MATH 265 HW2

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Question 1

Proof. First lets check the base case. For $n = 1$:

$$\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1$$

$1^{\frac{2}{3}} = 1$. Since $1 \geq 1$, the base case holds.
For forming up the inductive hypothesis, assume the statement is true for some $k \geq 1$,

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq k^{\frac{2}{3}}$$

Now we need to show the statement holds for $k + 1$, namely,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \geq (k+1)^{\frac{2}{3}}$$

From inductive hypothesis, we can add $\frac{1}{\sqrt{k+1}}$ both side:

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}}$$

Now we need to show:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

Using approximation of power functions (Taylor Expansion), we can approximate RHS:

$$(k+1)^{\frac{2}{3}} \approx k^{\frac{2}{3}} + \frac{2}{3k^{\frac{1}{3}}}$$

When k is large, we can approximate LHS:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \approx k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}}$$

For, $\frac{1}{k^{\frac{1}{2}}} \leq \frac{2}{3k^{\frac{1}{3}}}$ since the former diminishes slower. Hence,

$$k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}} \geq k^{\frac{2}{3}} + \frac{2}{3k^{\frac{2}{3}}} \Rightarrow k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

By mathematical induction, the statement $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq n^{\frac{2}{3}}$ is true for all $n in \mathbb{N}$.

$\square$

Question 2