THE UNIVERSITY OF MELBOURNE

Student number	

Semester 2 Assessment, 2021

School of Mathematics and Statistics

MAST10006 Calculus 2

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 19 pages (including this page) with 12 questions and 110 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- No calculators are permitted. No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- Marks may be awarded for correct use of appropriate mathematical techniques; accuracy and validity of any calculations or algebraic manipulations; clear justification or explanation of techniques and rules used; clear communication of mathematical ideas through diagrams; and use of correct mathematical notation and terminology.
- Label all important features, axes, axis intercepts and asymptotes in all graphs.
- Formulas from the Calculus 2 formula sheet may be used without further justification. Other formulas should be justified or proved before use.
- If you are writing answers on the exam or masked exam and need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning and Submitting

- **You must not leave Zoom supervision to scan your exam.** Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4.
- Submit your scanned exam as a single PDF file and carefully review the submission in Gradescope. Scan again and resubmit if necessary. Do not leave Zoom supervision until you have confirmed orally with the supervisor that you have received the Gradescope confirmation email.
- **You must not submit or resubmit after having left Zoom supervision.**

In questions 1 and 2, you must state if you use standard limits, limit laws, continuity, l'Hôpital's rule, sandwich theorem, or convergence tests for series.

Question 1 (13 marks)

Let $f(x) = \begin{cases} \tan(x)\log\big(\sin(x)\big) & x > 0 \\ 0 & x \leq 0 \end{cases}$

(a) Use the definition of continuity to determine if the function f is continuous at $x = 0$.

Space for question 1(a) continued

(b) Use the divergence test to determine if the series

$$\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$$

converges, or explain why the divergence test cannot be used.

Question 2 (11 marks) Determine if each of the following series converge or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2(n!)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4n + \log(2 + \frac{1}{n})}{3n^2 + 5n}$

Question 3 (5 marks)

(a) Use the exponential definition of the hyperbolic functions to prove the hyperbolic identity

$$\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y), \qquad x, y \in \mathbb{R}.$$

(b) Hence or otherwise, sketch the graph of

$$y = \cosh(x) - \tanh(2)\sinh(x).$$

Question 4 (12 marks)

(a) Evaluate the integral $\int \sqrt{1 - 9x^2}\, dx$

(b) Use the complex exponential to evaluate the integral $\int e^{3x} \cos 4x \, dx$

Question 5 (7 marks) Consider the ODE

$$\frac{dy}{dx} = \cosh^2(y), \qquad x, y \in \mathbb{R}.$$

(a) Find the general solution of the ODE.

(b) Sketch the family of solutions. Include solutions which pass through each of the following points:

i. $(0, 0)$ ii. $(1, 0)$ iii. $(2, 0)$

(c) Find the solution satisfying the initial condition $y(3) = 0$. What is the domain and range of the solution?

Question 6 (7 marks) Consider the ODE

$$\frac{dy}{dx} = \frac{y + x\arctan(x)\sqrt{x^2 - y^2}}{x}, \qquad x > 0, \quad |y| \leq x.$$

(a) Use the substitution $u = \dfrac{y}{x}$ and show that the ODE reduces to

$$\frac{du}{dx} = \arctan(x)\sqrt{1 - u^2}.$$

(b) Find the general solution of the original ODE.

Question 7 (9 marks) Consider the ODE

$$\frac{dy}{dx} = -(y+1)(y-3)(y-7), \qquad x \geq 0, y \in \mathbb{R}$$

(a) Find the equilibrium solutions of the ODE and determine their stability.

(b) On the same set of axes, sketch the equilibrium solutions, and sketch the solution $y(x)$ for each of the following initial conditions:

$$y(0) = 2; \qquad y(0) = 6; \qquad y(0) = 9$$

(c) For which value(s) of $y(0)$ is $\lim_{x\to\infty} y(x) > 0$?

Question 8 (9 marks) A 500L tank initially contains 100L of pure water. Beginning at time $t = 0$, water containing 0.5 g/L of pollutants flows into the tank at a rate of 2L/minute, and the well-stirred solution is drained out of the tank at a rate of 1L/minute.

The amount $x(t)$ (in grams, g) of pollutant in the tank at time t minutes satisfies the ODE

$$\frac{dx}{dt} = 1 - \frac{x}{100 + t}.$$

Find the concentration of pollutant in the tank at the moment the tank overflows.

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Question 9 (10 marks) Consider the ODE

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + 9y = \sin(\omega t)$$

where $\alpha \in \mathbb{R}$ and $\omega > 0$ are constants.

(a) Let $\alpha = 6$ and $\omega = 1$. Find the general solution of the ODE.

Space for question 9(a) continued

(b) For what value(s) of α and ω would a particular solution $y_p(t)$ of the inhomogeneous ODE have the form

$$y_p(t) = at\cos(\omega t) + bt\sin(\omega t)$$

where $a, b \in \mathbb{R}$? Explain your reasoning.

Question 10 (7 marks) An object of mass 1kg is attached to a spring hanging vertically from a fixed support. The spring has spring constant $k = 50$ N m^{-1}. In equilibrium, the spring is stretched a distance s m. Assume that the gravitational constant is $g = 10$ m s^{-2}.

The system is subject to a damping force with damping constant $\beta = 10$ N s m^{-1}. In addition, a constant downward external force of $f = 100$ N is exerted on the object.

Let $y(t)$ be the displacement in metres of the object below the system's equilibrium position at time t seconds.

At $t = 0$ the object is released from rest, 0.2m below its equilibrium position.

(a) Use Newton's 2nd law to show that the equation of motion for the system is

$$y'' + 10y' + 50y = 100.$$

Include a diagram of the system at a time when the object is below its equilibrium position and moving up, with all forces shown and labelled.

(b) State the initial conditions, in terms of y and y'.

(c) Let $v(t) = y'(t)$ be the velocity of the object. Differentiate the equation of motion to obtain a second-order ODE for $v(t)$.

(d) Show that $v'(0) = 90$.

Question 11 (13 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \operatorname{sech}\left(\sqrt{x^2 + y^2}\right)$$

and the surface S with equation $z = f(x, y)$.

(a) Find the equation of the level curve of f at $z = c$.

(b) Sketch the level curve for each of the following values of c, or explain why it is not possible.

$$c = 0, \qquad c = \frac{1}{2}, \qquad c = 1$$

(c) Find the equation of the cross section of S in the y-z plane, and sketch it.

(d) Sketch the surface S.

(e) Prove that the direction of steepest increase of f at a point $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is in the direction of the vector $\mathbf{v} = (-x, -y)$. Interpret this geometrically.

Question 12 (7 marks) Let $g : \mathbb{R}^2 \to \mathbb{R}$ be given by $g(x, y) = (2y - y^2)e^{-x^2}$.

(a) Find and classify the stationary points of g.

(b) Give an example of a point $(x, y) \in \mathbb{R}^2$ where the gradient of g is non-zero and the direction of steepest increase of g is parallel to the y-axis. Explain your reasoning, or explain why it is not possible to find such a point.

End of Exam — Total Available Marks = 110