MATH 265 HW7

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Oct/18/2024

Question 1

(a)

Proof. Let $x > 1$ and $y \in \mathbb{Q}$. We aim to show that x^y is the l.u.b of the set

$$E(x, y) = \{x^t | t < y, t \in \mathbb{Q}\}$$

1. Upper Bound:
   For any $t \in \mathbb{Q}$ with $t < y$, given $x > 1$, x^t is increasing in t, we have:

   $$x^t < x^y$$

   Therefore, x^y is an upper bound of $E(x, y)$

2. Least Upper Bound (Supremum):
   Suppose there exists a real number $M < x^y$ that is also an upper bound of $E(x, y)$ Since $x^y - M > 0$, set $\epsilon = x^y - M$.
   Noting that rationals are dense in $\mathbb{R}$, there exists $t \in \mathbb{Q}$ s.t.

   $$y - \delta < t < y$$

   Where $\delta > 0$ is small enough to ensure $x^y - x^t < \epsilon$. Then:

   $$x^t > x^y - \epsilon = M$$

   This contradict the assumption that M is an upper bound of $E(x, y)$. Therefore, no number less than x^y can be an upper bound of $E(x, y)$, so x^y is the supremum of $E(x, y)$.

□

(b)

Proof. To show boundness, we need to show there exists an upperbound and lower bound in $E(x, y)$, when $y \in \mathbb{R}$

1. Upper bound:
   Since $x^t < x^y$ for all $t < y$, x^y is an upper bound of $E(x, y)$

2. Lower Bound:
   And we know that $x^t > 0$ for all t by given definition. Therefore, $E(x, y)$ is bounded below.

Hence $E(x, y)$ is bounded. □

(c)

Proof. To proof $x^{y+z} = x^y x^z$, we need to validate two directions:

1. First we need to show $x^{y+z} \leq x^y x^z$:
   Let $t \in \mathbb{Q}$ with $t < y + z$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $t_1, t_2 \in \mathbb{Q}$ s.t.:
   $$t_1 < y;\ t_2 < z;\ t = t_1 + t_2$$
   Then:
   $$x^t = x^{t_1+t_2} = x^{t_1} x^{t_2} < x^y x^z$$
   Therefore, every element x^t of $E(x, y + z) \leq x^y x^z$ Thus:
   $$x^{y+z} = \sup E(x, y + z) \leq x^y x^z$$

2. Then we need to show $x^{y+z} \geq x^y x^z$: For any $\epsilon > 0$, choost $t_1, t_2 \in \mathbb{Q}$ s.t.:
   $$y - \epsilon < t_1 < y,\ z - \epsilon < t_2 < z$$
   Then $t_1 + t_2 < y + z$, so $t_1 + t_2 \in E(x, y + z)$ Compute:
   $$x^{t_1} x^{t_2} = x^{t_1+t_2} \leq x^{y+z}$$
   Since $t_1 \to y$ and $t_2 \to z$, $x^{t_1} \to x^y$ and $x^{t_2} \to x^z$.
   Therefore:
   $$x^{y+z} \geq \lim_{t_1 \to y^-} \lim_{t_2 \to z^-} x^{t_1} x^{t_2} = x^y x^z - \delta$$
   where δ can be made arbitrarily small. Thus:
   $$x^{y+z} \geq x^y x^z$$

Combining both inequalities, we get $x^{y+z} = x^y x^z$
The property $x^{y+z} = x^y x^z$ implies that the function $f(y) = x^y$ is **injective** because it ensures that equal outputs correspond to equal inputs. Specifically by definition, if $f(y_1) = f(y_2)$, then $x^{y_1} = x^{y_2}$. Using the exponential property, we have $x^{y_1 - y_2} = 1$. Since $x > 1$ and the exponential function x^t is strictly increasing, the equation $x^t = 1$ holds only when $t = 0$. Therefore, $y_1 - y_2 = 0$, which means $y_1 = y_2$. This shows that f is injective because no two different inputs produce the same output. □

Question 2

Proof. The following proof follows the given proof outline:
Step i: For any $n \in \mathbb{N}$, $x - 1 \geq n(x^{\frac{1}{n}} - 1)$
Recall Bernoulli's inequality, it states that for any real number $r \geq 1$ and $s \geq -1$:

$$(1+s)^r \geq 1 + rs$$

Let $s = x^{\frac{1}{n}} - 1$ (NOTE: $s > 0$ as $x > 1$) and $r = n$:

$$(1 + x^{\frac{1}{n}} - 1)^n \geq 1 + n(x^{\frac{1}{n}} - 1)$$

$$\Rightarrow x \geq 1 + n(x^{\frac{1}{n}} - 1) \Rightarrow x - 1 \geq n(x^{\frac{1}{n}} - 1)$$

Step ii: If $t > 1$ and $n \in \mathbb{N}$ s.t. $n > \frac{x-1}{t-1}$, then $x^{\frac{1}{n}} < t$.
From *step i*, we get:

$$x^{\frac{1}{n}} - 1 \leq \frac{x-1}{n}$$

If $n > \frac{x-1}{t-1}$, then:

$$\frac{x-1}{n} < t - 1$$

Therefore,

$$x^{\frac{1}{n}} - 1 < t - 1 \Longrightarrow x^{\frac{1}{n}} < t$$

Step iii: If $y \in \mathbb{R}$ and $x^y < z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y+\frac{1}{n}} < z$:
Set $t = \frac{x^y}{z}$, note $\frac{x^y}{z} > 1$ by definition. Again using step ii, choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x-1}{t-1}$$

Then $x^{\frac{1}{n}} < t$, recall question 1(c) and multiply x^y on both side, we can get:

$$x^{y+\frac{1}{n}} = x^y \cdot x^{\frac{1}{n}} < x^y \cdot t = z$$

Step iv: If $y \in \mathbb{R}$ and $x^y > z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y+\frac{1}{n}} > z$:
Set $t = \frac{z}{x^y}$, note $\frac{z}{x^y} > 1$ by definition. Again using step ii, choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x-1}{t-1}$$

Then $x^{\frac{1}{n}} < t$, so $x^{-\frac{1}{n}} = \frac{1}{x^{\frac{1}{n}}} > \frac{1}{n}$. Recall question 1($c$) and multiply x^y on both side, we can get:

$$x^{y-\frac{1}{n}} = x^y \cdot x^{-\frac{1}{n}} < x^y \cdot \frac{1}{t} = z$$

Step v: Define $A(z) = \{w \in \mathbb{R} | x^w < z\}$. Let $y = \sup A(z)$. then $x^y = z$
First, since $x^w \to 0$, as $w \to -\infty$ and $x^w \to \infty$, as $w \to \infty$, there exist real numbers w s.t. $x^w < z$.
The set $A(z)$ is bounded above because $x^w \geq z$ for sufficiently large w.

Suppose $x^y < z$: by step (iii), there exists n s.t. $x^{y+\frac{1}{n}} < z$, contradicting the fact that $y = \sup A(z)$.
Similarly, suppose $x^y > z$: by step (iv), there exists n s.t. $x^{y-\frac{1}{n}} > z$, but $y - \frac{1}{n} < y$, again contradicting the fact that $y = \sup A(z)$ (i.e. y is l.u.b).
Consequently, neither $x^y < z$ not $x^y > x$ is possible; thus, $x^y = z$. $\square$

Question 3

By analyzed several terms of the given sequence, likely it is increasing and should converges to 2.

Proof. For the given sequence, we can define it recusively:

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n},\ for\ n \geq 1$$

Then we can set an induction to prove this sequence is increasing. I.e. We will prove $a_n < a_{n+1}$ for all $n \geq 1$
Base Case (n = 1):

$$a_1 = \sqrt{2},\ a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$$

Noticing that $a_1, a_2 > 0$, and $a_1^2 = 2, a_2^2 = 2\sqrt{2}$, so:

$$2 < 2\sqrt{2} \Leftrightarrow a_1^2 < a_2^2 \Rightarrow a_1 < a_2$$

Inductive steps:
Assume $a_n < a_{n+1}$ for some $n \geq 1$. We need to show $a_{n+1} < a_{n+2}$.
Noticing, a_{n+1} and a_{n+2} can be deonte as:

$$a_{n+1} = \sqrt{2a_n},\ a_{n+2} = \sqrt{2a_{n+1}}$$

Then, since $a_n > 0,\ \forall n \in \mathbb{N}$:

$$\sqrt{2a_n} < \sqrt{2a_{n+1}} \Leftrightarrow 2a_n < 2a_{n+1} \Leftrightarrow a_n < a_{n+1}$$

Hence, by our induction hypothesis, $a_{n+1} < a_{n+2}$. Therefore by induction, sequence $\{a_n\}$ is increasing.
Then, we need to show that (a_n) is bounded above. I.e., we will prove by induction that $a_n < 2$ for all $n > 1$
Base Case (n = 1):

$$a_1 = \sqrt{2} < 2$$

Inductive Step:
Assume $a_n < 2$ for some $n \geq 1$, we need to show $a_{n+1} < 2$:

From our previous definition, $a_{n+1} = \sqrt{2a_n}$. Since $a_n < 2$, we have $2a_n < 2 \times 2 = 4$, hence:

$$a_{n+1} = \sqrt{2a_n} < \sqrt{4} = 2$$

By induction again, $a_n < 2$ for all $n > 1$.
Since $\{a_n\}$ is increasing and bounded above by 2, by MCT, $\{a_n\}$ converges to some limit L s.t. $L \leq 2$
Let $L = \lim_{n\to\infty} a_n$. Taking limit both sides of the recursive formula:

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2L}$$

Then:

$$L = \sqrt{2L} \Longrightarrow L^2 = 2L \Longrightarrow L^2 - 2L = 0 \Longrightarrow L(L-2) = 0$$

So $L = 0, 2$, and since $a_n > 0$, limit $L \geq \sqrt{2} > 0$. so $L = 2$.
Lastly, the limit found for the given sequence is:

$$\lim_{n\to\infty} a_n = 2$$

$\square$