MATH 265 HW5

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Sep/26/2024

Question 1

(a)

We can analysis this question by the parity of $n \in \mathbb{Z}_+$.

1. When n is odd, $(-1)^n = -1$. Hence the term becomes:
$$-\left(1 - \frac{1}{n}\right) = -1 + \frac{1}{n}$$
Noticing that for all n, $\frac{1}{n} \leq 1$, hence:
$$-1 + \frac{1}{n} \leq 0$$
Hence, for all odd n, $s \leq 0, s \in S$.

2. When n is even, $(-1)^n = 1$. Hence the term becomes:
$$\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$$
When $n \to \infty$:
$$\frac{1}{n} \to 0 \Rightarrow 1 - \frac{1}{n} \to 1$$
Hence, for all odd n, $1 - \frac{1}{n} \leq 1$

Since every element $s \in S$, $s \leq 1$. By definition, 1 is an upper bound of S.

(b)

Proof. If M is an upper bound for S, then $M \geq s$, for every element $s \in S$.
Suppose $M < 1$ and M is an upper bound of S, it is sufficient for us to only consider the case where n is even.
Since $\frac{1}{2(1-M)} < 0$, by Archimedes Property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{2(1-M)} < n$, we can get:

$$\frac{1}{2(1-M)} < n \Rightarrow \frac{1}{1-M} < 2n \Rightarrow 1 < 2n(1-M) \Rightarrow \frac{1}{2n} < 1-M \Rightarrow M < 1-\frac{1}{2n}$$

In this case, we know that for n is even,

$$M < 1 - \frac{1}{2n} = (1 - \frac{1}{2n})(-1)^{2n}$$

Hence, there exists a $s \in S$ s.t. $M < (1 - \frac{1}{2n})(-1)^{2n}$ for some $n \in \mathbb{Z}_+$. This derived a contradiction, hence $M \geq 1$. $\square$

(c)

Proof. From part (a), 1 is am upper bound of S, and from part (b), no number less then 1 can be an upper bound. Thus by definition, the supremum of S is 1 (i.e. $\sup S = 1$). $\square$

Quesiton 2

Noting that by definition $A + B = \{a + b \mid a \in A, b \in B\}$

Proof. First we need to show that $\sup(A + B) \leq \sup A + \sup B$:
let $a \in A$ and $b \in B$. By definition of supremum, we have $a \leq \sup A$ and $b \leq \sup B$.
Therefore, for any $a \in A$ and $b \in B$:

$$a + b \leq \sup A + \sup B$$

Since $A + B = \{a + b \mid a \in A, b \in B\}$, every element in $A + B$ is less than or equal to $\sup A + \sup B$. Therefore, $\sup(A + B) \leq \sup A + \sup B$.
Then we can derived the equality from here:
Let $\epsilon > 0$, by definition of supremum, for set A and B:

- $\exists a_\epsilon \in A : \sup A - \epsilon < a_\epsilon < \sup A$
- $\exists b_\epsilon \in B : \sup B - \epsilon < b_\epsilon < \sup B$

Now consider $a_\epsilon + b_\epsilon \in A + B$. Then:

$$(\sup A - \epsilon) + (\sup B - \epsilon) < a_\epsilon + b_\epsilon \leq \sup(A + B)$$

$$\Rightarrow \sup A + \sup B - 2\epsilon < a_\epsilon + b_\epsilon \leq \sup(A + B)$$

$$\Rightarrow \sup A + \sup B - 2\epsilon < \sup(A + B)$$

$$\Rightarrow 0 \leq \frac{1}{2}(\sup A + \sup B - \sup(A + B)) < \epsilon$$

By previous step, WLOG, since $\sup(A+B) \leq \sup A + \sup B$, the left inequality holds. Reffering to one of the theorem in section *2.1*, we can get:

$$\frac{1}{2}(\sup A + \sup B - \sup(A + B)) = 0 \Rightarrow \sup A + \sup B = \sup(A + B)$$

$\square$

Question 3

Proof. Let $x > 1$ and consider the set $S = \{x^n : n \in \mathbb{Z}_+\}$.
For any $M \in \mathbb{R}$, we want to show that there exists $n \in \mathbb{Z}_+$ such that $x^n > M$
Taking natural logarithm on both sides of the inequality $x^n > M$, we get:

$$n \cdot \ln(x) > \ln(M)$$

Since $\ln(x) > 0$, noting $x > 1$, we can solve for n:

$$n > \frac{\ln(M)}{\ln(x)}$$

Note that RHS is a constant, let $N = \lceil \frac{\ln(M)}{\ln(x)} \rceil + 1$,
Hence, for $n = N$, we have:

$$x^n > M$$

For any $M \in \mathbb{R}$, we can found $n \in \mathbb{Z}_+$ s.t. $x^n > M$. Thus, the set is not bounded from above. □

Question 4

For example:

$$I_n = \left[1 + \frac{1}{n}, 3 + \frac{1}{n}\right)$$

For each positive integer n.
Property verificaiton:

Proof. **Part 1:** We show that $\bigcup_{n=1}^{\infty} I_n = (1, 4)$.
(i) $\bigcup_{n=1}^{\infty} I_n \subseteq (1, 4)$:
Let $x \in \bigcup_{n=1}^{\infty} I_n$. Then $x \in I_n$ for some n:

$$1 + \frac{1}{n} < x < 4 - \frac{1}{n}, \text{ Since } \frac{1}{n} > 0$$

When $n \to \infty$, it follows that $1 \leq x < 4$. Thus, $x \in (1, 4) \subseteq [1, 4)$, we can conclude that $\bigcup_{n=1}^{\infty} I_n \subseteq (1, 4)$.
(ii) $(1, 4) \subseteq \bigcup_{n=1}^{\infty} I_n$:
Let $x \in (1, 4)$. Define $\epsilon = \min\{x - 1, 4 - x\} > 0$. By *Archimedes Theorem*, Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.
Then, for all $n \geq N$, we have:

$$1 + \frac{1}{n} < 1 + \epsilon < x < 4 - \epsilon < 4 - \frac{1}{n}$$

So $x \in I_n$. Thus, $x \in \bigcup_{n=1}^{\infty} I_n$, and hence $(1, 4) \subseteq \bigcup_{n=1}^{\infty} I_n$.
Combining (i) and (ii), we obtain $\bigcup_{n=1}^{\infty} I_n = (1, 4)$.

Part 2: We show that $\bigcap_{n=1}^{\infty} I_n = [2, 3]$.

(i) $[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$:
Let $x \in [2, 3]$. For any $n \in \mathbb{N}$

$$1 + \frac{1}{n} \leq 2 \leq x \leq 3 \leq 4 - \frac{1}{n}$$

Thus, $x \in I_n$ for all n, implying $x \in \bigcap_{n=1}^{\infty} I_n$. Therefore, $[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$.
(ii) $\bigcap_{n=1}^{\infty} I_n \subseteq [2, 3]$:
Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then $x \in I_n$ for all n, i.e., $1 + \frac{1}{n} \leq x < 4 - \frac{1}{n}$ for all n. As $n \to \infty$, $1 + \frac{1}{n} \to 1$ and $4 - \frac{1}{n} \to 4$. Thus, $x \geq 2$ (since x cannot be less than 2 for all large n) and $x \leq 3$ (since x cannot be greater than 3 for all large n). Hence, $x \in [2, 3]$.
Combining (i) and (ii), we obtain $\bigcap_{n=1}^{\infty} I_n = [2, 3]$.

$\square$