# MATH 265 HW5

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Sep/26/2024

## Question 1

### (a)

We can analysis this question by the parity of $n \in \mathbb{Z}_+$.

1. When $n$ is odd, $(-1)^n = -1$. Hence the term becomes:

$$-\left(1 - \frac{1}{n}\right) = -1 + \frac{1}{n}$$

   Noticing that for all $n$, $0 \leq \frac{1}{n} \leq 1$, hence:

$$\frac{1}{n} \to 1 \Rightarrow -1 + \frac{1}{n} \leq 0$$

   Hence, for all odd $n$, $s \leq 0, s \in S$.

2. When $n$ is even, $(-1)^n = 1$. Hence the term becomes:

$$\left(1 - \frac{1}{n}\right) = 1 - \frac{1}{n}$$

   Noticing that for all $n$, $0 \leq \frac{1}{n} \leq 1$, hence:

$$\frac{1}{n} \to 0 \Rightarrow 1 - \frac{1}{n} \to 1$$

   Hence, for all odd $n$, $1 - \frac{1}{n} \leq 1$

Since every element $s \in S$, $s \leq 1$. By definition, 1 is an upper bound of $S$.

### (b)

*Proof.* If $M$ is an upper bound for $S$, then $M \geq s$, for every element $s \in S$. Suppose $M < 1$ and $M$ is an upper bound of $S$, it is sufficient for us to only consider the case where $n$ is even.

Since $\frac{1}{2(1-M)} < 0$, by Archimedes Property, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{2(1-M)} < n$, we can get:

$$\frac{1}{2(1-M)} < n \Rightarrow \frac{1}{1-M} < 2n \Rightarrow 1 < 2n(1-M) \Rightarrow \frac{1}{2n} < 1-M \Rightarrow M < 1-\frac{1}{2n}$$

In this case, we know that for $n$ is even,

$$M < 1 - \frac{1}{2n} = (1 - \frac{1}{2n})(-1)^{2n}$$

Hence, there exists a $s \in S$ s.t. $M < (1 - \frac{1}{2n})(-1)^{2n}$ for some $n \in \mathbb{Z}_+$. This derived a contradiction, hence $M \geq 1$. $\square$

### (c)

*Proof.* From part (a), 1 is am upper bound of $S$, and from part (b), no number less then 1 can be an upper bound. Thus by definition, the supremum of $S$ is 1 (i.e. $\sup S = 1$). $\square$

## Quesiton 2

Noting that by definition $A + B = \{a + b \mid a \in A, b \in B\}$

*Proof.* First we need to show that $\sup(A + B) \leq \sup A + \sup B$:
let $a \in A$ and $b \in B$. By definition of supremum, we have $a \leq \sup A$ and $b \leq \sup B$.
Therefore, for any $a \in A$ and $b \in B$:

$$a + b \leq \sup A + \sup B$$

Since $A + B = \{a + b \mid a \in A, b \in B\}$, every element in $A + B$ is less than or equal to $\sup A + \sup B$. Therefore, $\sup(A + B) \leq \sup A + \sup B$.
Then we can derived the equality from here:
Let $\epsilon > 0$, by definition of supremum, for set $A$ and $B$:

- $\exists a_\epsilon \in A : \sup A - \epsilon < a_\epsilon < \sup A$
- $\exists b_\epsilon \in B : \sup B - \epsilon < b_\epsilon < \sup B$

Now consider $a_\epsilon + b_\epsilon \in A + B$. Then:

$$(\sup A - \epsilon) + (\sup B - \epsilon) < a_\epsilon + b_\epsilon \leq \sup(A + B)$$

$$\Rightarrow \sup A + \sup B - 2\epsilon < a_\epsilon + b_\epsilon \leq \sup(A + B)$$

$$\Rightarrow \sup A + \sup B - 2\epsilon < \sup(A + B)$$

$$\Rightarrow 0 \leq \frac{1}{2}(\sup A + \sup B - \sup(A + B)) < \epsilon$$

By previous step, WLOG, since $\sup(A+B) \leq \sup A + \sup B$, the left inequality holds. Reffering to one of the theorem in section *2.1*, we can get:

$$\frac{1}{2}(\sup A + \sup B - \sup(A+B)) = 0 \Rightarrow \sup A + \sup B = \sup(A+B)$$

$\square$

## Question 3

*Proof.* Let $x > 1$ and consider the set $S = \{x^n : n \in \mathbb{Z}_+\}$.
For any $M \in \mathbb{R}$, we want to show that there exists $n \in \mathbb{Z}_+$ such that $x^n > M$
Taking natural logarithm on both sides of the inequality $x^n > M$, we get:

$$n \cdot \ln(x) > \ln(M)$$

Since $\ln(x) > 0$, noting $x > 1$, we can solve for $n$:

$$n > \frac{\ln(M)}{\ln(x)}$$

Note that RHS is a constant, let $N = \lceil \frac{\ln(M)}{\ln(x)} \rceil + 1$,
Hence, for $n = N$, we have:

$$x^n > M$$

For any $M \in \mathbb{R}$, we can found $n \in \mathbb{Z}_+$ s.t. $x^n > M$. Thus, the set is not bounded from above. $\square$

## Question 4

For example:

$$I_n = \left[1 + \frac{1}{n}, 3 + \frac{1}{n}\right)$$

For each positive integer $n$.
Property verificaiton:

*Proof.*

1. Prove that $\bigcup_{n=1}^{\infty} I_n = (1, 4)$:

   - Show that $\bigcup_{n=1}^{\infty} I_n \subseteq (1, 4)$: Let $x \in \bigcup_{n=1}^{\infty} I_n$. Then there exists $n \in \mathbb{N}$ s.t. $x \in I_n$, i.e.,

     $$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

     Since $\frac{1}{n} > 0$, it follows that:

     $$1 < x < 3 + \frac{1}{n} \leq 3 + 1 = 4$$

Therefore, $x \in (1,4)$, and thus:

$$\bigcup_{n=1}^{\infty} I_n \subseteq (1,4)$$

- Show that $(1,4) \subseteq \bigcup_{n=1}^{\infty} I_n$: Let $x \in (1,4)$. We need to find $n \in \mathbb{N}$ s.t. $x \in I_n$.
  - If $x \in (1,3)$: Then $x-1>0$. By Archimedes' Theorem, $\exists N \in \mathbb{N}$ s.t.
  $$\frac{1}{N} < x-1$$
  For $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < x-1$, so:
  $$1+\frac{1}{n} < x$$
  Since $x<3$, and $\frac{1}{n}>0$, we have:
  $$x < 3+\frac{1}{n}$$
  Therefore $x \in I_n$, for all $n > N$.
  - If $x \in [3,4)$: Then $4-x>0$. By Archimedes' Theorem, $\exists N \in \mathbb{N}$ s.t.
  $$\frac{1}{N} < 4-x$$
  For all $n > N$ $\frac{1}{n} \leq \frac{1}{N} < 4-x$, so:
  $$x < 4-\frac{1}{n}$$
  However, since the right endpoint of $I_n$ is $3+\frac{1}{n}$, and $3+\frac{1}{n} < 4-\frac{1}{n}$, we need to ensure $x < 3+\frac{1}{n}$.
  Observe that $x-3 \geq 0$ and $x-3<1$. By Archimedes theorem again, there exists $n \in \mathbb{N}$ s.t.
  $$\frac{1}{n} > x-3$$
  This implies":
  $$x-3 < \frac{1}{n} \Rightarrow x < 3+\frac{1}{n}$$
  Also, since $x \geq 3$ and $\frac{1}{n}>0$:
  $$x \geq 3 = 1+2 \leq 1+\frac{1}{n}+2, \text{ since } \frac{1}{n} < 1$$
  *Note:* $1+\frac{1}{n} \leq x$ *holds because* $1+\frac{1}{n} \leq 1+1 = 2 < x$
  Therefore, $x \in I_n$

Combining two steps, $\bigcup_{n=1}^{\infty} I_n = (1, 4)$

2. Prove that $\bigcap_{n=1}^{\infty} I_n = [2, 3]$:

   - Show that $[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$ Let $x \in [2, 3]$. For all $n \in \mathbb{N}$, since: $1 + \frac{1}{n} \leq 1 + 1 = 2 \leq x \leq 3$, and $x < 3 + \frac{1}{n}$ we have:

     $$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

     Therefore. $x \in I_n$, for all $n$. Thus,

     $$[2, 3] \subseteq \bigcap_{n=1}^{\infty} I_n$$

   - Show that $\bigcap_{n=1}^{\infty} I_n \subseteq [2, 3]$ Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then for all $n \in \mathbb{N}$:

     $$1 + \frac{1}{n} \leq x < 3 + \frac{1}{n}$$

     - Show $x \geq 2$: Suppose, $x < 2$, then $x - 1 < 1$. By Archimedes Theorem, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < x - 1$.
       For all $n \geq N$:
       $$1 + \frac{1}{n} < x$$
       which contradicts the fact that $x \geq 1 + \frac{1}{n}$, for all $n$. Therefore, $x \geq 2$.
     - Show $x \leq 3$: Suppose, $x > 3$, then $x - 3 > 0$. By Archimedes Theorem, there exists $N \in \mathbb{N}$ s.t. $\frac{1}{N} < x - 3$.
       For all $n \geq N$:
       $$x > 3 + \frac{1}{n}$$
       which contradicts the fact that $x < 1 + \frac{1}{n}$, for all $n$. Therefore, $x \leq 3$.

     Thus, $x \in [2, 3]$, so:
     $$\bigcap_{n=1}^{\infty} I_n \subseteq [2, 3]$$

   Combining both inclusions, we have:

   $$\bigcap_{n=1}^{\infty} I_n = [2, 3]$$

Overall $I_n = \left[1 + \frac{1}{n}, 3 + \frac{1}{n}\right)$ satisfy two given properties.

$\square$