Math 280: Midterm

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I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

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Point Summary:

Part A: Computation			Part B: Short Proofs			Part C: Long Proof
1	2	3	4	5	6	7
15	15	20	15	15	20	25
50 points			50 points			25 points
Total: 125 points						

Part A: Computations [50 points]

1. [15 points] All norms are the ∞ norm in this problem.
An optimization problem gives rise to a system of linear equations

$$Lx = b$$

where $||L|| = 4$ and $||L^{-1}|| = 1/2$.
However, you only have an estimate $\tilde{b} \approx b$ for b and end up solving

$$L\tilde{x} = \tilde{b}.$$

a) [5 points] Suppose the *relative* error of b is 0.1. Bound the *relative* error of x.

b) [5 points] Suppose $|b| = 40$ and hence the *absolute* error $|b - \tilde{b}|$ of b is at most 4. What is a bound for the *absolute* error on x?

c) [5 points] Continuing to assume $|b| = 40$, bound $|x|$.

Proof.

a) It's just condition number times relative error of b,

$$\frac{|x - \tilde{x}|}{|x|} \leq \kappa(L)\frac{|b - \tilde{b}|}{|b|} \leq 4 \times (1/2) \times 0.1 = 0.2.$$

b) Estimate directly:

$$\begin{aligned} |x - \tilde{x}| &= |L^{-1}(b - \tilde{b})|, \\ &\leq ||L^{-1}|||b - \tilde{b}|, \\ &= \frac{1}{2} \times 4, \\ &= 2. \end{aligned}$$

c) Also estimate directly:

$$|x| = |L^{-1}b| \leq ||L^{-1}|||b| = \frac{1}{2}|40| = 20.$$

a) (-2) missing part of condition number

b) (-3) tries to give an argument using (a) but uses $|b|$ instead of $|x|$
$(+5)$ for using (c) though it is a weaker bound, and not quite what I had in mind

c) see comments

2. [15 points]

a) [5 points] State the definition of a degree k spline.

b) [10 points] Determine whether or not the following is a *quadratic* spline:

$$f(x) = \begin{cases} (x+2)^2 - 18(x+2) + 20 & x < -2 \\ 4x^2 - 2x & -2 \le x \le 2 \\ 3(x-2)^2 + 4(x-2) + 12 & x > 2 \end{cases}$$

Proof.

a) It is a piecewise polynomial function, where each piece is a polynomial of degree at most k which is $k-1$ times continuously differentiable (in particular, it is continuous and the first $k-1$ derivatives of consecutive pieces agree at the nodes).

b) It's definitely piecewise, and polynomials of degree at most 2, so it could be a spline. Label the components as S_1, S_2, S_3.

For S_1, we have:

$$S_1'(x) = 8(x+2) - 18$$

x	$S_1(x)$	$S_1'(x)$
-2	20	-18

For S_2, we have:

$$S_2'(x) = 8x - 2$$

x	$S_2(x)$	$S_2'(x)$
-2	20	-18
2	12	14

And finally, for S_3, we have:

$$S_1'(x) = 6(x-2) - 4$$

x	$S_2(x)$	$S_2'(x)$
2	12	-4

The first derivatives of S_2 and S_3 don't agree at their common node, $x = 2$, so it's not a quadratic spline.

a) (+1) piecewise degree k polynomial

(+2) compatibility up to the

(+2) correct number of derivatives

b) see comments

3. [20 points] Consider sequences defined by a recurrence of the form

$$x_{n+1} = \frac{x_n^2}{2x_n - 3}.$$

a) [10 points] Interpret a sequence of the above form in terms of Newton's method (i.e. find a function f such that the above is the NM recursion associated to f).

b) [10 points] If $x_0 = 8$, use (a) to show that

$$\lim_{n\to\infty} x_n$$

converges, and evaluate it exactly.

Proof. For (a), we need to solve the following differential equation:

$$x_n - \frac{f}{f'} = \frac{x_n^2}{2x_n - 3},$$

which, after some rearrangement, and swapping x for x_n for simplicity, is the same as

$$\frac{f'(x)}{f(x)} = \frac{2x-3}{x(x-3)}.$$

Of course, one may notice right here that, on the RHS, the numerator is the derivative of the denominator already, and hence arrive at $f(x) = 2x^2 - 3x$.

Otherwise, treat it like a plain Calc II problem.

The partial fraction decomposition on the right is, by inspection, just $1/x + 1/(x-3)$. Then we can solve quite quickly:

$$C + \ln(f) = \ln(x) + \ln(x-3),$$

(gliding over the $|\cdot|$ and domain questions, since we just need this to make a guess). Raising e to each side, we might guess

$$f(x) = x(x-3) = x^2 - 3x.$$

This does indeed lead to the correct recursion.

(b) It is easy to see that the roots of $f(x)$ are 0 and 3, so these must be the possible answers this part, but only if it converges, which needs to be checked. However, we must also check that the limit converges.

Notice that $f'(x) = 2x - 3$ is always negative when $x < 3/2$ (resp. positive when $x > 3/2$), hence the function is monotonic on intervals of the form $[a, 3/2]$ (resp. $[3/2, a]$. Also, $f''(x) = 2$ is always positive. This means that NM will always converge on intervals of the form $[a, 3/2]$ (resp. $[3/2, a]$) if that interval has a root and the initial guess is to the left (resp. right) of the root in that interval, which would be 0 (resp. 3).

Our $x_0 = 8$ is in the 3 case.

a) (+4) setup

b) *not deducting points for skipping the IVT part of the criterion*

(+2) knows the theorem to use and then

(+3) checks increasing

(+3) checks concavity

(+2) determines it converges to the root left of the guess

(+3) finds root or roots under the assumption of convergences, but doesn't get to the core of the problem

Part B: Short Proofs [50 points]

4. [15 points] Let V be a finite-dimensional vector space with a chosen ordered basis; given $v \in V$, we write v_i for the ith component of v with respect to that basis.
The L^∞ norm on V, with respect to this basis, is defined by

$$|v|_\infty = \max_{i=1,\ldots,n} \{|v_i|\}$$

a) [10 points] Verify that $|\cdot|_\infty$ is a vector norm.

b) [5 points] Draw a picture of the unit disk for this norm on $\mathbb{R}^2$ with the standard basis.

Proof. a) Clearly nonnegative, and if $v \neq 0$ then it has at least one nonzero component, hence has a nonzero infinity norm. Scalars can be taken out of a max. The only hard step is the triangle inequality:

$$\begin{aligned} |x+y| &= \max_{i=1}^{n}\{|x_i + y_i|\} \\ &\leq \max_{i=1}^{n}\{|x_i| + |y_i|\} \\ &\leq \max_{i,j=1}^{n}\{|x_i| + |y_j|\} \\ &= \max_{i=1}^{n}\{|x_i|\} + \max_{j=1}^{n}\{|y_j|\} \\ &= |x| + |y| \end{aligned}$$

where the $2 \to 3$ uses the fact that introducing more elements to a sum only makes it larger, and $3 \to 4$ is the observation that we are adding positive numbers $|x_i|$ and $|y_j|$ which vary independently, hence is the sum of their individual maxima.

b) The disk is a square with side length 2 centered around zero.

a) (+2) nonnegative
   (+1) zero vector
   (+3) scalars
   (+4) triangle inequality
   (−1) if triangle inequality is missing a little detail [ended up not taking off this point]

b) see comments

5. [15 points] Let M be a Markov matrix: square, all its entries are nonnegative integers between 0 and 1, and the sum of the entries in each column is 1.

a) [5 points] Show that, if λ is an eigenvalue of M, then $|\lambda| \leq 1$.

b) [5 points] Assume that the limit $\lim_{k\to\infty} M^k v$ converges to a nonzero vector x. Show x is a (right) eigenvector of M and 1 the corresponding eigenvalue.

c) [5 points] Directly show that 1 is an eigenvalue of M, without using limits.

Proof. (a) and (c) are parts of the unused long proof, so studying that would have helped you here!

a) Also a problem on HW3, the Gershgorin circle theorem applied to M^T, since it has the same eigenvalues as M.

   For every diagonal entry M_{ii}, the corresponding gershgorin disk for M^T has radius

$$r_i = \sum_{j\neq i}^{n} M_{ji} = \left(\sum_{j=1}^{n} M_{ji}\right) - M_{ii} = 1 - M_{ii}.$$

   Every eigenvalue λ is in some such Gershgorin disk: to be in the disk means that

$$|\lambda - M_{ii}| \leq r_i = 1 - M_{ii},$$

   which is enough to bound $|\lambda|$:

$$\begin{aligned} |\lambda| &= |\lambda - M_{ii} + M_{ii}|, \\ &\leq |\lambda - M_{ii}| + |M_{ii}|, \\ &\leq r_i + M_{ii}, \\ &\leq 1 - M_{ii} + M_{ii}, \\ &= 1. \end{aligned}$$

b) In this case, we use continuity:

$$Mx = M \lim_{k\to\infty} M^k v = \lim_{k\to\infty} M^{k+1} v,$$

   but that's the same limit, just index differently, hence converges to x. So $Mx = x$, meaning x is an eigenvector (right) with eigenvalue 1.

c) Multiply on the left by $y = [1, ..., 1]$: this adds up entries in columns (i.e. sums the rows of the matrix), and hence results in $[1, ..., 1]$. Thus $yM = y$, and so there's a *left* eigenvector with eigenvalue 1, which means 1 is an eigenvalue of the matrix.

a) (+4) GCT, with or without picture

   (+4) pure norm approach

   (+1) transpose

b) see comments

c) (+2.5) correct left eigenvector

   (+2.5) points out relevance of column sum

   (+1) (partial credit only) mentions left-right/transpose eigenvalue correspondence

6. [20 points] Suppose $f(x)$ is continuous on an interval $[a, b]$, both $f(a)$ and $f(b)$ are nonzero, and the signs of $f(a)$ and $f(b)$ differ.

a) [5 points] Why does f have a root in $[a, b]$?

b) [5 points] Explain how to carry out the bisection method to produce a sequence x_k which converges to a root of f in $[a, b]$.

c) [10 points] Prove that $\lim_{k\to\infty} x_k$ from the bisection method converges.

   You may use the following fact without proof: if you have two monotonic sequences a_k and b_k contained in a closed interval $[a, b]$ and such that $\lim_{k\to\infty} a_k - b_k = 0$, then $\lim a_k$ and $\lim b_k$ both exist, are in $[a, b]$, and are equal.

Proof.

a) IVT

b) At each step, we have an input interval $[a_k, b_k]$ where the signs of f evaluated at the endpoints are different, starting from $[a_0, b_0] = [a, b]$.

   Compute the midpoint x_k. Check the sign of $f(x_k)$ – of course, you can end right away if $f(x_k) = 0$. That sign must be different than that of either $f(a_k)$ or $f(b_k)$, and hence one of the intervals $[a_k, x_k]$ or $[x_k, b_k]$ will be guaranteed to have a root of f by the IVT. This new interval is the input to the next step.

   The sequence x_k of midpoints is what converges to the root of f.

c) At each step, we replace a_k or b_k with the midpoint x_k, which is larger (resp. smaller), meaning that $a_k \le a_{k+1}$ (resp. $b_k \ge b_{k+1}$) at each step. So both sequences are monotonic and contained in a closed interval.

   Now, at each step the interval halves in size. In other words, if we let a_k and b_k be the left and right endpoints of the interval at each step, starting with $a_0 = a$ and $b_0 = b$, then $|a_k - b_k| = \frac{|a-b|}{2^k}$. This goes to zero as $k \to \infty$. So the fact applies to them: the limits exist and are equal.

   Moreover, $a_k < x_k < b_k$, so squeeze theorem tells us that $\lim_{k\to\infty} x_k$ exists as well, even coincides with the other two limits.

a) see comments

b) (-1) doesn't point out sign difference is guaranteed (need not cover the case that it's accidentally a zero)

c) (+2) monotonicity explained

   (+3) explains why $b_k - a_k$ goes to zero

   (+2) applies hint

   (+3) applies squeeze

Longer Proof [25 points]

(♞) [25 points] Suppose f is a twice-continuously differentiable function on $[a, b]$. Assume there is a constant B such that $|f'| > B > 0$ on $[a, b]$.
Let $(x_n)_{n=0}^{\infty}$ be the approximations obtained from applying Newton's method, which we assume are all in $[a, b]$. Further suppose $\lim_{n\to\infty} x_n$ converges to x in $[a, b]$
Let $e_n = x - x_n$ be the error of the nth approximation. Show that

a) [5 points] There is a constant C such that $|f''| < C$ on $[a, b]$,

b) [20 points] The errors satisfy

$$|e_{n+1}| \leq \frac{C}{2B}|e_n|^2.$$

c) [bonus] [+5 points] State what the corresponding error estimate looks like when the root has multiplicity two (hence is critical, for example) and how the proof/verification of that estimate differs from the proof in (b).

Proof. See study guide.

a) see comments

b) (+5) correct initial setup, then does algebra and then
   (+5) points out BLA or taylor shape, to then
   (+5) apply the remainder estimate (aka MVT)
   (+5) combines bounds carefully

c) (+2) bound is linear
   (+1.5) approaching(/or) division by zero obstructs the previous proof
   (+1.5) needs MVT on derivative