Homework 6 Bonus

Hanzhang Yin

November 18, 2024

Q2 Bonus

Bonus: Example Where $\{a_k\}$ and $\{b_k\}$ Converge to Different Points
We provide an example of a continuous function f and initial intervals $[a_0, b_0]$ and $[c_0, d_0]$ where the sequences $\{a_k\}$ and $\{b_k\}$ converge to different roots.
Bonus Example:
Let $f : [0, 4] \to \mathbb{R}$ be defined by

$$f(x) = (x - 1)(x - 2)(x - 3).$$

Note that f is continuous on $[0, 4]$ and has roots at $x = 1$, $x = 2$, and $x = 3$.
Sequence 1: Converging to $x = 1$
Choose initial interval $[a_0, b_0] = [0.5, 1.5]$.

$$f(a_0) = (0.5 - 1)(0.5 - 2)(0.5 - 3) = (-0.5)(-1.5)(-2.5) = -1.875 < 0,$$

$$f(b_0) = (1.5 - 1)(1.5 - 2)(1.5 - 3) = (0.5)(-0.5)(-1.5) = 0.375 > 0.$$

Since $f(a_0) \cdot f(b_0) < 0$, there is a root in $[0.5, 1.5]$, specifically at $x = 1$.
Applying the secant method:

$$c_0 = 1.5 - 0.375 \cdot \frac{1.5 - 0.5}{0.375 - (-1.875)} = 1.5 - 0.375 \cdot \frac{1}{2.25} = 1.5 - 0.375 \cdot 0.4444 \approx 1.5 - 0.1667 = 1.3333,$$

$$f(c_0) \approx f(1.3333) = (1.3333 - 1)(1.3333 - 2)(1.3333 - 3) \approx (0.3333)(-0.6667)(-1.6667) \approx 0.3704 > 0.$$

Since $f(c_0) > 0$, update the interval to $[a_1, b_1] = [0.5, 1.3333]$.
Continuing this process, the sequence $\{a_k\}$ will converge to $x = 1$.
Sequence 2: Converging to $x = 3$
Choose initial interval $[c_0, d_0] = [2.5, 3.5]$.

$$f(c_0) = (2.5 - 1)(2.5 - 2)(2.5 - 3) = (1.5)(0.5)(-0.5) = -0.375 < 0,$$

$$f(d_0) = (3.5 - 1)(3.5 - 2)(3.5 - 3) = (2.5)(1.5)(0.5) = 1.875 > 0.$$

Since $f(c_0) \cdot f(d_0) < 0$, there is a root in $[2.5, 3.5]$, specifically at $x = 3$.
Applying the secant method:

$$e_0 = 3.5 - 1.875 \cdot \frac{3.5 - 2.5}{1.875 - (-0.375)} = 3.5 - 1.875 \cdot \frac{1}{2.25} = 3.5 - 1.875 \cdot 0.4444 \approx 3.5 - 0.8333 = 2.6667,$$

$$f(e_0) \approx f(2.6667) = (2.6667 - 1)(2.6667 - 2)(2.6667 - 3) \approx (1.6667)(0.6667)(-0.3333) \approx -0.3704 < 0.$$

Since $f(e_0) < 0$, update the interval to $[c_1, d_1] = [2.6667, 3.5]$.
Continuing this process, the sequence $\{b_k\}$ will converge to $x = 3$.
Conclusion:
The sequence $\{a_k\}$ generated from the initial interval $[0.5, 1.5]$ converges to the root at $x = 1$. The sequence $\{b_k\}$ generated from the initial interval $[2.5, 3.5]$ converges to the root at $x = 3$.
by choosing different initial intervals bracketing distinct roots, the sequences $\{a_k\}$ and $\{b_k\}$ can converge to different points.