

CSC279 HW3

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Question 10

(PART A)

Parametrizing the Segment pq :

Let $p = (x_0, y_0)$ and $q = (x_1, y_1)$ be the endpoints of the segment pq . Any point $x(t)$ on pq can be expressed as:

$$x(t) = (1-t)p + tq = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \quad t \in [0, 1]$$

Dual Lines Corresponding to Points on pq :

The dual line $\hat{x}(t)$ Corresponding to $x(t)$ is:

$$y = A(t)x - B(t)$$

where:

$$A(t) = (1-t)x_0 + tx_1$$

$$B(t) = (1-t)y_0 + ty_1$$

Now we can express rewrite the equation of $\hat{x}(t)$ as:

$$\begin{aligned} y &= [x_0 + t(x_1 - x_0)]x - [y_0 + t(y_1 - y_0)] \\ &= x_0x - y_0 + t[(x_1 - x_0)x - (y_1 - y_0)] \end{aligned}$$

Let:

$$D = (x_1 - x_0)x - (y_1 - y_0)$$

Then:

$$y = x_0x - y_0 + tD$$

For a fixed x , y varies linearly with t from $y = x_0x - y_0$ (when $t = 0$) to $y = x_1x - y_1$ (when $t = 1$). The Union of all dual lines $\hat{x}(t)$ for $t \in [0, 1]$ is the set of all points (x, y) in the plane satisfying:

$$\min(y_0, y_1) \leq y - x \cdot \min(x_0, x_1) \leq \max(y_0, y_1)$$

Equivalently, by eliminating parameter t , we can get similar inequality:
 We aim to eliminate the parameter t from the parametric equations to describe the union of dual lines $\hat{x}(t)$.
 Solving for t :

$$t = \frac{y - x_0x + y_0}{D}$$

Note: The sign of D affects the inequality direction.
 If $D > 0$, then

$$0 \leq \frac{y - x_0x + y_0}{D} \leq 1 \Rightarrow 0 \leq y - x_0x + y_0 \leq D$$

If $D < 0$, then

$$0 \geq \frac{y - x_0x + y_0}{D} \geq 1 \Rightarrow D \leq y - x_0x + y_0 \leq 0$$

Both cases can be unified by the product inequality:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes the region between the lines $y = x_0x - y_0$ and $y = x_1x - y_1$, where the expressions $y - x_0x + y_0$ and $y - x_1x + y_1$ have opposite signs or are zero.

Hence, the union of all dual lines is:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes all points (x, y) that lie between $y = x_0x - y_0$, $y = x_1x - y_1$. which forms a region called a "double wedge".

(PART B)

Input:

$S = \{ s_i \mid i = 1 \text{ to } n \}$

output:

l_{\max} # Line has max. intersections point with other sections

Algorithm FindMaxIntersectingLine(S):

Initialize an empty list $L_{\text{dual_lines}}$.

For each segment s_i in S do:

Let $p_i = (x_{\{0i\}}, y_{\{0i\}})$ and $q_i = (x_{\{1i\}}, y_{\{1i\}})$

Compute the dual lines:

$L_{\{p_i\}}: y = x_{\{0i\}} x - y_{\{0i\}}$

$L_{\{q_i\}}: y = x_{\{1i\}} x - y_{\{01\}}$

Add $L_{\{p_i\}}$ and $L_{\{q_i\}}$ to $L_{\text{dual_lines}}$.

Construct the arrangement A of the lines in $L_{\text{dual_lines}}$:

TIME COMPLEXITY: $O(n^2 \log n)$

Use the line sweep algorithm to compute the A .

Store the faces, edges, and vertices of the A .

Initialize count $c_f = 0$ for a starting PLANE f_0 at INFINITY

Traverse arrangement A to label each face with the number

of double wedges covering it:

TIME COMPLEXITY: $O(n^2)$

For each edge e in A :

Determine which double wedges have boundaries along e

For each face f adjacent to e :

* $c_{\{f'\}}$ is the count of the adjacent face before crossing e

If crossing e enters a double wedge W_i , then $c_f = c_{\{f'\}} + 1$

If crossing e exits a double wedge W_i , then $c_f = c_{\{f'\}} - 1$

Keep track of the face f_{\max} with the maximum count c_{\max} during traversal

Let (a_{\max}, b_{\max}) be a point inside face f_{\max} .

Compute the line l_{\max} in the primal plane corresponding to (a_{\max}, b_{\max}) :

$l_{\max}: y = a_{\max} x - b_{\max}$

Return l_{\max}

Question 11

Input:

Simple polygon P given as a list of vertices $[v_1, v_2, \dots, v_n]$ in order

Output:

Whether there exists a line l such that P is monotone w.r.t. l

Algorithm DetermineMonotonicity(P):

BadIntervals = empty_list()

TIME COMPLEXITY: $O(n)$

For i from 1 to n :

$v_{\text{prev}} = P[(i - 2) \bmod n]$

$v = P[i - 1]$

$v_{\text{next}} = P[i \bmod n]$

 Compute vectors $e_1 = v - v_{\text{prev}}$

 Compute vectors $e_2 = v_{\text{next}} - v$

 Compute cross product $cp = e_1.x * e_2.y - e_1.y * e_2.x$

 If $cp < 0$ (vertex is concave):

 Compute angles $a_1 = \text{atan2}(e_1.y, e_1.x) \bmod 2\pi$

 Compute angles $a_2 = \text{atan2}(e_2.y, e_2.x) \bmod 2\pi$

 Let Interval = $[a_2, a_1]$ if $a_1 > a_2$ else $[a_2, a_1 + 2\pi]$

 Normalize Interval to $[0, 2\pi]$

 Add Interval to BadIntervals

TIME COMPLEXITY: $O(n \log n)$

Sort BadIntervals by their start angles

Merge overlapping intervals in BadIntervals to get a list of disjoint intervals

If the merged intervals cover $[0, 2\pi]$:

 return FALSE

return TRUE

A more general description of the Algorithm:

For each vertex i , determine if it is concave using the **counterclockwise (ccw)** test. If the ccw sign at the vertex differs from the majority, mark it as concave.

For each concave vertex, compute two vectors:

$$c_0 = v(i) - v(i - 1), \quad c_1 = v(i + 1) - v(i).$$

Calculate their angles a_0 and a_1 using the **atan2** function, normalized to $[0, 2\pi]$:

$$a_0 = (\text{atan2}(c_0.y, c_0.x) + 2\pi) \bmod 2\pi, \quad a_1 = (\text{atan2}(c_1.y, c_1.x) + 2\pi) \bmod 2\pi.$$

Store the interval $[a_1, a_0]$ in a list.

After processing all vertices, sort the intervals and merge any overlapping ones.

If the merged intervals cover the full 2π range, the polygon is **not monotone**.
Otherwise, it is **monotone**.

Question 12

Input:

```
RedPoints = [ (x1, y1), (x2, y2), ..., (xn, yn) ]  
BluePoints = [ (x1', y1'), (x2', y2'), ..., (xn', yn') ]
```

Output:

```
Coefficients (a, b, c) defining the parabola  $y = a x^2 + b x + c$   
or report "No solution exists" if impossible
```

Algorithm FindSeparatingParabola(RedPoints, BluePoints):

```
Initialize an empty list Constraints = []
```

```
# NOTE: e is a small positive number
```

```
# TIME COMPLEXITY:  $O(n)$ 
```

```
For each red point  $(x_i, y_i)$  in RedPoints do:
```

```
    Make inequality:
```

```
         $a (x_i^2) + b (x_i) + c - y_i < 0$ 
```

```
    Convert to standard LP form (inequalities with  $\leq$ ):
```

```
         $a (x_i^2) + b (x_i) + c - y_i \leq -e$ 
```

```
    Add this to Constraints
```

```
# TIME COMPLEXITY:  $O(n)$ 
```

```
For each blue point  $(x_j, y_j)$  in RedPoints do:
```

```
    Make inequality:
```

```
         $a x_j^2 + b x_j + c - y_j > 0$ 
```

```
    Convert to standard LP form (inequalities with  $\leq$ ):
```

```
         $-a x_j^2 - b x_j - c + y_j \leq -e$ 
```

```
    Add this to Constraints
```

```
ObjectiveFunction: Minimize 0
```

```
# TIME COMPLEXITY:  $O(n)$ 
```

```
Solution = LinearProgramming(Constraints, ObjectiveFunction)
```

```
If Solution is feasible then:
```

```
    Output "Parabola found with coefficients:"
```

```
    Output "a =", Solution.a
```

```
    Output "b =", Solution.b
```

```
    Output "c =", Solution.c
```

```
Output "No solution exists"
```

Question 13

For this question, we use an approximation to replace L_2 norm in order to use LP.

$$\begin{cases} r \geq 0 \\ w \geq 0 \\ \forall i \in \{1, \dots, n\}, \forall j \in \{1, \dots, k\} : \\ \quad (p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \geq r \cos \left(\frac{\pi}{k} \right) \\ \quad (p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \leq (r + w) \sec \left(\frac{\pi}{k} \right) \end{cases}$$

Variables

- c_x, c_y : Coordinates of the center c of the annulus
- $r \geq 0$: Inner Radius
- $w \geq 0$: Width of the annulus ($w = R - r$)

Objective Function:

$$\min w$$

Constraint:

1. $r \geq 0, w \geq 0$
2. To approximate L_1 norm, we pick k directions. For a regular k -gon inscribed in the unit circle, define angles:

$$\theta_j = \frac{2\pi j}{k}, \quad j = 1, 2, \dots, k$$

3. Compute unit vectors in these directions:

$$u_j = (\cos \theta_j, \sin \theta_j)$$

4. For each point $p_i = (p_{i,x}, p_{i,y})$ and each direction u_j :

- Inner Boundary Constraint:

$$(p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \geq r \cos \left(\frac{\pi}{k} \right)$$

- Outer Boundary Constraint:

$$(p_{i,x} - c_x) \cos \theta_j + (p_{i,y} - c_y) \sin \theta_j \leq (r + w) \sec \left(\frac{\pi}{k} \right)$$

The factors $\cos \left(\frac{\pi}{k} \right)$ and $\sec \left(\frac{\pi}{k} \right)$ adjust for the approximation error due to replacing the circle with a regular k -gon.