MATH 265 HW6

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Question 1

Proof. Let $\epsilon > 0$, Consider:

$$N = \max\left\{1, \left\lceil \frac{1}{\left(\frac{\sqrt{2}}{\sqrt{2}+\epsilon}\right)^2 - 1} \right\rceil\right\} \in \mathbb{N}$$

Then for $n > N$:

$$n > \frac{1}{\left(\frac{\sqrt{2}}{\sqrt{2}+\epsilon}\right)^2 - 1}$$

$$\Rightarrow \frac{1}{n} < \left(\frac{\sqrt{2}}{\sqrt{2}+\epsilon}\right)^2 - 1 \Rightarrow 1 + \frac{1}{n} < \left(\frac{\sqrt{2}}{\sqrt{2}+\epsilon}\right)^2$$

$$\Rightarrow \sqrt{1+\frac{1}{n}} < \frac{\sqrt{2}}{\sqrt{2}+\epsilon} \Rightarrow \frac{\sqrt{2}}{\sqrt{1+\frac{1}{n}}} < \sqrt{2} + \epsilon$$

$$\Rightarrow \left|\frac{\sqrt{2}}{\sqrt{1+\frac{1}{n}}} - \sqrt{2}\right| < \epsilon \Rightarrow \left|\frac{\sqrt{2n}}{\sqrt{n+1}} - \sqrt{2}\right| < \epsilon$$

Hence, by definition, $\lim_{n\to\infty} \frac{\sqrt{2n}}{\sqrt{n+1}} = \sqrt{2}$ $\square$

Question 2

Proof. Let $\epsilon > 0$, (Note: $\frac{1}{n^2} - 1 < \frac{-1}{n^2}$), Consider:

$$N = \max\left\{1, \lceil\sqrt{\epsilon}\rceil\right\} \in \mathbb{N}$$

Then for $n > N$:

$$n > \sqrt{\epsilon} \Rightarrow n^2 > \epsilon \Rightarrow \frac{1}{n^2} < \epsilon$$

NOTE: by definition of cos(x),

$$\frac{|cos(n)|}{n^2} \leq \frac{1}{n^2} < \epsilon \Rightarrow \frac{|cos(n)|}{n^2} < \epsilon \Rightarrow \left|\frac{cos(n)}{n^2}\right| < \epsilon$$

Hence, by definition, $\lim_{n\to\infty} \frac{cos(n)}{n^2} = 0$ $\square$

Question 3

Proof. For this question, we need to use the theorem. Noting that $(x_n) = (\frac{2^n}{n!})$ is a positive sequence. We want to show $L = \lim_{n\to\infty}(\frac{xn+1}{x_n})$ exists and $L < 1$.

$$\frac{x_{n+1}}{x_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{2}{n+1} = \lim_{n\to\infty} \frac{\frac{2}{n}}{1+\frac{1}{n}}$$

By proposition, $\lim_{n\to\infty} \frac{1}{n} = 0$, similarly, $\lim_{n\to\infty} \frac{2}{n} = 0$. Hence, by limit law,

$$L = \frac{\lim_{n\to\infty} \frac{2}{n}}{\lim_{n\to\infty}(1+\frac{1}{n})} = \frac{\lim_{n\to\infty} \frac{2}{n}}{\lim_{n\to\infty}(1) + \lim_{n\to\infty}(\frac{1}{n})} = \frac{0}{1+0} = 0$$

Since $L = 0 < 1$, by theorem, $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. $\square$

Question 4

(a)

Proof.

$$(x_n) = \{\frac{1}{n}\}_{n=1}^{\infty}$$

Verficiation:

$$\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

$\square$

(b)

Proof.

$$(x_n) = \{n\}_{n=1}^{\infty}$$

Verficiation:

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 + 0 = 1$$

$\square$