Homework 1 (small calculus review)

Reminder: we define the derivative of $f(x)$ at $x = a$ to be the slope of the best linear approximation to $f(x)$ at $x = a$, which means that $f(x)$ is differentiable at $x = a$ with derivative $f'(a)$ if, when we write,

$$f(x) = f(a) + f'(a)(x - a) + E(x)$$

the function $E(x)$ is continuous and "much smaller than linear" near $x = a$, meaning that

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0$$

Our notation $E(x) << x - a$ at $x = a$ means exactly the same thing as $E(x) = \mathcal{o}(x - a)$ as $x \to a$ in the book's notation (Section 1.2). The $<<$ notation makes it a little easier to remember which quantity is smaller/bigger, and is more visibly asymmetric. (The other important asymptotic notation we will use is big-O, written $\mathcal{O}$, which *is* symmetric, unlike its nearly-identical counterpart, $\mathcal{o}$).

1. Verify the quotient rule: if f, g are differentiable at $x = a$ and $g(a) \neq 0$, then f/g is differentiable at $x = a$ and its derivative is

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Hint 1: once you calculate the derivative of $\frac{1}{g}$, you can use the product rule to finish.

Hint 2: to find the derivative of $\frac{1}{g}$, consider expanding the identity $g\frac{1}{g} = 1$ in terms of BLA's and solve for the slope. Alternatively, find the derivative of $\frac{1}{x}$ and use the composition rule.

2. Suppose that we know $f(x) = 2 - (x - 3) + E(x)$ near $x = 3$, where $2 - (x - 3)$ is the BLA and that $|E(x)| < 1$ on the interval $(2, 4)$.

a) Let $g(x) = x^3 - 2x^2 + 3$. Determine the BLA for $g(f(x))$ at $x = 3$. Can you determine a bound for the error term associated to the BLA of $f \circ g$ on $(2, 4)$? If so, provide it; if not, what information would you need to be able to do so?

b) Suppose we have a function $h(x)$ differentiable at $x = 0$, and such that $h(0) = 3$. Can you bound the error of $f(h(x))$ near zero? If so, provide a bound; if not, what information would you need to be able to do so?

3. Let $P(t)$ be a function defined almost everywhere which is known to have the following properties:

$$P(1) = 4 \qquad P'(t) = tP(t)^2$$

a) Using a best quadratic approximation at $t = 1$, estimate $P(0)$ and $P(-1)$.

b) Determine $P''(t)$. Assuming the function is defined at $t = 0$, do you think the critical point at $t = 0$ is a minimum, a maximum, or neither? Explain. Hint: squares are always positive.

c) Looking at your answers to the previous parts, do you think your approximations are accurate? Do you think you have over- or under-estimated the true values? Briefly explain your reasoning (you do not need to provide a proof).