Homework 6 (root finding)

Hanzhang Yin

December 15, 2024

1. One view of the secant method: it is a coarser Newton's method. We've seen that it has some of the speed of Newton's method. One might also hope that it enjoys similar convergence properties.

Adapt the convergence proof for Newton's method to show that the secant method also always converges under the following assumptions about the function f on the interval $[a, b]$:

i) f is twice continuously differentiable

ii) $f' > 0$

iii) $f'' > 0$

iv) f has a root x in the interval

v) the two initial guesses x_0, x_1 are both to the right of the root.

Hint: you will have to use convexity in a slightly more interesting way than in NM – the graph of f does not lie above the secant line, but you can argue that the right (well, left!) piece still does.

Solution.

Proof. The secant method iterates according to the formula:

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

We proceed in steps to show the convergence.
Monotonicity and Boundedness
Claim: The sequence $\{x_n\}$ is strictly decreasing and bounded below by x^*.
We will prove the claim by induction

- **Base Case ($n = 1$):** By assumption, $x_0 > x^*$ and $x_1 > x^*$. WLOG, we can reorder x_0 and x_1 such that $x_0 > x_1 > x^*$. Hence, the base case holds.
- **Inductive Step:** Assume $x_{n-2} > x_{n-1} > x^*$. We show that $x_n > x^*$ and $x_n < x_{n-1}$.

  From the secant update:
  $$x_n = x_{n-1} - f(x_{n-1}) \cdot \frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}.$$

  - Since $x_{n-2} > x_{n-1}$, we have $x_{n-1} - x_{n-2} < 0$.
  - Since $f'(x) > 0$ on $[a, b]$, $f(x_{n-1}) > f(x_{n-2})$, so $f(x_{n-1}) - f(x_{n-2}) > 0$.
  - Therefore, the ratio $\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})} < 0$.
  - Since $f(x_{n-1}) > 0$ (as $x_{n-1} > x^*$ and f is increasing), the term subtracted from x_{n-1} is positive:
    $$x_n = x_{n-1} - (\text{Positive Number}) < x_{n-1}$$
    implying $x_n < x_{n-1}$.
  - To show $x_n > x^*$, assume $x_n \le x^*$. Then $f(x_n) \le f(x^*) = 0$, contradicting the fact that $f(x_n) > 0$ for $x_n > x^*$. Thus, $x_n > x^*$.

By induction, $\{x_n\}$ is strictly decreasing and bounded below by x^*.
Convergence of the Sequence
Claim: The sequence $\{x_n\}$ converges to x^*.
Since $\{x_n\}$ is strictly decreasing and bounded below by x^*, it converges to some limit $l \geq x^*$ by **MCT**. Suppose, for contradiction, that $l > x^*$.

- Since f is continuous and strictly increasing:

$$\lim_{n\to\infty} f(x_n) = f(l) > f(x^*) = 0.$$

- Consider the secant update:

$$x_{n+1} = x_n - \frac{f(x_n)}{s_n}, \quad \text{where } s_n = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

- Because f is convex ($f'' > 0$), the slope $s_n > f'(x^*) > 0$, so:

$$\left|\frac{f(x_n)}{s_n}\right| < \frac{f(x_n)}{f'(x^*)}.$$

- As $n \to \infty$, $f(x_n) \to f(l) > 0$, meaning the step sizes $x_n - x_{n+1}$ do not shrink to zero.
- This contradicts the convergence $x_n \to l$, as the step sizes must tend to zero for convergence.

Thus, $l = x^*$, and the sequence converges to x^*.
The convexity of f ensures that the secant line between any two points lies below the graph of f, preventing the iterates x_n from overshooting the root x^*. Thus, $x_n > x^*$ for all n. Under the given assumptions:

1. $\{x_n\}$ is strictly decreasing and bounded below by x^*,
2. By the monotone convergence theorem, $\{x_n\}$ converges to a limit $l \geq x^*$,
3. Assuming $l > x^*$ leads to a contradiction, hence $l = x^*$,
4. Convexity ensures no overshooting, maintaining $x_n > x^*$.

Therefore, the secant method converges to the root x^*. □

2. Another view of the secant method, discussed in class, is as a weighted bisection method. Here too, one might hope for a convergence guarantee, because BM is much more robust than NM in that regard.

Consider a modified secant method which at step k takes in endpoints a_k, b_k, calculates their weighted midpoint c_k and then returns two new endpoints a_{k+1}, b_{k+1}, one of which is c_k, to which IVT applies. These new endpoints are input to the next step.

Prove that if f is continuous on $[a, b] = [a_0, b_0]$ and the IVT applies to f on the interval, then the sequence c_k from the modified secant method converges to a root of f.

Hint: the reason for convergence is *not* the same as for bisection. This would require the stronger assumption that f is continuously differentiable. In fact:
[Bonus] Give an example where the sequences x_k and y_k converge to different points, so squeeze does not apply.

Solution.

Proof. Let f be a continuous function on the interval $[a_0, b_0]$ such that $f(a_0) \cdot f(b_0) < 0$. By the Intermediate Value Theorem (IVT), there exists at least one root x in (a_0, b_0) where $f(x) = 0$.
Modified Secant Method Algorithm:
At each iteration k:

1. **Compute the Weighted Midpoint by Secant Method:**

$$c_k = \frac{b_k f(a_k)}{f(a_k) - f(b_k)} - \frac{a_k f(b_k)}{f(a_k) - f(b_k)}$$

   This point c_k is the root of the secant line connecting $(a_k, f(a_k))$ and $(b_k, f(b_k))$ since IVT applies on each intervals within $[a_0, b_0]$.

2. **Update the Interval:**
   - Determine which subinterval $[a_k, c_k]$ or $[c_k, b_k]$ contains a sign change, i.e., where f changes sign.
   - Set (a_{k+1}, b_{k+1}) to be the endpoints of this subinterval.

Properties of the Sequences $\{a_k\}$ and $\{b_k\}$:
Without loss of generality, assume that c_k has the same sign as a_k. Then, set $a_{k+1} = c_k$ and $b_{k+1} = b_k$ following the bisection method update rule. Consequently, the sequences $\{a_k\}$ and $\{b_k\}$ are monotonic, as each endpoint is either updated to a new point within the interval or remains unchanged at each iteration. Formally speaking:

- **Monotonicity:**
  - $\{a_k\}$ is non-decreasing.
  - $\{b_k\}$ is non-increasing.
- **Boundedness:**
  - $\{a_k\} \subseteq [a_0, b_0]$.
  - $\{b_k\} \subseteq [a_0, b_0]$.
- **Convergence:**
  - Both sequences converge due to monotonicity and boundedness by **MCT**:

$$\lim_{k\to\infty} a_k = a, \quad \lim_{k\to\infty} b_k = b, \quad \text{with} \quad a \leq b.$$

The next step that we want to do is to argue that at least one of the element a or b converges to zero.
Case 1: $a = b$
The interval $[a_k, b_k]$ shrinks to the point $a = b = x$. Since $f(a_k) \cdot f(b_k) < 0$ for all k, and f is continuous, we have:

$$\lim_{k\to\infty} f(a_k) = f(a), \quad \lim_{k\to\infty} f(b_k) = f(b).$$

It must be that $f(a) = 0$, because otherwise $f(a)$ and $f(b)$ would have the same sign, contradicting $f(a_k) \cdot f(b_k) < 0$. Therefore, $c_k \in [a_k, b_k]$ converges to $x = a = b$, which is a root of f.
Case 2: $a < b$
The interval $[a_k, b_k]$ in this case does not shrink to a point.
Suppose for contradiction that neither a and b is a root of the function f. Then pick an arbitrary number k s.t. it is big enough that a_k and b_k are close to the a_∞ and b_∞. Since a and b are not roots, $f(a_k), f(b_k) \neq 0$ so that IVT can be applied. From IVT, we can find a point c_k s.t. $c_k \in [a_k, b_k]$ (i.e. $a_k < c_k < b_k$). From our assumption above, noting that (a_k) and (b_k) are monotonic sequence s.t. for every k, $a_k \leq b_k$. Also noticing that in this case, we assume $a < b$. Hence, we can draw a conclusion that:

$$a_k \approx a < b = b_k, \text{when k} \to \infty$$

Since c_k as the new midpoint is within $[a_n, b_n]$, apprximately $a \leq c_n \leq b$. Also, base on our bisection method update rule c_n is the next term of the sequence $\{a_k\}$ or $\{b_{k+1}\}$.
At this stage, WLOG, we can assume that c_k to be the next further term of sequence $\{a_k\}$, then $a_{k+1} > \lim_{k\to\infty} a_k = a$, we derived a contradiction.
***Case 3:**
Since *case 2* is not sound, suppose *case 1* is not the case, then surely one of a and b converges to the root (denote it as x). Lets assume $a = x$, WLOG, then, for $k \to \infty$, $f(a_k) \approx 0$. Then then next point of c_k, which is c_{k+1} can be represented as:

$$c_k = \frac{b_k f(a_k)}{f(a_k) - f(b_k)} - \frac{a_k f(b_k)}{f(a_k) - f(b_k)} = \frac{0}{0 - f(b_k)} - \frac{f(b_k)}{0 - f(b_k)} a_k \approx a_k$$

Then the "next term" is approximately a_k. By knowing that,

$$c_k = a_{k+1} \; or \; c_k = b_{k+1}$$

But we know from our assumption that,

$$c_k \approx a_{k+1} < a < b < b_{k+1}$$

so $c_k \neq b_{k+1}$. Then applying limit on both side for equation $c_k = a_{k+1}$, then:

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} a_{k+1} \Rightarrow c = a$$

By continuously updating the midpoint, root $x = a$ ultimately. □

3. Suppose $f(x)$ and $g(x)$ are functions with a common root $x = a$.

a) Prove that a solution to the homotopy continuation initial value problem

$$x'(t) = -\frac{H_t}{H_x} \qquad x(0) = a$$

is the constant function $x = a$.

b) Give an example where the solution above is *not* unique.

Hint: see handout for a picture of (a). Think about how it could be adapted (b); you can even use the tool to help you construct an example.

Solution.

Proof. **Part (a): Proving $x(t) = a$ is a Solution**
Let us define the homotopy $H(x,t)$ as

$$H(x,t) = (1-t)f(x) + tg(x).$$

Since $f(a) = g(a) = 0$, it follows that $H(a,t) = 0$ for all $t \in [0,1]$.
We need to show that $x(t) = a$ satisfies the differential equation

$$x'(t) = -\frac{H_t}{H_x}, \quad x(0) = a.$$

Computing the Partial Derivatives:
First, compute H_t and H_x:

$$H_t(x,t) = -f(x) + g(x),$$
$$H_x(x,t) = (1-t)f'(x) + tg'(x).$$

Evaluate these at $x = a$:

$$H_t(a,t) = -f(a) + g(a) = -0 + 0 = 0.$$
$$H_x(a,t) = (1-t)f'(a) + tg'(a).$$

Note that $H_x(a,t)$ may not be zero unless both $f'(a)$ and $g'(a)$ are zero.
Computing $x'(t)$ at $x = a$:
Substitute $x(t) = a$ into the differential equation:

$$x'(t) = -\frac{H_t(a,t)}{H_x(a,t)} = -\frac{0}{H_x(a,t)} = 0.$$

Therefore,

$$x'(t) = 0, \quad x(0) = a.$$

This implies that $x(t) = a$ for all $t \in [0,1]$.
Conclusion:
The constant function $x(t) = a$ is a solution to the homotopy continuation initial value problem. □

Proof. **Part (b): Example Where the Solution is Not Unique**
We will construct specific functions $f(x)$ and $g(x)$ with a common root at $x = a$ such that the initial value problem

$$x'(t) = -\frac{H_t}{H_x}, \quad x(0) = a,$$

has multiple solutions.
Example Functions:
Let

$$f(x) = (x-a)^{1/3}, \quad g(x) = -(x-a)^{1/3}.$$

Both functions have a root at $x = a$:

$$f(a) = g(a) = 0.$$

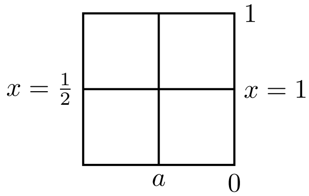
Constructing the Homotopy:
Define

$$H(x,t) = (1-t)f(x) + tg(x) = (1-t)(x-a)^{1/3} + t(-(x-a)^{1/3}) = (1-2t)(x-a)^{1/3}.$$

Compute H_t and H_x:

$$H_t(x,t) = -f(x) + g(x) = -(x-a)^{1/3} - (x-a)^{1/3} = -2(x-a)^{1/3},$$

$$H_x(x,t) = (1-2t)\cdot\frac{1}{3}(x-a)^{-2/3}.$$



When t shifts from $0 \to 1$, at a certain point for t the homotopy will coincide with the x-axis completely. This leads a shift from the solution $x = a$ when $t = 0$ to the solution x equals to every possible points on the x-axis between $x = \frac{1}{2}$ to $x = 1$. This clearly makes the solution **NOT Unique** during the approximation. $\square$