# Homework 7 (differential equations)

Hanzhang Yin

November 20, 2024

Unless otherwise specified, you may assume that any differential equations defined below are sufficiently nice that the existence and uniqueness theorems apply.

**1.** In this problem, we'll use homotopy continuation to give a proof of the quadratic formula. It may be helpful to use the homotopy continuation notebook to plot the problem.

Let $f(x) = x^2 + ax + b$ a quadratic function where $a, b$ are real numbers such that $a^2 > 4b$ and $a \neq 0$. Let $g(x) = f(x) - f(0) = x^2 + ax$.

a) Find the two real roots of $g$.

b) Determine the standard homotopy $H(x, t)$.

c) Use the homotopy to write down the standard homotopy differential equation

$$x'(t) = -\frac{H_t}{H_x}.$$

d) Make the change of variable $y = 2x + a$ and solve the resulting differential equation. Don't forget the initial condition $y(0) = 2x(0) + a$, where $x(0)$ is one of the two roots from (a).

e) Determine the paths $x(t)$ and the values $x(1)$, which should match the usual quadratic formula.

f) Explain the paths in terms of the quadratic formula. Point out what this proof and the standard proof of the quadratic formula have in common.

**Solution.**

**2.** Suppose $f$ is differentiable on $(a, b)$. Show that the mean value theorem implies the fundamental theorem of the derivative:

MVT: given $x, y$ in $(a, b)$ there is some $z$ in $(a, b)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(z).$$

FTD: if $f' = 0$ on $(a, b)$ then $f$ is constant.

Observe that the FTD is a simple example of the uniqueness theorem for differential equations.

**Solution.**

**3.** Define a mystery function $f$ by the initial value problem:

$$f'' = f \qquad f(0) = 0\ ,\ f'(0) = 1,$$

and $g = f'$. From this definition, verify the following identity:

$$f(x + y) = f(x)g(y) + f(y)g(x)$$

without explicitly determining $f$ and $g$.

This is the pythagorean identity for the hyperbolic trig functions. The solution should have *no* geometry (unless you want to develop it from first principles).

**Solution.**

**4.** Derivatives aren't very different for complex numbers. In particular, it's still true that

$$\frac{d}{dx}e^{rx} = re^{rx}$$

when $r$ is a complex number. The usual differential equation $f' = f, f(0) = 1$ still defines $e^x$ and the existence and uniqueness theorems apply in the complex setting.

Other exponent identities like $(e^x)^r = e^{rx}$ and $e^{x+a} = e^x e^a$ also work (can be established with the same proofs, using existence and uniqueness, if you want).

a) Verify the following identities

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}.$$

b) Combine those identities into

$$e^{ix} = \cos(x) + i\sin(x).$$

c) Use that to establish De Moivre's formula,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx).$$

d) Take $n = 2$ in De Moivre's formula and observe that it furnishes a quick simultaneous proof of the double-angle identities for sin and cos (and an easy way to remember them). Remark: in spite of what you may have heard in other courses, calculators don't use power series to evaluate trig functions, but rather (more or less) from a version of this identity.

**Solution.**

**5.** Consider the differential equation

$$x' = 6x^2 - 5\arctan(x), \qquad x(0) = 1.$$

a) Write down the first three terms of its taylor series around $t = 0$.

b) Estimate $x(1)$.

c) Estimate $x(0.5)$.

d) Use your estimate for $x(0.5)$ to estimate the first three terms of a taylor series for $x(t)$ centered at $t = 0.5$.

e) Estimate $x(1)$ again, but using this second series. Compare to (b). What do you think is happening?

**Solution.**

**6.** Consider the system of differential equations:

$$X' = -5X + 9Y$$
$$Y' = -4X + 7Y$$

with initial condition $X(0) = 1$ and $Y(0) = 1$.
Here's a Jordan decomposition

$$\begin{bmatrix} -5 & 9 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

where

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}^{-1}$$

a) Solve this differential equation exactly.

b) Use (a) to determine $X(2)$ and $Y(2)$.

c) Estimate $X(2)$ and $Y(2)$ using Euler's method two ways: one step and two steps.

**Solution.**

**7.** [Bonus] Let $A$ and $B$ be $n \times n$ matrices.

a) Prove that

$$e^{A+B} = e^A e^B$$

when $AB = BA$.

b) Give an example where

$$e^{A+B} \neq e^A e^B.$$

**Solution.**