MATH 265 HW8 Nov. 8, 2024

1. (2 points) Let (x_n) be a sequence such that $|x_{n+1} - x_n| \le 2^{-n}$ for all $n \in \mathbb{N}$. Prove that (x_n) is a Cauchy sequence.

For (x_n), WTS that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n, m > N$, $|x_m - x_n| < \varepsilon$.

Given $|x_{n+1} - x_n| \le 2^{-n}$ $\forall n \in \mathbb{N}$, consider the telescoping sum for $m > n$:

$$|x_m - x_n| = \left| \sum_{k=n}^{m-1} (x_{k+1} - x_k) \right|$$

$$\le \sum_{k=n}^{m-1} |x_{k+1} - x_k| \quad \text{By } \Delta\text{-inequality}$$

$$\le \sum_{k=n}^{m-1} 2^{-k}$$

$$\sum_{k=n}^{m-1} 2^{-k} = 2^{-n} \sum_{k=0}^{m-n-1} 2^{-k} = 2^{-n} \left(\frac{1 - 2^{-(m-n)}}{1 - 1/2} \right)$$

$$= 2^{-n} \left(1 - 2^{-(m-n)}\right) \cdot 2$$

$$= 2^{1-n} \left(1 - 2^{-(m-n)}\right)$$

Since $0 < 1 - 2^{-(m-n)} < 1$, we have:

$$|x_m - x_n| \le 2^{1-n}$$

Choose $N = \lceil \log_2 (2/\varepsilon) \rceil$

Then for $n, m > N$

$$|x_m - x_n| \le 2^{1-n} \le 2^{1-N} < \varepsilon$$

Hence, (x_n) is Cauchy $\boxtimes$

2. (2 points) Consider the series:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Prove that for any real number x this series is convergent.

Hint First show the following lemma: For any natural number k, $n! \geq \frac{k^n}{k!}$ for all $n \geq k-1$. Then choose $k \in \mathbb{N}$ with $k > |x|$ and apply Cauchy criterion.

Start with : $n=1$, $k=2$

proof

Now by the given hint,

Let $x \in \mathbb{R}$, choose a $k \in \mathbb{N}$ s.t.

$$k > |x| \Rightarrow \frac{|x|}{k} < 1$$

Using the lemma, $\forall n \in \mathbb{N}$ s.t. $n \geq k-1$,

$$n! \geq \frac{k^n}{k!} \Rightarrow \frac{1}{n!} \leq \frac{k!}{k^n}$$

Therefore,

$$\left|\frac{x^n}{n!}\right| \leq \frac{|x|^n}{n!} \leq \frac{|x|^n \cdot k!}{k^n} = k! \left(\frac{|x|}{k}\right)^n$$

Now we WTS the series is Cauchy, to apply "Cauchy convergent theorem"

Recall

A series $\sum_{n=0}^{\infty} a_n$ conv. iff, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq m \geq N$:

$$\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

For $m, n \in \mathbb{N}$ with $n \geq m$, Consider partial sum:

$$S_{n,m} = \sum_{i=m+1}^{n} \frac{x^i}{i!}$$

this is a constant.

We have:

$$|S_{n,m}| \le \sum_{i=m+1}^{n} \left|\frac{x^i}{i!}\right| \le \sum_{i=m+1}^{n} k! \left(\frac{|x|}{k}\right)^i$$

Since $\frac{|x|}{k} < 1$, the geometric series $\sum_{i=m+1}^{\infty} \left(\frac{|x|}{k}\right)^i$ converges.

Therefore, we can get:

$$\sum_{i=m+1}^{n} \left(\frac{|x|}{k}\right)^i \le \sum_{i=m+1}^{\infty} \left(\frac{|x|}{k}\right)^i = \frac{\left(\frac{|x|}{k}\right)^m}{1-\frac{|x|}{k}}$$

$$\Rightarrow \sum_{i=m+1}^{n} k! \left(\frac{|x|}{k}\right)^i \le k! \cdot \frac{\left(\frac{|x|}{k}\right)^{m+1}}{1-\frac{|x|}{k}} \quad \left(\text{let } r = \frac{|x|}{k}\right)$$

For any $\varepsilon > 0$, we can choose $\delta = \max\left\{ \left\lceil \log_r\left(\frac{\varepsilon}{k!}(1-r)\right) - 1 \right\rceil, k-1 \right\}$

$$\Rightarrow k! \cdot \frac{r^m}{1-r} < \varepsilon,$$

Thus, $\forall n \ge m$, $\left|\sum_{i=m+1}^{n} \frac{x^i}{i!}\right| < \varepsilon$

By Cauchy convergent criterion, since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is Cauchy, it is convergent.

$\boxtimes$

proof of lemma (sketch)

Base Case ($n=1$):

$$n! = 1! = 1 \quad \& \quad \frac{k^1}{k!}$$

Note $k \in \mathbb{N}$, $k! \geq k \cdot (k-1)! \geq k \Rightarrow 1 \geq \frac{k}{k!}$, so,

$$1! = 1 \geq \frac{k}{k!} = \frac{k^1}{k!}$$

Base case proved.

Inductive step:

Assume the lemma holds for some n s.t. $n! \geq \frac{k^n}{k!}$

WTS that $(n+1)! \geq \frac{k^{n+1}}{k!}$, starting with LHS with inductive hypothesis:

$$(n+1)! = (n+1) \cdot n! \geq (n+1) \cdot \frac{k^{n+1}}{k!}$$

given the lemma holds for $n \geq k-1 \iff n+1 \geq k$, hence,

$$(n+1)! = (n+1) n! \geq (n+1) \cdot \frac{k^n}{k!} \geq k \cdot \frac{k^n}{k!} = \frac{k^{n+1}}{k!}$$

By induction, the lemma holds $\forall n \in \mathbb{N}$.

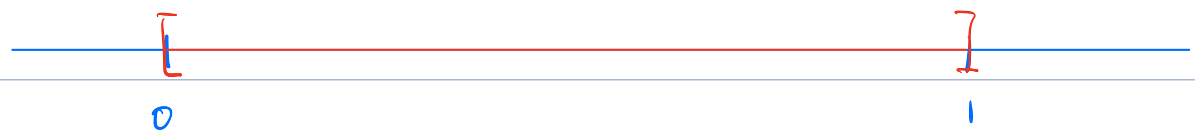
$\blacksquare$

3. (3 points) Let $E \subset [0,1]$ be the set of all real numbers $0 \leq x \leq 1$ such that a decimal representation of x has at least one entry which is a 7.

(a) Draw a 'rough' sketch of what the set E looks like on a number line between 0 and 1.

(b) Prove that $\frac{1}{2}$ is a cluster point of E. (In fact, every number between 0 and 1 is a cluster point of E.)

(a)



Number in Set E is dense in $[0,1]$.

(b).

To show $1/2$ is a cluster pt, we need to prove for every open interval of $1/2$, $\exists$ at least one pt included from E other then $1/2$.

Let $\varepsilon > 0$, Consider $(1/2 - \varepsilon, 1/2 + \varepsilon)$.

We can construct a number $x_n \in E$ s.t. $x_n \neq 1/2$ and $x \in (1/2 - \varepsilon, 1/2 + \varepsilon)$. Consider:

$$x_n = 0.5\underbrace{0\ldots0}_{n \text{ zeros}}7 \quad \text{For any } n \in \mathbb{N}.$$

By def, $x_n \in E$ and has 7 at $n+2$ decimal place.
Then,

$$|x_n - 1/2| = 7 \times 10^{-(n+2)} = \frac{7}{10^{n+2}}$$

NOTE: $\lim_{n \to \infty} \frac{7}{10^{n+2}} = 0$

By our construction, seq. $(x_n) \in E$ also, Consider

$$\Rightarrow \lim_{n\to\infty} \left|x_n - \tfrac{1}{2}\right| = \lim_{n\to\infty} \frac{7}{10^{n+2}} = 0.$$

$$\Rightarrow \lim_{n\to\infty} |x_n| - \lim_{n\to\infty} \left|\tfrac{1}{2}\right| = 0$$

$$\Rightarrow \lim_{n\to\infty} |x_n| = \tfrac{1}{2}$$

By Thm:

> $c \in \mathbb{R}$ is a cluster point of A iff $\exists$ a seq. (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ $\forall n \in \mathbb{N}$.

Hence, $1/2$ is a cluster point.

$\boxed{\text{M}}$

4. (3 points) Let

$$f(x) = \frac{x^4 - 2x^3 + x^2 - 1}{3x^6 + x^3 + 1}.$$

Use an $\epsilon - \delta$ argument to show that

$$\lim_{x \to -1} f(x) = 1.$$

Observe.

$$|f(x) - 1| = \left| \frac{x^4 - 2x^3 + x^2 - 1}{3x^6 + x^3 + 1} - 1 \right|$$

$$= \left| \frac{x^4 - 2x^3 + x^2 - 1 - 3x^6 - x^3 - 1}{3x^6 + x^3 + 1} \right| = \left| \frac{-3x^6 + x^4 - 3x^3 + x^2 - 2}{3x^6 + x^3 + 1} \right|$$

$$= \left| \frac{-3x^3(x^3+1) + (x^4 + x^2 - 2)}{3x^6 + x^3 - 1} \right|$$

If $|x+1| < 2$, then (NOTE: $|x+1| < 2 \Rightarrow |x| < 3$)

Let $M_1, M_2 > 0$.

①

$$|-3x^3(x^3+1)| = 3|x|^3 \cdot |x+1| \cdot |x^2 - x + 1|$$

$$\leq 3|x|^3 \cdot |x+1| \cdot (|x|^2 + |x| + 1)$$

$$\leq 3 \cdot 3^3 \cdot 2 \cdot (3^2 + 3 + 1) = M_1$$

Similarly.

② $|x^4 + x^2 - 2| = |(x^2+2)(x^2-1)| \leq (|x|^2 + 2) \cdot |x+1| \cdot (|x| + 1)$.

$$\leq (3^2 + 2) \cdot 2 \cdot (3 + 1) = M_2$$

And obviously that,

③ $|3x^6 + x^3 + 1| = 3(x^3 + 1/6)^2 + 11/12 \geq 11/12$

Hence, we can bound $|f(x)-1|$ by ①, ②, ③ s.t.

$$|f(x)-1| \leq \frac{M_1+M_2}{11/12} = 12/11 \cdot (M_1+M_2)$$

Let $\varepsilon>0$, choose $\delta = \min\left\{2, \frac{\varepsilon}{12/11 \cdot (M_1+M_2)}\right\}$

Then, if $0<|x-2|<\delta$, then $\left|\frac{x^4-2x^3+x^2-1}{3x^6+x^3+1} - 1\right| < \varepsilon$

$\boxed{\text{Ⅲ}}$

* Another proof Observe:

$$|f(x)-1| = \left| \frac{x^4-2x^3+x^2-1}{3x^6+x^3+1} - 1 \right|$$

$$= \left| \frac{x^4-2x^3+x^2-1-3x^6-x^3-1}{3x^6+x^3+1} \right| = \left| \frac{-3x^6+x^4-3x^3+x^2-2}{3x^6+x^3+1} \right|$$

$$= \left| \frac{-3x^3(x^3+1)+(x^4+x^2-2)}{3x^6+x^3-1} \right|$$

Suppose $\exists \delta_1$ s.t. $\forall x \in V_{\delta_1}(-1)$, $|x|<2$, $|x+1|<\delta$

①

$$\left|-3x^3(x^3+1)\right| = 3|x|^3 \cdot |x+1| \cdot |x^2-x+1|$$

$$\leq 3|x|^3 \cdot |x+1| \cdot (|x|^2-|x|+1)$$

$$\leq 3 \cdot 2^3 \cdot \delta \cdot (2^2-2+1)$$

$$= M_1 \delta$$

② $|x^4+x^2-2| = |(x^2+2)(x^2-1)| \leq (|x|^2+2) \cdot |x+1| \cdot (|x|-1)$.

$$\leq (2^2+2) \cdot \delta \cdot (2-1) = M_2 \delta$$

③ $|3x^6+x^3+1| = 3(x^3+\frac{1}{6})^2 + \frac{11}{12} \geq \frac{11}{12}$

By ①, ②, ③ we can pick $\delta = \min\left\{1, \frac{\varepsilon}{\frac{12}{11}(M_1+M_2)}\right\}$

$\Rightarrow |f(x)-1| \leq \frac{12}{11} \cdot (M_1+M_2)\delta = \varepsilon$