

MATH 265 HW5

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Question 1

(a)

We can analysis this question by the parity of $n \in \mathbb{Z}_+$.

1. When n is odd, $(-1)^n = -1$. Hence the term becomes:

$$-\left(1 - \frac{1}{n}\right) = -1 + \frac{1}{n}$$

Noticing that for all n , $\frac{1}{n} \leq 1$, hence:

$$-1 + \frac{1}{n} \leq 0$$

Hence, for all odd n , $s \leq 0, s \in S$.

2. When n is even, $(-1)^n = 1$. Hence the term becomes:

$$\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$$

When $n \rightarrow \infty$:

$$\frac{1}{n} \rightarrow 0 \Rightarrow 1 - \frac{1}{n} \rightarrow 1$$

Hence, for all odd n , $1 - \frac{1}{n} \leq 1$

Since every element $s \in S$, $s \leq 1$. By definition, 1 is an upper bound of S .

(b)

Proof. If M is an upper bound for S , then $M \geq s$, for every element $s \in S$.

Suppose $M < 1$ and M is an upper bound of S , it is sufficient for us to only consider the case where n is even.

Since $\frac{1}{2(1-M)} < 0$, by Archimedes Property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{2(1-M)} < n$, we can get:

$$\frac{1}{2(1-M)} < n \Rightarrow \frac{1}{1-M} < 2n \Rightarrow 1 < 2n(1-M) \Rightarrow \frac{1}{2n} < 1-M \Rightarrow M < 1 - \frac{1}{2n}$$

In this case, we know that for n is even,

$$M < 1 - \frac{1}{2n} = (1 - \frac{1}{2n})(-1)^{2n}$$

Hence, there exists a $s \in S$ s.t. $M < (1 - \frac{1}{2n})(-1)^{2n}$ for some $n \in \mathbb{Z}_+$. This derived a contradiction, hence $M \geq 1$. \square

(c)

Proof. From part (a), 1 is an upper bound of S , and from part (b), no number less than 1 can be an upper bound. Thus by definition, the supremum of S is 1 (i.e. $\sup S = 1$). \square

Question 2

Noting that by definition $A + B = \{a + b \mid a \in A, b \in B\}$

Proof. First we need to show that $\sup(A + B) \leq \sup A + \sup B$:
let $a \in A$ and $b \in B$. By definition of supremum, we have $a \leq \sup A$ and $b \leq \sup B$.

Therefore, for any $a \in A$ and $b \in B$:

$$a + b \leq \sup A + \sup B$$

Since $A + B = \{a + b \mid a \in A, b \in B\}$, every element in $A + B$ is less than or equal to $\sup A + \sup B$. Therefore, $\sup(A + B) \leq \sup A + \sup B$.

Then we can derive the equality from here:

Let $\epsilon > 0$, by definition of supremum, for set A and B :

- $\exists a_\epsilon \in A : \sup A - \epsilon < a_\epsilon < \sup A$
- $\exists b_\epsilon \in B : \sup B - \epsilon < b_\epsilon < \sup B$

Now consider $a_\epsilon + b_\epsilon \in A + B$. Then:

$$\begin{aligned} (\sup A - \epsilon) + (\sup B - \epsilon) &< a_\epsilon + b_\epsilon \leq \sup(A + B) \\ \Rightarrow \sup A + \sup B - 2\epsilon &< a_\epsilon + b_\epsilon \leq \sup(A + B) \\ \Rightarrow \sup A + \sup B - 2\epsilon &< \sup(A + B) \\ \Rightarrow 0 &\leq \frac{1}{2}(\sup A + \sup B - \sup(A + B)) < \epsilon \end{aligned}$$

By previous step, WLOG, since $\sup(A + B) \leq \sup A + \sup B$, the left inequality holds. Referring to one of the theorems in section 2.1, we can get:

$$\frac{1}{2}(\sup A + \sup B - \sup(A + B)) = 0 \Rightarrow \sup A + \sup B = \sup(A + B)$$

\square

Question 3

Proof. Let $x > 1$ and consider the set $S = \{x^n : n \in \mathbb{Z}_+\}$.

For any $M \in \mathbb{R}$, we want to show that there exists $n \in \mathbb{Z}_+$ such that $x^n > M$

Taking natural logarithm on both sides of the inequality $x^n > M$, we get:

$$n \cdot \ln(x) > \ln(M)$$

Since $\ln(x) > 0$, noting $x > 1$, we can solve for n :

$$n > \frac{\ln(M)}{\ln(x)}$$

Note that RHS is a constant, let $N = \lceil \frac{\ln(M)}{\ln(x)} \rceil + 1$,

Hence, for $n = N$, we have:

$$x^n > M$$

For any $M \in \mathbb{R}$, we can find $n \in \mathbb{Z}_+$ s.t. $x^n > M$. Thus, the set is not bounded from above. \square

Question 4

For example:

$$I_n = \left[1 + \frac{1}{n}, 3 + \frac{1}{n}\right)$$

For each positive integer n .

Property verification:

Proof.

1. Prove that $\bigcup_{n=1}^{\infty} I_n = (1, 4)$:

- Show that $\bigcup_{n=1}^{\infty} I_n \subseteq (1, 4)$: Let $x \in \bigcup_{n=1}^{\infty} I_n$. Then there exists $n \in \mathbb{N}$ s.t. $x \in I_n$, i.e.,

$$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

Since $\frac{1}{n} > 0$, it follows that:

$$1 < x < 3 + \frac{1}{n} \leq 3 + 1 = 4$$

Therefore, $x \in (1, 4)$, and thus:

$$\bigcup_{n=1}^{\infty} I_n \subseteq (1, 4)$$

- Show that $(1, 4) \subseteq \bigcup_{n=1}^{\infty} I_n$: Let $x \in (1, 4)$. We need to find $n \in \mathbb{N}$ s.t. $x \in I_n$.

- If $x \in (1, 3)$: Then $x - 1 > 0$. By Archimedes' Theorem, $\exists N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < x - 1$$

For $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < x - 1$, so:

$$1 + \frac{1}{n} < x$$

Since $x < 3$, and $\frac{1}{n} > 0$, we have:

$$x < 3 + \frac{1}{n}$$

Therefore $x \in I_n$, for all $n > N$.

- If $x \in [3, 4)$: Then $4 - x > 0$. By Archimedes' Theorem, $\exists N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < 4 - x$$

For all $n > N$ $\frac{1}{n} \leq \frac{1}{N} < 4 - x$, so:

$$x < 4 - \frac{1}{n}$$

However, since the right endpoint of I_n is $3 + \frac{1}{n}$, and $3 + \frac{1}{n} < 4 - \frac{1}{n}$, we need to ensure $x < 3 + \frac{1}{n}$.

Observe that $x - 3 \geq 0$ and $x - 3 < 1$. By Archimedes theorem again, there exists $n \in \mathbb{N}$ s.t.

$$\frac{1}{n} > x - 3$$

This implies":

$$x - 3 < \frac{1}{n} \Rightarrow x < 3 + \frac{1}{n}$$

Also, since $x \geq 3$ and $\frac{1}{n} > 0$:

$$x \geq 3 = 1 + 2 \leq 1 + \frac{1}{n} + 2, \text{ since } \frac{1}{n} < 1$$

Note: $1 + \frac{1}{n} \leq x$ holds because $1 + \frac{1}{n} \leq 1 + 1 = 2 < x$
Therefore, $x \in I_n$

Combining two steps, $\bigcup_{n=1}^{\infty} I_n = (1, 4)$

2. Prove that $\bigcap_{n=1}^{\infty} I_n = [2, 3]$:

- Show that $[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$. Let $x \in [2, 3]$. For all $n \in \mathbb{N}$, since: $1 + \frac{1}{n} \leq 1 + 1 = 2 \leq x \leq 3$, and $x < 3 + \frac{1}{n}$ we have:

$$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

Therefore. $x \in I_n$, for all n . Thus,

$$[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$$

- Show that $\bigcap_{n=1}^{\infty} I_n \subseteq [2, 3]$. Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then for all $n \in \mathbb{N}$:

$$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

- Show $x \geq 2$: Suppose, $x < 2$, then $x - 1 < 1$. By Archimedes Theorem, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < x - 1$. For all $n \geq N$:

$$1 + \frac{1}{n} < x$$

which contradicts the fact that $x \geq 1 + \frac{1}{n}$, for all n . Therefore, $x \geq 2$.

- Show $x \leq 3$: Suppose, $x > 3$, then $x - 3 > 0$. By Archimedes Theorem, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < x - 3$. For all $n \geq N$:

$$x > 3 + \frac{1}{n}$$

which contradicts the fact that $x < 3 + \frac{1}{n}$, for all n . Therefore, $x \leq 3$.

Thus, $x \in [2, 3]$, so:

$$\bigcap_{n=1}^{\infty} I_n \subseteq [2, 3]$$

Combining both inclusions, we have:

$$\bigcap_{n=1}^{\infty} I_n = [2, 3]$$

Overall $I_n = [1 + \frac{1}{n}, 3 + \frac{1}{n})$ satisfy two given properties.

□