Homework 7 Bonus

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December 4, 2024

1. [Bonus] Let A and B be $n \times n$ matrices.

a) Prove that
$$e^{A+B} = e^A e^B$$
when $AB = BA$.

b) Give an example where
$$e^{A+B} \neq e^A e^B.$$

Solution.

(a)

Proof. The matrix exponential e^M of a square matrix M is defined by the power series:

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

Given that A and B commute, we can manipulate the exponential of their sum.

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!}$$

Since A and B commute, we can apply the binomial thm to expand $(A+B)^k$:

$$(A+B)^k = \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j}$$

Sub it back, we can get:

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} A^j B^{k-j} = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{\binom{k}{j}}{k!} A^j B^{k-j}$$

Noting that,

$$\frac{\binom{k}{j}}{k!} = \frac{1}{j!(k-j)!}$$

So by rearranging,

$$e^A e^B = \left(\sum_{m=0}^{\infty} \frac{A^m}{m!} \right) \left(\sum_{mn=0}^{\infty} \frac{B^n}{n!} \right) = \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \frac{\binom{k}{j}}{k!} A^j B^{k-j} = e^{A+B}$$

$\square$

(b)

Let A and B be the 2×2 matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Verify $AB \neq BA$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $AB \neq BA$, A and B do not commute.
Noting that both A and B are nilpotent ($A^2 = 0$, $B^2 = 0$),

$$e^A = I + A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad e^B = I + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Now, we compute $e^A e^B$

$$e^A e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then, we compute e^{A+B}

$$A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (A+B)^2 = I.$$

Using the series expansion of the exponential, by definition, we can get:

$$e^{A+B} = \cosh(1) \cdot I + \sinh(1) \cdot (A+B).$$

Numerical approximation ($\cosh(1) \approx 1.5431$, $\sinh(1) \approx 1.1752$):

$$e^{A+B} \approx \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix}.$$

But, by comparison

$$e^A e^B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq e^{A+B} \approx \begin{bmatrix} 1.5431 & 1.1752 \\ 1.1752 & 1.5431 \end{bmatrix}.$$

Which match our expectation.