MATH 280 HW2

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Question 1

(a)

The degree 5 Taylor series approximation for $\sin x$ around $x = \frac{\pi}{4}$ is:

$$\sin(x) \approx \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) - \frac{1}{2}\sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{6}\cos\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{24}\sin\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right)^4$$

Substituting the values for $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ both as $\frac{\sqrt{2}}{2}$, we get:

$$\sin(x) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x - \frac{\pi}{4}\right)^4$$

The integral form of the remainder for the Taylor series of $\sin x$ around $x = \frac{\pi}{4}$ and truncated after the fifth degree is:

$$R_5(x) = \int_{\frac{\pi}{4}}^{x} \frac{\sin(t)}{120}(x - t)^5 \, dt$$

(b)

$$\sin(0) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(0 - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(0 - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(0 - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(0 - \frac{\pi}{4}\right)^4 = 0.001963321$$

$$\sin(\frac{\pi}{2}) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)^4 = 0.9984927$$

$$\sin(1) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(1 - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(1 - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(1 - \frac{\pi}{4}\right)^4 = 0.84146840$$

(c)

$$R_5(0) = \int_{\frac{\pi}{4}}^{0} \frac{\sin(t)}{120}(0-t)^5 \, dt = 0.00020235$$

$$R_5(\frac{\pi}{2}) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(t)}{120}(\frac{\pi}{2}-t)^5 \, dt = 0.000243632$$

$$R_5(1) = \int_{\frac{\pi}{4}}^{1} \frac{\sin(t)}{120}(1-t)^5 \, dt = 9.879 \times 10^{-8}$$

These values indicate extremely small errors, showcasing the high accuracy of the Taylor series approximation when truncated at the fifth degree.

- **At** $x = 0$: The error bound of 0.00020235 indicates that the approximation is nearly exact.

- **At** $x = \frac{\pi}{2}$: An error bound of 0.00024362 suggests that the approximation is extremely close to the actual value 1.

- **At** $x = 1$: The error bound of 9.879×10^{-8} indicates the approximation is indistinguishable from the true value $\sin(1)$.

These bounds confirm that the fifth-order Taylor series provides an excellent approximation for $\sin x$ around $x = \frac{\pi}{4}$, particularly near this point and at commonly evaluated points within the function's periodic range.

Question 2

(a)

$$f(X) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given $Y = (1,3)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.

For linear function of the form $f(X) = AX + b$, the Jacobian J_f is simply A, hence:

$$\sum_{f(y)} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4.6 & 4.2 \\ 2.2 & 1.6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 30.6 & 13 \\ 13 & 5.4 \end{bmatrix}$$

(b)

$$f(X) = \begin{bmatrix} X_1^2 + 3X_1X_2 - 5 \\ X_2 - X_1 \end{bmatrix}$$

Given $Y = (2, 1)$, and $\sum_y = \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix}$.

For Jacobian Matrix calculation, here is the requried partial derivatives:

$$\frac{\partial}{\partial X_1}(X_1^2 + 3X_1X_2 - 5) = 2X_1 + 3X_2$$

$$\frac{\partial}{\partial X_2}(X_1^2 + 3X_1X_2 - 5) = 3X_1$$

$$\frac{\partial}{\partial X_1}(X_2 - X_1) = -1$$

$$\frac{\partial}{\partial X_2}(X_2 - X_1) = 1$$

At $Y = (2, 1)$, we have:

$$J_f = \begin{bmatrix} 2(2) + 3(1) & 3(2) \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix}$$

Hence,

$$\sum_{f(y)} = \begin{bmatrix} 7 & 6 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 1 \\ 1 & 0.3 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7.4 & 8.8 \\ 0.8 & -0.7 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 194.6 & 1.4 \\ 1.4 & -1.5 \end{bmatrix}$$

Question 3

Given, $\sum_{x,y} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$.

(a)

$$f(X, Y) = X + Y$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [1, 1]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sigma_X^2 + \sigma_Y^2$$

(b)

$$f(X, Y) = XY$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [Y, X]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} Y & X \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} Y \\ X \end{bmatrix} = Y^2 \sigma_X^2 + X^2 \sigma_Y^2$$

(c)

$$f(X, Y) = \frac{X}{Y}$$

The Jacobian matrix of f w.r.t. X and Y is:

$$J_f = [\frac{1}{Y}, -\frac{X}{Y^2}]$$

Thus, covariance matrix $\sum_{f(x,y)}$ is:

$$\sum_{f(x,y)} = \begin{bmatrix} \frac{1}{Y} & -\frac{X}{Y^2} \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix} \begin{bmatrix} \frac{1}{Y} \\ -\frac{X}{Y^2} \end{bmatrix} = \frac{\sigma_X^2}{Y^2} + \frac{X^2 \sigma_Y^2}{Y^4}$$

Bonus Question