

MATH 265 HW7

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Question 1

(a)

Proof. Let $x > 1$ and $y \in \mathbb{Q}$. We aim to show that x^y is the l.u.b of the set

$$E(x, y) = \{x^t | t < y, t \in \mathbb{Q}\}$$

1. Upper Bound:

For any $t \in \mathbb{Q}$ with $t < y$, given $x > 1$, x^t is increasing in t , we have:

$$x^t < x^y$$

Therefore, x^y is an upper bound of $E(x, y)$

2. Least Upper Bound (Supremum):

By the consequence of the Density Theorem: For $M \in \mathbb{R}$ satisfying $M < x^y$, there exists $t \in \mathbb{Q}$ such that $M < x^t < x^y$.

Suppose M is an upper bound of set $E(x, y)$ and $M < x^y$, by the consequence listed upwards, we can always find an x^t s.t. $x^t > M$. This contradicts the assumption that M is an upper bound of $E(x, y)$. Therefore, no number less than x^y can be an upperbound of $E(x, y)$. Hence, $x^y = \sup E(x, y)$.

□

(b)

Proof. Given $y \in \mathbb{R}$, since \mathbb{Q} is dense in \mathbb{R} , $\exists y' \in \mathbb{Q}$ s.t. $y < y'$. Hence,

$$E(x, y) \subseteq E(x, y')$$

Which is equivalent saying by definition, for all $t < y$,

$$0 < x^t < x^{y'}$$

Thus, $E(x, y)$ is bounded for $y \in \mathbb{R}$.

□

(c)

Proof. First, we need to show that let $t \in \mathbb{Q}$ with $t < y+z$, there exists $t_1, t_2 \in \mathbb{Q}$ s.t. $t = t_1 + t_2$, $t_1 < y$, $t_2 < z$.

By density theorem, $\exists t_1 \in \mathbb{Q}$ s.t.

$$y - \epsilon < t_1 < y \quad (1)$$

we pick $\epsilon = y + z - t$ for the following proof. let $t_2 = t - t_1$ (NOTE: Since $t, t_1 \in \mathbb{Q}$, then $t_2 \in \mathbb{Q}$).

Then we sub $t_1 = t - t_2$ and our defined ϵ into (1), we can get:

$$\begin{aligned} y - \epsilon < t - t_2 < y &\Rightarrow y - y - z + t < t - t_2 < y \Rightarrow t - z < t - t_2 < y \\ &\Rightarrow -z < -t_2 < y - t \Rightarrow t_2 < z < y - t \end{aligned}$$

By looking at the left inequality, we finished our proof.

Using our proved statement upwards, to proof $x^{y+z} = x^y x^z$, we need to validate two directions:

1. First we need to show $x^{y+z} \leq x^y x^z$:

Let $t \in \mathbb{Q}$ with $t < y + z$. Since \mathbb{Q} is dense in \mathbb{R} , there exists $t_1, t_2 \in \mathbb{Q}$ s.t.:

$$t_1 < y; \quad t_2 < z; \quad t = t_1 + t_2$$

Then:

$$x^t = x^{t_1+t_2} = x^{t_1} x^{t_2} < x^y x^z$$

Therefore, every element x^t of $E(x, y + z) \leq x^y x^z$. Thus:

$$x^{y+z} = \sup E(x, y + z) \leq x^y x^z$$

2. Then we need to show $x^{y+z} \geq x^y x^z$: For any $t < y, u < z$, $t, u \in \mathbb{Q}$, by 1 (c), we can get:

$$x^{t+u} \leq x^{y+z} \Leftrightarrow x^t x^u \leq x^{y+z}$$

(NOTE: $x^{t+u} \in E(x, y + z)$)

We can rearrange this inequality into 2 forms:

(a)

$$x^u \leq x^{-t} x^{y+z} \Rightarrow x^z \leq x^{-t} x^{y+z}$$

Since x^z is l.u.b by definition and $x^{-t} x^{y+z}$ is an upperbound.

(b)

$$x^t \leq x^{-u} x^{y+z} \Rightarrow x^z \leq x^{-u} x^{y+z}$$

Similarly for this case.

Therefore, combining two inequalities,

$$x^z x^y \leq (x^{-t})(x^{-u})x^{y+z} \leq x^{y+z} \Rightarrow x^{y+z} \leq x^z x^y$$

Combining both inequalities, we get $x^{y+z} = x^y x^z$

The property $x^{y+z} = x^y x^z$ implies that the function $f(y) = x^y$ is **injective**. Specifically by definition, if $f(y_1) = f(y_2)$, then $x^{y_1} = x^{y_2}$. Using the exponential property proved in (c), we have $x^{y_1 - y_2} = 1$. Since $x > 1$ and the exponential function x^t is strictly increasing, the equation $x^t = 1$ holds only when $t = 0$. Therefore, $y_1 - y_2 = 0$, which means $y_1 = y_2$. This shows that f is injective because no two different inputs produce the same output. \square

Question 2

Proof. The following proof follows the given proof outline:

Step i: For any $n \in \mathbb{N}$, $x - 1 \geq n(x^{\frac{1}{n}} - 1)$

Recall Bernoulli's inequality, it states that for any real number $r \geq -1$ and $s \geq -1$:

$$(1 + s)^r \geq 1 + rs$$

Let $s = x^{\frac{1}{n}} - 1$ (NOTE: $s > 0$ as $x > 1$) and $r = n$:

$$(1 + x^{\frac{1}{n}} - 1)^n \geq 1 + n(x^{\frac{1}{n}} - 1)$$

$$\Rightarrow x \geq 1 + n(x^{\frac{1}{n}} - 1) \Rightarrow x - 1 \geq n(x^{\frac{1}{n}} - 1)$$

Step ii: If $t > 1$ and $n \in \mathbb{N}$ s.t. $n > \frac{x-1}{t-1}$, then $x^{\frac{1}{n}} < t$.

From *step i*, we get:

$$x^{\frac{1}{n}} - 1 \leq \frac{x - 1}{n}$$

If $n > \frac{x-1}{t-1}$, then:

$$\frac{x - 1}{n} < t - 1$$

Therefore,

$$x^{\frac{1}{n}} - 1 < t - 1 \Rightarrow x^{\frac{1}{n}} < t$$

Step iii: If $y \in \mathbb{R}$ and $x^y < z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y+\frac{1}{n}} < z$:

Set $t = \frac{x^y}{z}$, note $\frac{x^y}{z} > 1$ by definition. Again using step (ii), choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x - 1}{t - 1}$$

Then $x^{\frac{1}{n}} < t$, recall question 1(c) and multiply x^y on both side, we can get:

$$x^y \cdot x^{\frac{1}{n}} = x^{y+\frac{1}{n}} < x^y \cdot t = z$$

Step iv: If $y \in \mathbb{R}$ and $x^y > z$, then there exists $n \in \mathbb{N}$ s.t. $x^{y-\frac{1}{n}} > z$:

Set $t = \frac{z}{x^y}$, note $\frac{z}{x^y} > 1$ by definition. Again using step (ii), choose $n \in \mathbb{N}$ s.t.:

$$n > \frac{x - 1}{t - 1}$$

Then $x^{\frac{1}{n}} < t$, so $\frac{1}{x^{\frac{1}{n}}} = x^{-\frac{1}{n}} > \frac{1}{t}$. Recall question 1(c) and multiply x^y on both side, we can get:

$$x^y \cdot x^{-\frac{1}{n}} = x^{y-\frac{1}{n}} > x^y \cdot \frac{1}{t} = z$$

Step v: Define $A(z) = \{w \in \mathbb{R} | x^w < z\}$. Let $y = \sup A(z)$. then $x^y = z$

Case 1: Suppose $x^y < z$: by step (iii), there exists n s.t. $x^{y+\frac{1}{n}} < z$. This indicates $y + \frac{1}{n} \in A(z)$. Noting that $x^{y+\frac{1}{n}} > x^y$, we can get:

$$z > x^{y+\frac{1}{n}} > x^y$$

This contradicting the fact that $y = \sup A(z)$.

Case 2: Similarly, suppose $x^y > z$: by step (iv), there exists n s.t. $x^{y-\frac{1}{n}} > z$. But for $y - \frac{1}{n}$, we can find a $w_0 \in A(z)$ s.t.

$$y - \frac{1}{n} < w < z \Rightarrow x^{y-\frac{1}{n}} < x^w < x^z$$

This contradicting with the result that we have in step (iv)

Consequently, neither $x^y < z$ nor $x^y > z$ is possible; thus, $x^y = z$. □

Question 3

By analyzed several terms of the given sequence, likely it is increasing and should converges to 2.

Proof. For the given sequence, we can define it recursively:

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n}, \text{ for } n \geq 1$$

First, we need to set up an induction to prove this sequence is increasing. I.e.

We will prove $a_n < a_{n+1}$ for all $n \geq 1$

Base Case ($n = 1$):

$$a_1 = \sqrt{2}, a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$$

Noticing that $a_1, a_2 > 0$, and $a_1^2 = 2, a_2^2 = 2\sqrt{2}$, so:

$$2 < 2\sqrt{2} \Leftrightarrow a_1^2 < a_2^2 \Rightarrow a_1 < a_2$$

Inductive steps:

Assume $a_n < a_{n+1}$ for some $n \geq 1$. We need to show $a_{n+1} < a_{n+2}$.

Noticing, a_{n+1} and a_{n+2} can be deonte as:

$$a_{n+1} = \sqrt{2a_n}, a_{n+2} = \sqrt{2a_{n+1}}$$

Then, since $a_n > 0, \forall n \in \mathbb{N}$:

$$\sqrt{2a_n} < \sqrt{2a_{n+1}} \Leftrightarrow 2a_n < 2a_{n+1} \Leftrightarrow a_n < a_{n+1}$$

Hence, by our induction hypothesis, $a_{n+1} < a_{n+2}$.

Therefore by induction, sequence $\{a_n\}$ is increasing.

Then, we need to show that (a_n) is bounded above. (I.e., we will prove by induction that $a_n < 2$ for all $n \geq 1$)

Base Case ($n = 1$):

$$a_1 = \sqrt{2} < 2$$

Inductive Step:

Assume $a_n < 2$ for some $n \geq 1$, we need to show $a_{n+1} < 2$:

From our previous definition, $a_{n+1} = \sqrt{2a_n}$, we can get:

$$a_{n+1} = \sqrt{2a_n} < \sqrt{4} = 2$$

By induction again, $a_n < 2$ for all n .

Since $\{a_n\}$ is increasing and bounded above by 2, by MCT, $\{a_n\}$ converges to some limit L s.t. $L \leq 2$

Let $L = \lim_{n \rightarrow \infty} a_n$. Taking limit both sides of the recursive formula:

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{2a_n} = \sqrt{2L}$$

Then:

$$L = \sqrt{2L} \implies L^2 = 2L \implies L^2 - 2L = 0 \implies L(L - 2) = 0$$

So $L = 0, 2$, and since $a_n > 0$, limit $L \geq \sqrt{2} > 0$. so $L = 2$.

Lastly, the limit found for the given sequence is:

$$\lim_{n \rightarrow \infty} a_n = 2$$

□