MATH 280 HW1

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Question 1

Proof. Define the function $h(x) = \frac{f(x)}{g(x)}$, by the def. of the derivative with limit process:

$$h'(a) = \lim_{x \to a} = \frac{h(x) - h(a)}{x - a}$$

Expanding $h(x)$ and $h(a)$:

$$h'(a) = \lim_{x \to a} = \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a}$$

$$\Rightarrow h'(a) = \lim_{x \to a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)(x - a)}$$

Applying $f(x) \approx f(a) + f'(a)(x - a) + E_f(x)$ and $g(x) \approx g(a) + g'(a)(x - a) + E_g(x)$, we can rearrange numerator as:

$$f(x)g(a) - f(a)g(x) \approx (f(a) + f'(a)(x-a) + E_f(x))g(a) + f(a)(g(a) + g'(a)(x-a) + E_g(x))$$
$$= f(a)g(a) + f'(a)g(a)(x-a) + E_f(x)g(a) - f(a)g(a) - f(a)g'(a)(x-a) - f(a)E_g(x)$$
$$= f'(a)g(a)(x - a) - f(a)g'(a)(x - a) + E_f(x)g(a) - f(a)E_g(x)$$

Applying the definition of $E_f(x)$ and $E_g(x)$:

$$\lim_{x \to a} \frac{E_f(x)g(a) - f(a)E_g(x)}{g(x)g(a)g(x - a)} = 0$$

we can get overall:

$$h'(a) = \lim_{x \to a} \frac{f'(a)g(a)(x - a) - f(a)g'(a)(x - a) + E_f(x)g(a) - f(a)E_g(x)}{g(x)g(a)g(x - a)}$$

Since $g(x) \approx g(a)$ when $x = a$:

$$\Rightarrow \left(\frac{f}{g}\right)' = h'(a) = \lim_{x \to a} \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

$\square$

Question 2

(a)

Given,

- $f(x) = 2 - (x-3) + E_f(x)$, near $x = 3$, with $|E_f(x) < 1|$ one the interval $[2, 4]$.
- $g(x) = x^3 - 2x^2 + 3$; $g(2) = 8 - 4(2) + 3 = 3, g'(2) = 12 - 8 = 4$.

Proof. From the initial assumptions, here is the approximation of $f(x)$ near $x = 3$:

$$f(x) = 2 - (x-3) + E_f(x)$$

Value of g at $f(3)$:

$$f(3) = 2 - (x3 - 3) + E_f(3) = 2 + E_f(3)$$

Noticing that $|E_f(x)| < 1$, then

$$g(f(3)) = g(2 + E_f(3)) = g(2) = 3$$

Then, we can use the Taylor Expansion around $f(3) = 2$. The first order Taylor expansion for $g(f(x))$ around $x = 3$ would be:

$$g(f(x)) \approx g(2) + g'(2) \cdot (f(x) - 2)$$

Then we can sub $f(x)$ in:

$$g(f(x)) \approx 3 + 4(2 + E_f(x) - (x-3) - 2)$$
$$= 3 + 4(-x + 3 + E_f(x))$$
$$= 3 + 4(-x + 3) + 4E_f(x)$$
$$= 15 - 4x + 4E_f(x)$$

Given $E(x) << (x - a)$ and considering the linearity of $g'(x) : E_{f \circ g} = 4E_f(x)$. As $E_f(x)$ is much smaller $(x-3)$, and given $|E_f(x)| < 1 : |E_{f \circ g}| \leq 4$. $\square$

(b)

Given $h(x)$ is differentiable at $x = 0$ and $h(0) = 3$, WTS the bound the error of $f(h(x))$ near zero.

Proof. To precisely bound the error of $f(h(x))$ near zero, we need two more information:

- Explicit form of $E_f(x)$
- Rate of Change of $h(x)$

By knowing these two conditions, we can bound $E_f(h(x))$ by the following approach:
By definition of Taylor Expansion, we can subsitute

$$h(x) = 3 + h'(0)x + o(x)$$

Where $o(x)$ is the higher order terms that become negligibly small faster than linear.
Then, suppose $E_f(x) = k(x-3)^n$, then:

$$E_f(h(x)) = k(h(x) - 3)^n = k(h'(0)x + o(x))^n$$

For evaluation we will only focus on how the "linear" terms in x contribute to $E_f(h(x))$.
Assume $E_f(x) \sim (x-3)^n$ and $h(x) = 3 + h'(0)x + o(x)$, the bound of $E_f(h(x))$ when $x = 0$ can be written in the form:

$$|E_f(h(x))| \leq |k| \cdot |h'(0)x + o(x)|^n$$

□

Question 3

Given, $P(t)$ s.t. $P(1) = 4, P'(t) = tP(t)^2$

(a) Quadratic Approximation

Proof. Given,

$$P'(1) = 4, P'(t) = tP(t)^2$$

For $P^{'}(t)$ and $P^{''}(t)$, we can get (using the product rule):

$$P'(1) = 1 \times 4^2 = 16$$

$$P^{''}(t) = \frac{d}{dt}[tP(t)^2] = P(t)^2 + 2tP(t)P^{'}(t)$$

Sub $t = 1$ into $P^{''}(t)$:

$$P^{''}(t) = 4^2 + 2 \times 1 \times 4 \times 16 = 16 + 128 = 144$$

Then we can apply the Taylor Expansion around $t = 1$, include the small error term $E(t-1)$ s.t. $E(t-1) << t-1$:

$$P(t) \approx P(1) + P'(1)(t-1) + \frac{1}{2}P^{''}(1)(t-1)^2 + E(t-1)$$

$$P(t) \approx 4 + 16(t-1) + 72(t-1)^2 + E(t-1)$$

Then, we can estimate $P(0)$ and $P(-1)$. Subsitute $t = 0$ and $t = -1$, we can get:

$$P(0) = 4 + 16(-1) + 72(1)^2 = 4 - 16 + 72 = 60$$

$$P(-1) = 4 + 16(-2) + 72(4) = 4 - 32 + 288 = 260$$

Noting that $E(t-1)$ is negligible near $t = 1$. □

(b) Analysis Critical Point at $t = 0$

Noting that in here we can apply the 2nd derivative test.

Proof. From (a), we know that,

$$P^{''}(0) = P(0)^2 \approx 60^2 = 3600$$

Since $P^{''}(t) > 0$ for $t = 0$, critical point at $t = 0$ is a local minimum point. □

(c)

I think in this case the estimation using quadratic approximation for $P(0)$ and $P(-1)$ might likely underestimate the exact values. Such phenomenon aries is because of the quadratic model might not fully capture the exponential growth. Specifically, since $P'(t) = tP(t)^2$, the rate of change of $P(t)$ depends on the product of t and $P(t)^2$. This might lead to rapid changes in the function's value, particularly when t & $P(t)$ are significantly large. Hence, quadratic approximation might fails to accurately predict the behavior of the function $P(t)$ and underestimate its actual value due to its lack of complexity.