THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2022

School of Mathematics and Statistics

MAST10006 Calculus 2

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 21 pages (including this page) with 11 questions and 117 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- If you are writing answers on the exam or masked exam and need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

Question 1 (11 marks)

Evaluate the following limits, if they exist. In this question you must state if you use any standard limits, limit laws, continuity, l'Hôpital's rule or the Sandwich Theorem.

(a) $\displaystyle\lim_{x\to 2} \frac{\log(\pi x)}{1-x^2}$

(b) $\displaystyle\lim_{x\to 2} \left(\frac{\sin(\pi x)}{x-2\cos(\pi x)}\right)$

(c) $\lim_{x\to\infty}\left(\dfrac{\tanh(x^2)}{x^2}\right)$

Question 2 (11 marks)

For the following series, indicate if they are convergent or divergent. Justify your answer with any relevant tests that you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{2n(n+1)+\sin(n)}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+1}{n^2-1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n\, n!}{n^n}$

Question 3 (8 marks)

Use the complex exponential to evaluate the following:

(a) $\dfrac{d^{10}}{dx^{10}}\left(e^x \cos(x)\right)$

(b) $\displaystyle\int e^x \sin(2x)dx$

Question 4 (16 marks)

(a) Show that

$$\int x\cosh(nx)\,\mathrm{d}x = \frac{x\sinh(nx)}{n} - \frac{\cosh(nx)}{n^2} + C$$

for $n \in \mathbb{N}$.

(b) Hence or otherwise, use a hyperbolic substitution to evaluate

$$\int \operatorname{arccosh}(\sqrt{x^2+1})\,dx.$$

(c) Use the definition of $\cosh(x)$ in terms of exponentials to show that

$$2\cosh(2x)\cosh(3x) = \cosh(5x) + \cosh(x).$$

(d) Evaluate $\displaystyle\int e^{2x}\cosh(2e^x)\cosh(3e^x)\,\mathrm{d}x$.
You can use results from earlier parts of this question.

Question 5 (7 marks)

(a) Find the equilibrium solutions of the differential equation

$$\frac{dy}{dx} = \cos^2(y) + x^2\cos^2(y), \quad \text{for } -\pi \leq y \leq \pi$$

Recall that equilibrium solutions are solutions where $y(x)$ is a constant function.

(b) Solve the initial value problem

$$\frac{dy}{dx} = \cos^2(y) + x^2\cos^2(y), \quad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

given $y(0) = \frac{\pi}{4}$.

Question 6 (7 marks)

(a) Give an example of an autonomous first-order differential equation, $\frac{dy}{dx} = f(y)$ with infinitely many equilibrium solutions. Briefly explain why your example has infinitely many equilibrium solutions.

(b) Give an example of a first order ODE, $\frac{dy}{dx} + \mathcal{P}(x)y = \mathcal{Q}(x)$ such that $I = x^3$ is a possible integrating factor. You do not need to solve the example that you provide.

(c) Draw a phase plot for an autonomous first-order ODE $\frac{dp}{dt} = f(p)$ so that the ODE has exactly one stable equilibrium solution, exactly one unstable equilibrium solution and exactly one semi-stable equilibrium solution. You do not need to explicitly write the rule down for f.

Question 7 (13 marks)

Consider the inhomogeneous linear second order ODE

$$y'' + ky' + y = x^2$$

where $k \in \mathbb{R}$ is a parameter.

(a) For which values of k does the characteristic equation of the associated homogeneous ODE have two real roots, one real repeated root, a pair of complex conjugate roots?

(b) Find a particular solution of the inhomogeneous ODE, treating $k \in \mathbb{R}$ as an unknown constant. Is there a value of k such that $y = x^2$ is a particular solution?

(c) Using your answer from earlier parts, or otherwise, find the general solution to the inhomogeneous ODE:

$$y'' + y' + y = x^2$$

Question 8 (10 marks)

A mass of weight 1kg is attached to one end of a spring and hung from a ceiling.

(a) Draw a diagram of the mass-spring system, while it is below the equilibrium position at a point in time when the mass is moving upwards. Include arrows indicating the gravitational force, damping force and spring force acting on the mass.

(b) The system is described by the differential equation

$$\ddot{y} + 2\dot{y} + 9y = 0.$$

Describe the behaviour of the mass in the long run, and note whether there is any oscillatory behaviour. You do not need to solve the ODE.

(c) Suppose we apply a force $f(t) = \sin(\omega t)$ to the mass so the system is now described by the differential equation

$$\ddot{y} + 2\dot{y} + 9y = \sin(\omega t)$$

where $\omega > 0$. For what values of ω will the mass oscillate with an amplitude that is bounded (i.e., the amplitude of oscillation does not get arbitrarily large)?

Question 9 (14 marks)

Consider a function given by the rule:

$$f(x, y) = \sqrt{1 - x^2 - y^2}.$$

(a) State the largest possible domain on which f is defined and explain why f is continuous at every point on this domain.

(b) Find equations for the level curves of f for $z = 0, \frac{1}{2}, 1$ and sketch them on the same set of axes. Clearly label each graph so as to be able to distinguish the level curves, and label any axis intercepts.

(c) Find equations for the x-z and y-z cross-sections and sketch their graphs, and label any axis intercepts.

(d) Hence sketch the graph the function f.

Question 10 (9 marks)

Consider the function of two variables $f(x, y) = e^x \cos(y)$.

(a) Compute the directional derivative of f at $(0, 0)$ in the direction of the vector $(1, 1)$.

(b) Find an equation for the tangent plane to the surface $z = f(x, y)$ at the point $(0, \frac{\pi}{4}, \frac{1}{\sqrt{2}})$.

(c) Let $x(t) = t^2\cos(t)$ and $y(t) = t^2\sin(t)$. Setting $z(t) = f(x(t), y(t))$, compute the derivative $\frac{dz}{dt}$ using the chain rule.

Question 11 (11 marks)

Consider the function $f : D \to \mathbb{R}$ where the domain $D = [-1, 1] \times [-1, 1]$ is a square and f is specified by

$$f(x, y) = \cosh(x) \sin(\pi y) \ .$$

(a) Determine the location of all stationary points that lie within the domain D.

(b) Compute the Hessian function $H(x, y)$ associated with f at a general point (x, y).

(c) For each stationary point you computed in part (a) determine whether it is a maximum, a minimum or a saddle point. Justify your reasoning.

End of Exam — Total Available Marks = 117

MAST10006 Calculus 2 Formulae Sheet

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$
$\cot^2 x + 1 = \operatorname{cosec}^2 x$ $\qquad$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\cos(2x) = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh(2x) = \cosh^2 x + \sinh^2 x$
$\cos(2x) = 2\cos^2 x - 1$ $\qquad$ $\cosh(2x) = 2\cosh^2 x - 1$
$\cos(2x) = 1 - 2\sin^2 x$ $\qquad$ $\cosh(2x) = 1 + 2\sinh^2 x$
$\sin(2x) = 2\sin x\cos x$ $\qquad$ $\sinh(2x) = 2\sinh x\cosh x$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ $\qquad$ $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\sin(x+y) = \sin x\cos y + \cos x\sin y$ $\qquad$ $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1}) \qquad \operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$$

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0) \qquad \lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0) \qquad \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R},\ a > 1)$$