MATH 265 HW4

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Question 1

Proof. Given w is a lower bound of S, by definition:

$$w \leq s, \quad \forall s \in S.$$

Since $w \in S$, there exists a $s \in S$ s.t. $s = w$. Then,

$$w \leq s = w \implies w = s.$$

Suppose there exists another lower bound $v \in \mathbb{R}$ of S such that $v > w$. However, since $w \in S$, it must satisfy:

$$v \leq w.$$

we derived a contradiction. Therefore, no such v exists, and w must be the g.l.b. of S $\square$

Question 2

Proof. First, we want to show $\inf(A) = m$ is an upper bound of $-A$.
Let $\inf(A) = m$, by definition of infimum, $m \leq a$ for all $a \in A$.

$$\Rightarrow -m \geq -a, \forall a \in A$$

By our definition of set $-A$, this means $-m$ (i.e. $\inf(A)$) is an upper bound of $-A$.
Following that, now we can show $-\inf(A)$ is the l.u.b of $-A$:
Assume there exists an upper bound $u \in -A$ s.t. $u < -\inf(A) = -m$. Then:

$$u < -m \rightarrow -u > m$$

Since $\inf(A) = m$, for $\epsilon = -u - m > 0$, there exists $a_\epsilon \in A$ s.t.

$$a_\epsilon < m + \epsilon = m + (-u - m) = -u \Rightarrow -a_\epsilon > u$$

Hence, u is NOT an upper bound of $-A$, and $-m \leq u$ always. This shows that $-\inf(A) = \sup(-A)$. $\square$

Question 3

(a)

Proof. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, any upper bound of A or B is also an upper bound of $A \cap B$.
Let $\sup A = u$. By definition, u is the least upper bound of A, hence u is an upper bound for $A \cap B$. Similarly, let $\sup B = w$. Then w is an upper bound for $A \cap B$.
Since $\sup(A \cap B)$ is the least upper bound of $A \cap B$, it must satisfy:

$$\sup(A \cap B) \leq u = \sup A \quad \text{and} \quad \sup(A \cap B) \leq w = \sup B.$$

Therefore,

$$\sup A, \sup B \geq \sup(A \cap B).$$

$\square$

(b)