MATH 265 Homework 7

Due Oct 24

Instructions:

- Please scan your work and upload it to Gradescope by **the end of Oct 24**.

1 Non-Graded Questions

Textbook Section 3.3: Questions 1, 2, 4, 7, 8, 9

2 Graded Questions

1. (4 points) Let $x > 1$. Recall that $x^{\frac{1}{q}}$ is defined for any $q \in \mathbb{N}$, and $x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p$ for any $p \in \mathbb{Z}$, $q \in \mathbb{N}$, *i.e.*, x^y is defined for any $y \in \mathbb{Q}$. It is also not hard to see that $x^{y+z} = x^y x^z$ for y, $z \in \mathbb{Q}$.

   In this question, you'll prove that the exponential is a valid operation for real numbers and prove some of its properties. For all parts, fix $x > 1$.

   (a) For $y \in \mathbb{R}$, define the set
   $$E(x, y) = \{x^t \mid t < y,\ t \in \mathbb{Q}\}.$$
   Suppose that y is rational. Prove that
   $$x^y = \sup E(x, y).$$

   (b) For $y \in \mathbb{R}$ (not necessarily rational), show that $E(x, y)$ is bounded.

   (c) Let $x > 1$. For $y \in \mathbb{R}$, we define $x^y = \sup E(x, y)$. Prove that for y, $z \in \mathbb{R}$,
   $$x^{y+z} = x^y x^z.$$
   Explain why this implies that the function $f : \mathbb{R} \to \mathbb{R}_{>0}$ given by $f(y) = x^y$ is injective.
   Hint To prove $x^{y+z} = x^y x^z$, you may need to show the following property: Let $t \in \mathbb{Q}$ with $t < y + z$. There exist t_1, $t_2 \in \mathbb{Q}$ such that $t = t_1 + t_2$, $t_1 < y$ and $t_2 < z$.
   To prove this, you may consider t_1 to be a rational number satisfying $y - \varepsilon \le t_1 \le y$ where $\varepsilon = y + z - t$. Please fill in the details.

2. (3 points) Let $x > 1$. In this question, you'll prove that the function $f : \mathbb{R} \to \mathbb{R}_{>0}$ given by $f(y) = x^y$ is invertible. In particular, you should prove that for any $z > 0$ there is a unique y such that $x^y = z$. (We write $y = \log_x(z)$.) This proof is tricky, so a rough outline to follow has been provided for you. You should explain each step in the outline in more detail.

   Proof Outline:

   (i) For any $n \in \mathbb{N}$, we have $x - 1 \ge n\left(x^{\frac{1}{n}} - 1\right)$.
   Hint Use Bernoulli's Inequality.

   (ii) If $t > 1$ and $n \in \mathbb{N}$ such that $n > \frac{x-1}{t-1}$, then $x^{\frac{1}{n}} < t$.

   (iii) If $y \in \mathbb{R}$ is such that $x^y < z$, then there exists $n \in \mathbb{N}$ such that $x^{y+\frac{1}{n}} < z$.

   (iv) If $y \in \mathbb{R}$ is such that $x^y > z$, then there exists $n \in \mathbb{N}$ such that $x^{y+\frac{1}{n}} > z$.

   (v) Define the set $A(z) = \{w \in \mathbb{R} \mid x^w < z\}$. Let $y = \sup A(z)$. Then $x^y = z$.

3. (3 points) Consider the sequence
   $$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots.$$
   Does it converge or diverge? Prove your claim. Find the limit if you claim that it converges.