MATH 265 HW1

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Question 1

Proof. Define a function f as:

$$f : \mathbb{N} \to \{1, 4, 7, 10, 13, \cdots\} : n \mapsto 3n - 2$$

Now we need to show f is *one to one* and *onto*.

- Prove f is *one to one*

  Proof. To prove f is *one to one*, we must show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
  Suppose $f(n_1) = f(n_2)$. Then:

  $$3n_1 - 2 = 3n_2 - 2$$

  $$\Rightarrow 3n_1 = 3n_2 \Rightarrow n_1 = n_2$$

  Since $n_1 = n_2$, f is injective. □

- Prove f is *onto*

  Proof. To prove f is *onto*, we need to show for every $m \in \{1, 4, 7, 10, 13, \cdots\}$, there exists an $n \in \mathbb{N}$ s.t. $f(n) = m$.
  Take any $m \in \{1, 4, 7, 10, 13, \cdots\}$. We want to find n such that $3n-2 = m$.
  Solving for n:

  $$3n - 2 = m \Rightarrow 3n = m + 2 \Rightarrow n = \frac{m+2}{3}$$

  Since m is given of the form $3k + 1$ for some integer k. Therefore, $n \in \mathbb{N}$, and f is onto. □

  Since f is *one to one* and *onto*, it is a bijection. Hence, $f(n) = 3n - 2$ is a explicit bijection from the set $\mathbb{N}$ to $\{1, 4, 7, 10, 13, \cdots\}$.

  □

Question 2

Proof. Define a function f as:

$$f : (0,1) \to \mathbb{R}_{>0} : n \mapsto \frac{n}{1-n}$$

Now we need to show f is *one to one* and *onto*.

- Prove f is *one to one*

  Proof. To prove f is *one to one*, we must show that if $f(n_1) = f(n_2)$, then $n_1 = n_2$.
  Suppose $f(n_1) = f(n_2)$. Then:

  $$\frac{n_1}{1-n_1} = \frac{n_2}{1-n_2}$$
  $$\Rightarrow n_1(1-n_2) = n_2(1-n_1)$$
  $$\Rightarrow n_1 - n_1 n_2 = n_2 - n_2 n_1$$
  $$\Rightarrow n_1 = n_2$$

  Thus, f is one to one. □

- Prove f is *onto*

  Proof. To prove f is *onto*, we need to show for every $m \in \mathbb{R}_{>0}$, there exists an $n \in (0,1)$ s.t. $f(n) = m$.
  Take any $m > 0$. We want to find n such that $\frac{n}{1-n} = m$. Solving for n:

  $$n = m(1-n) \Rightarrow n + nm = m \Rightarrow n(1+m) = m \Rightarrow n = \frac{m}{1+m}$$

  Note that $m > 0$, this implies $0 < \frac{m}{1+m} < 1$, hence $n \in (0,1)$.
  Thus, for every $m > 0$, there is an $n \in (0,1)$ s.t. $f(n) = m$. (i.e. f is onto) □

  Since f is *one to one* and *onto*, it is a bijection. Hence, $f(n) = \frac{n}{1-n}$ is a explicit bijection from the set $(0,1)$ to $\mathbb{R}_{>0}$.

  □

Question 3

Proof. Let $x \in [0,1)$ s.t. $g(x) = w, w \in \mathbb{R}_{>0}$, modified the function f from Q2 to g as:

$$g : [0,1) \to \mathbb{R}_{>0} : g(x) = \begin{cases} \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1, & \text{if } x = 0 \\ \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2^{n+1}}} = \frac{1}{2^{n+1}-1}, & \text{if } x = \frac{1}{2^n}, n \in \mathbb{N}^+ \\ \frac{x}{1-x}, & \text{if } x \neq \frac{1}{2^n}, n \in \mathbb{N} \end{cases}$$

Now we need to show g is *one to one* and *onto*.

- Prove g is *one to one*

  Proof. To prove g is *one to one*, we must show that if $g(x_1) = g(x_2)$, then $x_1 = x_2$.
  Case 1: $x_1 = 0$ and $x_2 \neq 0$:

  - If $x_1 = 0$, then $g(x_1) = 1$
  - If $x_2 \neq 0$, $g(x_2)$ will either be in form $\frac{2n+1}{1-\frac{1}{2n+1}}$ or $\frac{x_2}{1-x_2}$, both are not equal to 1. Thus, $g(x_1) \neq g(x_2)$.

  Case 2: $x_1 = \frac{1}{2^n}, x_2 = \frac{1}{2^m}$ s.t. $n \neq m$

  - For $x_1 = \frac{1}{2^n}$, we have $g(x_1) = \frac{2n+1}{1-\frac{1}{2n+1}}$.
  - For $x_2 = \frac{1}{2^m}$, we have $g(x_2) = \frac{2m+1}{1-\frac{1}{2m+1}}$.

  Since $n \neq m$, $g(x_1) \neq g(x_2)$.
  Case 3: $x_1 \neq \frac{1}{2^n}$ and $x_2 \neq \frac{1}{2^m}$ with $x_1 \neq x_2$
  If $x_1, x_2 \notin \left\{\frac{1}{2^n}\right\}$, then $g(x_1) = \frac{x_1}{1-x_1}$ and $g(x_2) = \frac{x_2}{1-x_2}$. Since $x_1 \neq x_2$, $\frac{x_1}{1-x_1} \neq \frac{x_2}{1-x_2}$. Hence, $g(x_1) \neq g(x_2)$.
  Combining all cases, $g(x)$ is injective. □

- Prove g is *onto*
  To prove g is *onto*, we need to show for every $w \in \mathbb{R}_{>0}$, there exists an $x \in [0,1)$ s.t. $g(x) = w$.

  Proof.
  - For any $w = 1$, we have $g(0) = 1$.
  - For any $w = \frac{2n+1}{1-\frac{1}{2n+1}}$, there exists and $x = \frac{1}{2^n}$ such that $g(x) = w$.
  - For any other $w > 0$, there exists an $x \in (0,1) \backslash \{\frac{1}{2^n}\}$ s.t. $g(x) = \frac{x}{1-x} = w$, we can get:
  $$x = \frac{w}{1+w}$$
  Since $w > 0, x \in (0,1)$.

  Overall, for any $w > 0$, there is an $x \in [0,1)$ s.t. $g(x) = w$. Hence, $g(x)$ is surjective. □

Since g is *one to one* and *onto*, it is a bijection. □

Question 4

(a) Show that $g \circ f$ is injective if both of f and g are injective.

Proof. Given that $f : A \to B$ and $g : B \to C$ are injective. We can write $g \circ f$ as $g \circ f : A \to C$, defined as $(g \circ f)(x) = g(f(x))$.
Assume: $(g \circ f)(x) = (g \circ f)(y)$, for some $x, y \in A$,

1. By the definition of composite function, we can get $g(f(x)) = g(f(y))$
2. Given, g is injective, $f(x) = f(y)$
3. Also given, f is injective, $x = y$

Hence, $g \circ f$ is injective. □

(b) Show that $g \circ f$ is surjective if both of f and g are surjective.

Proof. Given that $f : A \to B$ and $g : B \to C$ are surjective. We can write $g \circ f$ as $g \circ f : A \to C$, defined as $(g \circ f)(x) = g(f(x))$.
Let an arbitary value $z \in C$. We need to find some $x \in A$ such that $(g \circ f)(x) = z$

1. Given g is surjective, hence there exists a $y \in B$ such that $g(y) = z$
2. Given f is surjective, hence there exists a $x \in B$ such that $f(x) = y$
3. Hence, $(g \circ f)(x) = g(f(x)) = g(y) = z$

Hence, $g \circ f$ is surjective. □