Homework 6 (root finding)

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1. One view of the secant method: it is a coarser Newton's method. We've seen that it has some of the speed of Newton's method. One might also hope that it enjoys similar convergence properties.

Adapt the convergence proof for Newton's method to show that the secant method also always converges under the following assumptions about the function f on the interval $[a, b]$:

i) f is twice continuously differentiable

ii) $f' > 0$

iii) $f'' > 0$

iv) f has a root x in the interval

v) the two initial guesses x_0, x_1 are both to the right of the root.

Hint: you will have to use convexity in a slightly more interesting way than in NM – the graph of f does not lie above the secant line, but you can argue that the right (well, left!) piece still does.

Solution.

2. Another view of the secant method, discussed in class, is as a weighted bisection method. Here too, one might hope for a convergence guarantee, because BM is much more robust than NM in that regard.

Consider a modified secant method which at step k takes in endpoints a_k, b_k, calculates their weighted midpoint c_k and then returns two new endpoints a_{k+1}, b_{k+1}, one of which is c_k, to which IVT applies. These new endpoints are input to the next step.

Prove that if f is continuous on $[a, b] = [a_0, b_0]$ and the IVT applies to f on the interval, then the sequence c_k from the modified secant method converges to a root of f.

Hint: the reason for convergence is *not* the same as for bisection. This would require the stronger assumption that f is continuously differentiable. In fact:
[Bonus] Give an example where the sequences x_k and y_k converge to different points, so squeeze does not apply.

Solution.

3. Suppose $f(x)$ and $g(x)$ are functions with a common root $x = a$.

a) Prove that a solution to the homotopy continuation initial value problem

$$x'(t) = -\frac{H_t}{H_x} \qquad x(0) = a$$

is the constant function $x = a$.

b) Give an example where the solution above is *not* unique.

Hint: see handout for a picture of (a). Think about how it could be adapted (b); you can even use the tool to help you construct an example.

Solution.