MATH 265 HW3

Hanzhang Yin

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Question 1

Proof. proof by contradiction: suppose $a < b$.
By assumption, we have $a \geq b - \epsilon$, for all $\epsilon > 0$.
define $d = b - a$. Since $a < b$, by definition, $d = b - a \in \mathbb{P} > 0$.
Now we pick $\epsilon = \frac{d}{2}$. Since $d > 0$, $\epsilon = \frac{d}{2} > 0$.
We know that:

$$a \geq b - \epsilon$$

Sub $\epsilon = \frac{d}{2}$ in, we get:

$$a \geq b - \frac{d}{2}$$

Rewrite the expression in terms of a and b (Note: $d = b - a$):

$$a \geq b - \frac{b-a}{2} \Rightarrow a \geq \frac{2b - (b-a)}{2} = \frac{b+a}{2}$$

Multiply both side by 2 (using A2 and M1), we get:

$$2a \geq b + a$$

Then, use the A4, subtract a from both sides:

$$2a - a \geq b + a - a$$

Then, use A1 and A3, we can get:

$$a \geq b + (a - a) \Rightarrow a \geq b + 0 \rightarrow a \geq b$$

Hence we derived a contradiction. Using the field axioms, the given assumption implies $a \geq b$. $\square$

Question 2

Proof. Recall the hint that:

$$a^2 - b^2 = (a + b)(a - b)$$

Sub $b = -a$ into the identity, we can get:

$$a^2 - (-a^2) = (a + (-a))(a - (-a))$$

Using the axoim of additive inverses (A4): $a + (-a) = 0$.

$$\Rightarrow a^2 - (-a^2) = (0)(a - (-a))$$

By the Theorem of we know that $a \cdot 0 = 0$ for any $x \in \mathbb{R}$. Thus we have,

$$\Rightarrow a^2 - (-a^2) = 0 \Rightarrow a^2 = (-a^2)$$

$\square$

Question 3

Proof. First, we want to prove:

$$\left(\frac{1}{2}(a+b)\right)^2 \leq \frac{1}{2}(a^2+b^2)$$

for all $a, b \in \mathbb{R}$.
First using axiom D (distributive property) twice, we can expand LHS:

$$\left(\frac{1}{2}(a+b)\right)^2 = \frac{1}{4}(a+b)^2 = \frac{1}{4}(a^2+2ab+b^2)$$

Now we reform the inequality and WTS that:

$$\frac{1}{4}(a^2+2ab+b^2) \leq \frac{1}{2}(a^2+b^2)$$

Multiply both sides by 4, then use M2, we can get:

$$(\frac{1}{4}\cdot 4)\cdot(a^2+2ab+b^2) \leq (\frac{1}{4}\cdot\frac{1}{2})\cdot(a^2+b^2) \Rightarrow a^2+2ab+b^2 \leq 2(a^2+b^2)$$

Now we can rearrange terms using A1, A2, and D axiom:

$$a^2+2ab+b^2 \leq 2a^2+2b^2 \Rightarrow a^2+2ab+b^2-2a^2-2b^2 \leq 0$$

This simplifies to:

$$\Rightarrow -a^2+2ab-b^2 \leq 0$$

By D axiom again, we can factor out the (-1) as:

$$\Rightarrow (-1)(a^2-2ab+b^2) \leq 0$$

We know that $a^2-2ab+b^2=(a-b)^2$, hence:

$$\Rightarrow (-1)(a-b)^2 \leq 0$$

Therefore, the inequality always holds for $a, b \in \mathbb{R}$.
Then we want to show this equality holds iff $a = b$. Noting that since $(a-b)^2 \geq 0$, for any $a, b \in \mathbb{R}$, by A4, its additive inverse $(-1)(a-b)^2 \leq 0$ must also hold. Since we have simplified $\left(\frac{1}{2}(a+b)\right)^2 \leq \frac{1}{2}(a^2+b^2)$ to $(-1)(a-b)^2 \leq 0$. Hence,

$$(a-b)^2 = 0 \Leftrightarrow a-b=0 \Leftrightarrow a=b$$

Therefore, the given inequality holds if and only if for $a = b$. □