

CSC279 HW3

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Collaborator

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Question 10

(PART A)

Parametrizing the Segment pq :

Let $p = (x_0, y_0)$ and $q = (x_1, y_1)$ be the endpoints of the segment pq . Any point $x(t)$ on pq can be expressed as:

$$x(t) = (1-t)p + tq = ((1-t)x_0 + tx_1, (1-t)y_0 + ty_1), \quad t \in [0, 1]$$

Dual Lines Corresponding to Points on pq :

The dual line $\hat{x}(t)$ Corresponding to $x(t)$ is:

$$y = A(t)x - B(t)$$

where:

$$A(t) = (1-t)x_0 + tx_1$$

$$B(t) = (1-t)y_0 + ty_1$$

Now we can express rewrite the equation of $\hat{x}(t)$ as:

$$\begin{aligned} y &= [x_0 + t(x_1 - x_0)]x - [y_0 + t(y_1 - y_0)] \\ &= x_0x - y_0 + t[(x_1 - x_0)x - (y_1 - y_0)] \end{aligned}$$

Let:

$$D = (x_1 - x_0)x - (y_1 - y_0)$$

Then:

$$y = x_0x - y_0 + tD$$

For a fixed x , y varies linearly with t from $y = x_0x - y_0$ (when $t = 0$) to $y = x_1x - y_1$ (when $t = 1$). The Union of all dual lines $\hat{x}(t)$ for $t \in [0, 1]$ is the set of all points (x, y) in the plane satisfying:

$$\min(y_0, y_1) \leq y - x \cdot \min(x_0, x_1) \leq \max(y_0, y_1)$$

Equivalently, by eliminating parameter t , we can get similar inequality:
 We aim to eliminate the parameter t from the parametric equations to describe the union of dual lines $\hat{x}(t)$.
 Solving for t :

$$t = \frac{y - x_0x + y_0}{D}$$

Note: The sign of D affects the inequality direction.
 If $D > 0$, then $0 \leq \frac{y - x_0x + y_0}{D} \leq 1 \Rightarrow 0 \leq y - x_0x + y_0 \leq D$.
 If $D < 0$, then $0 \geq \frac{y - x_0x + y_0}{D} \geq 1 \Rightarrow D \leq y - x_0x + y_0 \leq 0$.
 Both cases can be unified by the product inequality:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes the region between the lines $y = x_0x - y_0$ and $y = x_1x - y_1$, where the expressions $y - x_0x + y_0$ and $y - x_1x + y_1$ have opposite signs or are zero.
 Hence, the union of all dual lines is:

$$(y - x_0x + y_0)(y - x_1x + y_1) \leq 0$$

This inequality describes all points (x, y) that lie between $y = x_0x - y_0$, $y = x_1x - y_1$. which forms a region called a "double wedge".

(PART B)

Input:

$S = \{ s_i \mid i = 1 \text{ to } n \}$

output:

l_{\max} # Line has max. intersections point with other sections

Algorithm FindMaxIntersectingLine(S):

Initialize an empty list $L_{\text{dual_lines}}$.

For each segment s_i in S do:

Let $p_i = (x_{\{0i\}}, y_{\{0i\}})$ and $q_i = (x_{\{1i\}}, y_{\{1i\}})$

Compute the dual lines:

$L_{\{p_i\}}: y = x_{\{0i\}} x - y_{\{0i\}}$

$L_{\{q_i\}}: y = x_{\{1i\}} x - y_{\{01\}}$

Add $L_{\{p_i\}}$ and $L_{\{q_i\}}$ to $L_{\text{dual_lines}}$.

Construct the arrangement A of the lines in $L_{\text{dual_lines}}$:

TIME COMPLEXITY: $O(n^2 \log n)$

Use the line sweep algorithm to compute the A .

Store the faces, edges, and vertices of the A .

Initialize count $c_f = 0$ for a starting PLANE f_0 at INFINITY

Traverse arrangement A to label each face with the number

of double wedges covering it:

TIME COMPLEXITY: $O(n^2)$

For each edge e in A :

Determine which double wedges have boundaries along e

For each face f adjacent to e :

* $c_{\{f'\}}$ is the count of the adjacent face before crossing e

If crossing e enters a double wedge W_i , then $c_f = c_{\{f'\}} + 1$

If crossing e exits a double wedge W_i , then $c_f = c_{\{f'\}} - 1$

Keep track of the face f_{\max} with the maximum count c_{\max} during traversal

Let (a_{\max}, b_{\max}) be a point inside face f_{\max} .

Compute the line l_{\max} in the primal plane corresponding to (a_{\max}, b_{\max}) :

$l_{\max}: y = a_{\max} x - b_{\max}$

Return l_{\max}

Question 11

Input:

Simple polygon P given as a list of vertices $[v_1, v_2, \dots, v_n]$ in order

Output:

Whether there exists a line l such that P is monotone w.r.t. l

Algorithm DetermineMonotonicity(P):

BadIntervals = empty_list()

TIME COMPLEXITY: $O(n)$

For i from 1 to n :

$v_{\text{prev}} = P[(i - 2) \bmod n]$

$v = P[i - 1]$

$v_{\text{next}} = P[i \bmod n]$

 Compute vectors $e_1 = v - v_{\text{prev}}$

 Compute vectors $e_2 = v_{\text{next}} - v$

 Compute cross product $cp = e_1.x * e_2.y - e_1.y * e_2.x$

 If $cp < 0$ (vertex is concave):

 Compute angles $a_1 = \text{atan2}(e_1.y, e_1.x) \bmod 2\pi$

 Compute angles $a_2 = \text{atan2}(e_2.y, e_2.x) \bmod 2\pi$

 Let Interval = $[a_2, a_1]$ if $a_1 > a_2$ else $[a_2, a_1 + 2\pi]$

 Normalize Interval to $[0, 2\pi)$

 Add Interval to BadIntervals

TIME COMPLEXITY: $O(n \log n)$

Sort BadIntervals by their start angles

Merge overlapping intervals in BadIntervals to get a list of disjoint intervals

If the merged intervals cover $[0, 2\pi)$:

 return FALSE

return TRUE

Question 12

Input:

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RedPoints = [ (x1, y1), (x2, y2), ..., (xn, yn) ]  
BluePoints = [ (x1', y1'), (x2', y2'), ..., (xn', yn') ]
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Output:

Coefficients (a, b, c) defining the parabola $y = a x^2 + b x + c$
or report "No solution exists" if impossible

Algorithm FindSeparatingParabola(RedPoints, BluePoints):

Initialize an empty list Constraints = []

NOTE: e is a small positive number

TIME COMPLEXITY: $O(n)$

For each red point (x_i, y_i) in RedPoints do:

Make inequality:

$$a x_i^2 + b x_i + c - y_i < 0$$

Convert to standard LP form (inequalities with \leq):

$$a x_i^2 + b x_i + c - y_i \leq -e$$

Add this to Constraints

TIME COMPLEXITY: $O(n)$

For each blue point (x_j, y_j) in RedPoints do:

Make inequality:

$$a x_j^2 + b x_j + c - y_j > 0$$

Convert to standard LP form (inequalities with \leq):

$$-a x_j^2 - b x_j - c + y_j \leq -e$$

Add this to Constraints

ObjectiveFunction: Minimize 0

TIME COMPLEXITY: $O(n)$

Solution = LinearProgramming(Constraints, ObjectiveFunction)

If Solution is feasible then:

Output "Parabola found with coefficients:"

Output "a =", Solution.a

Output "b =", Solution.b

Output "c =", Solution.c

Output "No solution exists"

Question 13

For this question, we use given L_1 norm to approximate distance in the plane.

Variables

- c_x, c_y : Coordinates of the center c of the annulus
- $s \geq 0$: Inner Radius
- $t \geq 0$: Outer Radius
- For each point $p_i = (x_i, y_i)$:
 - $x_i^+ > 0 : |x_i - c_x|$
 - $y_i^+ > 0 : |y_i - c_y|$

Objective Function:

$$\min (t - s)$$

Constraint:

For all $i = 1$ to n :

1. Non-negatively:

$$s \geq 0, t \geq 0, x_i^+ \geq 0, y_i^+ \geq 0$$

2. Absolute value constraints:

$$\begin{cases} x_i - c_x \leq x_i^+ \\ -(x_i - c_x) \leq x_i^+ \\ y_i - c_y \leq y_i^+ \\ -(y_i - c_y) \leq y_i^+ \end{cases}$$

3. Distance constraints:

$$s \leq x_i^+ + y_i^+ \leq t$$