

# MATH 265 HW7

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## Question 1

(a)

*Proof.* Let  $x > 1$  and  $y \in \mathbb{Q}$ . We aim to show that  $x^y$  is the l.u.b of the set

$$E(x, y) = \{x^t | t < y, t \in \mathbb{Q}\}$$

1. Upper Bound:

For any  $t \in \mathbb{Q}$  with  $t < y$ , given  $x > 1$ ,  $x^t$  is increasing in  $t$ , we have:

$$x^t < x^y$$

Therefore,  $x^y$  is an upper bound of  $E(x, y)$

2. Least Upper Bound (Supremum):

By the consequence of the Density Theorem: For  $M \in \mathbb{R}$  satisfying  $M < x^y$ , there exists  $t \in \mathbb{Q}$  such that  $M < x^t < x^y$ .

Suppose  $M$  is an upper bound of set  $E(x, y)$  and  $M < x^y$ , by the consequence listed upwards, we can always find an  $x^t$  s.t.  $x^t > M$ . This contradicts the assumption that  $M$  is an upper bound of  $E(x, y)$ . Therefore, no number less than  $x^y$  can be an upperbound of  $E(x, y)$ . Hence,  $x^y = \sup E(x, y)$ .

□

(b)

*Proof.* Given  $y \in \mathbb{R}$ , since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ,  $\exists y' \in \mathbb{Q}$  s.t.  $y < y'$ . Hence,

$$E(x, y) \subseteq E(x, y')$$

Which is equivalent saying by definition, for all  $t < y$ ,

$$0 < x^t < x^{y'}$$

Thus,  $E(x, y)$  is bounded for  $y \in \mathbb{R}$ .

□

(c)

*Proof.* First, we need to show that let  $t \in \mathbb{Q}$  with  $t < y+z$ , there exists  $t_1, t_2 \in \mathbb{Q}$  s.t.  $t = t_1 + t_2$ ,  $t_1 < y$ ,  $t_2 < z$ .

By density theorem,  $\exists t_1 \in \mathbb{Q}$  s.t.

$$y - \epsilon < t_1 < y \quad (1)$$

we pick  $\epsilon = y + z - t$  for the following proof. let  $t_2 = t - t_1$  (NOTE: Since  $t, t_1 \in \mathbb{Q}$ , then  $t_2 \in \mathbb{Q}$ ).

Then we sub  $t_1 = t - t_2$  and our defined  $\epsilon$  into (1), we can get:

$$\begin{aligned} y - \epsilon < t - t_2 < y &\Rightarrow y - y - z + t < t - t_2 < y \Rightarrow t - z < t - t_2 < y \\ &\Rightarrow -z < -t_2 < y - t \Rightarrow t_2 < z < y - t \end{aligned}$$

By looking at the left inequality, we finished our proof.

Using our proved statement upwards, to proof  $x^{y+z} = x^y x^z$ , we need to validate two directions:

1. First we need to show  $x^{y+z} \leq x^y x^z$ :

Let  $t \in \mathbb{Q}$  with  $t < y + z$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there exists  $t_1, t_2 \in \mathbb{Q}$  s.t.:

$$t_1 < y; \quad t_2 < z; \quad t = t_1 + t_2$$

Then:

$$x^t = x^{t_1+t_2} = x^{t_1} x^{t_2} < x^y x^z$$

Therefore, every element  $x^t$  of  $E(x, y + z) \leq x^y x^z$ . Thus:

$$x^{y+z} = \sup E(x, y + z) \leq x^y x^z$$

2. Then we need to show  $x^{y+z} \geq x^y x^z$ : For any  $t < y, u < z$ ,  $t, u \in \mathbb{Q}$ , by 1 (c), we can get:

$$x^{t+u} \leq x^{y+z} \Leftrightarrow x^t x^u \leq x^{y+z}$$

(NOTE:  $x^{t+u} \in E(x, y + z)$ )

We can rearrange this inequality into 2 forms:

(a)

$$x^u \leq x^{-t} x^{y+z} \Rightarrow x^z \leq x^{-t} x^{y+z}$$

Since  $x^z$  is l.u.b by definition and  $x^{-t} x^{y+z}$  is an upperbound.

(b)

$$x^t \leq x^{-u} x^{y+z} \Rightarrow x^y \leq x^{-u} x^{y+z}$$

Similarly for this case.

Therefore, combining two inequalities,

$$x^z x^y \leq (x^{-t})(x^{-u})x^{y+z} \leq x^{y+z} \Rightarrow x^{y+z} \leq x^y x^z$$

Overall, combining inequalities we proved in both cases, we get  $x^{y+z} = x^y x^z$ . The property  $x^{y+z} = x^y x^z$  implies that the function  $f(y) = x^y$  is **injective**. Specifically by definition, if  $f(y_1) = f(y_2)$ , then  $x^{y_1} = x^{y_2}$ . Using the exponential property proved in (c), we have  $x^{y_1 - y_2} = 1$ . Since  $x > 1$  and the exponential function  $x^t$  is strictly increasing, the equation  $x^t = 1$  holds only when  $t = 0$ . Therefore,  $y_1 - y_2 = 0$ , which means  $y_1 = y_2$ . This shows that  $f$  is injective because no two different inputs produce the same output.  $\square$

## Question 2

*Proof. The following proof follows the given proof outline:*

**Step i:** For any  $n \in \mathbb{N}$ ,  $x - 1 \geq n(x^{\frac{1}{n}} - 1)$

Recall Bernoulli's inequality, it states that for any real number  $r \geq -1$  and  $s \geq -1$ :

$$(1 + s)^r \geq 1 + rs$$

Let  $s = x^{\frac{1}{n}} - 1$  (NOTE:  $s > 0$  as  $x > 1$ ) and  $r = n$ :

$$(1 + x^{\frac{1}{n}} - 1)^n \geq 1 + n(x^{\frac{1}{n}} - 1)$$

$$\Rightarrow x \geq 1 + n(x^{\frac{1}{n}} - 1) \Rightarrow x - 1 \geq n(x^{\frac{1}{n}} - 1)$$

**Step ii:** If  $t > 1$  and  $n \in \mathbb{N}$  s.t.  $n > \frac{x-1}{t-1}$ , then  $x^{\frac{1}{n}} < t$ .

From *step i*, we get:

$$x^{\frac{1}{n}} - 1 \leq \frac{x - 1}{n}$$

If  $n > \frac{x-1}{t-1}$ , then:

$$\frac{x - 1}{n} < t - 1$$

Therefore,

$$x^{\frac{1}{n}} - 1 < t - 1 \Rightarrow x^{\frac{1}{n}} < t$$

**Step iii:** If  $y \in \mathbb{R}$  and  $x^y < z$ , then there exists  $n \in \mathbb{N}$  s.t.  $x^{y+\frac{1}{n}} < z$ :

Set  $t = \frac{x^y}{z}$ , note  $\frac{x^y}{z} > 1$  by definition. Again using step (ii), choose  $n \in \mathbb{N}$  s.t.:

$$n > \frac{x - 1}{t - 1}$$

Then  $x^{\frac{1}{n}} < t$ , recall question 1(c) and multiply  $x^y$  on both side, we can get:

$$x^y \cdot x^{\frac{1}{n}} = x^{y+\frac{1}{n}} < x^y \cdot t = z$$

**Step iv:** If  $y \in \mathbb{R}$  and  $x^y > z$ , then there exists  $n \in \mathbb{N}$  s.t.  $x^{y-\frac{1}{n}} > z$ :

Set  $t = \frac{z}{x^y}$ , note  $\frac{z}{x^y} > 1$  by definition. Again using step (ii), choose  $n \in \mathbb{N}$  s.t.:

$$n > \frac{x - 1}{t - 1}$$

Then  $x^{\frac{1}{n}} < t$ , so  $\frac{1}{x^{\frac{1}{n}}} = x^{-\frac{1}{n}} > \frac{1}{t}$ . Recall question 1(c) and multiply  $x^y$  on both side, we can get:

$$x^y \cdot x^{-\frac{1}{n}} = x^{y-\frac{1}{n}} > x^y \cdot \frac{1}{t} = z$$

**Step v:** Define  $A(z) = \{w \in \mathbb{R} | x^w < z\}$ . Let  $y = \sup A(z)$ . then  $x^y = z$

*Case 1:* Suppose  $x^y < z$ : by step (iii), there exists  $n$  s.t.  $x^{y+\frac{1}{n}} < z$ . This indicates  $y + \frac{1}{n} \in A(z)$ . Noting that  $x^{y+\frac{1}{n}} > x^y$ , we can get:

$$z > x^{y+\frac{1}{n}} > x^y$$

This contradicting the fact that  $y = \sup A(z)$ .

*Case 2:* Similarly, suppose  $x^y > z$ : by step (iv), there exists  $n$  s.t.  $x^{y-\frac{1}{n}} > z$ . But for  $y - \frac{1}{n}$ , we can find a  $w_0 \in A(z)$  s.t.

$$y - \frac{1}{n} < w < z \Rightarrow x^{y-\frac{1}{n}} < x^w < x^z$$

This contradicting with the result that we have in step (iv)

Consequently, neither  $x^y < z$  nor  $x^y > z$  is possible; thus,  $x^y = z$ . □

### Question 3

By analyzed several terms of the given sequence, likely it is increasing and should converges to 2.

*Proof.* For the given sequence, we can define it recursively:

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n}, \text{ for } n \geq 1$$

First, we need to set up an induction to prove this sequence is increasing. I.e.

We will prove  $a_n < a_{n+1}$  for all  $n \geq 1$

*Base Case* ( $n = 1$ ):

$$a_1 = \sqrt{2}, a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}}$$

Noticing that  $a_1, a_2 > 0$ , and  $a_1^2 = 2, a_2^2 = 2\sqrt{2}$ , so:

$$2 < 2\sqrt{2} \Leftrightarrow a_1^2 < a_2^2 \Rightarrow a_1 < a_2$$

*Inductive steps:*

Assume  $a_n < a_{n+1}$  for some  $n \geq 1$ . We need to show  $a_{n+1} < a_{n+2}$ .

Noticing,  $a_{n+1}$  and  $a_{n+2}$  can be deonte as:

$$a_{n+1} = \sqrt{2a_n}, a_{n+2} = \sqrt{2a_{n+1}}$$

Then, since  $a_n > 0, \forall n \in \mathbb{N}$ :

$$\sqrt{2a_n} < \sqrt{2a_{n+1}} \Leftrightarrow 2a_n < 2a_{n+1} \Leftrightarrow a_n < a_{n+1}$$

Hence, by our induction hypothesis,  $a_{n+1} < a_{n+2}$ .

Therefore by induction, sequence  $\{a_n\}$  is increasing.

Then, we need to show that  $(a_n)$  is bounded above. (I.e., we will prove by induction that  $a_n < 2$  for all  $n \geq 1$ )

*Base Case ( $n = 1$ ):*

$$a_1 = \sqrt{2} < 2$$

*Inductive Step:*

Assume  $a_n < 2$  for some  $n \geq 1$ , we need to show  $a_{n+1} < 2$ :

From our previous definition,  $a_{n+1} = \sqrt{2a_n}$ , we can get:

$$a_{n+1} = \sqrt{2a_n} < \sqrt{4} = 2$$

By induction again,  $a_n < 2$  for all  $n$ .

Since  $\{a_n\}$  is increasing and bounded above by 2, by MCT,  $\{a_n\}$  converges to some limit  $L$  s.t.  $L \leq 2$

Let  $L = \lim_{n \rightarrow \infty} a_n$ . Taking limit both sides of the recursive formula:

$$L = \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \sqrt{2a_n} = \sqrt{2L}$$

Then:

$$L = \sqrt{2L} \implies L^2 = 2L \implies L^2 - 2L = 0 \implies L(L - 2) = 0$$

So  $L = 0, 2$ , and since  $a_n > 0$ , limit  $L \geq \sqrt{2} > 0$ . so  $L = 2$ .

Lastly, the limit found for the given sequence is:

$$\lim_{n \rightarrow \infty} a_n = 2$$

□

## Question 4

*Proof.*

Let  $(x_n)$  be a monotone sequence.

$\Rightarrow$ : By one of the *theorem* in 10/21's lecture, any subsequence of a convergent sequence converges to the same limit. So  $(x_n)$  has a convergent subsequence.

$\Leftarrow$ : Assume  $(x_n)$  has a subsequence  $(x_{n_k})$  that converges to some limit  $L$ .

Since  $(x_n)$  is monotone, it is either non-decreasing or non-increasing. Without loss of generality, suppose  $(x_n)$  is non-decreasing.

1. **Upper Bound:** Because  $(x_{n_k}) \rightarrow L$ , for all  $n \geq N, n, N \in \mathbb{N}$ , we have  $x_n \leq L$  (where  $N$  is large enough), we have  $x_n < L$ .

2. **Monotonicity and limit:** Since  $(x_n)$  is non-decreasing by our set up and bounded above by  $L$ , by MCT, it must converge to its supremum. That is,

$$\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\} \leq L$$

But, since given that there exists a subsequence  $(x_{n_k})$  converges to  $L$ , the supremum of  $\{x_n\}$  must be  $L$ .

Therefore,  $(x_n)$  converges to  $L$ .

By proving both direction is valid, we finish the proof.  $\square$

## Question 5

*Proof.* Let  $(x_n)$  be a monotone sequence.

$\Rightarrow$ : This direction is identical to the previous proof, so we skip.

$\Leftarrow$ : Following with the same setup,

Assume  $(x_n)$  has a subsequence  $(x_{n_k})$  that converges to some limit  $L$ .

Since  $(x_n)$  is monotone, it is either non-decreasing or non-increasing. Without loss of generality, suppose  $(x_n)$  is non-decreasing.

1. **Upper Bound:** Since  $(x_{n_k}) \rightarrow L$ , by definition, for any  $\epsilon > 0$ , there exists  $K$  s.t. for all  $k \geq K$ ,

$$|x_{n_k} - L| < \epsilon \implies x_{n_k} > L - \epsilon$$

2. **Monotonicity:** Because  $(x_n)$  is non-decreasing, for all  $n \geq n_K$ , we have  $x_{n_K} \leq x_n$ .  
Since  $x_{n_K} > L - \epsilon$ ,

$$x_n \geq x_{n_K} > L - \epsilon, \text{ for all } n \geq n_K$$

3. **boundness of  $x_n$ :** Also, since  $(x_n)$  is bounded above by any terms beyond  $x_{n_K}$  (NOTE: It is non-decreasing), and  $x_{n_K} \rightarrow L$ , we have

$$x_n \leq L + \epsilon \text{ for all } n$$

Therefore, for all  $n \geq n_K$ ,

$$L - \epsilon < x_n \leq L + \epsilon$$

This implies that

$$|x_n - L| < \epsilon \text{ for all } n > n_K$$

Therefore, by the limit definition, we have  $\lim_{n \rightarrow \infty} x_n = L$ . Thus,  $(x_n)$  converges to  $L$ .  $\square$

Now, let's recall Bolzano-Weierstrass Theorem and Monotone Convergence Theorem:

*Monotone Convergence Theorem:* Every bounded monotonic sequence converges.

*Bolzano-Weierstrass Theorem:* Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

To prove MCT using Bolzano-Weierstrass Theorem and our previous result, we can approach in this way:

*Proof.* Let  $(x_n)$  be a bounded monotonic sequence. WLOG,  $(x_n)$  is non-decreasing. Since  $(x_n)$  is bounded, by *Bolzano-Weierstrass Theorem*, there exists a convergent subsequence  $(x_{n_k})$  s.t.

$$\lim_{k \rightarrow \infty} x_{n_k} = L$$

From our previous result since  $(x_n)$  is monotonic and has a converging subsequence  $(x_{n_k})$ , sequence  $(x_n)$  converges to the same limit  $L$ . Therefore, every bounded monotonic sequence converges.  $\square$