# MATH 265 HW7

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Oct/18/2024

## Question 1

### (a)

*Proof.* Let $x > 1$ and $y \in \mathbb{Q}$. We aim to show that $x^y$ is the l.u.b of the set

$$E(x, y) = \{x^t | t < y, t \in \mathbb{Q}\}$$

1. Upper Bound:
   For any $t \in \mathbb{Q}$ with $t < y$, given $x > 1$, $x^t$ is increasing in $t$, we have:

   $$x^t < x^y$$

   Therefore, $x^y$ is an upper bound of $E(x, y)$

2. Least Upper Bound (Supremum):
   By the consequence of the Density Theorem: For $M \in \mathbb{R}$ satisfying $M < x^y$, there exists $t \in \mathbb{Q}$ such that $M < x^t < x^y$.
   Suppose $M$ is an upper bound of set $E(x, y)$ and $M < x^y$, by the consequence listed upwards, we can always find an $x^t$ s.t. $x^t > M$. This contradict the assumption that $M$ is an upper bound of $E(x, y)$. Therefore, no number less than $x^y$ can be an upperbound of $E(x, y)$. Hence, $x^= \sup E(x, y)$.

$\square$

### (b)

*Proof.* Given $y \in \mathbb{R}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists y' \in \mathbb{Q}$ s.t. $y < y'$. Hence,

$$E(x, y) \subseteq E(x, y')$$

Which is equivilant saying by definition, for all $t < y$,

$$0 < x^t < x^{y'}$$

Thus, $E(x, y)$ is bounded for $y \in \mathbb{R}$. $\square$

## (c)

*Proof.* First, we need to show that let $t \in \mathbb{Q}$ with $t < y+z$, there exists $t_1, t_2 \in \mathbb{Q}$ s.t. $t = t_1 + t_2,\ t_1 < y,\ t_2 < z$.
By densitiy theorem, $\exists t_1 \in \mathbb{Q}$ s.t.

$$y - \epsilon < t_1 < y \ (1)$$

we pick $\epsilon = y + z - t$ for the following proof. let $t_2 = t - t_1$ (NOTE: Since $t, t_1 \in \mathbb{Q}$, then $t_2 \in \mathbb{Q}$).
Then we sub $t_1 = t - t_2$ and our defined $\epsilon$ into (1), we can get:

$$y - \epsilon < t - t_2 < y \Rightarrow y - y - z + t < t - t_2 < y \Rightarrow t - z < t - t_2 < y$$

$$\Rightarrow -z < -t_2 < y - t \Rightarrow t_2 < z < y - t$$

By looking at the left inequality, we finished our proof.
Using our proved statement upwards, to proof $x^{y+z} = x^y x^z$, we need to validate two directions:

1. First we need to show $x^{y+z} \leq x^y x^z$:
   Let $t \in \mathbb{Q}$ with $t < y + z$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $t_1, t_2 \in \mathbb{Q}$ s.t.:

   $$t_1 < y;\ t_2 < z;\ t = t_1 + t_2$$

   Then:

   $$x^t = x^{t_1+t_2} = x^{t_1} x^{t_2} < x^y x^z$$

   Therefore, every element $x^t$ of $E(x, y + z) \leq x^y x^z$. Thus:

   $$x^{y+z} = \sup E(x, y + z) \leq x^y x^z$$

2. Then we need to show $x^{y+z} \geq x^y x^z$: For any $t < y, u < z,\ t, u \in \mathbb{Q}$, by 1 (c), we can get:

   $$x^{t+u} \leq x^{y+z} \Leftrightarrow x^t x^u \leq x^{y+z}$$

   (NOTE: $x^{t+u} \in E(x, y + z)$)
   We can rearrange this inequality into:

   $$x^u \leq x^{-t} x^{y+z} \Rightarrow x^z \leq x^{-t} x^{y+z}$$

   $$\Rightarrow x^t \leq (x^z)^{-1} x^{y+z} \Rightarrow x^y \leq (x^z)^{-1} x^{y+z}$$

   $$\Rightarrow x^y x^z \leq x^{y+z}$$

Overall, combining inequalities we proved in both cases, we get $x^{y+z} = x^y x^z$.
The property $x^{y+z} = x^y x^z$ implies that the function $f(y) = x^y$ is **injective**. Specifically by definition, if $f(y_1) = f(y_2)$, then $x^{y_1} = x^{y_2}$. Using the exponential property proved in (c), we have $x^{y_1 - y_2} = 1$. Since $x > 1$ and the exponential function $x^t$ is strictly increasing, the equation $x^t = 1$ holds only when $t = 0$. Therefore, $y_1 - y_2 = 0$, which means $y_1 = y_2$. This shows that $f$ is injective because no two different inputs produce the same output. □

## Question 2

*Proof. The following proof follows the given proof outline:*
**Step i:** For any $n \in \mathbb{N}$, $x - 1 \geq n(x^{\frac{1}{n}} - 1)$
Recall Bernoulli's inequality, it states that for any real number $r \geq 1$ and $s \geq -1$:

$$(1+s)^r \geq 1 + rs$$

Let $s = x^{\frac{1}{n}} - 1$ (NOTE: $s > 0$ as $x > 1$) and $r = n$:

$$(1 + x^{\frac{1}{n}} - 1)^n \geq 1 + n(x^{\frac{1}{n}} - 1)$$

$$\Rightarrow x \geq 1 + n(x^{\frac{1}{n}} - 1) \Rightarrow x - 1 \geq n(x^{\frac{1}{n}} - 1)$$

**Step ii:** If $t > 1$ and $n \in \mathbb{N}$ s.t. $n > \frac{x-1}{t-1}$, then $x^{\frac{1}{n}} < t$.
From *step i*, we get:

$$x^{\frac{1}{n}} - 1 \leq \frac{x-1}{n}$$

If $n > \frac{x-1}{t-1}$, then:

$$\frac{x-1}{n} < t - 1$$

Therefore,

$$x^{\frac{1}{n}} - 1 < t - 1 \Longrightarrow x^{\frac{1}{n}} < t$$

**Step iii:** If $y \in \mathbb{R}$ and $x^y < z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y+\frac{1}{n}} < z$:
Set $t = \frac{x^y}{z}$, note $\frac{x^y}{z} > 1$ by definition. Again using step (ii), choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x-1}{t-1}$$

Then $x^{\frac{1}{n}} < t$, recall question 1($c$) and multiply $x^y$ on both side, we can get:

$$x^y \cdot x^{\frac{1}{n}} = x^{y+\frac{1}{n}} < x^y \cdot t = z$$

**Step iv:** If $y \in \mathbb{R}$ and $x^y > z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y-\frac{1}{n}} > z$:
Set $t = \frac{z}{x^y}$, note $\frac{z}{x^y} > 1$ by definition. Again using step (ii), choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x-1}{t-1}$$

Then $x^{\frac{1}{n}} < t$, so $\frac{1}{x^{\frac{1}{n}}} = x^{-\frac{1}{n}} > \frac{1}{t}$. Recall question 1($c$) and multiply $x^y$ on both side, we can get:

$$x^y \cdot x^{-\frac{1}{n}} = x^{y-\frac{1}{n}} > x^y \cdot \frac{1}{t} = z$$

**Step v:** Define $A(z) = \{w \in \mathbb{R} | x^w < z\}$. Let $y = \sup A(z)$. then $x^y = z$
*Case 1:* Suppose $x^y < z$: by step (iii), there exists $n$ s.t. $x^{y+\frac{1}{n}} < z$. This indicates $y + \frac{1}{n} \in A(z)$. This contradicting the fact that $y = \sup A(z)$.

*Case 2:* Similarly, suppose $x^y > z$: by step (iv), there exists $n$ s.t. $x^{y-\frac{1}{n}} > z$. But for $y - \frac{1}{n}$, we can find a $w_0 \in A(z)$ s.t.

$$y - \frac{1}{n} < w_0 \Rightarrow x^{y-\frac{1}{n}} < x^w < z$$

This contradict with the result that we have in step (iv).
Consequently, neither $x^y < z$ nor $x^y > z$ is possible; thus, $x^y = z$. □

## Question 3

By analyzed several terms of the given sequence, likely it is increasing and should converges to 2.

*Proof.* For the given sequence, we can define it recusively:

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n},\ for\ n \geq 1$$

First, we need to set up an induction to prove this sequence is increasing. I.e. We will prove $a_n < a_{n+1}$ for all $n \geq 1$
*Base Case (n = 1):*

$$a_1 = \sqrt{2},\ a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$$

Noticing that $a_1, a_2 > 0$, and $a_1^2 = 2, a_2^2 = 2\sqrt{2}$, so:

$$2 < 2\sqrt{2} \Leftrightarrow a_1^2 < a_2^2 \Rightarrow a_1 < a_2$$

*Inductive steps:*
Assume $a_n < a_{n+1}$ for some $n \geq 1$. We need to show $a_{n+1} < a_{n+2}$.
Noticing, $a_{n+1}$ and $a_{n+2}$ can be deonte as:

$$a_{n+1} = \sqrt{2a_n},\ a_{n+2} = \sqrt{2a_{n+1}}$$

Then, since $a_n > 0,\ \forall n \in \mathbb{N}$:

$$\sqrt{2a_n} < \sqrt{2a_{n+1}} \Leftrightarrow 2a_n < 2a_{n+1} \Leftrightarrow a_n < a_{n+1}$$

Hence, by our induction hypothesis, $a_{n+1} < a_{n+2}$.
Therefore by induction, sequence $\{a_n\}$ is increasing.
Then, we need to show that $(a_n)$ is bounded above. (I.e., we will prove by induction that $a_n < 2$ for all $n \geq 1$)
*Base Case (n = 1):*

$$a_1 = \sqrt{2} < 2$$

*Inductive Step:*

Assume $a_n < 2$ for some $n \geq 1$, we need to show $a_{n+1} < 2$:
From our previous definition, $a_{n+1} = \sqrt{2a_n}$, we can get:

$$a_{n+1} = \sqrt{2a_n} < \sqrt{4} = 2$$

By induction again, $a_n < 2$ for all $n \geq 1$.
Since $\{a_n\}$ is increasing and bounded above by 2, by MCT, $\{a_n\}$ converges to some limit $L$ s.t. $L \leq 2$
Let $L = \lim_{n\to\infty} a_n$. Taking limit both sides of the recursive formula:

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2L}$$

Then:

$$L = \sqrt{2L} \Longrightarrow L^2 = 2L \Longrightarrow L^2 - 2L = 0 \Longrightarrow L(L-2) = 0$$

So $L = 0, 2$, and since $a_n > 0$, limit $L \geq \sqrt{2} > 0$. Lastly, $L = 2$ only.
The limit found for the given sequence is:

$$\lim_{n\to\infty} a_n = 2$$

$\square$

# Question 4

*Proof.*
Let $(x_n)$ be a monotone sequence.
$\Leftarrow$:
By one of the theorem in "10/21" lecture, since $(x_n)$ in convergent to $x$, then $(x_{n_k})$ is convergent to $x$ for any subsequence of $(x_n)$. So we are done.
$\Rightarrow$:
Assume $(x_n)$ has a subsequence $(x_{n_k})$ that converges to some limit $L$.
Since $(x_n)$ is monotone, it is either non-decreasing or non-increasing. Without loss of generality, suppose $(x_n)$ is non-decreasing.
We want to show that for $\epsilon > 0$, $\exists N > 0$ s.t. $n \geq N_\epsilon$:

$$L - \epsilon < x_n < L + \epsilon$$

First we show the right inequality of our goal ($x_n < L + \epsilon$):
Let $k > N$, then $x_k \leq x_{n_k}$. Hence, we can get:

$$x_k \leq x_{n_k} < L + \epsilon$$

Then, we need to show the left inequality of our goal ($x_n > L - \epsilon$):
Let $k \geq n_N$, then $x_k \geq x_{n_N}$. Hence, we can get:

$$x_k \geq x_{n_N} < L - \epsilon$$

Combining the previous 2 cases, choose $M := \max\{n_N, N\} = n_N$. Then for $n = k \geq M$

$$L - \epsilon < x_n \leq L + \epsilon \Longrightarrow |x_n - L| < \epsilon$$

Therefore, by definition, we have $\lim_{n\to\infty} x_n = L$. Thus, $(x_n)$ convertges to $L$. $\square$

## Question 5

First, lets recall Bolzano-Weierstrass and Monotone Convergence Theorem:
*Monotone Convergence Theorem:* A monotone sequence of real numbers is convergent *if and only if* it is bounded.
*Bolzano-Weierstrass Theorem:* Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.
To prove MCT using Bolzano-Weierstrass Theorem and our previous result, referencing the result that we have in $Q4$, we can approach in this way:

*Proof.*
$\Leftarrow$:
Let $(x_n)$ be a bounded monotonic sequence. WLOG, $(x_n)$ is non-decreasing.
Since $(x_n)$ is bounded, by *Bolzano-Weierstrass Theorem*, there exists a convergent subsequence $(x_{n_k})$ s.t.

$$\lim_{k\to\infty} x_{n_k} = L$$

From our previous result since $(x_n)$ is monotonic and has a converging subsequence $(x_{n_k})$, sequence $(x_n)$ converges to the same limit $L$.
Therefore, a bounded monotonic sequence converges.
$\Rightarrow$:
This direction is obviously true. Consider $\epsilon = 1$ for $(x_n) \to L$ (convergent sequence ). $\exists N_\epsilon \in \mathbb{N}$ s.t. $n \geq N_\epsilon$:

$$|x_n - L| < 1 \Leftrightarrow |x_n| < |L| + 1$$

Then we can pick a value $M$ to bound sequence $(x_n)$ as:

$$M = \max\{|x_1|, \cdots, |x_{N_{\epsilon-1}}|, |L| + 1\}$$

Hence, $|x_n| \leq M$, a convergent monotonic sequence bounded. $\square$