MATH 280 HW2

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Question 1

(a)

The degree 5 Taylor series approximation for $\sin x$ around $x = \frac{\pi}{4}$ is:

$$\sin(x) \approx \sin\left(\frac{\pi}{4}\right)+\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2-\frac{1}{6}\cos\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^3+\frac{1}{24}\sin\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^4$$

Substituting the values for $\sin\left(\frac{\pi}{4}\right)$ and $\cos\left(\frac{\pi}{4}\right)$ both as $\frac{\sqrt{2}}{2}$, we get:

$$\sin(x) \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x-\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x-\frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{48}\left(x-\frac{\pi}{4}\right)^4$$

The integral form of the remainder for the Taylor series of $\sin x$ around $x = \frac{\pi}{4}$ and truncated after the fifth degree is:

$$R_5(x) = \int_{\frac{\pi}{4}}^{x} \frac{\sin(t)}{120}(x-t)^5\,dt$$

(b)

$$\sin(0) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(0-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(0-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(0-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(0-\frac{\pi}{4}\right)^4 = 0.001963321$$

$$\sin(\frac{\pi}{2}) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)^4 = 0.9984927$$

$$\sin(1) \approx \frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(1-\frac{\pi}{4}\right)-\frac{\sqrt{2}}{4}\left(1-\frac{\pi}{4}\right)^2-\frac{\sqrt{2}}{12}\left(1-\frac{\pi}{4}\right)^3+\frac{\sqrt{2}}{48}\left(1-\frac{\pi}{4}\right)^4 = 0.84146840$$

(c)

$$R_5(0) = \int_{\frac{\pi}{4}}^{0} \frac{\sin(t)}{120}(0-t)^5\,dt = 0.00020235$$

$$R_5(\frac{\pi}{2}) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(t)}{120}(\frac{\pi}{2}-t)^5\,dt = 0.000243632$$

$$R_5(1) = \int_{\frac{\pi}{4}}^{1} \frac{\sin(t)}{120}(1-t)^5\,dt = 9.879 \times 10^{-8}$$

These values indicate extremely small errors, showcasing the high accuracy of the Taylor series approximation when truncated at the fifth degree.

- **At** $x = 0$: The error bound of 0.00020235 indicates that the approximation is nearly exact.
- **At** $x = \frac{\pi}{2}$: An error bound of 0.00024362 suggests that the approximation is extremely close to the actual value 1.
- **At** $x = 1$: The error bound of 9.879×10^{-8} indicates the approximation is indistinguishable from the true value sin(1).

These bounds confirm that the fifth-order Taylor series provides an excellent approximation for $\sin x$ around $x = \frac{\pi}{4}$, particularly near this point and at commonly evaluated points within the function's periodic range.

Question 2

(a)

$$f(X) = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b)

$$f(X) = \begin{bmatrix} X_1^2 + 3X_1X_2 - 5 \\ X_2 - X_1 \end{bmatrix}$$