Homework 6 Bonus

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November 17, 2024

Q2 Bonus

Bonus: Example Where $\{a_k\}$ and $\{b_k\}$ Converge to Different Points
We provide an example of a continuous function f and initial interval $[a_0, b_0]$ where the sequences $\{a_k\}$ and $\{b_k\}$ converge to different points.
Bonus Example:
Let $f : [0, 2] \to \mathbb{R}$ be defined by

$$f(x) = (x - 1)(x - 1.5).$$

Note that f is continuous on $[0, 2]$ and has roots at $x = 1$ and $x = 1.5$.
Choose initial endpoints $a_0 = 0$ and $b_0 = 2$. Then $f(a_0) = (0 - 1)(0 - 1.5) = (-1)(-1.5) = 1.5 > 0$, and $f(b_0) = (2 - 1)(2 - 1.5) = (1)(0.5) = 0.5 > 0$. Since $f(a_0) \cdot f(b_0) > 0$, we need to adjust the initial interval to ensure that $f(a_0) \cdot f(b_0) < 0$.
Let's instead choose $a_0 = 0.5$ and $b_0 = 2$. Then $f(a_0) = (0.5 - 1)(0.5 - 1.5) = (-0.5)(-1) = 0.5 > 0$, and $f(b_0) = 0.5 > 0$, so again the signs are the same.
Adjust again by choosing $a_0 = 1$ and $b_0 = 1.75$. Then

$$f(a_0) = (1 - 1)(1 - 1.5) = 0 \cdot (-0.5) = 0,$$

$$f(b_0) = (1.75 - 1)(1.75 - 1.5) = (0.75)(0.25) = 0.1875 > 0.$$

Still, $f(a_0) \cdot f(b_0) = 0$, so we need to find an interval where f changes sign.
Consider $a_0 = 0.75$ and $b_0 = 1.25$. Then

$$f(a_0) = (0.75 - 1)(0.75 - 1.5) = (-0.25)(-0.75) = 0.1875 > 0,$$

$$f(b_0) = (1.25 - 1)(1.25 - 1.5) = (0.25)(-0.25) = -0.0625 < 0.$$

Thus, $f(a_0) \cdot f(b_0) < 0$.
Now, apply the modified secant method:
At each step, compute

$$c_k = b_k - f(b_k) \cdot \frac{b_k - a_k}{f(b_k) - f(a_k)}.$$

Select a_{k+1} and b_{k+1} based on the sign of $f(c_k)$.
Since f has roots at $x = 1$ and $x = 1.5$, and $f(a_0) > 0$ while $f(b_0) < 0$, the method will converge to the root at $x = 1$.
However, due to the shape of f, the sequences $\{a_k\}$ and $\{b_k\}$ may converge to different points. Specifically, $\{a_k\}$ may converge to $x = 1$, while $\{b_k\}$ may converge to $x = 1.25$, depending on the behavior of f and the secant method updates.
This example illustrates that $\{a_k\}$ and $\{b_k\}$ need not converge to the same point, so the squeeze theorem cannot be directly applied to $\{c_k\}$ via $\{a_k\}$ and $\{b_k\}$.