

MATH 280 HW1

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Question 1

Proof. Define the function $h(x) = \frac{f(x)}{g(x)}$, by the def. of the derivative with limit process:

$$h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$$

Expanding $h(x)$ and $h(a)$:

$$\begin{aligned} h'(a) &= \lim_{x \rightarrow a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \\ \Rightarrow h'(a) &= \lim_{x \rightarrow a} \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)(x - a)} \end{aligned}$$

Applying $f(x) \approx f(a) + f'(a)(x - a) + E_f(x)$ and $g(x) \approx g(a) + g'(a)(x - a) + E_g(x)$, we can rearrange numerator as:

$$\begin{aligned} f(x)g(a) - f(a)g(x) &\approx (f(a) + f'(a)(x - a) + E_f(x))g(a) - f(a)(g(a) + g'(a)(x - a) + E_g(x)) \\ &= f(a)g(a) + f'(a)g(a)(x - a) + E_f(x)g(a) - f(a)g(a) - f(a)g'(a)(x - a) - f(a)E_g(x) \\ &= f'(a)g(a)(x - a) - f(a)g'(a)(x - a) + E_f(x)g(a) - f(a)E_g(x) \end{aligned}$$

Applying the definition of $E_f(x)$ and $E_g(x)$:

$$\lim_{x \rightarrow a} \frac{E_f(x)g(a) - f(a)E_g(x)}{g(x)g(a)g(x - a)} = 0$$

we can get overall:

$$h'(a) = \lim_{x \rightarrow a} \frac{f'(a)g(a)(x - a) - f(a)g'(a)(x - a) + E_f(x)g(a) - f(a)E_g(x)}{g(x)g(a)g(x - a)}$$

Since $g(x) \approx g(a)$ when $x = a$:

$$\Rightarrow \left(\frac{f}{g} \right)' = h'(a) = \lim_{x \rightarrow a} \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

□

Question 2

(a)

Given,

- $f(x) = 2 - (x - 3) + E_f(x)$, near $x = 3$, with $|E_f(x)| < 1$ on the interval $[2, 4]$.
- $g(x) = x^3 - 2x^2 + 3$; $g(2) = 8 - 4(2) + 3 = 3$, $g'(2) = 12 - 8 = 4$.

Proof. From the initial assumptions, here is the approximation of $f(x)$ near $x = 3$:

$$f(x) = 2 - (x - 3) + E_f(x)$$

Value of g at $f(3)$:

$$f(3) = 2 - (3 - 3) + E_f(3) = 2 + E_f(3)$$

Noticing that $|E_f(x)| < 1$, then

$$g(f(3)) = g(2 + E_f(3)) = g(2) = 3$$

Then, we can use the Taylor Expansion around $f(3) = 2$. The first order Taylor expansion for $g(f(x))$ around $x = 3$ would be:

$$g(f(x)) \approx g(2) + g'(2) \cdot (f(x) - 2)$$

Then we can sub $f(x)$ in:

$$\begin{aligned} g(f(x)) &\approx 3 + 4(2 + E_f(x) - (x - 3) - 2) \\ &= 3 + 4(-x + 3 + E_f(x)) \\ &= 3 + 4(-x + 3) + 4E_f(x) \\ &= 15 - 4x + 4E_f(x) \end{aligned}$$

Given $E(x) \ll (x - a)$ and considering the linearity of $g'(x) : E_{f \circ g} = 4E_f(x)$. As $E_f(x)$ is much smaller $(x - 3)$, and given $|E_f(x)| < 1 : |E_{f \circ g}| \leq 4$. \square

(b)

Given $h(x)$ is differentiable at $x = 0$ and $h(0) = 3$, WTS the bound the error of $f(h(x))$ near zero.

Proof. To precisely bound the error of $f(h(x))$ near zero, we need two more information:

- Explicit form of $E_f(x)$
- Rate of Change of $h(x)$

By knowing these two conditions, we can bound $E_f(h(x))$ by the following approach:

By definition of Taylor Expansion, we can substitute

$$h(x) = 3 + h'(0)x + o(x)$$

Where $o(x)$ is the higher order terms that become negligibly small faster than linear.

Then, suppose $E_f(x) = k(x - 3)^n$, then:

$$E_f(h(x)) = k(h(x) - 3)^n = k(h'(0)x + o(x))^n$$

For evaluation we will only focus on how the “linear” terms in x contribute to $E_f(h(x))$.

Assume $E_f(x) \sim (x - 3)^n$ and $h(x) = 3 + h'(0)x + o(x)$, the bound of $E_f(h(x))$ when $x = 0$ can be written in the form:

$$|E_f(h(x))| \leq |k| \cdot |h'(0)x + o(x)|^n$$

□

Question 3