

# MATH 265 HW2

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## Question 1

*Proof.* First let's check the base case. For  $n = 1$ :

$$\sum_{i=1}^1 \frac{1}{\sqrt[3]{i}} = \frac{1}{\sqrt[3]{1}} = 1$$

$1^{\frac{2}{3}} = 1$ . Since  $1 \geq 1$ , the base case holds.

For forming up the inductive hypothesis, assume the statement is true for some  $k \geq 1$ ,

$$\sum_{i=1}^k \frac{1}{\sqrt[3]{i}} \geq k^{\frac{2}{3}}$$

Now we need to show the statement holds for  $k + 1$ , namely,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt[3]{i}} \geq (k+1)^{\frac{2}{3}}$$

From inductive hypothesis, we can add  $\frac{1}{\sqrt[3]{k+1}}$  both side:

$$\sum_{i=1}^k \frac{1}{\sqrt[3]{i}} + \frac{1}{\sqrt[3]{k+1}} \geq (k)^{\frac{2}{3}} + \frac{1}{\sqrt[3]{k+1}}$$

Now we need to show:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt[3]{k+1}} \geq (k+1)^{\frac{2}{3}}$$

By factorization, we can get:

$$\Rightarrow \frac{k}{\sqrt[3]{k}} + \frac{1}{\sqrt[3]{k+1}} \geq \frac{k}{\sqrt[3]{k+1}} + \frac{1}{\sqrt[3]{k+1}} > (k+1)^{\frac{2}{3}}$$

Since  $\frac{k}{\sqrt[3]{k}} > \frac{1}{\sqrt[3]{k+1}}$  always holds as  $k \in \mathbb{N}$ .

Hence, by mathematical induction, the statement  $\sum_{i=1}^k \frac{1}{\sqrt[3]{i}} \geq k^{\frac{2}{3}}$  is true for all  $n \in \mathbb{N}$ .  $\square$

## Question 2

*Proof.* First lets check the base case. For  $n = 0$ :

$$x_0 = 3, x_1 = \frac{1}{8} \cdot (3)^2 + 2 = \frac{9}{8} + 2 = \frac{25}{8} = 3.125$$

$x_0 < x_1 < 4$ , the base case proved.

Assume that for some  $n = k, k \geq 1$ ,  $x_k < x_{k+1} < 4$ . WTS  $x_{k+1} < x_{k+2} < 4$ .

From the recurrence relation:

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2$$

Using the inductive hypothesis, note that  $x_{k+1} < 4$ :

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2 < \frac{1}{8}(4^2) + 2 = 2 + 2 = 4$$

By mathematical induction, the statement  $x_n < x_{n+1} < 4$  is true for all  $n \in \mathbb{N} \cup \{0\}$ .  $\square$

## Question 3

*Proof.* First lets check the base case. For  $k = 1$ :

$$F_{m+1} = F_{m-1}F_1 + F_mF_2$$

By definition of Fibonacci Sequence,  $F_1 = F_2 = 1$ .

$$F_{m+1} = F_{m-1} + F_m$$

This is true by definition, hence the base case proved.

Using strong induction, assume the following statement is true for all Fibonacci Sequence up to  $k$ . (i.e.  $1, \dots, k$ )

$$F_{m+k} = F_{m-1}F_k + F_mF_{k+1}$$

We need to prove that this statement holds for  $k + 1$ , namely,

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_mF_{k+2}$$

Using the Fibonacci sequence's definition, we express  $F_{k+2}$  and  $F_{m+k+1}$  as:

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{m+k+1} = F_{m+k} + F_{m+k-1}$$

Substitute the values from the inductive hypothesis into the definition  $F_{m+k+1}$ :

$$F_{m+k+1} = (F_{m-1}F_k + F_mF_{k+1}) + (F_{m-1}F_{k-1} + F_mF_k)$$

Combine and reorganize terms:

$$F_{m+k+1} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$$

By Fibonacci definition:

$$\begin{aligned} F_{m+k+1} &= F_{m-1}F_{k+1} + F_m(F_{k+1} + F_k) \\ &= F_{m-1}F_{k+1} + F_mF_{k+2} \end{aligned}$$

By mathematical induction, the statement is true for all  $k, m \in \mathbb{N}$  with  $m \geq 2$ .  $\square$

## Question 4

*Proof.* Let  $P(x) = a_nx^n + \dots + a_1x + a_0$ , define  $P(X) \in \mathbb{Z}[x]$  s.t.  $a_i \in \mathbb{Z}$ . Define height of  $P(x)$  as:

$$h(P) := n + \sum_{i=0}^n |a_i|$$

Let  $c$  be an arbitrary constant that is large enough. The number of  $P(x)$  satisfying  $h(P) \leq c$  is finite.

Hence we can define,

$$\mathbb{Z}_n := \{P(x) \in \mathbb{Z}[x] : h(P(x)) \leq n\}$$

In here,  $\mathbb{Z}_n$  is finite. Let  $I$  be a countable set. Then,  $\forall P(x) \in \mathbb{Z}[x]$

$$Z[x] = \bigcup_{i \in I} P_i(x)$$

The countable union of finite sets is countable, hence  $Z[x]$  is countable.

For each polynomial  $P_i(x) \in Z[x]$ , there exists  $n$  roots, hence, we can denote the element  $A_i$  in the set of algebraic number  $A$  as:

$$A_i = \{x : P_i(x) = 0 \text{ s.t. } |A_i| = \deg(P_i)\}$$

Hence,  $A_i$  is also countable since we can find a constant  $c$  again s.t.  $|A_i| < c$ .

$$A = \bigcup_{i \in I} A_i$$

Again, since  $A_i$  is countable, the countable union of the finite set is countable. Therefore, algebraic numbers are countably infinite.  $\square$