265F24midterm1_Sol

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Math 265: Real Analysis I

Midterm Exam 1

Thursday, September 26, 2024

NAME (please print legibly): ______________________

Your University ID Number: ______________________

Indicate your instructor with a check in the appropriate box:

Wei-Cheng Huang	MW 10:25 - 11:40 AM	
Woongbae Park	MW 12:30 - 1:45 PM	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 10 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- **Show all work and justify all answers**, unless specified otherwise. Correct answers with insufficient work will not be given full credit.
- A blank page for scratch work is provided at the end of the exam. **Work on this page will not be graded.** Please show your work on the page containing the relevant question.
- Clearly circle all final answers.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

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YOUR SIGNATURE: ______________________

1. (20 points)
For each part, provide an example satisfying the description, and justify your answer by proving it.

(a) Find a bijection of $A = \{x : 1 < x < \pi\}$ onto $B = \{y : -1 < y < 1\}$.

Define $f : A \to B$ by $f(x) = \frac{x-1}{\pi-1} \times 2 - 1$

1-1: Suppose $f(x_1) = f(x_2)$. Then $\frac{x_1-1}{\pi-1}\cdot 2 - 1 = \frac{x_2-1}{\pi-1}\cdot 2 - 1$

$\Rightarrow x_1 = x_2$

Onto: Let $y \in B$. Consider $x = \frac{\pi-1}{2}(y+1) + 1$

$\because -1 < y < 1 \quad \therefore 1 < x < \pi \Rightarrow x \in A$.

$f(x) = y$.

(b) Let X be a nonempty set. Find an injection from X to its power set $\mathcal{P}(X)$.

Define $f : X \to \mathcal{P}(X)$ by $f(x) = \{x\}$.

Suppose $f(x_1) = f(x_2)$

$\Rightarrow \{x_1\} = \{x_2\}$

$\Rightarrow x_1 = x_2$

(Let $a \in \{x_1\} = \{x_2\}$
Then $a = x_1$ & $a = x_2 \Rightarrow x_1 = x_2$)

(c) Find an example such that $f^{-1}(f(E)) \neq E$, where A and B are sets, $E \subseteq A$, and $f : A \to B$ is a function. In other word, provide an explicit example of sets A, B, E, and a function f satisfying $f^{-1}(f(E)) \neq E$.

$\left(\begin{array}{c} f: A \to B \\ \cup\!\shortmid \\ E \end{array} \right)$

Consider $A = \{a, b\}$, $B = \{c\}$, $E = \{a\}$

$f : A \to B$: $f(x) = c$ for $x \in A$.

Then $f(E) = B$, $f^{-1}(f(E)) = A \neq E$.

2. (15 points) Using Mathematical induction to show that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all $n \in \mathbb{N}$.

$n=1$: $1 \cdot 2 = \frac{1\cdot 2\cdot 3}{3} \Rightarrow$ "=" holds.

Suppose $n = k$, "=" holds.

$n = k+1$: $1\times 2 + 2\cdot 3 + \cdots + k(k+1) + (k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$

$= (k+1)(k+2)\left(\frac{k+3}{3}\right)$

$= \frac{(k+1)(k+2)(k+3)}{3} \Rightarrow$ "=" holds for $n=k+1$.

$\Rightarrow$ "=" holds for $n \in \mathbb{N}$.

3. (15 points) Prove that the set of functions $f : \{123, 456\} \to \mathbb{N}$ is countable.

Let $A := \{f \mid f : \{123, 456\} \to \mathbb{N}\}$

Define $\phi : A \to \mathbb{N}\times\mathbb{N}$ by $\phi(f) = (f(123), f(456))$

Claim ϕ is bijective.

Pf 1-1: $\phi(f) = \phi(g)$

$\Rightarrow (f(123), f(456)) = (g(123), g(456))$

$\Rightarrow f(123) = g(123)$ & $f(456) = g(456)$

$\Rightarrow f = g$.

Onto: Suppose $(m, n) \in \mathbb{N}\times\mathbb{N}$.

Consider $f : \{123, 456\} \to \mathbb{N}$ defined by $f(123) = m$, $f(456) = n$.

Then $f \in A$ & $\phi(f) = (m, n)$.

$\because \phi$ is bijective & $\mathbb{N}\times\mathbb{N}$ is countable.

$\therefore A$ is countable.

4. (20 points) **(a)** Show that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

for all $a, b, c \in \mathbb{R}$.

$a^2+b^2+c^2 = \frac{(a^2+b^2)+(b^2+c^2)+(c^2+a^2)}{2}$

$\geq \frac{2\sqrt{a^2b^2} + 2\sqrt{b^2c^2} + 2\sqrt{c^2a^2}}{2}$

$= |ab| + |bc| + |ca|$

$\geq ab + bc + ca$

(b) Show that

$$a^3 + b^3 + c^3 \geq 3abc$$

for all $a, b, c > 0$. (Hint : Use the inequality in part (a).)

By (a) with $a^{\frac{3}{2}}, b^{\frac{3}{2}}, c^{\frac{3}{2}}$,

$\underbrace{(a^{\frac{3}{2}})^2 + (b^{\frac{3}{2}})^2 + (c^{\frac{3}{2}})^2}_{= a^3+b^3+c^3} \geq a^{\frac{3}{2}}b^{\frac{3}{2}} + b^{\frac{3}{2}}c^{\frac{3}{2}} + c^{\frac{3}{2}}a^{\frac{3}{2}}$

$\geq 3\sqrt[3]{a^3b^3c^3} = 3abc$

each term are >0.

5. (15 points)

(a) Show that if $a, b \in \mathbb{R}$, then $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.

Case 1 $\max\{a, b\} = a$. Then $a - b \geq 0$

$\Rightarrow \frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b+a-b) = a$

Case 2 $\max\{a, b\} = b$. Then $b - a \geq 0$

$\Rightarrow \frac{1}{2}(a+b+|a-b|) = \frac{1}{2}(a+b-(a-b)) = b$

(b) Show that if $a, b \in \mathbb{R}$, $|a + b| = |a| + |b|$ if and only if $ab \geq 0$.

($\Rightarrow$) $|a+b| = |a| + |b|$

$\Rightarrow |a+b|^2 = (|a|+|b|)^2$

$\Rightarrow a^2 + 2ab + b^2 = a^2 + 2|ab| + b^2$

$\Rightarrow ab = |ab| \Rightarrow ab \geq 0$.

($\Rightarrow$) Case 1 $a \geq 0$, $b \geq 0 \Rightarrow a+b \geq 0$

$\Rightarrow |a+b| = a+b$

$|a|+|b| = a+b$

Case 2 $a \leq 0$, $b \leq 0 \Rightarrow a+b \leq 0$

$\Rightarrow |a+b| = -(a+b)$

$|a|+|b| = -a-b$

$\Rightarrow |a+b| = |a|+|b|$

6. (15 points) Show that if A, B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Also show that

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}.$$

Suppose M_1, M_2 are bounds for A & B, resp.

Let $M = \max\{M_1, M_2\}$

Then for $x \in A \cup B$

If $x \in A$, then $|x| \leq M_1 \leq M$

If $x \in B$, then $|x| \leq M_2 \leq M$

$\Rightarrow A \cup B$ is bounded.

Let $N = \sup(A \cup B)$, $N_1 = \sup A$, $N_2 = \sup B$.

$\sup\{N_1, N_2\}$ is an upper bound of $A \cup B$ by the argument above.

$\Rightarrow N \leq \sup\{N_1, N_2\}$

N is an upper bound of $A \cup B$

$\Rightarrow N$ is – – – of A & B.

$\Rightarrow N \geq N_1$, $N \geq N_2$

$\Rightarrow N \geq \sup\{N_1, N_2\}$

$\Rightarrow N = \sup\{N_1, N_2\}$.

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