# MATH 265 HW2

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## Question 1

*Proof.* First lets check the base case. For $n = 1$:

$$\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1$$

$1^{\frac{2}{3}} = 1$. Since $1 \geq 1$, the base case holds.
For forming up the inductive hypothesis, assume the statement is true for some $k \geq 1$,

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq k^{\frac{2}{3}}$$

Now we need to show the statement holds for $k+1$, namely,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \geq (k+1)^{\frac{2}{3}}$$

From inductive hypothesis, we can add $\frac{1}{\sqrt{k+1}}$ both side:

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}}$$

Now we need to show:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

Using approximation of power functions (Taylor Expansion), we can approximate RHS:

$$(k+1)^{\frac{2}{3}} \approx k^{\frac{2}{3}} + \frac{2}{3k^{\frac{1}{3}}}$$

When $k$ is large, we can approximate LHS:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \approx k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}}$$

For, $\frac{1}{k^{\frac{1}{2}}} \leq \frac{2}{3k^{\frac{1}{3}}}$ since the former diminishes slower. Hence,

$$k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}} \geq k^{\frac{2}{3}} + \frac{2}{3k^{\frac{2}{3}}} \Rightarrow k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

By mathematical induction, the statement $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq n^{\frac{2}{3}}$ is true for all $nin\mathbb{N}$.
$\square$

## Question 2

*Proof.* First lets check the base case. For $n = 0$:

$$x_0 = 3, x_1 = \frac{1}{8} \cdot (3)^2 + 2 = \frac{9}{8} + 2 = \frac{25}{8} = 3.125$$

$x_0 < x_1 < 4$, the base case proved.
Assume that for some $n = k, k \geq 1$, $x_k < x_{k+1} < 4$. WTS $x_{k+1} < x_{k+1=2} < 4$.
From the recurrence relation:

$$x_{k+2} = \frac{1}{8} x_{k+1}^2 + 2$$

Using the inductive hypothesis, note that $x_{k+1} < 4$:

$$x_{k+2} = \frac{1}{8} x_{k+1}^2 + 2 < \frac{1}{8}(4^2) + 2 = 2 + 2 = 4$$

By mathematical induction, the statement $x_n < x_{n+1} < 4$ is true for all $n \in \mathbb{N} \cup \{0\}$. $\square$

## Question 3

*Proof.* First lets check the base case. For $k = 1$:

$$F_{m+1} = F_{m-1}F_1 + F_m F_2$$

By definition of Fibonacci Sequence, $F_1 = F_2 = 1$.

$$F_{m+1} = F_{m-1} + F_m$$

This is true by definition, hence the base case proved.
Assume that for some $k \geq 1$, the given identity holds:

$$F_{m+k} = F_{m-1}F_k + F_m F_{k+1}$$

We need to prove that this statement holds for $k+1$, namely,

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_m F_{k+2}$$

Using the Fibonacci sequence's definition, we express $F_{k+2}$ and $F_{m+k+1}$ as:

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{m+k+1} = F_{m+k} + F_{m+k-1}$$

Substitute the values from the inductive hypothesis into $F_{m+k+1}$:

$$F_{m+k+1} = (F_{m-1}F_k + F_m F_{k+1}) + (F_{m-1}F_{k-1} + F_m F_k)$$

Combine and reorganize terms:

$$F_{m+k+1} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$$

By Finbonacci definition:

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_m(F_{k+1} + F_k)$$

$$= F_{m-1}F_{k+1} + F_m F_{k+2}$$

By mathematical induction, the statement is true for all $k.m \in \mathbb{N}$ with $m \geq 2$. $\square$

## Question 4

*Proof.* Let $A_n := \{\text{polynomials of degree } n \text{ over } \mathbb{Z}\}$
Since $\mathbb{Z}$ is countably infinite and product of countably infinite set is countably infinite, we want to show the bijection between:

$$A_n \hookrightarrow \prod_{i=0}^{n} \mathbb{Z}_i : a_n x^n + \cdots + a_1 x + a_0 \mapsto (a_n, \cdots, a_1, a_0)$$

- Prove $A_n$ is *one to one*

  *Proof.* Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ To prove $A_n$ is *one to one*, we must show that if $A_n(p(x_1)) = A_n(p(x_2))$, then $p(x_1) = p(x_2)$.
  Suppose $A_n(p(x_1)) = A_n(p(x_2))$. Then:

  $$(a_{n_1}, \cdots, a_{1_1}, a_{0_1}) = (a_{n_2}, \cdots, a_{1_2}, a_{0_2})$$

By definition of tuple equality,

$$a_{n_1} = a_{n_2}, \cdots, a_{0_1} = a_{0_2}$$

Since the coefficient of $p(x_1)$ and $p(x_2)$ are identical, the function $A_n$ is injective. $\square$

- Prove $A_n$ is *onto*

  *Proof.* To prove $A_n$ is *onto*, we must show that for every tuple $(a_n, \cdots, a_0) \in \prod_{i=0}^{n} \mathbb{Z}_i$, there exists a polynomial $p(x) \in A_n$ s.t. $f(p(x)) = (a_n, \cdots, a_0)$. For any given tuple $(a_n, \cdots, a_0)$, we can find a polynomial

  $$p(x) = a_n x^n + \cdots + a_1 x + a_0 \ s.t. \ f(p(x)) = (a_n, \cdots, a_0)$$

  Hence, $A_n$ is surjective by definition $\square$

Note that $\mathbb{Z}$ is countably infinite, there exists a bijection between $\mathbb{Z}$ and $\mathbb{N}$. We also know that $\mathbb{N} \times \mathbb{N}$ is countably infinite. Hence, we can get:

$$A_n \hookrightarrow \prod_{i=0}^{n} \mathbb{Z}_i \hookrightarrow \mathbb{N} \times \mathbb{N}$$

Therefore, algebraic numbers is countably infinite. $\square$