Homework 3 (error and iterative methods, markov chains)

name

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Recall that a Markov matrix is an $n \times n$ matrix whose *columns* represent the probabilities of transitioning between n states: the columns sum to 1 and entry M_{ij} is the probability of transitioning from state j to state i. Some sources use the transpose of this matrix.

1. Apply iterative methods (our generalized Jacobi method) to estimate a solution to

$$\begin{bmatrix} 4 & 5 \\ 3 & 5 \end{bmatrix} x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

a) Pick any splitting matrix besides A^{-1}, and any initial guess besides the actual solution, and determine the result after two iterations.

b) Determine the quantity $\delta = ||I - Q^{-1}A||$ (notation from class/book; Q is the splitting matrix).

c) Determine the actual solution by inverting the matrix.

d) Compare the actual solution to your approximate solution from (a) using the ∞ norm. Then, compare to the error estimate theorem from class. What do you notice?

Solution.

2. In our steady-state calculation for a Markov matrix M, we determined that all Markov matrices have 1 as an eigenvalue by iterative methods. Iterative methods requires some technical assumptions that we did not discuss. This problem walks you through verifying this fact without without them.

a) Show that a Markov matrix M has 1 as a left eigenvalue (i.e. an eigenvalue of M^T). Hint: the sum of the rows is 1 in a Markov matrix – what left vector multiplication would produce such a row sum? Is it an eigenvector?

b) Show that A and A^T have the same minimal polynomial. Hint: check that $p(A)^T = p(A^T)$ for any polynomial p.

c) Combine the previous two facts to conclude that A has an eigenvector with eigenvalue 1.

Solution.

3. Recall that we assumed a matrix has a full-rank eigenspace and a unique largest eigenvalue in order to locate its largest eigenvalue (and associated eigenvector) by iterative methods.
We saw above that every Markov matrix M has 1 as an eigenvalue.

1. Prove every eigenvalue of M has norm at most 1. Hint: use (2b) and the ∞ norm, or, equivalently, the Gershgorin circle theorem.
2. Suppose that M is 2×2 with full rank eigenspace, but 1 is a repeated eigenvalue. What does this mean for the original matrix?
3. Suppose that M is 2×2 and has 1 as its only eigenvalue, but its eigenspace is not full rank. What can you say about this situation? Hint: use the ∞-norm and examine the left hand side of $|Mv| \leq ||M|||v| = |v|$ more carefully.

Solution.

4. (bonus) Generalize (3c) to show that a Markov matrix with no zeros has a *unique* eigenvector whose associated eigenvalue has norm 1.

Solution.

5. (ungraded bonus) If you know some graph theory, discuss the implications for a Markov matrix whose state diagram, like the weather one we drew in class, is (directed) connected and such that each state has a nonzero probability of remaining the same. Hint: could the fact above apply to a large power of M?

Solution.