Homework 4 (interpolation)

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1. Let M be a Markov matrix (sum of entries in a column is 1) with all diagonal entries nonzero. Show that the only possible eigenvalue with norm 1 is 1, and that any other eigenvalue has strictly smaller norm. Hint: apply the gershgorin circle theorem to M^T.

Solution.

Proof. *NOTE: $\mathbb{C}$: Complex Number, and $Re()$: Real Number part.
Let M be an $n \times n$ Markov matrix with all diagonal entries $M_{ii} > 0$. Since M is column stochastic, its transpose M^T is row stochastic, i.e., $\sum_{j=1}^{n} M_{ij}^T = 1$ for all i. By the Gershgorin Circle Theorem, every eigenvalue λ of M^T (and hence of M) lies within at least one disc D_i centered at M_{ii} with radius $R_i = 1 - M_{ii}$:

$$D_i = \{\lambda \in \mathbb{C} : |\lambda - M_{ii}| \leq 1 - M_{ii}\}.$$

Suppose λ is an eigenvalue with $|\lambda| = 1$. Then,

$$|\lambda - M_{ii}| \leq 1 - M_{ii}.$$

But since $|\lambda| = 1$ and $M_{ii} > 0$,

$$|\lambda - M_{ii}| \geq |\lambda| - M_{ii} = 1 - M_{ii}.$$

Thus,

$$|\lambda - M_{ii}| = 1 - M_{ii},$$

which means λ lies on the boundary of D_i. Expanding,

$$\begin{aligned}
|\lambda - M_{ii}|^2 &= (1 - M_{ii})^2, \\
|\lambda|^2 - 2M_{ii}\,\mathrm{Re}(\lambda) + M_{ii}^2 &= 1 - 2M_{ii} + M_{ii}^2, \\
1 - 2M_{ii}\,\mathrm{Re}(\lambda) + M_{ii}^2 &= 1 - 2M_{ii} + M_{ii}^2.
\end{aligned}$$

Subtracting $1 - 2M_{ii} + M_{ii}^2$ from both sides yields:

$$-2M_{ii}\,\mathrm{Re}(\lambda) = -2M_{ii}.$$

Since $M_{ii} > 0$, dividing both sides by $-2M_{ii}$ gives:

$$\mathrm{Re}(\lambda) = 1.$$

With $|\lambda| = 1$ and $\mathrm{Re}(\lambda) = 1$, it follows that $\lambda = 1$. Therefore, the only eigenvalue of M with modulus 1 is 1, and all other eigenvalues satisfy $|\lambda| < 1$. □

2. [Book 6.4.14] Determine whether the following is a natural cubic spline:

$$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & x \in [-1,0] \\ 3 + 5x + 3x^2 & x \in [0,1] \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 & x \in [1,2] \end{cases}$$

Solution.

Proof. To determine whether $f(x)$ is a natural cubic spline on $[-1,2]$, we need to check the following criteria:

1. $f(x)$ must be twice continuously differentiable on $[-1,2]$.
2. The second derivatives at the endpoints must be zero: $f''(-1) = f''(2) = 0$ (natural boundary conditions).

To start with, we simplify each piece of $f(x)$:
1. For $x \in [-1,0]$:

$$f(x) = 2(x+1) + (x+1)^3$$

2. For $x \in [0,1]$:

$$f(x) = 3 + 5x + 3x^2.$$

3. For $x \in [1,2]$:

$$\begin{aligned} f(x) &= 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3 \\ &= 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3. \end{aligned}$$

First, check continuity at the knots $x = 0$ and $x = 1$:
- At $x = 0$:

$$\lim_{x \to 0^-} f(x) = 2(0+1) + (0+1)^3 = 2 + 1 = 3,$$

$$\lim_{x \to 0^+} f(x) = 3 + 5(0) + 3(0)^2 = 3.$$

So, $f(x)$ is continuous at $x = 0$.
- At $x = 1$:

$$\lim_{x \to 1^-} f(x) = 3 + 5(1) + 3(1)^2 = 11,$$

$$\lim_{x \to 1^+} f(x) = 11 + 11(1-1) + 3(1-1)^2 - (1-1)^3 = 11.$$

So, $f(x)$ is continuous at $x = 1$.
Now, Compute the first derivative in each interval:
For $x \in [-1,0]$:

$$f'(x) = 3 \cdot 3(x+1)^2 = 3(x+1)^2 + 2$$

2. For $x \in [0,1]$:

$$f'(x) = 5 + 6x.$$

3. For $x \in [1,2]$:

$$f'(x) = 11 + 6(x-1) - 3(x-1)^2.$$

Then, check the continuity of the first derivative at $x = 10$:
- From the left:

$$\lim_{x \to 0^-} f'(x) = 3(0+1)^2 + 2 = 5$$

- From the right:

$$\lim_{x \to 0^+} f'(x) = 5 + 6(0) = 5.$$

Lastly, check the continuity of the first derivative at $x = 1$:
- From the left:

$$\lim_{x \to 1^-} f'(x) = 5 + 6(1) = 11.$$

- From the right:

$$\lim_{x \to 1^+} f'(x) = 11 + 6(1 - 1) - 3(x - 1)^2 = 11$$

By cubic spline's definition, the function is twice continuously differentiable on the interval, statisfying natural boundart conditions. Therefore, $f(x)$ is a cubic spline. □

3. [Book 6.4.25] Determine coefficients a, b, c, d, which make the following a cubic spline:

$$f(x) = \begin{cases} x^3 & -1 \le x \le 0 \\ a + bx + cx^2 + dx^3 & 0 \le x \le 1 \end{cases}$$

Solution.

Proof. Define $f(x)$ and its derivatives:
For $x \in [-1, 0]$:

$$f(x) = x^3, f^{'}(x) = 3x^2, f^{''}(x) = 6x$$

For $x \in [0, 1]$:

$$f(x) = a + bx + cx^2 + dx^3, f^{'}(x) = b + 2cx + 3dx^2, f^{''}(x) = 2c + 6dx$$

Continuity at $x = 0$:
By definition, function countinuity means: $f(0^-) = f(0^+)$:
Calculate each side:

$$f(0^-) = (0)^3 = 0, f(0^+) = a + b(0) + c(0)^2 + d(0)^3 = a$$

Set them equal so that $a = 0$.
By definition, first derivative countinuity means: $f'(0^-) = f'(0^+)$:
Calculate each side:

$$f(0^-) = 3(0)^2 = 0, f(0^+) = b + 2c(0) + 3d(0)^2 = b$$

Set them equal so that $b = 0$.
By definition, second derivative countinuity countinuity means: $f''(0^-) = f''(0^+)$:
Calculate each side:

$$f(0^-) = 6(0)^3 = 0, f(0^+) = 2c + 6d(0) = 2c$$

Set them equal so that $2c = 0 \Rightarrow c = 0$.
Now we need to determine d using the spline's definition:
Since $a = b = c = 0$, the functiona for $x \in [0, 1]$ simplifies to:

$$f(x) = dx^3$$

Eventhough the function is not defined beyond $x = 1$, we want to ensure the Smoothness of the spline. In general, d can be any real number, since the question is lack of boundness conditions.
Assuming we want $f(x)$ to be continuous at $x = 1$, and since $f(x) = x^3$ on $[-1, 0]$, it is reasonable externing this to $[0, 1]$ by letting $d = 1$.
Therefore, the coefficients are:

$$a = 0, b = 0, c = 0, d = 1$$

This makes $f(x) = x^3$ on both intervals, ensuring that the function and its derivatives are continuous across the entire domain $[-1, 1]$. Overall, the answer will be:

$$f(x) = \begin{cases} x^3 & -1 \le x \le 0 \\ x^3 & 0 \le x \le 1 \end{cases}$$

□

4. Let $f(x) = \arctan(x)$

a) Suppose you interpolated $f(x)$ by a degree 3 polynomial using the Chebyshev nodes as x values [you do not need to calculate the interpolating polynomial]. Estimate the error associated to this interpolation.

b) Using a taylor series around 0, write down a degree 5 approximation to $f(x)$.

c) With Taylor's form of the remainder, estimate the error associated to the interpolation in (b). (you may use a computer to calculate the 6th derivative, but you must bound it on your own, explaining your work carefully)

d) Compare your error estimates (a) and (c). Which seems better, and why do you think this might be the case? Hint: taylor series are a little like interpolating just at a single point, using derivatives at just that point to provide extra constraints.

Solution.
(a):

Proof. By definition, for poly. interpolation, the error at a point x is given by:

$$|f(x) - P_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i) \right|$$

Using the Chebyshev nodes, for $n = 3$ (degree of 3 polynomial), the Chebyshev nodes on the interval $[-1, 1]$ are:

$$x_k = \cos\left(\frac{2k+1}{2(n+1)\pi}\right), \ k = 0, 1, 2, 3$$

Computerd nodes:

$$x_0 = \cos(\frac{\pi}{8}) \approx 0.924$$

$$x_1 = \cos(\frac{3\pi}{8}) \approx 0.383$$

$$x_2 = \cos(\frac{5\pi}{8}) \approx -0.383$$

$$x_2 = \cos(\frac{7\pi}{8}) \approx -0.924$$

For Chebyshev nodes on $[-1, 1]$, the term $\prod_{i=0}^{n} |x - x_i|$ is bounded by:

$$\prod_{i=0}^{n} |x - x_i| \leq \frac{1}{2^n}$$

Now we need to find an upper bound M for $|f^{(4)}(x)|$ on $[-1, 1]$.
Compute $f^{(4)}(x)$:
First derivative:

$$f^{(1)}(x) = \frac{1}{1 - x^2}$$

Second derivative:

$$f^{(2)}(x) = \frac{d}{dx}(\frac{1}{1 - x^2}) = -\frac{2x}{(1 + x^2)^2}$$

Third derivative:

$$f^{(3)}(x) = \frac{d}{dx}(-\frac{2x}{(1 + x^2)^2}) = -\frac{2(1 + x^2)^2 - 8x^2(1 + x^2)}{(1 + x^2)^4} = -\frac{2(1 - 3x^2)}{(1 + x^2)^3}$$

Fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx}(-\frac{2(1-3x^2)}{(1+x^2)^3}) = \frac{2(6x(1+x^2)^3 - 3(1-3x^2)(3)(1+x^2)^2(2x))}{(1+x^2)^6} = -\frac{24x(x^2-1)}{(1+x^2)^4}$$

Thus, the absolute value of the fourth derivative is:

$$|f^{(4)}(x)| = \left| -\frac{24x(x^2-1)}{(1+x^2)^4} \right|$$

To find an upper bound M for $|f^{(4)}(x)|$ on $[-1,1]$, we analyze the numerator and the denominator separately:

- **Numerator Bounding Analysis:**
  - For $x \in [-1,1]$, $|x| \leq 1$.
  - Also, $|x^2-1| \leq 1$, because $x^2 \leq 1$, which implies $|x^2-1| \leq 1$.
- **Denominator Bounding Analysis:**
  - For $x \in [-1,1]$, $1+x^2 \geq 1$.
  - Therefore, $(1+x^2)^4 \geq 1^4 = 1$.

Combining these results, we have:

$$|f^{(4)}(x)| = \left| \frac{24x(x^2-1)}{(1+x^2)^4} \right| \leq \frac{24 \cdot 1 \cdot 1}{1} = 24$$

Thus, the maximum value of $|f^{(4)}(x)|$ on $[-1,1]$ is:

$$M = 24.$$

Lastly, we apply the error formula:

$$|f(x) - P_3(x)| \leq \frac{M}{4!} \cdot \prod_{k=0}^{3} |x - x_k| \leq \frac{24}{24} \cdot \frac{1}{8} = \frac{1}{8} = 0.125$$

The maximum interpolation error when approximating $f(x) = arctan(x)$ on $[-1,1]$ using a degree 3 polynomial with Chebyshev nodes is bounded by 0.125. □

(b):

Proof. Recall that Taylor Series expansion of $arctan(x)$ around $x = 0$ is:

$$arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1},\ |x| \leq 1$$

For Degree of 5 polynomial approximation, we can have:

$$f(x) \approx x - \frac{x^3}{3} + \frac{x^5}{5} + O(x^6)$$

NOTE: $O(x^6)$ is the remaining higher order term of the expansion that can be ignored in our calculation. □

(c):

Proof. By Taylor Remainder Theorem, the remainder $R_5(x)$ for a degree 5 Taylor polynomial is:

$$R_5 = \frac{f^{(6)}(\xi)}{6!}\ x^6$$

By calculator, $f^{(6)}(x)$ is:

$$f^{(6)}(x) = -\frac{240x(3x^4 - 10x^2 + 3)}{(x^2+1)^6}$$

To apply the theorem, we need to find an upper bound M $|f^{(6)}(x)|$ on $[-1, 1]$.

$$|f^{(6)}(x)| = \left|-\frac{240x(3x^4 - 10x^2 + 3)}{(x^2+1)^6}\right| = \frac{240|x||3x^4 - 10x^2 + 3|}{(x^2+1)^6}$$

Bounding $|x|$:

$$|x| \leq 1 \ for \ x \in [-1, 1]$$

Bounding $|3x^4 - 10x^2 + 3|$:
Let $g(x) = 3x^4 - 10x^2 + 3$. To find the maximum of $|g(x)|$ on $[-1, 1]$, we analyze its critical points and endpoints.
1. Find critical points:

$$g'(x) = 12x^3 - 20x = 4x(3x^2 - 5) = 0$$

$$\Rightarrow x = 0 \ or \ x = \pm\sqrt{\frac{5}{3}} \approx \pm 1.291$$

Only $x = 0$ lies within $[-1, 1]$.
2. Evaluate $g(x)$ at critical and end points:

$$g(0) = 3(0)^4 - 10(0)^2 + 3 = 3$$

$$g(1) = 3(1)^4 - 10(1)^2 + 3 = 3 - 10 + 3 = -4 \Rightarrow |g(1)| = 4$$

$$g(-1) = 3(-1)^4 - 10(-1)^2 + 3 = 3 - 10 + 3 = -4 \Rightarrow |g(1)| = 4$$

Thus, $|g(x)| \leq 4 \ for \ x \in [-1, 1]$
Bounding the denominator $(x^2 + 1)^6$: On $[-1, 1]$:

$$1 \leq x^2 + 1 \leq 2 \quad \Rightarrow \quad 1^6 \leq (x^2+1)^6 \leq 2^6 = 64$$

Lastly, we combine the bounds:

$$|f^{(6)}(x)| = \frac{240|x||3x^4 - 10x^2 + 3|}{(x^2+1)^6} \leq \frac{240 \times 1 \times 4}{1} = 960 \quad \text{at } x = \pm 1$$

However, evaluating at $x = \pm 1$:

$$|f^{(6)}(\pm 1)| = \frac{240 \times 1 \times 4}{(1+1)^6} = \frac{960}{64} = 15$$

Since $|f^{(6)}(x)|$ attains its maximum at $x = \pm 1$, we set:

$$M = 15$$

Substituting $M = 15$ and $6! = 720$ into the remainder formula:

$$|R_5(x)| \leq \frac{15}{720}|x|^6 = \frac{1}{48}|x|^6 \approx 0.0208|x|^6$$

Since $|x| \leq 1$ on $[-1, 1]$:

$$|R_5(x)| \leq \frac{1}{48} \approx 0.0208$$

$\square$

(d):

In comparing the error estimates from parts (a) and (c), the Taylor series approximation (part c) yields a significantly smaller error bound of $|R_5(x)| \le 0.0208$ over the interval $[-1, 1]$, compared to the Chebyshev interpolation (part a) which has an error bound of $|f(x) - P_3(x)| \le 0.125$. This higher accuracy of the Taylor approximation arises because it utilizes a higher-degree polynomial (degree 5 versus degree 3) and incorporates derivative information at a extra critica point ($x = 0$), allowing for a more precise local fit. In contrast, Chebyshev interpolation distributes interpolation nodes across the entire interval to minimize the maximum error uniformly but does not use information from function's derivative, resulting a larger overall error bound. Therefore, the Taylor series provides a better error estimate in this case due to its enhanced local accuracy near the expansion point.

5. Determine a quadratic spline approximation $S(x)$ to $f(x) = \arctan(x)$ with nodes $-1, 0, 1$.

Solution.

Proof. First we need to define $S(x)$ as a peicewise quadratic function:

$$S(x) \begin{cases} S_1(x) = a_1x^2 + b_1x + c_1, \ for \ x \in [-1, 0], \\ S_2(x) = a_2x^2 + b_2x + c_2, \ for \ x \in [0, 1], \end{cases}$$

Then we apply the interpolation conditions, computing the function vlaues at the nodes:

$$f(-1) = arctan(-1) = -\frac{\pi}{4}$$

$$f(0) = arctan(0) = 0$$

$$f(1) = arctan(1) = \frac{\pi}{4}$$

interpolation at $x = -1$:

$$S_1(-1) = a_1(-1)^2 + b_1(-1) + c_1 = -\frac{\pi}{4}$$

interpolation at $x = 0$: For both $S_1(x)$ and $S_2(x)$:

$$S_1(0) = c_1 = 0, \ S_2(0) = c_2 = 0$$

interpolation at $x = 1$:

$$S_2(-1) = a_2(1)^2 + b_2(1) + c_2 = \frac{\pi}{4}$$

To solve for the needed coefficients, we first apply the continuity conditions at $x = 0$,
Continuity of the function at $x = 0$:

$$S_1(0) = S_2(0) \Longrightarrow c_1 = c_2 = 0$$

Continuity of the First Derivative at $x = 0$:

$$S_1'(x) = 2a_1x + b_1$$

$$S_2'(x) = 2a_2x + b_2$$

At $x = 0$:

$$S_1'(x) = b_1, \ S_2'(x) = b_2$$

Set them equal:

$$b_1 = b_2 = b$$

After that, we apply second continuity conditions at endpoints,
Compute the function's second derivatives at the endpoints:

$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = -\frac{2x}{(1+x^2)^2}$$

At $x = -1$:

$$f''(-1) = -\frac{2(-1)}{(1+(-1)^2)^2} = \frac{2}{4} = \frac{1}{2}$$

Set:

$$S_1''(x) = 2a_1 = f''(-1) \Longrightarrow a_1 = \frac{1}{4}$$

At $x = 1$:

$$f''(1) = -\frac{2(1)}{(1+1^2)^2} = -\frac{2}{4} = -\frac{1}{2}$$

Set:

$$S_2''(x) = 2a_2 = f''(1) \Longrightarrow a_2 = -\frac{1}{4}$$

Now we can solve for the remaining coefficients,
Equation from $x = -1$:

$$S_1(-1) = a_1(-1)^2 + b_1(-1) + c_1 = \frac{1}{4}(1) - b_1 + 0 = -\frac{\pi}{4}$$

$$\Rightarrow \frac{1}{4} - b_1 = -\frac{\pi}{4} \Longrightarrow b_1 = \frac{1}{4} + \frac{\pi}{4} = \frac{1+\pi}{4}$$

Equation from $x = 1$:

$$S_2(-1) = a_2(-1)^2 + b_2(-1) + c_2 = \frac{1}{4}(1) - b_2 + 0 = \frac{\pi}{4}$$

$$\Rightarrow -\frac{1}{4} - b_2 = \frac{\pi}{4} \Longrightarrow b_2 = \frac{1}{4} + \frac{\pi}{4} = \frac{1+\pi}{4}$$

Since $b_1 = b_2 = b$, this is consistent.
Now, we can Substituting the coefficients into $S_1(x)$ and $S_2(x)$: For $x \in [-1, 0]$:

$$S_1(x) = a_1x^2 + bx + c_1 = \frac{1}{4}x^2 + \frac{1+\pi}{4}x + 0$$

$$S_2(x) = a_2x^2 + bx + c_2 = -\frac{1}{4}x^2 + \frac{1+\pi}{4}x + 0$$

Lastly, to ensure the validity of the approximation, we need to do some verification:
At $x = -1$:

$$S_1(-1) = \frac{1}{4}(-1)^2 + \frac{1+\pi}{4}(-1) = \frac{1}{4} - \frac{1+\pi}{4} = -\frac{\pi}{4}$$

which matches $f(-1) = -\frac{\pi}{4}$
At $x = 0$:

$$S_1(0) = S_2(0) = 0$$

Matches $f(0) = 0$
At $x = 1$:

$$S_2(1) = -\frac{1}{4}(1)^2 + \frac{1+\pi}{4}(1) = -\frac{1}{4} + \frac{1+\pi}{4} = \frac{\pi}{4}$$

Matches $f(1) = \frac{\pi}{4}$
And for checking the first derivative continuity at $x = 0$;:

$$S_1'(x) = 2a_1x + b = \frac{1}{2}x + \frac{1+\pi}{4}, \; S_1'(0) = \frac{1+\pi}{4}$$

$$S_2'(x) = 2a_2x + b = -\frac{1}{2}x + \frac{1+\pi}{4}, \; S_2'(0) = \frac{1+\pi}{4}$$

$S_1'(x) = S_2'(x)$, which ensures the validity.
The quadratic spline approximation $S(x)$ is:

$$S(x) = \begin{cases} \frac{1}{4}x^2 + \frac{1+\pi}{4}x, & \text{for } x \in [-1, 0], \\ -\frac{1}{4}x^2 + \frac{1+\pi}{4}x, & \text{for } x \in [0, 1]. \end{cases}$$

$\square$

6. Let $f(x) = 4x^2 - 4^x$.

1. Using the intermediate value theorem, show that $f(x)$ has at least one root in $[-1, 0]$ and another in $[0, 1.5]$.
2. Interpolate $f(x)$ by a degree 3 polynomial using nodes $x = -1/2, 0, 1/2$.
3. Use the interpolation to estimate the roots of $f(x)$ in those intervals.

Solution.
(1):

Proof. For interval $[-1, 0]$, compute $f(-1)$ and $f(0)$.
At $x = -1$:

$$f(-1) = 4(-1)^2 - 4^{-1} = 4(1) - \frac{1}{4} = 4 - \frac{1}{4} = \frac{15}{4} > 0$$

At $x = 0$:

$$f(0) = 4(0)^2 - 4^0 = 0 - 1 = -1 < 0$$

Since $f(-1) > 0$ and $f(0) < 0$, and $f(x)$ is continuous on $[-1, 0]$, by IVT, there is at least one root in $[-1, 0]$.
For interval $[0, 1.5]$, compute $f(0)$ and $f(1.5)$.
At $x = 0$:

$$f(0) = -1 < 0$$

At $x = 1.5$:

$$f(1.5) = 4(1.5)^2 - 4^{1.5} = 4(2.25) - 4^{1.5} = 9 - 4^{1.5} = 9 - 8 = 1 > 0$$

Since $f(0) < 0$ and $f(1.5) > 0$, and $f(x)$ is continuous on $[0, 1.5]$, by IVT, there is at least one root in $[0, 1.5]$. $\square$

(2):

Proof. As an additional condition for interpolation, we use the derivative at $x = 0$:

$$f'(0) = 8x - 4^x \ln(4) = -\ln 4$$

Now, we need to compute $f(x)$ at the nodes:

$$f\left(-\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$f(0) = -1$$

$$f\left(\frac{1}{2}\right) = 1 - 2 = -1$$

After that, we can set up the interpolation conditions:

$$P\left(-\frac{1}{2}\right) = \frac{1}{2}, \; P(0) = -1$$

$$P\left(\frac{1}{2}\right) = -1, P'(0) = -\ln 4$$

Now we can write the equations to solve for coefficients

$$-\frac{a}{8} + \frac{b}{4} - \frac{c}{2} + d = \frac{1}{2}$$

.

$$\frac{a}{8} + \frac{b}{4} + \frac{c}{2} - 1 = -1 \Rightarrow \frac{a}{8} + \frac{b}{4} + \frac{c}{2} = 0$$

$$d = -1, c = -\ln 4$$

Simplify equations:
Equation (1):

$$-\frac{a}{8} + \frac{b}{4} + \frac{\ln 4}{2} - 1 = \frac{1}{2}.$$

Multiply both sides by 8:

$$-a + 2b + 4\ln 4 - 8 = 4.$$

Simplify:

$$-a + 2b + 4\ln 4 = 12. \quad (1a)$$

Equation (3):

$$\frac{a}{8} + \frac{b}{4} - \frac{\ln 4}{2} = 0.$$

Multiply both sides by 8:

$$a + 2b - 4\ln 4 = 0. \quad (3a)$$

Then, we can solve for a and b:
Add equations (1a) and (3a):

$$(-a + a) + (2b + 2b) + (4\ln 4 - 4\ln 4) = 12 + 0 \implies 4b = 12 \implies b = 3.$$

Substitute $b = 3$ into (3a):

$$a + 2(3) - 4\ln 4 = 0 \implies a = 4\ln 4 - 6.$$

Since $\ln 4 = 2\ln 2$, we have:

$$a = 8\ln 2 - 6, \quad c = -2\ln 2, \quad d = -1.$$

Lastly, the Final Interpolating Polynomial will be

$$P(x) = (8\ln 2 - 6)x^3 + 3x^2 - 2\ln 2\, x - 1.$$

□

(3):
From Part 2, the interpolating polynomial is:

$$P(x) = (8\ln 2 - 6)x^3 + 3x^2 - 2\ln 2\, x - 1.$$

Using $\ln 2 \approx 0.6931$, the polynomial becomes:

$$P(x) = -0.4552\, x^3 + 3x^2 - 1.3862\, x - 1.$$

Estimating the Root in $[-1, 0]$
Evaluate $P(x)$ at the endpoints:

$$\begin{cases} P(-1) = -0.4552(-1)^3 + 3(-1)^2 - 1.3862(-1) - 1 \approx 3.8414 > 0, \\ P(0) = -1 < 0. \end{cases}$$

Since $P(-1) > 0$ and $P(0) < 0$, by the Intermediate Value Theorem, there is a root x_1 in $[-1, 0]$. Using the Bisection Method:

1. First midpoint: $x_{\text{mid}} = -0.5 \Rightarrow P(-0.5) \approx 0.5 > 0 \Rightarrow$ root is in $[-0.5, 0]$.
2. Second midpoint: $x_{\text{mid}} = -0.25 \Rightarrow P(-0.25) \approx -0.46 < 0 \Rightarrow$ root is in $[-0.5, -0.25]$.
3. Third midpoint: $x_{\text{mid}} = -0.375 \Rightarrow P(-0.375) \approx -0.034 < 0 \Rightarrow$ root is in $[-0.5, -0.375]$.
4. Fourth midpoint: $x_{\text{mid}} = -0.4375 \Rightarrow P(-0.4375) \approx 0.22 > 0 \Rightarrow$ root is in $[-0.4375, -0.375]$.

By continuing this process, we estimate:

$$x_1 \approx -0.41.$$

Estimating the Root in $[0, 1.5]$:
Evaluate $P(x)$ at $x = 0$ and $x = 1$:

$$\begin{cases} P(0) = -1 < 0, \\ P(1) \approx 0.1586 > 0. \end{cases}$$

There is a root x_2 in $[0, 1]$.
Using the Bisection Method:

1. First midpoint: $x_{\text{mid}} = 0.5 \Rightarrow P(0.5) \approx -1 < 0 \Rightarrow$ root is in $[0.5, 1]$.
2. Second midpoint: $x_{\text{mid}} = 0.75 \Rightarrow P(0.75) \approx -0.544 < 0 \Rightarrow$ root is in $[0.75, 1]$.
3. Third midpoint: $x_{\text{mid}} = 0.875 \Rightarrow P(0.875) \approx -0.151 < 0 \Rightarrow$ root is in $[0.875, 1]$.
4. Fourth midpoint: $x_{\text{mid}} = 0.9375 \Rightarrow P(0.9375) \approx 0.0025 > 0 \Rightarrow$ root is in $[0.875, 0.9375]$.

By continuing this process, we estimate:

$$x_2 \approx 0.93.$$

Conclusion:
Using the interpolating polynomial $P(x)$, we estimate the roots of $f(x)$ in the specified intervals:

$$\begin{cases} \text{Root in } [-1, 0]: & x_1 \approx -0.41, \\ \text{Root in } [0, 1.5]: & x_2 \approx 0.93. \end{cases}$$