MATH 240 HW1

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Question 1

Proof. Define a function f as:

$$f : \mathbb{N} \to \{1, 4, 7, 10, 13, \cdots\} : n \rightarrowtail 3n - 2$$

Now we need to show f is *one to one* and *onto*.

- Prove f is *one to one*

  Proof. To prove f is *one to one*, we must show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
  Suppose $f(n_1) = f(n_2)$. Then:

  $$3n_1 - 2 = 3n_2 - 2$$

  $$\Rightarrow 3n_1 = 3n_2 \Rightarrow n_1 = n_2$$

  Since $n_1 = n_2$, f is injective. □

- Prove f is *onto*

  Proof. To prove f is *onto*, we need to show for every $m \in \{1, 4, 7, 10, 13, \cdots\}$, there exists an $n \in \mathbb{N}$ s.t. $f(n) = m$.
  Take any $m \in \{1, 4, 7, 10, 13, \cdots\}$. We want to find n such that $3n - 2 = m$.
  Solving for n:

  $$3n - 2 = m \Rightarrow 3n = m + 2 \Rightarrow n = \frac{m + 2}{3}$$

  Since m is of the form $3k - 2$ for some integer k, and $n = k$ is an integer.
  Therefore, $n \in \mathbb{N}$, and f is onto. □

  Since f is *one to one* and *onto*, it is a bijection. Hence, $f(n) = 3n - 2$ is a explicit bijection from the set $\mathbb{N}$ to $\{1, 4, 7, 10, 13, \cdots\}$.

  □

Question 2

Proof. Define a function f as:

$$f : (0,1) \to \mathbb{R}_{>0} : n \rightarrowtail \frac{n}{1-n}$$

Now we need to show f is *one to one* and *onto*.

- Prove f is *one to one*

  Proof. To prove f is *one to one*, we must show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
  Suppose $f(n_1) = f(n_2)$. Then:

  $$\frac{n_1}{1-n_1} = \frac{n_2}{1-n_2}$$
  $$\Rightarrow x_1(1-x_2) = x_2(1-x_1)$$
  $$\Rightarrow x_1 - x_1x_2 = x_2 - x_2x_1$$
  $$\Rightarrow x_1 = x_2$$

  Thus, f is one to one. □

- Prove f is *onto*

  Proof. To prove f is *onto*, we need to show for every $m \in \mathbb{R}_{>0}$, there exists an $n \in (0,1)$ s.t. $f(n) = m$.
  Take any $m > 0$. We want to find n such that $\frac{n}{1-n} = m$. Solving for n:

  $$n = m(1-n) \Rightarrow n + nm = m \Rightarrow n(1+m) = m \Rightarrow n = \frac{m}{1+m}$$

  Note that $m > 0$, this implies $0 < \frac{m}{1+m} < 1$, hence $n \in (0,1)$.
  Thus, for every $m > 0$, there is an $n \in (0,1)$ s.t. $f(n) = m$. (i.e. f is onto) □

  Since f is *one to one* and *onto*, it is a bijection. Hence, $f(n) = \frac{n}{1-n}$ is a explicit bijection from the set $(0,1)$ to $\mathbb{R}_{>0}$.

  □

Question 3

Proof. Define a function f as:

$$f : (0,1) \to \mathbb{R}_{>0} : n \rightarrowtail \frac{n}{1-n}$$

Now we need to show f is *one to one* and *onto*.

- Prove f is *one to one*

  Proof. To prove f is *one to one*, we must show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
  Suppose $f(n_1) = f(n_2)$. Then:

  $$\frac{n_1}{1-n_1} = \frac{n_2}{1-n_2}$$
  $$\Rightarrow x_1(1-x_2) = x_2(1-x_1)$$
  $$\Rightarrow x_1 - x_1x_2 = x_2 - x_2x_1$$
  $$\Rightarrow x_1 = x_2$$

  Thus, f is one to one. $\square$

- Prove f is *onto*

  Proof. To prove f is *onto*, we need to show for every $m \in \mathbb{R}_{>0}$, there exists an $n \in (0,1)$ s.t. $f(n) = m$.
  Take any $m > 0$. We want to find n such that $\frac{n}{1-n} = m$. Solving for n:

  $$n = m(1-n) \Rightarrow n + nm = m \Rightarrow n(1+m) = m \Rightarrow n = \frac{m}{1+m}$$

  Note that $m > 0$, this implies $0 < \frac{m}{1+m} < 1$, hence $n \in (0,1)$.
  Thus, for every $m > 0$, there is an $n \in (0,1)$ s.t. $f(n) = m$. (i.e. f is onto) $\square$

  Since f is *one to one* and *onto*, it is a bijection. Hence, $f(n) = \frac{n}{1-n}$ is a explicit bijection from the set $(0,1)$ to $\mathbb{R}_{>0}$.

  $\square$