MATH 265 HW2

Hanzhang Yin

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Question 1

Proof. First lets check the base case. For $n = 1$:

$$\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1$$

$1^{\frac{2}{3}} = 1$. Since $1 \geq 1$, the base case holds.
For forming up the inductive hypothesis, assume the statement is true for some $k \geq 1$,

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq k^{\frac{2}{3}}$$

Now we need to show the statement holds for $k + 1$, namely,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \geq (k+1)^{\frac{2}{3}}$$

From inductive hypothesis, we can add $\frac{1}{\sqrt{k+1}}$ both side:

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \geq (k)^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}}$$

Now we need to show:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

Using an approximation of power functions (Taylor Expansion), we can approximate RHS ($O\left(\frac{1}{k^{\frac{1}{3}}}\right)$ is the higher order terms):

$$(k+1)^{\frac{2}{3}} = k^{\frac{2}{3}} + \frac{2}{3k^{\frac{1}{3}}} + \cdots + O\left(\frac{1}{k^{\frac{1}{3}}}\right) \geq k^{\frac{2}{3}} + \frac{2}{3}\frac{1}{k^{\frac{1}{3}}}$$

When k is large, we can approximate LHS w.r.t the rate of change of the function:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}}$$

Note, $\frac{1}{k^{\frac{1}{2}}} \leq \frac{2}{3k^{\frac{1}{3}}}$ since the former diminishes slower when k is large. Hence,

$$k^{\frac{2}{3}} + \frac{1}{k^{\frac{1}{2}}} \geq k^{\frac{2}{3}} + \frac{2}{3k^{\frac{2}{3}}} \Rightarrow k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}}$$

By mathematical induction, the statement $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq n^{\frac{2}{3}}$ is true for all $n \in \mathbb{N}$. □

Question 2

Proof. First lets check the base case. For $n = 0$:

$$x_0 = 3, x_1 = \frac{1}{8} \cdot (3)^2 + 2 = \frac{9}{8} + 2 = \frac{25}{8} = 3.125$$

$x_0 < x_1 < 4$, the base case proved.
Assume that for some $n = k, k \geq 1$, $x_k < x_{k+1} < 4$. WTS $x_{k+1} < x_{k+1=2} < 4$.
From the recurrence relation:

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2$$

Using the inductive hypothesis, note that $x_{k+1} < 4$:

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2 < \frac{1}{8}(4^2) + 2 = 2 + 2 = 4$$

By mathematical induction, the statement $x_n < x_{n+1} < 4$ is true for all $n \in \mathbb{N} \cup \{0\}$. □

Question 3

Proof. First lets check the base case. For $k = 1$:

$$F_{m+1} = F_{m-1}F_1 + F_mF_2$$

By definition of Fibonacci Sequence, $F_1 = F_2 = 1$.

$$F_{m+1} = F_{m-1} + F_m$$

This is true by definition, hence the base case proved.
Assume that for some $k \geq 1$, the given identity holds:

$$F_{m+k} = F_{m-1}F_k + F_m F_{k+1}$$

We need to prove that this statement holds for $k+1$, namely,

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_m F_{k+2}$$

Using the Fibonacci sequence's definition, we express F_{k+2} and F_{m+k+1} as:

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{m+k+1} = F_{m+k} + F_{m+k-1}$$

Substitute the values from the inductive hypothesis into F_{m+k+1}:

$$F_{m+k+1} = (F_{m-1}F_k + F_m F_{k+1}) + (F_{m-1}F_{k-1} + F_m F_k)$$

Combine and reorganize terms:

$$F_{m+k+1} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$$

By Finbonacci definition:

$$\begin{aligned} F_{m+k+1} &= F_{m-1}F_{k+1} + F_m(F_{k+1} + F_k) \\ &= F_{m-1}F_{k+1} + F_m F_{k+2} \end{aligned}$$

By mathematical induction, the statement is true for all $k.m \in \mathbb{N}$ with $m \geq 2$. $\square$

Question 4

Proof. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$, define $\mathbb{Z}[x] \in P(X)$ s.t. $a_i \in \mathbb{Z}$
Define height of $P(x)$ as:

$$h(P) := n + \sum_{i=0}^{n} |a_i|$$

The number of $P(x)$ satisfying $h(P) \leq c$ is finite.
Hence we can define,

$$\mathbb{Z}_n := \{P(x) \in \mathbb{Z}[x] : h(P(x)) \leq n\}$$

In here, $\mathbb{Z}_n$ is finite. Then, $\forall P(x) \in \mathbb{Z}[x]$

$$Z[x] = \bigcup_{n=0}^{\infty} \mathbb{Z}_n$$

The finite union of countable sets is countable, hence $Z[x]$ is countable.
For each polynomial in set $Z[x]$, there exists at most n roots, hence, we can denote the set of algebraic number A as:

$$A = \bigcup_{n=0}^{\infty} \bigcup_{n=0}^{\infty} \mathbb{Z}_n = \bigcup_{n=0}^{\infty} Z[x]$$

Again, since $Z[x]$ is countable, the finite union of the countable set is countable.
Therefore, algebraic numbers are countably infinite. $\square$