MATH 280 HW2

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Question 1

(a)

The degree 5 Taylor series approximation for $\sin x$ around $x = \frac{\pi}{4}$ is:

$$\sin(x) \approx \sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2-\frac{1}{6}\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3+\frac{1}{24}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^4$$

Substituting the values for $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ both as $\frac{\sqrt{2}}{2}$, we get:

$$\sin(x) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4$$

The integral form of the remainder for the Taylor series of $\sin x$ around $x = \frac{\pi}{4}$ and truncated after the fifth degree is:

$$R_5(x) = \int_{\frac{\pi}{4}}^{x} \frac{\sin(t)}{120}(x-t)^5\,dt$$

(b)

$$\sin(0) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(0-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(0-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(0-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(0-\frac{\pi}{4}\right)^4 = 0.001963321$$

$$\sin(\frac{\pi}{2}) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^4 = 0.9984927$$

$$\sin(1) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(1-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(1-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(1-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(1-\frac{\pi}{4}\right)^4 = 0.84146840$$

(c)

$$R_5(0) = \int_{\frac{\pi}{4}}^{0} \frac{\sin(t)}{120}(0-t)^5\,dt = 0.00020235$$

$$R_5(\frac{\pi}{2}) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(t)}{120}(\frac{\pi}{2}-t)^5\,dt = 0.000243632$$

$$R_5(1) = \int_{\frac{\pi}{4}}^{1} \frac{\sin(t)}{120}(1-t)^5\,dt = 9.879 \times 10^{-8}$$

These values indicate extremely small errors, showcasing the high accuracy of the Taylor series approximation when truncated at the fifth degree.

- **At** $x = 0$: The error bound of 0.00020235 indicates that the approximation is nearly exact.
- **At** $x = \frac{\pi}{2}$: An error bound of 0.00024362 suggests that the approximation is extremely close to the actual value 1.
- **At** $x = 1$: The error bound of 9.879×10^{-8} indicates the approximation is indistinguishable from the true value sin(1).

These bounds confirm that the fifth-order Taylor series provides an excellent approximation for $\sin x$ around $x = \frac{\pi}{4}$, particularly near this point and at commonly evaluated points within the function's periodic range.

Question 2

(a)

$$f(X) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given $Y = (1,3)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.

For linear function of the form $f(X) = AX + b$, the Jacobian J_f is simply A, hence:

$$\sum_{f(y)} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & 4.2 \\ 2.2 & 1.6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 30.6 & 13 \\ 13 & 5.4 \end{bmatrix}$$

(b)

$$f(X) = \begin{bmatrix} X_1^2 + 3X_1X_2 - 5 \\ X_2 - X_1 \end{bmatrix}$$

Given $Y = (2, 1)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.
For Jacobian Matrix calculation, here is the requried partial derivatives:

$$\frac{\partial}{\partial X_1}(X_1^2 + 3X_1X_2 - 5) = 2X_1 + 3X_2$$

$$\frac{\partial}{\partial X_2}(X_1^2 + 3X_1X_2 - 5) = 3X_1$$

$$\frac{\partial}{\partial X_1}(X_2 - X_1) = -1$$

$$\frac{\partial}{\partial X_2}(X_2 - X_1) = 1$$

At $Y = (2, 1)$, we have:

$$J_f = \begin{bmatrix} 2(2) + 3(1) & 3(2) \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\sum_{f(y)} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.4 & 8.8 \\ 0.8 & -0.7 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 194.6 & 1.4 \\ 1.4 & -1.5 \end{bmatrix}$$

Question 3

Given, $\sum_{x,y} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$.

(a)

$$f(X, Y) = X + Y$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [1, 1]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_X^2 + \sigma_Y^2$$

(b)

$$f(X,Y) = XY$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [Y, X]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} = Y^2\sigma_X^2 + X^2\sigma_Y^2$$

(c)

$$f(X,Y) = \frac{X}{Y}$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [\frac{1}{Y}, -\frac{X}{Y^2}]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} \frac{1}{Y} & -\frac{X}{Y^2} \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} \frac{1}{Y} \\ -\frac{X}{Y^2} \end{bmatrix} = \frac{\sigma_X^2}{Y^2} + \frac{X^2\sigma_Y^2}{Y^4}$$

Bonus Question

Given the viral equation in ideal gas:

$$P_i V = n_i RT$$

In non-ideal gas case, we might introduce compressibility factors $Z_{i,mix}$ for each gas:

$$P_i V = n_i RT Z_{i,mix}$$

Propose a generalization of the virial equation of state to a two-component mixture

Virial Equation for a Single Component Gas:
The virial equation of state for a single component gas expands the ideal gas law to account for interactions between molecules. It is expressed as:

$$PV = nRT\left(1 + B_2(T)\frac{n}{V} + B_3(T)\left(\frac{n}{V}\right)^2 + \cdots\right)$$

where $B_2(T), B_3(T), \ldots$ are the virial coefficients that depend on temperature and represent interactions among molecules.
Extension to Two-Component Mixtures:
For a mixture of gases, the virial equation can be generalized as:

$$PV = nRT\left(1 + \left(B_{11}\frac{n_1}{V} + B_{22}\frac{n_2}{V} + 2B_{12}\frac{n_1 n_2}{V^2}\right) + \cdots\right)$$

Here, $B_{11}(T)$, $B_{22}(T)$ are virial coefficients correspond to the interactions within the same type of molecules; $B_{12}(T)$ is the virial coefficient that considers cross-interactions between two gas species.
Note that x_1 and x_2 are the mole fractions of gas 1 and gas 2, and $n = n_1 + n_2$ is the total number of moles in the mixture.

Simplification Under Ideal Conditions

If both gases are ideal:

- Interactions between molecules in their own gas species are negligible. Only cross-interactions between different gas species are left.
- This implies $B_{11} = B_{22} = \ldots = 0$.

Following the same definition previously, the virial equation reduces to:

$$PV = nRT\left(1 + 2B_{12}\frac{n_1 n_2}{V^2} + O(n_1, n_2)\right)$$

NOTE: $O(n_1, n_2)$: Higher order polynomial terms.
For simplicity, since higher order terms from the generalized viral equation does not contribute significant impact to the calculation, it is sufficient to consider the equation as:

$$PV \approx nRT\left(1 + 2B_{12}\frac{n_1 n_2}{V^2}\right)$$