MATH 265 HW3

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Question 1

Proof. proof by contradiction: suppose $a < b$.
By assumption, we have $a \leq b - \epsilon$, for all $\epsilon > 0$.
define the differences d as $d = b - a$. Since $a < B$, by definition, $d = b - a \in \mathbb{P} > 0$.
Now we pick $\epsilon = \frac{d}{2}$. Since $d > 0$, $\epsilon = \frac{d}{2} > 0$.
We know that:

$$a \leq b - \epsilon$$

Sub $\epsilon = \frac{d}{2}$ in, we get:

$$a \leq b - \frac{d}{2}$$

Rewrite the expression in terms of a and b (Note: $d = b - a$):

$$a \leq b - \frac{b-a}{2} \Rightarrow a \leq \frac{2b - (b-a)}{2} = \frac{b+a}{2}$$

Multiply both side by 2 (using A2 and M1), we get:

$$2a \geq b + a$$

Then, use the A4, subtract a from both sides:

$$2a - a \geq b + a - a$$

Then, use A1 and A3, we can get:

$$a \geq b + (a - a) \Rightarrow a \geq b + 0 \rightarrow a \geq b$$

Hence we derived a contradiction. Using the field axioms, the given condition implies $a \geq b$. □

Question 2

Proof. Recall the hint that:

$$a^2 - b^2 = (a+b)(a-b)$$

Sub $b = -a$ into the identity, we can get:

$$a^2 - (-a^2) = (a + (-a))(a - (-a))$$

Using the axoim of additive inverses (A4): $a + (-a) = 0$.

$$\Rightarrow a^2 - (-a^2) = (0)(a - (-a))$$

By the Theorem of we know that $a \cdot 0 = 0$ for any $x \in \mathbb{R}$. Thus we have,

$$\Rightarrow a^2 - (-a^2) = 0 \Rightarrow a^2 = (-a^2)$$

□

Question 3

Proof. we want to prove:

$$\left(\frac{1}{2}(a+b)^2\right) \leq \frac{1}{2}(a^2 + b^2)$$

iff for all $a, b \in \mathbb{R}$, $a = b$.
First using axiom D (distributive property) twice, we can expand LHS:

$$\left(\frac{1}{2}(a+b)^2\right) = \frac{1}{4}(a+b)^2 = \frac{1}{4}(a^2 + 2ab + b^2)$$

Now we reform the inequality and WTS that:

$$\frac{1}{4}(a^2 + 2ab + b^2) \leq \frac{1}{2}(a^2 + b^2)$$

Multiply both sides by 4, then use M2, we can get:

$$(\frac{1}{4} \cdot 4) \cdot (a^2 + 2ab + b^2) \leq (\frac{1}{4} \cdot \frac{1}{2}) \cdot (a^2 + b^2) \Rightarrow a^2 + 2ab + b^2 \leq 2(a^2 + b^2)$$

Now we can rearrange terms using A1, A2, and D axiom:

$$a^2 + 2ab + b^2 \leq 2a^2 + 2b^2 \Rightarrow a^2 + 2ab + b^2 - 2a^2 - 2b^2 \leq 0$$

This simplifies to:

$$\Rightarrow -a^2 + 2ab - b^2 \leq 0$$

By D axiom again, we can factor out the (-1) as:

$$\Rightarrow (-1)(a^2 - 2ab + b^2) \leq 0$$

We know that $a^2 - 2ab + b^2 = (a-b)^2$, hence:

$$\Rightarrow (-1)(a-b)^2 \leq 0$$

Noting that since $(a-b)^2 \geq 0$, for any $a, b \in \mathbb{R}$, by A4, its additive inverse $(-1)(a-b)^2 \leq 0$ must also hold.
Hence, we can get this equality that:

$$(a-b)^2 = 0 \Rightarrow a - b = 0 \Rightarrow a = b$$

Therefore, the inequality $\left(\frac{1}{2}(a+b)^2\right) \leq \frac{1}{2}(a^2+b^2)$ holds iff $a = b$ s.t. $a, b \in \mathbb{R}$. $\square$