Homework 2 (power series, VES, error propagation)

1. Taylor's theorem and remainder.

a) Write the degree 5 power series approximation for $\sin x$ at $x = \pi/4$. Also state the integral form of the remainder (Section 1.1, Theorem 5).

b) Use your approximation to estimate $\sin(0)$, $\sin \pi/2$ and $\sin 1$.

c) Use your remainder to bound the error on your estimates. What do you think about the bounds?

2. Suppose we have measured $Y = (2, 1)$ with some error described by the covariance matrix $\Sigma_Y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$. Use the error propagation equation to calculate or estimate $\Sigma_{f(Y)}$ for each of the following:

a) $f(X) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $f(X) = \begin{bmatrix} X_1^2 + 3X_1X_2 - 5 \\ X_2 - X_1 \end{bmatrix}$

3. Use the error propagation equation to give general expressions for $\Sigma_{f(x,y)}$ when x and y are uncorrelated, meaning $\Sigma_{x,y} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$, in terms of X, Y, σ_x, σ_y.

a) $f(X, Y) = X + Y$.

b) $f(X, Y) = XY$.

c) $f(X, Y) = X/Y$.

Bonus. The virial equation of state is for single-component gases, not mixtures. For an ideal mixture of ideal gases, one can break up the ideal gas law using partial pressures:

$$P_i V = n_i RT$$

where n_i is the number of molecules of the ith constituent and P_i is the partial pressure of the ith consitutent (its molar proportion times total pressure). These sum to the ideal gas law.

For example, if there are two components,

$$P_1 V = n_1 RT$$

$$P_2 V = n_2 RT$$

and $P_1 + P_2 = P$, the total pressure, and $n_1 + n_2 = n$, the total number of molecules.

In the non-ideal case, one might be inclined to introduce compressibility factors $Z_{i,mix}$ for each gas (depending on the mixture),

$$P_i V = n_i RT Z_{i,mix}.$$

($*$) Propose a generalization of the virial equation of state to a two-component mixture. In other words, a power series expression for the Z_1, Z_2 with physical interpretations of their coefficients.

($**$) Assuming the gases are individually ideal, how could you simplify your equation of state?