265F24midterm2_Sol

Math 265: Real Analysis I

Midterm Exam 2

Thursday, October 31, 2024

NAME (please print legibly): ______________________

Your University ID Number: ______________________

Indicate your instructor with a check in the appropriate box:

Wei-Cheng Huang	MW 10:25 - 11:40 AM	
Woongbae Park	MW 12:30 - 1:45 PM	

- You have 75 minutes to work on this exam.
- You are responsible for checking that this exam has all 9 pages.
- No calculators, phones, electronic devices, books, notes are allowed during the exam.
- **Show all work and justify all answers**, unless specified otherwise. Correct answers with insufficient work will not be given full credit.
- A blank page for scratch work is provided at the end of the exam. **Work on this page will not be graded.** Please show your work on the page containing the relevant question.
- Clearly circle all final answers.

Please **COPY** the HONOR PLEDGE and **SIGN:**

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

HONOR PLEDGE:

____

____

____

YOUR SIGNATURE:______________________

1. (15 points) Prove that there exists a sequence of rational numbers which converge to the number π.
(Hint: The Density Theorem)

For $n \in \mathbb{N}$, by the density thm, $\exists\, r_n \in \mathbb{Q}$ s.t.
$\pi < r_n < \pi + \frac{1}{n}$.
$\because \lim_{n\to\infty}(\pi) = \pi = \lim_{n\to\infty}(\pi + \frac{1}{n})$
$\therefore$ By the squeeze thm, $\lim_{n\to\infty}(r_n) = \pi$.

2. (20 points) Let (x_n) be a sequence of positive numbers. If $I_n = [-\frac{1}{n}, x_n]$, $n \in \mathbb{N}$, is a nested sequence of closed bounded intervals.
(a) Prove that (x_n) is decreasing.
(b) By (a), the sequence (x_n) is convergent. Let $x = \lim(x_n)$. Write

$$\bigcap_{n=1}^{\infty} I_n$$

into a single interval. Justify your answer.

(a) For $n \in \mathbb{N}$, since $I_{n+1} \subseteq I_n$ & $x_{n+1} \in I_{n+1}$,
$x_{n+1} \in I_n \Rightarrow x_{n+1} \leq x_n$.
$\Rightarrow (x_n)$ is decreasing.

(b) <u>Claim</u> $\bigcap_{n=1}^{\infty} I_n = [0, x]$.
<u>pf</u> "$\supseteq$" Let $y \in [0, x]$. Then $0 \leq y \leq x$
$\because (x_n)$ is decreasing & $x = \lim(x_n)$
$\therefore x = \inf(x_n)$ by the monotone convergence thm.
$\Rightarrow x \leq x_n \quad \forall n \in \mathbb{N}$.
$\Rightarrow -\frac{1}{n} \leq y \leq x \leq x_n \quad \forall n \in \mathbb{N}$.
$\Rightarrow y \in I_n \quad \forall n \in \mathbb{N}$
$\Rightarrow y \in \bigcap_{n=1}^{\infty} I_n$
"$\subseteq$" Let $y \in \bigcap_{n=1}^{\infty} I_n$. Then $-\frac{1}{n} \leq y \leq x_n \quad \forall n \in \mathbb{N}$.
$\Rightarrow 0 = \sup(-\frac{1}{n}) \leq y \leq \inf(x_n) = x$
since y is an upper bound for $(-\frac{1}{n})$
& a lower bound for (x_n)
$\Rightarrow y \in [0, 1]$.

3. (15 points) Find the following limit and use the definition of a limit to justify your answer.

$$\lim_{n\to\infty}\left(\frac{n^2}{n^2 + (-1)^{n+1}}\right).$$

<u>Claim</u> $\lim_{n\to\infty}\left(\frac{n^2}{n^2+(-1)^{n+1}}\right) = 1$

<u>pf</u> $\left|\frac{n^2}{n^2+(-1)^{n+1}} - 1\right| = \frac{1}{n^2+(-1)^{n+1}} \leq \frac{1}{n^2-1}$, if $n \geq 2$.
Let $\varepsilon > 0$. Consider $N_\varepsilon := \left\lceil \left(\frac{1}{\varepsilon}+1\right)^{\frac{1}{2}} \right\rceil$.
For $n > N_\varepsilon$. $n^2 > \frac{1}{\varepsilon} + 1$
$\Rightarrow \frac{1}{n^2-1} < \varepsilon$
$\Rightarrow \left|\frac{n^2}{n^2+(-1)^{n+1}} - 1\right| < \varepsilon$.
Therefore, $\lim_{n\to\infty}\left(\frac{n^2}{n^2+(-1)^{n+1}}\right) = 1$

4. (20 points) Let (x_n) be a sequence of positive real numbers that converges to $x > 0$ and $x_n \neq x$ for all $n \in \mathbb{N}$. Find the following limit and justify your answer.

$$\lim_{n\to\infty} \frac{\sqrt{x_n + x} - \sqrt{2x}}{\sqrt{x_n} - \sqrt{x}}.$$

$$\frac{\sqrt{x_n+x} - \sqrt{2x}}{\sqrt{x_n} - \sqrt{x}} = \frac{\cancel{x_n - x}^{\,\sqrt{x_n}+\sqrt{x}}}{\cancel{(\sqrt{x_n}-\sqrt{x})}(\sqrt{x_n+x}+\sqrt{2x})}$$
$$= \frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n+x}+\sqrt{2x}}$$

$\because (x_n) \to x \quad \therefore (\sqrt{x_n+x}) \to \sqrt{2x}$
$(\sqrt{x_n}) \to \sqrt{x}$
$\because \sqrt{2x} + \sqrt{2x} \neq 0$
$\therefore$ By limit laws,
$$\lim_{n\to\infty} \frac{\sqrt{x_n+x} - \sqrt{2x}}{\sqrt{x_n} - \sqrt{x}} = \frac{\sqrt{x}+\sqrt{x}}{\sqrt{2x}+\sqrt{2x}} = \frac{1}{\sqrt{2}}$$

5. (15 points) Define a sequence (x_n) of real numbers by

$$x_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Using Monotone Convergence Theorem, show that (x_n) is convergent.
(*Hint : Use* $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ *for* $k \geq 2$.)

For $n \in \mathbb{N}$, $x_{n+1} = x_n + \frac{1}{(n+1)} \geq x_n$
$\Rightarrow (x_n)$ is increasing.
$x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$
$\leq 1 + \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)n}$
$= 1 + \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n-1}} - \frac{1}{n}$
$= 2 - \frac{1}{n} \leq 2$.
$\Rightarrow (x_n)$ is bounded above.
Therefore, (x_n) is conv. by Monotone Convergence Thm.

6. (15 points) For a sequence (x_n) of real numbers and for any subsequence (x_{n_k}) of (x_n), show that

$$\limsup(x_n) \geq \limsup(x_{n_k}).$$

Let S & S' be the set of subsequential limits of (x_n) & (x_{n_k}), resp.
Then $S \supseteq S'$
$\Rightarrow \limsup(x_n) = \sup S \geq \sup S'$
$= \limsup(x_{n_k})$.

(scratchwork page)

(scratchwork page)