O(\log n)$ time.
   - **Reasoning:** Perform binary search over edges to find the closest point where the minimum distance occurs.

2. **Problem 6:** q outside P; find the closest point to q on P.
   - **Solution:** $O(\log n)$ time.
   - **Reasoning:** Same as Problem 5, use binary search to find the closest point on edges.

Question 7 - 8 Reasoning:
The farthest point from q can lie anywhere along the perimeter of the convex polygon P, necessitating a check of all edges and vertices. The distance function for farthest points is not unimodal, preventing the use of efficient binary or ternary search methods. Hence, without preprocessing, we need $\Omega(n)$ time to do them.

1. **Problem 7:** q inside P; find the farthest point from q on P.
   - **Solution:** $\Omega(n)$ time.
   - **Reasoning:** Requires examining all edges using rotating calipers for antipodal points.

2. **Problem 8:** q outside P; find the farthest point from q on P.
   - **Solution:** $\Omega(n)$ time.
   - **Reasoning:** Similar to Problem 7, needs $\Omega(n)$ time to check all edges.

PROBLEM 16

Proof. For a n-polygon,

□

PROBLEM 17

```
import random
import matplotlib.pyplot as plt
from collections import deque

Input:
    - P = {p1, p2, ..., pn}: Set of n points in the plane
    - r: Radius of each circle Ci
    - l: Radius of the target circle Q, where l > r

Output:
    - G: Set of "good" points

Algorithm FindGoodPoints(p_1, ..., p_n, r, l):

1. Build the Delaunay triangulation (DT) of the points {p_1, ..., p_n}.
    - Time complexity: O(n log n)

2. Initialize an empty list GoodPoints.

3. For each point p_i in {p_1, ..., p_n}:
    - Time complexity: O(n)
    a. Initialize an empty list of intervals I_i.

    b. For each neighbor p_j of p_i in DT:
        - Time complexity: O(1).
        i. Compute d = distance between p_i and p_j.

        ii. If d <= 2l:
            - Compute theta_j = angle between vector (p_j - p_i) and the x-axis.
            - Compute phi_j = arccos(d / (2 * (l + r))).
                (Ensure the argument of arccos is within [-1, 1].)

            - If phi_j is real:
                - Add the interval [theta_j - phi_j, theta_j + phi_j] to I_i.

    c. Sort the intervals in I_i.
        - Time complexity: O(1), every Voronoi point have constant neighbor.
        - Merge overlapping intervals to get the union.
```

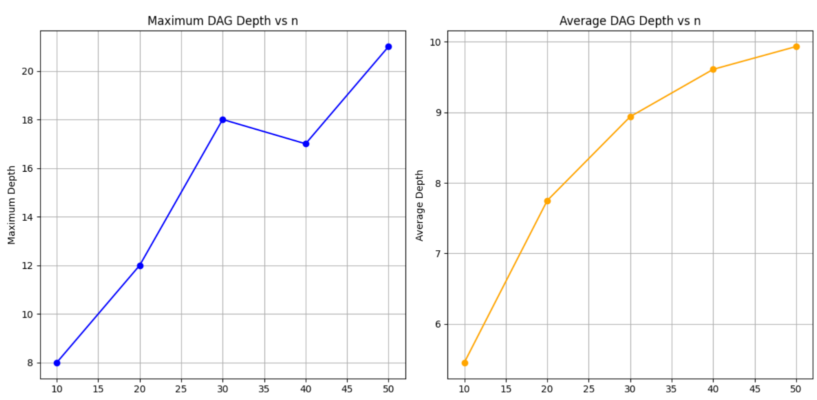
```
    d. If the union of intervals in I_i covers [0, 2pi]:
        - p_i is not good.
    e. Else:
        - p_i is good.
        - Add p_i to GoodPoints.

4. Return GoodPoints.
```

PROBLEM 18

Result:

Processing n	Max Depth	Avg Depth
10	8	5.45273631840796
20	12	7.750312109862672
30	18	8.939478067740144
40	17	9.610121836925961
50	21	9.93361327734453

Table 1: Depth statistics for different values of n



Short Analysis:
The result I got is reasonable, with the average depths increasing in *logn* as expected, indicating that the algorithm effectively maintains a balanced DAG structure. Although the maximum depths are somewhat higher than theoretical predictions, they remain within an acceptable range considering the algorithm's randomness and the potential for local depth increases during edge legalization.
Implementation:

The following code of randomized Delaunay triangulation algorithm with history DAG was implemented in Python:

```
# ----------------------------
# Data Structures Definitions
# ----------------------------

class Point:
    def __init__(self, x, y):
        self.x = x
        self.y = y

class SegmentWrapper:
    """
    Helper class to uniquely identify a segment irrespective of point order.
    """
    def __init__(self, p1, p2):
        # Ensure consistent ordering
        if (p1.x, p1.y) < (p2.x, p2.y):
            self.p1, self.p2 = p1, p2
        else:
            self.p1, self.p2 = p2, p1

    def __eq__(self, other):
        return (self.p1.x, self.p1.y) == (other.p1.x, other.p1.y) and \
               (self.p2.x, self.p2.y) == (other.p2.x, other.p2.y)

    def __hash__(self):
        return hash(((self.p1.x, self.p1.y), (self.p2.x, self.p2.y)))

    def __repr__(self):
        return f"Segment(({self.p1.x}, {self.p1.y}) - ({self.p2.x}, {self.p2.y}))"

class Triangle:
    def __init__(self, p1, p2, p3):
        self.vertices = [p1, p2, p3]  # Points
        self.edges = [
            SegmentWrapper(p1, p2),
            SegmentWrapper(p2, p3),
            SegmentWrapper(p3, p1)
        ]
        self.neighbors = {}  # edge -> adjacent triangle
        self.children = []  # For history DAG

    def contains_point(self, point):
        return point in self.vertices
```

```
    def __repr__(self):
        verts = ', '.join([f"({p.x}, {p.y})" for p in self.vertices])
        return f"Triangle({verts})"

class HistoryDAGNode:
    def __init__(self, triangle):
        self.triangle = triangle  # Triangle object
        self.children = []        # List of HistoryDAGNode

    def add_child(self, child_node):
        self.children.append(child_node)

# ----------------------------
# Delaunay Triangulation Class
# ----------------------------

class DelaunayTriangulation:
    def __init__(self, points):
        self.points = points  # List of Point objects
        self.segments = {}    # SegmentWrapper -> list of adjacent Triangles
        self.triangles = []   # List of Triangle objects
        self.history_root = None

    def create_super_triangle(self):
        """
        Create a super-triangle that encompasses all the points.
        """
        min_x = min(p.x for p in self.points)
        max_x = max(p.x for p in self.points)
        min_y = min(p.y for p in self.points)
        max_y = max(p.y for p in self.points)

        dx = max_x - min_x
        dy = max_y - min_y
        delta_max = max(dx, dy) * 100  # Make it large enough

        # Create three points that form a super-triangle
        p1 = Point(min_x - delta_max, min_y - delta_max)
        p2 = Point(min_x + 2 * delta_max, min_y - delta_max)
        p3 = Point(min_x - delta_max, max_y + 2 * delta_max)

        super_triangle = Triangle(p1, p2, p3)
        self.triangles.append(super_triangle)
        self.history_root = HistoryDAGNode(super_triangle)
```

```
        # Add segments of the super-triangle
        for edge in super_triangle.edges:
            self.segments.setdefault(edge, []).append(super_triangle)

    def locate_containing_triangle(self, point):
        """
        Traverse the history DAG to locate the triangle containing the point.
        """
        node = self.history_root
        while node.children:
            found = False
            for child in node.children:
                if self.point_in_triangle(point, child.triangle):
                    node = child
                    found = True
                    break
            if not found:
                break  # Point not found in any child; fallback
        # Now, node.triangle should contain the point
        return node.triangle, node

    @staticmethod
    def point_in_triangle(p, triangle):
        """
        Check if point p is inside the given triangle using barycentric coordinates.
        """
        def sign(p1, p2, p3):
            return (p1.x - p3.x) * (p2.y - p3.y) - \
                   (p2.x - p3.x) * (p1.y - p3.y)

        b1 = sign(p, triangle.vertices[0], triangle.vertices[1]) < 0.0
        b2 = sign(p, triangle.vertices[1], triangle.vertices[2]) < 0.0
        b3 = sign(p, triangle.vertices[2], triangle.vertices[0]) < 0.0

        return ((b1 == b2) and (b2 == b3))

    def insert_point(self, point):
        """
        Insert a single point into the triangulation.
        """
        containing_triangle, containing_node = self.locate_containing_triangle(point)

        # Remove the containing triangle
        self.triangles.remove(containing_triangle)

        # Create new triangles by connecting the point
```

```
        # to the vertices of the containing triangle
        t1 = Triangle(point, containing_triangle.vertices[0],
            containing_triangle.vertices[1])
        t2 = Triangle(point, containing_triangle.vertices[1],
            containing_triangle.vertices[2])
        t3 = Triangle(point, containing_triangle.vertices[2],
            containing_triangle.vertices[0])

        # Set neighbors
        self.set_neighbors(t1, containing_triangle)
        self.set_neighbors(t2, containing_triangle)
        self.set_neighbors(t3, containing_triangle)

        # Add new triangles
        self.triangles.extend([t1, t2, t3])

        # Update segments
        for t in [t1, t2, t3]:
            for edge in t.edges:
                self.segments.setdefault(edge, []).append(t)

        # Update history DAG
        child_nodes = [
            HistoryDAGNode(t1),
            HistoryDAGNode(t2),
            HistoryDAGNode(t3)
        ]
        for child in child_nodes:
            containing_node.add_child(child)

        # Edge Legalization
        for t in [t1, t2, t3]:
            for edge in t.edges:
                if point in [edge.p1, edge.p2]:
                    self.legalize_edge(edge, t, point)

    def set_neighbors(self, new_triangle, old_triangle):
        """
        Set the neighboring triangles for the new triangle.
        """
        for edge in new_triangle.edges:
            if edge in self.segments:
                for neighbor in self.segments[edge]:
                    if neighbor != old_triangle and neighbor != new_triangle:
                        new_triangle.neighbors[edge] = neighbor
                        neighbor.neighbors[edge] = new_triangle
```

```python
def legalize_edge(self, edge, triangle, point):
    """
    Legalize an edge to restore the Delaunay condition.
    """
    if edge not in triangle.neighbors:
        return  # Boundary edge

    neighbor = triangle.neighbors[edge]
    if neighbor is None:
        return

    # Find the opposite point in the neighbor triangle
    opposite_point = [v for v in neighbor.vertices
                      if v not in [edge.p1, edge.p2]][0]

    if self.in_circumcircle(opposite_point, triangle):
        # Perform edge flip
        new_triangles = self.edge_flip(triangle, neighbor, edge,
                                       point, opposite_point)
        for new_t in new_triangles:
            for e in new_t.edges:
                if point in [e.p1, e.p2]:
                    self.legalize_edge(e, new_t, point)

def edge_flip(self, t1, t2, edge, point, opposite_point):
    """
    Flip the shared edge between two triangles.
    """
    # Remove old triangles
    self.triangles.remove(t1)
    self.triangles.remove(t2)

    # Remove edge from segments
    self.segments[edge].remove(t1)
    self.segments[edge].remove(t2)
    if not self.segments[edge]:
        del self.segments[edge]

    # Create new edge
    new_edge = SegmentWrapper(point, opposite_point)

    # Create new triangles
    new_t1 = Triangle(point, edge.p1, opposite_point)
    new_t2 = Triangle(point, opposite_point, edge.p2)
```

```python
        # Update segments
        for t in [new_t1, new_t2]:
            for e in t.edges:
                self.segments.setdefault(e, []).append(t)

        # Update neighbors
        self.update_neighbors_after_flip(t1, t2, new_t1, new_t2, edge, new_edge)

        # Add new triangles
        self.triangles.extend([new_t1, new_t2])

        # Update history DAG
        parent_node = HistoryDAGNode(None)
        t1_node = self.find_history_node(self.history_root, t1)
        t2_node = self.find_history_node(self.history_root, t2)
        parent_node.add_child(t1_node)
        parent_node.add_child(t2_node)
        new_t1_node = HistoryDAGNode(new_t1)
        new_t2_node = HistoryDAGNode(new_t2)
        parent_node.add_child(new_t1_node)
        parent_node.add_child(new_t2_node)

        return [new_t1, new_t2]

    def update_neighbors_after_flip(self, t1, t2, new_t1, new_t2, old_edge, new_edge):
        """
        Update neighbor relationships after an edge flip.
        """
        # Set neighbors for new_t1
        new_t1.neighbors[new_edge] = new_t2
        new_t2.neighbors[new_edge] = new_t1

        # Update other neighbors
        for e in new_t1.edges:
            if e != new_edge:
                for neighbor in self.segments[e]:
                    if neighbor != new_t1:
                        new_t1.neighbors[e] = neighbor
                        neighbor.neighbors[e] = new_t1

        for e in new_t2.edges:
            if e != new_edge:
                for neighbor in self.segments[e]:
                    if neighbor != new_t2:
                        new_t2.neighbors[e] = neighbor
                        neighbor.neighbors[e] = new_t2
```

```
def find_history_node(self, node, triangle):
    """
    Find the history DAG node corresponding to the given triangle.
    """
    if node.triangle == triangle:
        return node
    for child in node.children:
        result = self.find_history_node(child, triangle)
        if result is not None:
            return result
    return None

def in_circumcircle(self, point, triangle):
    """
    Check if a point is inside the circumcircle of a triangle.
    """
    ax, ay = triangle.vertices[0].x - point.x, triangle.vertices[0].y - point.y
    bx, by = triangle.vertices[1].x - point.x, triangle.vertices[1].y - point.y
    cx, cy = triangle.vertices[2].x - point.x, triangle.vertices[2].y - point.y

    det = (ax * (by * (cx**2 + cy**2) - cy * (bx**2 + by**2)) -
           ay * (bx * (cx**2 + cy**2) - cx * (bx**2 + by**2)) +
           (ax**2 + ay**2) * (bx * cy - cx * by))

    return det > 0

def build_triangulation(self):
    """
    Build the triangulation by inserting all points.
    """
    self.create_super_triangle()
    for point in self.points:
        self.insert_point(point)
    self.remove_super_triangle()

def remove_super_triangle(self):
    """
    Remove any triangles that share a vertex with the super-triangle.
    """
    # Super-triangle vertices
    super_vertices = set(self.history_root.triangle.vertices)
    self.triangles = [t for t in self.triangles
                      if not any(v in super_vertices for v in t.vertices)]

def calculate_dag_depths(self):
```

```
        """
        Calculate maximum and average depths of the history DAG.
        """
        depths = []
        queue = deque([(self.history_root, 0)])

        while queue:
            node, depth = queue.popleft()
            if not node.children:
                depths.append(depth)
            else:
                for child in node.children:
                    queue.append((child, depth + 1))

        max_depth = max(depths) if depths else 0
        avg_depth = sum(depths) / len(depths) if depths else 0
        return max_depth, avg_depth

    def plot_triangulation(self):
        """
        Plot the triangulation.
        """
        plt.figure(figsize=(8, 8))
        for triangle in self.triangles:
            x_coords = [vertex.x for vertex in triangle.vertices + [triangle.vertices[0]]]
            y_coords = [vertex.y for vertex in triangle.vertices + [triangle.vertices[0]]]
            plt.plot(x_coords, y_coords, 'k-')
        plt.scatter([p.x for p in self.points],
                    [p.y for p in self.points], color='red', s=10)
        plt.axis('equal')
        plt.title('Delaunay Triangulation')
        plt.show()
```