Midterm Correction

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Question Correction:

1 Part A

Question 1

1.0.1 (a)

For this question, I made a misconception on the inequality relationship between absolute error and relative error. Although I have correctly identitfied it out, I sub the incorrect values in it during calculations
Corrections:
NOTE: $\kappa(A) = ||L|| \cdot ||L^{-1}||$

$$\frac{|x-\tilde{x}|}{|x|} \leq \kappa(L)\frac{|b-\tilde{b}|}{|b|} \leq 4 \times \left(\frac{1}{2}\right) \times 0.1 = 0.2.$$

1.0.2 (b)

For this question, I have confused and forget the relationship between absolute and relative error. (Which should not happened...)
Corrections:
The system of equations given is $Lx = b$. where L is a matrix, x is the solution vector, and b is the right-hand side vector. However, we only have an estimate $\tilde{b}$ for b, so we ended up solving $L\tilde{x} = \tilde{b}$, where $\tilde{x}$ is the approx. solution corresponding to $\tilde{b}$.
To analyze the effect of the error b on x, we subtract the two equations $Lx = b, L\tilde{x} = \tilde{b}$, giving that:

$$L(x-\tilde{x}) = b - \tilde{b} \Rightarrow x - \tilde{x} = L^{-1}(b-\tilde{b})$$

To bound the error, we take the norm of both sides here:

$$\Rightarrow |x-\tilde{x}| = |L^{-1}(b-\tilde{b})|$$

By triangle inequality and definition of matrix norm, we can further have:

$$\begin{aligned}\Rightarrow |x-\tilde{x}| &\leq ||L^{-1}|||b-\tilde{b}| \\ &= \frac{1}{2} \times 4, \\ &= 2.\end{aligned}$$

1.0.3 (c)

Similarly to (b), I have failed recognize using the given system of linear equation is sufficient during examination.
Corrections:
Since $Lx = b$, we can express x as:

$$x = L^{-1}b$$

To bound $|x|$, we take the norm:

$$|x| = |L^{-1}b|$$

Using the traingle inequality and definition of matrix norm $|L^{-1}b| \leq ||L^{-1}|||b|$, we get:

$$|x| \leq ||L^{-1}|||b|$$

Substitute $\|L^{-1}\| = \frac{1}{2}$ and $|b| = 40$:

$$|x| \leq \frac{1}{2} \times 40 = 20$$

Question 2

1.0.4 (a)

Forgot to specify that degree k spline must be polynomial. Here I will provide a more formal definition as a correction:
Corrections:
A degree k spline on an interval $[a, b]$ is a function $S(x)$ composed of polynomial segments $S_i(x)$ of degree k, defined on subintervals $[x_i, x_{i+1}]$, where $a = x_0 < x_1 < \cdots < x_n = b$.
The spline satisfies the following properties:

- Piecewise Polynomial: On each subinterval $[x_i, x_{i+1}]$, $S(x)$ is a polynomial of degree k:

  $$S(x) = S_i(x) \quad \text{for } x \in [x_i, x_{i+1}]$$

- Continuity of Function and Derivatives: The function $S(x)$ is continuous on $[a, b]$, and its derivatives up to order $k - 1$ are also continuous across the subinterval boundaries (knots):

  $$S(x_i^-) = S(x_i^+), \quad S'(x_i^-) = S'(x_i^+), \quad \ldots, \quad S^{(k-1)}(x_i^-) = S^{(k-1)}(x_i^+)$$

  for each knot x_i, where $S(x_i^-)$ and $S(x_i^+)$ denote the left-hand and right-hand limits at x_i, respectively.

Question 3

1.0.5 (b)

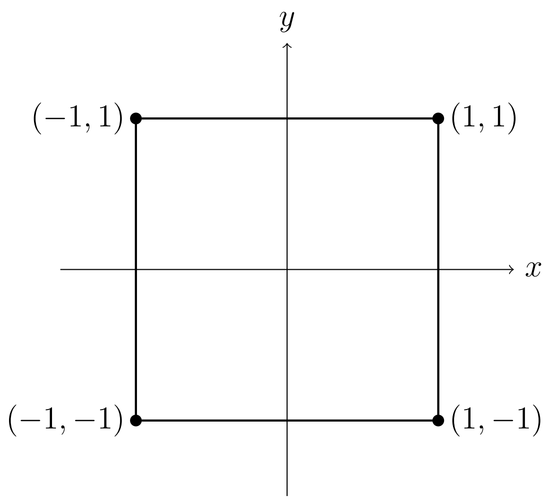
For this question, I have no time to perform further steps as the time runs up, so that in this case I could only make a somehow plausible guess after all.
Corrections:

2 Part B

Question 4

2.0.1 (b)

I have made a misconception between the graph of L_2 norm and ∞ norm. During exam I don't have sufficient time to think carefully, but after revisit this question I found out the graph is straight-forward.
Corrections:
For the L^∞ norm on $\mathbb{R}^2$, the unit disk forms a square with vertices at $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$, as it includes all points where $\max(|x|, |y|) \leq 1$.



Question 5

2.0.2 (b)

For this question, I misuse the characteristics of Markov matrix and matrix norm, and forgot the continuity of matrix-vector multiplication w.r.t. the limits.
Corrections:
We use the continuity of matrix-vector multiplication with respect to limits in here for further analysis
are given that $\lim_{k\to\infty} M^k v = x$ for some nonzero vector x, meaning repeated applications of M to v converge to x.
Continuity of Matrix-Vector Multiplication: Since matrix-vector multiplication is continuous, we have:

$$Mx = M \lim_{k\to\infty} M^k v = \lim_{k\to\infty} M^{k+1} v = x$$

Thus, $Mx = x$, showing that x is an eigenvector of M with eigenvalue 1.

2.0.3 (c)

For this question, since the question does not specify whether the markov matrix have the column or the row sum to 1, so that I assumed the former and showed that there is a right

eigenvector with eigenvalue 1. I think both will work but since it diverged with what the answer key said, here is the modified answer.
Corrections:
Let $u = [1, 1, \ldots, 1]$, a row vector with each entry equal to 1.
When we multiply y on the left by M, we get yM. This operation sums each column of M, resulting in y again:

$$uM = u$$

Since $uM = u$, u is a left eigenvector of M with eigenvalue 1. Therefore, 1 is an eigenvalue of M.

Question 6

2.0.4 (c)

For this question, I think it is sufficient to use sequence a_k and b_k is increasing and decreasing resp. without stating out as given they are given monotonic. Hence ended with my a circular reasoning within my proof.
Corrections:
In each step of the bisection method, we start with an interval $[a_k, b_k]$ where $f(a_k)$ and $f(b_k)$ have opposite signs. We compute the midpoint $x_k = \frac{a_k+b_k}{2}$. If $f(x_k) = 0$, then x_k is the root, and we stop. Otherwise, depending on the sign of $f(x_k)$, we replace either a_k or b_k with x_k to ensure the new interval $[a_{k+1}, b_{k+1}]$ still contains the root.
By construction, a_k is updated only when $f(x_k)$ has the same sign as $f(a_k)$. In this case, we set $a_{k+1} = x_k$, meaning a_k is non-decreasing. Similarly, b_k is updated only when $f(x_k)$ has the same sign as $f(b_k)$. In this case, we set $b_{k+1} = x_k$, making b_k non-increasing. Therefore, a_k forms a non-decreasing sequence and b_k forms a non-increasing sequence, with both sequences bounded within $[a_0, b_0]$.
Each iteration halves the interval size, so $|a_k - b_k| = \frac{|a_0-b_0|}{2^k}$, which approaches zero as $k \to \infty$. Given the fact that if two monotonic sequences a_k and b_k are contained in a closed interval and $\lim_{k\to\infty}(a_k - b_k) = 0$, then $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$ and these limits are equal.
Since $a_k \le x_k \le b_k$ at each step, the Squeeze Theorem implies that x_k also converges to the same limit as a_k and b_k. Therefore, $\lim_{k\to\infty} x_k$ exists, lies in $[a, b]$, and is equal to the root of f.

Question 7 - Long Proof Part (b)

For this question, I did not state clear with a little vagueness about how to use Taylor's theorem in this proof, and made a small typo in exponent listed in the expanded polynomial. This probably due to that I missed some details that need to remembered in the proof detial.
Corrections:
To show that $|e_{n+1}| \le \frac{C}{2B} e_n^2$, we start by expanding the error e_{n+1} in terms of e_n, using Newton's method.
Using Newton's update formula, we have:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since $e_{n+1} = x - x_{n+1}$, we can rewrite e_{n+1} as:

$$e_{n+1} = x - \left(x_n - \frac{f(x_n)}{f'(x_n)}\right) = (x - x_n) + \frac{f(x_n)}{f'(x_n)}.$$

Substituting $e_n = x - x_n$, we get:

$$e_{n+1} = e_n + \frac{f(x_n)}{f'(x_n)}.$$

To approximate $f(x_n)$ in terms of e_n, we use Taylor's theorem to expand $f(x_n)$ around x (the root we are converging to), where $f(x) = 0$. Taylor's theorem gives:

$$f(x_n) = f(x) + (x_n - x)f'(x) + \frac{f''(\zeta_n)}{2}(x_n - x)^2,$$

for some ζ_n between x and x_n. Since $f(x) = 0$, this simplifies to:

$$f(x_n) = (x_n - x)f'(x) + \frac{f''(\zeta_n)}{2}(x_n - x)^2.$$

Substituting $e_n = x - x_n$, we get:

$$f(x_n) = -f'(x_n)e_n + \frac{f''(\zeta_n)}{2}e_n^2.$$

Now, we substitute this expression for $f(x_n)$ into the formula for e_{n+1}:

$$e_{n+1} = e_n + \frac{-f'(x_n)e_n + \frac{f''(\zeta_n)}{2}e_n^2}{f'(x_n)}.$$

Simplifying, we find:

$$e_{n+1} = e_n - \frac{f'(x_n)}{f'(x_n)}e_n + \frac{f''(\zeta_n)}{2f'(x_n)}e_n^2.$$

The terms involving e_n cancel out, leaving:

$$e_{n+1} = \frac{f''(\zeta_n)}{2f'(x_n)}e_n^2.$$

Taking the absolute value of both sides, we obtain:

$$|e_{n+1}| = \left|\frac{f''(\zeta_n)}{2f'(x_n)}\right| e_n^2.$$

By the assumptions of the theorem, $|f'| > B$ on $[a, b]$, so $|f'(x_n)| > B$. Also, since f'' is bounded by C on $[a, b]$, we have $|f''(\zeta_n)| < C$ for all ζ_n in $[a, b]$. Therefore:

$$|e_{n+1}| \leq \frac{C}{2B}e_n^2.$$

Write Your Own:

I am willing to design a problem related to *Secant Method* that we have discussed currently.

Problem

Consider the function $f(x) = x^3 - 2x^2 + x - 3$.

(a) Explain how the secant method differs from the Newton method in finding roots of nonlinear equations.

(b) Prove that the order of convergence of the secant method is approximately ϕ, where $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio). Provide the derivation for this order of convergence.

Solution

Part (a): Differences Between Secant Method and Newton-Raphson Method

- **Derivative Requirement:**
  - **Newton-Raphson Method:** Requires the calculation of the derivative $f'(x)$ at each iteration.
  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
  - **Secant Method:** Does not require the derivative. Instead, it approximates the derivative using two previous function values.
  $$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$
- **Initial Guesses:**
  - **Newton-Raphson Method:** Requires a single initial guess x_0.
  - **Secant Method:** Requires two initial guesses x_0 and x_1 to start the iteration.
- **Convergence Rate:**
  - **Newton-Raphson Method:** Has quadratic convergence ($p = 2$) near the root if the function is sufficiently smooth.
  - **Secant Method:** Has a convergence rate of approximately $p \approx 1.618$ (superlinear but less than quadratic).
- **Computational Costs:**

- **Newton-Raphson Method:** Requires evaluation of both $f(x)$ and $f'(x)$ at each step, which can be computationally expensive if $f'(x)$ is complex.
- **Secant Method:** Only requires evaluation of $f(x)$, saving computational resources when $f'(x)$ is difficult to compute.

Part (a): Differences Between Secant Method and Newton's Method

- **Derivative Requirement:**
  - **Newton's Method:** Requires the calculation of the derivative $f'(x)$ at each iteration.
  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
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- **Initial Guesses:**
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- **Computational Effort:**
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  - **Secant Method:** Only requires evaluation of $f(x)$, saving computational resources when $f'(x)$ is difficult to compute.

Part (b): Order of Convergence of the Secant Method

Objective: Show that the order of convergence p of the secant method is approximately $\phi = \frac{1+\sqrt{5}}{2}$ (the golden ratio).
Proof: Suppose that we are solving the equation $f(x) = 0$ using the secant method. Let the iterations

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, 3, \ldots, \ (1)$$

be successful and approach a solution α, $f(\alpha) = 0$, as $n \to \infty$. We want to find how find it converges. I.e. We want to find the exponent p such that:

$$|x_{n+1} - \alpha| \approx C|x_n - \alpha|^p,$$

Equation (1) expresses x_{n+1} as a function of x_n and x_{n-1} iteratively. Let $x_n = \alpha + \epsilon_n$. Since $x_n \to \alpha$, the sequence of errors ϵ_n approaches 0 as $n \to \infty$. Hence, in terms of α and ϵ_n, the formula becomes

$$\epsilon_{n+1} = \epsilon_n - \frac{f(\alpha + \epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1})} \ (2)$$

Assume that $f(x)$ is a three times differentiable function and $f'(\alpha)$, $f''(\alpha) \neq 0$. The Taylor Expansion of the formula will be:

$$f(\alpha + \epsilon) = f(\alpha) + f'(\alpha)\epsilon + \frac{f''(\alpha)}{2}\epsilon^2 + R_2(\epsilon).$$

NOTE: $f(\alpha) = 0$, ϵ is small, and $R_2(\epsilon)$ is the remainder term.
Since $R_2(\epsilon)$ vanishes at $\epsilon = 0$ at a faster rate than ϵ^2, we neglect the terms of order higher than ϵ^2, we have the approximation:

$$f(\alpha + \epsilon) \approx f'(\alpha)\epsilon + \frac{f''(\alpha)}{2}\epsilon^2.$$

For clarity of the following proof, we let:

$$N = \frac{f''(\alpha)}{2f'(\alpha)},$$

and use the approximate equalities:

$$f(\alpha + \epsilon_n) \approx f'(\alpha)\epsilon_n(1 + N\epsilon_n),$$

$$f(\alpha + \epsilon_n) - f(\alpha + \epsilon_{n-1}) \approx f'(\alpha)(\epsilon_n - \epsilon_{n-1})(1 + N(\epsilon_n + \epsilon_{n-1}))$$

to simplify equation (2):

$$\begin{aligned}
\epsilon_{n+1} &\approx \epsilon_n - \frac{\epsilon_n f'(\alpha)(1 + N\epsilon_n)(\epsilon_n - \epsilon_{n-1})}{f'(\alpha)(\epsilon_n - \epsilon_{n-1})(1 + M(\epsilon_n + \epsilon_{n-1}))} \\
&= \epsilon_n - \frac{\epsilon_n(1 + N\epsilon_n)}{1 + N(\epsilon_n + \epsilon_{n-1})} \\
&= \frac{\epsilon_{n-1}\epsilon_n N}{1 + N(\epsilon_n + \epsilon_{n-1})} \\
&\approx \epsilon_{n-1}\epsilon_n N.
\end{aligned}$$

At this stage, we have obtained a relation for the errors:

$$\epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_n\epsilon_{n-1}, \ (3)$$

where the terms of order higher than ϵ are neglected.
Compare this to the corresponding formula for Newton's method:

$$\epsilon_{n+1} \approx \frac{f''(\alpha)}{2f'(\alpha)}\epsilon_n^2.$$

Formula (3) tells us that, as $n \to \infty$, the error tends to zero faster than linear function and yet not quadratically.

$$\epsilon_{n+1} \approx C|\epsilon_n|^p$$

If $\epsilon_{n+1} \approx C|\epsilon_n|^p$ then

$$C|\epsilon_n|^p \approx |N|\,|\epsilon_n||\epsilon_{n-1}|,$$

$$|\epsilon_n|^{p-1} \approx \frac{|N|}{C}|\epsilon_{n-1}|,$$

$$|\epsilon_n| \approx \left(\frac{|N|}{C}\right)^{\frac{1}{p-1}} |\epsilon_{n-1}|^{\frac{1}{p-1}}.$$

Therefore, $C = \left(\frac{|N|}{C}\right)^{\frac{1}{p-1}}$ and $p = \frac{1}{p-1} \Rightarrow p^2 - p - 1 = 0$. Because $p > 0$, the condition on p gives

$$\Rightarrow p = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Referring back to the relationship we have on C, we can conclude that:

$$C^p = |N| \quad \text{or} \quad C = |N|^{1/p} = \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{p-1}.$$

Lastly, we can conclude that for the secant method (NOTE: $\phi = p$ as shown)

$$|x_{n+1} - \alpha| \approx \left|\frac{f''(\alpha)}{2f'(\alpha)}\right|^{\frac{\sqrt{5}-1}{2}} |x_n - \alpha|^{\phi}.$$

Q.E.D.