MATH 265 HW2

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Question 1

Proof. First lets check the base case. For $n = 1$:

$$\sum_{i=1}^{1} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} = 1$$

$1^{\frac{2}{3}} = 1$. Since $1 \geq 1$, the base case holds.
For forming up the inductive hypothesis, assume the statement is true for some $k \geq 1$,

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq k^{\frac{2}{3}}$$

Now we need to show the statement holds for $k + 1$, namely,

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \geq (k+1)^{\frac{2}{3}}$$

From inductive hypothesis, we can add $\frac{1}{\sqrt{k+1}}$ both side:

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \geq (k)^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}}$$

Now we need to show:

$$k^{\frac{2}{3}} + \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}} \Rightarrow \frac{1}{\sqrt{k+1}} \geq (k+1)^{\frac{2}{3}} - k^{\frac{2}{3}}$$

Let, $f(k) = (k+1)^{-\frac{1}{2}} - (k+1)^{\frac{2}{3}} + k^{\frac{2}{3}}$:

$$f'(k) = -\frac{1}{2}(k+1)^{-\frac{3}{2}} - \frac{2}{3}(k+1)^{-\frac{1}{3}} + \frac{2}{3}(k)^{-\frac{1}{3}}$$

As $k \to \infty$, since all three terms are converging to 0, $f'(k) \to 0$.

$$f''(k) = \frac{3}{4}(k+1)^{-\frac{5}{2}} + \frac{2}{9}(k+1)^{-\frac{4}{3}} - \frac{2}{9}(k)^{-\frac{4}{3}}$$

For $k \geq 1$, noting that $\frac{2}{9}(k+1)^{-\frac{4}{3}} - \frac{2}{9}(k)^{-\frac{4}{3}} \geq 0$, hence $f''(k) \geq 0$.
Since $k \to \infty, f'(k) \to 0$ and $f''(0) \geq 0$. This indicates $f'(k) < 0$ for all $k \geq 1$.
Hence the function $f(k)$ is decreasing for $k \geq 1$.
Referring back to $f(k)$, we can approximate its behavior using taylor series:

$$f(k) = (k+1)^{-\frac{1}{2}} - \left[(k+1)^{\frac{2}{3}} + k^{\frac{2}{3}}\right] = (k+1)^{-\frac{1}{2}} - \frac{2}{3}(k-1)^{\frac{1}{3}} + O(k^{-\frac{4}{3}})$$

Hence, when $k \to +\infty, f(k) \geq 0 \Rightarrow \forall k \geq 1, f(k) \geq 0$. The original inquality holds for all $k \geq 0$. By mathematical induction, the statement $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \geq n^{\frac{2}{3}}$ is true for all $n \in \mathbb{N}$. $\square$

Question 2

Proof. First lets check the base case. For $n = 0$:

$$x_0 = 3, x_1 = \frac{1}{8} \cdot (3)^2 + 2 = \frac{9}{8} + 2 = \frac{25}{8} = 3.125$$

$x_0 < x_1 < 4$, the base case proved.
Assume that for some $n = k, k \geq 1$, $x_k < x_{k+1} < 4$. WTS $x_{k+1} < x_{k+1=2} < 4$.
From the recurrence relation:

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2$$

Using the inductive hypothesis, note that $x_{k+1} < 4$:

$$x_{k+2} = \frac{1}{8}x_{k+1}^2 + 2 < \frac{1}{8}(4^2) + 2 = 2 + 2 = 4$$

By mathematical induction, the statement $x_n < x_{n+1} < 4$ is true for all $n \in \mathbb{N} \cup \{0\}$. $\square$

Question 3

Proof. First lets check the base case. For $k = 1$:

$$F_{m+1} = F_{m-1}F_1 + F_mF_2$$

By definition of Fibonacci Sequence, $F_1 = F_2 = 1$.

$$F_{m+1} = F_{m-1} + F_m$$

This is true by definition, hence the base case proved.
Assume that for some $k \geq 1$, the given identity holds:

$$F_{m+k} = F_{m-1}F_k + F_m F_{k+1}$$

We need to prove that this statement holds for $k+1$, namely,

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_m F_{k+2}$$

Using the Fibonacci sequence's definition, we express F_{k+2} and F_{m+k+1} as:

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{m+k+1} = F_{m+k} + F_{m+k-1}$$

Substitute the values from the inductive hypothesis into F_{m+k+1}:

$$F_{m+k+1} = (F_{m-1}F_k + F_m F_{k+1}) + (F_{m-1}F_{k-1} + F_m F_k)$$

Combine and reorganize terms:

$$F_{m+k+1} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$$

By Finbonacci definition:

$$F_{m+k+1} = F_{m-1}F_{k+1} + F_m(F_{k+1} + F_k)$$

$$= F_{m-1}F_{k+1} + F_m F_{k+2}$$

By mathematical induction, the statement is true for all $k.m \in \mathbb{N}$ with $m \geq 2$. $\square$

Question 4

Proof. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$, define $P(X) \in \mathbb{Z}[x]$ s.t. $a_i \in \mathbb{Z}$
Define height of $P(x)$ as:

$$h(P) := n + \sum_{i=0}^{n} |a_i|$$

The number of $P(x)$ satisfying $h(P) \leq c$ is finite.
Hence we can define,

$$\mathbb{Z}_n := \{P(x) \in \mathbb{Z}[x] : h(P(x)) \leq n\}$$

In here, $\mathbb{Z}_n$ is finite. Let I be a countable set. Then, $\forall P(x) \in \mathbb{Z}[x]$

$$Z[x] = \bigcup_{i \in I} P_i(x)$$

The countable union of finite sets is countable, hence $Z[x]$ is countable.
For each polynomial $P_i(x) \in Z[x]$, there exists n roots, hence, we can denote the element A_i in the set of algebraic number A as:

$$A_i = \{x : P_i(x) = 0\} \text{ s.t. } |A_i| = \deg(P_i)$$

Hence, A_i is also countable since we can find a constant c s.t. $|A_i| < c$.

$$A = \bigcup_{i \in I} A_i$$

Again, since A_i is countable, the countable union of the finite set is countable.
Therefore, algebraic numbers are countably infinite. $\square$