Midterm Correction

Hanzhang Yin

November 5, 2024

Question Correction:

1 Part A

Question 1

1.0.1 (a)

For this question, I made a misconception on the inequality relationship between absolute error and relative error. Although I have correctly identitfied it out, I sub the incorrect values in it during calculations

Corrections:

NOTE: $\kappa(A) = ||L|| \cdot ||L^{-1}||$

$$\frac{|x - \tilde{x}|}{|x|} \leq \kappa(L)\frac{|b - \tilde{b}|}{|b|} \leq 4 \times \left(\frac{1}{2}\right) \times 0.1 = 0.2.$$

1.0.2 (b)

For this question, I have confused and forget the relationship between absolute and relative error. (Which should not happened...)

Corrections:

The system of equations given is $Lx = b$. where L is a matrix, x is the solution vector, and b is the right-hand side vector. However, we only have an estimate $\tilde{b}$ for b, so we ended up solving $L\tilde{x} = \tilde{b}$, where $\tilde{x}$ is the approx. solution corresponding to $\tilde{b}$.

To analyze the effect of the error b on x, we subtract the two equations $Lx = b, L\tilde{x} = \tilde{b}$, giving that:

$$L(x - \tilde{x}) = b - \tilde{b} \Rightarrow x - \tilde{x} = L^{-1}(b - \tilde{b})$$

To bound the error, we take the norm of both sides here:

$$\Rightarrow |x - \tilde{x}| = |L^{-1}(b - \tilde{b})|$$

By triangle inequality and definition of matrix norm, we can further have:

$$\begin{aligned} \Rightarrow |x - \tilde{x}| &\leq ||L^{-1}|||b - \tilde{b}| \\ &= \frac{1}{2} \times 4, \\ &= 2. \end{aligned}$$

1.0.3 (c)

Similarly to (b), I have failed recognize using the given system of linear equation is sufficient during examination.

Corrections:

Since $Lx = b$, we can express x as:

$$x = L^{-1}b$$

To bound $|x|$, we take the norm:

$$|x| = |L^{-1}b|$$

Using the traingle inequality and definition of matrix norm $|L^{-1}b| \leq ||L^{-1}|||b|$, we get:

$$|x| \leq ||L^{-1}|||b|$$

Substitute $\|L^{-1}\| = \frac{1}{2}$ and $|b| = 40$:

$$|x| \leq \frac{1}{2} \times 40 = 20$$

Question 2

1.0.4 (a)

Forgot to specify that degree k spline must be polynomial. Here I will provide a more formal definition as a correction:

Corrections:

A degree k spline on an interval $[a, b]$ is a function $S(x)$ composed of polynomial segments $S_i(x)$ of degree k, defined on subintervals $[x_i, x_{i+1}]$, where $a = x_0 < x_1 < \cdots < x_n = b$.

The spline satisfies the following properties:

- Piecewise Polynomial: On each subinterval $[x_i, x_{i+1}]$, $S(x)$ is a polynomial of degree k:

  $$S(x) = S_i(x) \quad \text{for } x \in [x_i, x_{i+1}]$$

- Continuity of Function and Derivatives: The function $S(x)$ is continuous on $[a, b]$, and its derivatives up to order $k - 1$ are also continuous across the subinterval boundaries (knots):

  $$S(x_i^-) = S(x_i^+), \quad S'(x_i^-) = S'(x_i^+), \quad \ldots, \quad S^{(k-1)}(x_i^-) = S^{(k-1)}(x_i^+)$$

  for each knot x_i, where $S(x_i^-)$ and $S(x_i^+)$ denote the left-hand and right-hand limits at x_i, respectively.

Question 3

1.0.5 (b)

For this question, I have no time to perform further steps as the time runs up, so that in this case I could only make a somehow plausible guess after all.

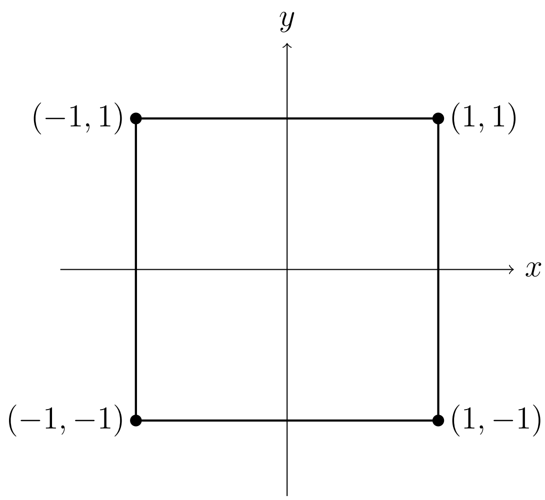
Corrections:

2 Part B

Question 4

2.0.1 (b)

I have made a misconception between the graph of L_2 norm and ∞ norm. During exam I don't have sufficient time to think carefully, but after revisit this question I found out the graph is straight-forward.
Corrections:
For the L^∞ norm on $\mathbb{R}^2$, the unit disk forms a square with vertices at $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$, as it includes all points where $\max(|x|, |y|) \leq 1$.



Question 5

2.0.2 (b)

For this question, I misuse the characteristics of Markov matrix and matrix norm, and forgot the continuity of matrix-vector multiplication w.r.t. the limits.
Corrections:
We use the continuity of matrix-vector multiplication with respect to limits in here for further analysis
are given that $\lim_{k\to\infty} M^k v = x$ for some nonzero vector x, meaning repeated applications of M to v converge to x.
Continuity of Matrix-Vector Multiplication: Since matrix-vector multiplication is continuous, we have:

$$Mx = M \lim_{k\to\infty} M^k v = \lim_{k\to\infty} M^{k+1} v = x$$

Thus, $Mx = x$, showing that x is an eigenvector of M with eigenvalue 1.

2.0.3 (c)

For this question, since the question does not specify whether the markov matrix have the column or the row sum to 1, so that I assumed the former and showed that there is a right

eigenvector with eigenvalue 1. I think both will work but since it diverged with what the answer key said, here is the modified answer.
Corrections:
Let $u = [1, 1, \ldots, 1]$, a row vector with each entry equal to 1.
When we multiply y on the left by M, we get yM. This operation sums each column of M, resulting in y again:

$$uM = u$$

Since $uM = u$, u is a left eigenvector of M with eigenvalue 1. Therefore, 1 is an eigenvalue of M.

Question 6

2.0.4 (c)

For this question, I think it is sufficient to use sequence a_k and b_k is increasing and decreasing resp. without stating out as given they are given monotonic. Hence ended with my a circular reasoning within my proof.
Corrections:
In each step of the bisection method, we start with an interval $[a_k, b_k]$ where $f(a_k)$ and $f(b_k)$ have opposite signs. We compute the midpoint $x_k = \frac{a_k+b_k}{2}$. If $f(x_k) = 0$, then x_k is the root, and we stop. Otherwise, depending on the sign of $f(x_k)$, we replace either a_k or b_k with x_k to ensure the new interval $[a_{k+1}, b_{k+1}]$ still contains the root.
By construction, a_k is updated only when $f(x_k)$ has the same sign as $f(a_k)$. In this case, we set $a_{k+1} = x_k$, meaning a_k is non-decreasing. Similarly, b_k is updated only when $f(x_k)$ has the same sign as $f(b_k)$. In this case, we set $b_{k+1} = x_k$, making b_k non-increasing. Therefore, a_k forms a non-decreasing sequence and b_k forms a non-increasing sequence, with both sequences bounded within $[a_0, b_0]$.
Each iteration halves the interval size, so $|a_k - b_k| = \frac{|a_0-b_0|}{2^k}$, which approaches zero as $k \to \infty$. Given the fact that if two monotonic sequences a_k and b_k are contained in a closed interval and $\lim_{k\to\infty}(a_k - b_k) = 0$, then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$ and these limits are equal.
Since $a_k \leq x_k \leq b_k$ at each step, the Squeeze Theorem implies that x_k also converges to the same limit as a_k and b_k. Therefore, $\lim_{k\to\infty} x_k$ exists, lies in $[a, b]$, and is equal to the root of f.

Question 7 - Long Proof Part (b)

For this question, I did not state clear with a little vagueness about how to use Taylor's theorem in this proof, and made a small typo in exponent listed in the expanded polynomial. This probably due to that I missed some details that need to remembered in the proof detial.
Corrections:
To show that $|e_{n+1}| \leq \frac{C}{2B} e_n^2$, we start by expanding the error e_{n+1} in terms of e_n, using Newton's method.
Using Newton's update formula, we have:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since $e_{n+1} = x - x_{n+1}$, we can rewrite e_{n+1} as:

$$e_{n+1} = x - \left(x_n - \frac{f(x_n)}{f'(x_n)} \right) = (x - x_n) + \frac{f(x_n)}{f'(x_n)}.$$

Substituting $e_n = x - x_n$, we get:

$$e_{n+1} = e_n + \frac{f(x_n)}{f'(x_n)}.$$

To approximate $f(x_n)$ in terms of e_n, we use Taylor's theorem to expand $f(x_n)$ around x (the root we are converging to), where $f(x) = 0$. Taylor's theorem gives:

$$f(x_n) = f(x) + (x_n - x)f'(x) + \frac{f''(\zeta_n)}{2}(x_n - x)^2,$$

for some ζ_n between x and x_n. Since $f(x) = 0$, this simplifies to:

$$f(x_n) = (x_n - x)f'(x) + \frac{f''(\zeta_n)}{2}(x_n - x)^2.$$

Substituting $e_n = x - x_n$, we get:

$$f(x_n) = -f'(x_n)e_n + \frac{f''(\zeta_n)}{2}e_n^2.$$

Now, we substitute this expression for $f(x_n)$ into the formula for e_{n+1}:

$$e_{n+1} = e_n + \frac{-f'(x_n)e_n + \frac{f''(\zeta_n)}{2}e_n^2}{f'(x_n)}.$$

Simplifying, we find:

$$e_{n+1} = e_n - \frac{f'(x_n)}{f'(x_n)}e_n + \frac{f''(\zeta_n)}{2f'(x_n)}e_n^2.$$

The terms involving e_n cancel out, leaving:

$$e_{n+1} = \frac{f''(\zeta_n)}{2f'(x_n)}e_n^2.$$

Taking the absolute value of both sides, we obtain:

$$|e_{n+1}| = \left| \frac{f''(\zeta_n)}{2f'(x_n)} \right| e_n^2.$$

By the assumptions of the theorem, $|f'| > B$ on $[a, b]$, so $|f'(x_n)| > B$. Also, since f'' is bounded by C on $[a, b]$, we have $|f''(\zeta_n)| < C$ for all ζ_n in $[a, b]$. Therefore:

$$|e_{n+1}| \leq \frac{C}{2B}e_n^2.$$

Write Your Own: