# MATH 280 HW2

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## Question 1

### (a)

The degree 5 Taylor series approximation for $\sin x$ around $x = \frac{\pi}{4}$ is:

$$\sin(x) \approx \sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2-\frac{1}{6}\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3+\frac{1}{24}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^4$$

Substituting the values for $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ both as $\frac{\sqrt{2}}{2}$, we get:

$$\sin(x) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4$$

The integral form of the remainder for the Taylor series of $\sin x$ around $x = \frac{\pi}{4}$ and truncated after the fifth degree is:

$$R_5(x) = \int_{\frac{\pi}{4}}^{x} \frac{\sin(t)}{120}(x-t)^5\,dt$$

### (b)

$$\sin(0) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(0-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(0-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(0-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(0-\frac{\pi}{4}\right)^4 = 0.001963321$$

$$\sin(\frac{\pi}{2}) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^4 = 0.9984927$$

$$\sin(1) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(1-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(1-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(1-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(1-\frac{\pi}{4}\right)^4 = 0.84146840$$

### (c)

$$R_5(0) = \int_{\frac{\pi}{4}}^{0} \frac{\sin(t)}{120}(0-t)^5\,dt = 0.00020235$$

$$R_5(\frac{\pi}{2}) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(t)}{120}(\frac{\pi}{2}-t)^5\,dt = 0.000243632$$

$$R_5(1) = \int_{\frac{\pi}{4}}^{1} \frac{\sin(t)}{120}(1-t)^5\,dt = 9.879 \times 10^{-8}$$

These values indicate extremely small errors, showcasing the high accuracy of the Taylor series approximation when truncated at the fifth degree.

- **At** $x = 0$: The error bound of 0.00020235 indicates that the approximation is nearly exact.
- **At** $x = \frac{\pi}{2}$: An error bound of 0.00024362 suggests that the approximation is extremely close to the actual value 1.
- **At** $x = 1$: The error bound of $9.879 \times 10^{-8}$ indicates the approximation is indistinguishable from the true value sin(1).

These bounds confirm that the fifth-order Taylor series provides an excellent approximation for $\sin x$ around $x = \frac{\pi}{4}$, particularly near this point and at commonly evaluated points within the function's periodic range.

## Question 2

### (a)

$$f(X) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given $Y = (1, 3)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.
For linear function of the form $f(X) = AX + b$, the Jacobian $J_f$ is simply $A$, hence:

$$\sum_{f(y)} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & 4.2 \\ 2.2 & 1.6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 30.6 & 13 \\ 13 & 5.4 \end{bmatrix}$$

### (b)

$$f(X) = \begin{bmatrix} X_1^2 + 3X_1X_2 - 5 \\ X_2 - X_1 \end{bmatrix}$$

Given $Y = (2, 1)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.
For Jacobian Matrix calculation, here is the requried partial derivatives:

$$\frac{\partial}{\partial X_1}(X_1^2 + 3X_1X_2 - 5) = 2X_1 + 3X_2$$

$$\frac{\partial}{\partial X_2}(X_1^2 + 3X_1X_2 - 5) = 3X_1$$

$$\frac{\partial}{\partial X_1}(X_2 - X_1) = -1$$

$$\frac{\partial}{\partial X_2}(X_2 - X_1) = 1$$

At $Y = (2, 1)$, we have:

$$J_f = \begin{bmatrix} 2(2) + 3(1) & 3(2) \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\sum_{f(y)} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.4 & 8.8 \\ 0.8 & -0.7 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 194.6 & 1.4 \\ 1.4 & -1.5 \end{bmatrix}$$

## Question 3

Given, $\sum_{x,y} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$.

### (a)

$$f(X, Y) = X + Y$$

The Jacobian matrix of $f$ w.r.t. $X$ and $Y$ is:

$$J_f = [1, 1]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_X^2 + \sigma_Y^2$$

## (b)

$$f(X,Y) = XY$$

The Jacobian matrix of $f$ w.r.t. $X$ and $Y$ is:

$$J_f = [Y, X]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} = Y^2\sigma_X^2 + X^2\sigma_Y^2$$

## (c)

$$f(X,Y) = \frac{X}{Y}$$

The Jacobian matrix of $f$ w.r.t. $X$ and $Y$ is:

$$J_f = [\frac{1}{Y}, -\frac{X}{Y^2}]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} \frac{1}{Y} & -\frac{X}{Y^2} \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} \frac{1}{Y} \\ -\frac{X}{Y^2} \end{bmatrix} = \frac{\sigma_X^2}{Y^2} + \frac{X^2\sigma_Y^2}{Y^4}$$

# Bonus Question

Given the viral equation in ideal gas:

$$P_iV = n_iRT$$

In non-ideal gas case, we might introduce compressibility factors $Z_{i,mix}$ for each gas:

$$P_iV = n_iRTZ_{i,mix}$$

## Propose a generalization of the virial equation of state to a two-component mixture

**Virial Equation for a Single Component Gas:**
The virial equation of state for a single component gas expands the ideal gas law to account for interactions between molecules. It is expressed as:

$$P = \frac{nRT}{V}\left(1 + \frac{B_2(T)}{V} + \frac{B_3(T)}{V^2} + \cdots\right)$$

where $B_2(T), B_3(T), \ldots$ are the virial coefficients that depend on temperature and represent interactions among molecules.
**Extension to Two-Component Mixtures:**
For a mixture of gases, the virial equation can be generalized as:

$$P = \frac{(n_1 + n_2)RT}{V} \left(1 + \frac{B_{12}(T)}{V} + \frac{C_{123}(T)}{V^2} + \cdots \right)$$

Here, $B_{12}(T)$ is the second virial coefficient that considers cross-interactions between different gas species and is approximated as:

$$B_{12}(T) = x_1^2 B_{11}(T) + 2x_1 x_2 B_{12}(T) + x_2^2 B_{22}(T)$$

where $x_1$ and $x_2$ are the mole fractions of gas 1 and gas 2, respectively.

## Simplification Under Ideal Conditions

If both gases are ideal:

- Interactions between molecules and between different species are negligible.
- This implies $B_2 = B_3 = \ldots = 0$.

Thus, the virial equation reduces to:

$$P = \frac{nRT}{V}$$

where $n = n_1 + n_2$, indicating that the pressure is determined purely by the ideal gas contributions of both components, unaffected by intermolecular interactions.