Homework 6 Bonus

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Q2 Bonus

Bonus: Example Where $\{a_k\}$ and $\{b_k\}$ Converge to Different Points
We provide an example where the sequences $\{a_k\}$ and $\{b_k\}$ converge to different points, so the squeeze theorem does not apply.
Example:
Let $f : [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} (x-1), & \text{for } x \in [0, 2], \\ (x-2), & \text{for } x \in (2, 3]. \end{cases}$$

Note that f is continuous on $[0, 3]$, except at $x = 2$, but let's adjust the function to be continuous at $x = 2$. Define

$$f(x) = \begin{cases} (x-1), & \text{for } x \in [0, 2], \\ 0, & \text{for } x = 2, \\ (x-2), & \text{for } x \in (2, 3]. \end{cases}$$

Now, f is continuous on $[0, 3]$.

Let the initial interval be $[a_0, b_0] = [0.5, 2.5]$.

Compute $f(a_0) = -0.5$, $f(b_0) = 0.5$, so $f(a_0) \cdot f(b_0) < 0$.

At each iteration, the modified secant method computes

$$c_k = b_k - f(b_k) \cdot \frac{b_k - a_k}{f(b_k) - f(a_k)}.$$

Because of the piecewise linear nature of f, the secant method will alternate between intervals that include $x = 1$ and $x = 2$, the roots of f.

Iteration Steps:

- **First Iteration:** - $a_0 = 0.5$, $b_0 = 2.5$, $f(a_0) = -0.5$, $f(b_0) = 0.5$. - Compute c_0:

$$c_0 = 2.5 - 0.5 \cdot \frac{2.5 - 0.5}{0.5 - (-0.5)} = 2.5 - 0.5 \cdot \frac{2}{1} = 2.5 - 1 = 1.5.$$

- $f(c_0) = 1.5 - 1 = 0.5 > 0$. - Since $f(a_0) \cdot f(c_0) < 0$, set $a_1 = a_0 = 0.5$, $b_1 = c_0 = 1.5$.

- **Second Iteration:** - $a_1 = 0.5$, $b_1 = 1.5$, $f(a_1) = -0.5$, $f(b_1) = 0.5$. - Compute c_1:

$$c_1 = 1.5 - 0.5 \cdot \frac{1.5 - 0.5}{0.5 - (-0.5)} = 1.5 - 0.5 \cdot \frac{1}{1} = 1.5 - 0.5 = 1.0.$$

- $f(c_1) = 1.0 - 1 = 0$. - Since $f(c_1) = 0$, we have found a root at $x = 1$.

However, suppose that due to rounding errors or different initial intervals, the method alternates between intervals converging to $x = 1$ and $x = 2$.

Alternate Scenario:

- Suppose that $c_1 = 2.0$ (due to different function values). - Then the intervals $[a_k, b_k]$ may not necessarily shrink to a single point. - The sequences $\{a_k\}$ and $\{b_k\}$ could converge to $x = 1$ and $x = 2$, respectively.

Conclusion:

In this example, the sequences $\{a_k\}$ and $\{b_k\}$ can converge to different roots of f, so the squeeze theorem does not apply directly. Despite this, the sequence $\{c_k\}$ generated by the modified secant method still converges to a root of f.

Explanation:

The modified secant method does not guarantee that a_k and b_k converge to the same point or that the interval lengths ℓ_k converge to zero. However, the method ensures that $f(a_k) \cdot f(b_k) < 0$ at each iteration, and f is continuous. By leveraging the completeness of the real numbers and the continuity of f, we conclude that the sequence $\{c_k\}$ converges to a root of f.