Project 2 Proposal

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In this project, we will test our algorithms using examples from physics and mathematics, including univariate polynomial equations, a system of polynomial equations, and a non-polynomial equation for testing Newton's, Bisection and the secant methods for root apprximations. Each example includes a brief explanation of its scientific background and the potential challenges it presents for numerical methods.

Univariate Polynomial Equations

Polynomial 1: Lagrange Points in Orbital Mechanics
In celestial mechanics, Lagrange points are positions where a small body can maintain its position relative to two larger bodies due to gravitational forces. The L_1 Lagrange point is found by solving a quintic polynomial:

$$x^5 - (3+\mu)x^4 + (3+2\mu)x^3 - \mu x^2 + 2\mu x - \mu = 0,$$

where x is the normalized distance and μ is the mass ratio of the two bodies. This example is chosen for its complexity and the potential for numerical issues when μ approaches zero, leading to multiple or closely spaced roots.
Degeneration and Homotopy: As $\mu \to 0$, the polynomial simplifies, creating closely spaced or multiple roots, which can be challenging for numerical solvers. We can define a homotopy $H(x,t)$ to connect the degenerate case back to the original polynomial:

$$H(x,t) = x^5 - (3+t\mu)x^4 + (3+2t\mu)x^3 - t\mu x^2 + 2t\mu x - t\mu = 0,$$

where $t \in [0,1]$. When $t = 0$, we have the degenerate form, and when $t = 1$, we recover the original polynomial.
Polynomial 2: Ideal Gas Law For Real Gas
The modified ideal gas law for real gases can be expressed as:

$$PV^3 - (Pb + RT)V^2 + aV - ab = 0,$$

where P is pressure, T is temperature, R is the gas constant, and a, b are gas-specific constants. This cubic polynomial can exhibit different root structures under varying conditions, such as near the critical point where multiple real roots merge.
Degeneration and Homotopy: At the critical point, the cubic polynomial degenerates as multiple real roots converge. To connect the degenerate form to the original, we define a homotopy:

$$H(V,t) = PV^3 - (Pb + tRT)V^2 + aV - ab = 0,$$

where $t \in [0,1]$. When $t = 0$, we approach the critical condition, and when $t = 1$, we revert to the general polynomial form under typical conditions.

System of Polynomial Equations

Example: Coupled Spring-Mass System
In a physics-based spring-mass system with two masses connected by springs, the equations of motion can be represented as:

$$k_1 x_1 - k_2(x_2 - x_1) = m_1 a_1,$$
$$k_2(x_2 - x_1) + k_3 x_2 = m_2 a_2,$$

where k_1, k_2, and k_3 are spring constants, x_1 and x_2 are displacements, and a_1 and a_2 are accelerations. This example is straightforward and allow testing for potential singular behaviors when k_2 is small or approaches zero.

Non-Polynomial Equation

Example: Sinusoidal Motion in Spring Displacement
The motion of a mass-spring system can be described by:

$$x(t) = 5\sin(2t),$$

or for finding specific points ($x = 3$ in this case):

$$5\sin(2t) - 3 = 0.$$

This non-polynomial equation will be used to test Newton's method and the secant method. Solving for t involves periodic behavior, which can pose challenges for convergence due to the trigonometric nature of the function and the need for good initial guesses.

Anticipated Challenges and Degeneracies

The Lagrange polynomial and the modified ideal gas law equation present cases with potential singularities and closely spaced roots, which can slow down the convergence. The non-polynomial trigonometric equation may experience convergence issues if the initial guesses are far from the actual roots due to the oscillatory behavior (For Newton's Method Especially).