MATH 265 Homework 9

Due Nov 21

Instructions:

- Please scan your work and upload it to Gradescope by **the end of Nov 21**.

1 Non-Graded Questions

Textbook Section 4.2 Questions 2bd, 5, 8, 9, 10
Textbook Section 5.1: Problems 3, 8, 9, 12.

2 Graded Questions

1. (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose that $\lim_{x\to 0} f(x) = L$ for some $L \in \mathbb{R}$.
   (a) Prove that $L = 0$.
   (b) Prove that for any $c \in \mathbb{R}$, f has a limit at c, and that this limit is equal to $f(c)$.
   (c) Let $a = f(1)$. Prove that for all $x \in \mathbb{R}$, $f(x) = ax$.
   Hint Prove this first for rational x, and then for irrational x.
2. (2 points) Suppose $A \subseteq \mathbb{R}$ and $f, g : A \to \mathbb{R}$ are both continuous on A. Let
   $$S = \{x \in A : f(x) \leq g(x)\}.$$
   Suppose $c \in A$ is a cluster point of S. Prove that $c \in S$.
3. (2 points) Suppose $f : [a, b] \to \mathbb{R}$ is increasing and satisfies the intermediate value property (for each $k \in \mathbb{R}$ such that $f(a) \leq k \leq f(b)$, there exists a $c \in [a, b]$ such that $f(c) = k$). Prove that f is continuous on $[a, b]$.