THE UNIVERSITY OF MELBOURNE

Student number

Semester 1 Assessment, 2021

School of Mathematics and Statistics

MAST10006 Calculus 2

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 24 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- No calculators are permitted.

Instructions to Students

- If you have a printer, print the exam one-sided. If using an electronic PDF reader to read the exam, it must be disconnected from the internet. Its screen must be visible in Zoom. No mathematical or other software on the device may be used. No file other than the exam paper may be viewed.
- Ask the supervisor if you want to use the device running Zoom.

Writing

- There are 13 questions with marks as shown. The total number of marks available is 119.
- Write your answers in the boxes provided on the exam that you have printed or the masked exam template that has been previously made available. If you need more space, you can use blank paper. Note this in the answer box, so the marker knows. The extra pages can be added to the end of the exam to scan.
- If you have been unable to print the exam and do not have the masked template write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning

- Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Make sure that you upload the correct PDF file and that your PDF file is readable.

Submitting

- **You must submit while in the Zoom room.** No submissions will be accepted after you have left the Zoom room.
- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Wait for Gradescope email confirming your submission. Tell your supervisor when you have received it.

Question 1 (12 marks)

Evaluate the following limits if they exist, or explain why they do not exist. In this question you must state if you use any standard limits, limit laws, continuity, l'Hôpital's rule or the Ssandwich Theorem.

(a)

$$\lim_{x\to\infty}(\sin(x^2) - 1)$$

(b)

$$\lim_{x\to 0}\frac{\sin(x^2) - x^2}{\cos(x^2) - 1}$$

(c)

$$\lim_{x\to\infty} \frac{\sin(x^2) - x^2}{x^2 - 1}$$

(d)

$$\lim_{n\to\infty} \frac{2n^3 + n + 5}{6n^3 + 4n^2 + 1} \quad (n \in \mathbb{N})$$

Question 2 (8 marks)

Determine whether or not the following series converge. In this question, you must state any tests for convergence or divergence that you use.

(a)

$$\sum_{n=1}^{\infty} \frac{n^2+n+1}{2n^3-1}$$

(b)

$$\sum_{n=1}^{\infty} \frac{n^3}{n!}$$

Question 3 (7 marks)

Consider the family of sequences

$$f_n = \left(\frac{xn}{xn+1}\right)^n, \quad (n \in \mathbb{N})$$

where x is a real constant.
Note: This is a *family* of sequences as (f_n) denotes a different sequence for each value of x.

(a) Find all values of x for which the sequence (f_n) converges. You should state any standard limits that you use.

(b) Let D be the set of values of x found in part (a). Define a function $f : D \to \mathbb{R}$ given by

$$f(x) = \lim_{n\to\infty} f_n.$$

This means the value of the function at a particular x value is the limit of the sequence for that particular value of x.
Is f continuous at every point in D? Briefly justify your answer.

Question 4 (7 marks)

(a) Use the definition of $\cosh(\theta)$ to show that

$$\cosh^4(\theta) = \frac{1}{8}\cosh(4\theta) + \frac{1}{2}\cosh(2\theta) + \frac{3}{8}.$$

(b) Hence evaluate the following integral:

$$\int \cosh^2(\theta)(1 + \sinh^2(\theta))\,\mathrm{d}\theta.$$

Question 5 (7 marks)

(a) Calculate $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{2021}$. Express your answer in the form $a + ib$, where a, $b \in \mathbb{R}$.

(b) Calculate the 2021st derivative of the function $f(t) = e^{\frac{t}{\sqrt{2}}} \cos(\frac{t}{\sqrt{2}})$.

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Question 6 (13 marks)

Compute the following integrals, stating the technique that you use.

(a)

$$\int x^2 \cosh(x^3 + 2)\,\mathrm{d}x$$

(b)

$$\int \frac{x^4 + 2x + 1}{(x-1)(x^2+1)}\,\mathrm{d}x.$$

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Space for 6(b).

(c)

$$\int (s^2 + 2s + 2)^{-3/2} \, \mathrm{d}s$$

Hint: Use the substitution $s = \tan\theta - 1$.

Question 7 (8 marks)

Consider the ODE

$$xy^2\frac{dy}{dx} = x^6 \operatorname{cosec}\left(\frac{2y}{x}\right) + y^3 \qquad \text{with } -\frac{\pi}{2} < \frac{y}{x} < \frac{\pi}{2},\ \ x \neq 0 \tag{1}$$

(a) Make the substitution $u = \frac{y}{x}$ to obtain the ODE

$$u^2\frac{du}{dx} = x^2 \operatorname{cosec}(2u) \qquad \text{with } -\frac{\pi}{2} < u < \frac{\pi}{2},\ \ x \neq 0. \tag{2}$$

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(b) Find the general solution of the ODE (2) and hence find the general solution to the ODE (1).
You do not need to express u in terms of x when solving ODE (2), and you do not need to express y in terms of x in the solution to ODE (1).
You do not need to specify the domain of the solution or the constant from integration.

Question 8 (11 marks)

Consider the ODE

$$\frac{dy}{dx} = y^4 - 16y^2. \tag{3}$$

(a) Find all the equilibrium solutions.

(b) Sketch a phase plot showing the equilibrium solutions, and label the points where the derivative is at a maximum or minimum.

(c) Determine the stability of the equilibrium solutions.

(d) Sketch the family of solutions for $x \geq 0$. Your sketch should contain the equilibrium solutions, and the solutions with the initial conditions $y(0) = -1$, $y_2(0) = 1$, and $y_3(0) = 5$. What is the long term behaviour for when the initial condition is $y(0) = 1$? (**Do not try to solve the ODE!**)

Question 9 (10 marks)

Consider the second order ODE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 16y = \cosh(2x). \tag{4}$$

(a) Compute the general solution of the homogeneous ODE

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 16y = 0. \tag{5}$$

(b) Find a particular and the general solution of the inhomogeneous ODE (4).
Hint: Use the definition of $\cosh x$ in terms of exponentials.

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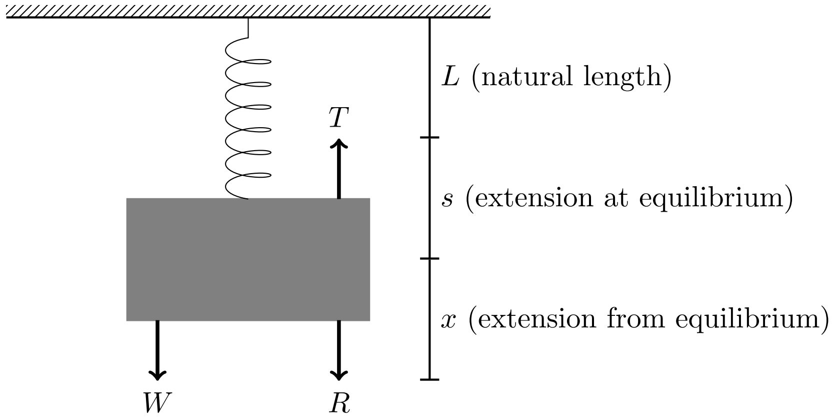
Question 10 (7 marks)

A mass is attached to one end of a spring and hung from a ceiling.



W is the force due to gravity, R is the damping force and T is the spring force.

(a) Suppose that the mass on the spring is 2 kg, the damping constant is $2\sqrt{2}$ kg/s and the spring constant is 1 kg/s^2. Starting from Newton's Second Law, show that the displacement of the mass from the equilibrium position can be described by the differential equation

$$\ddot{x} + \sqrt{2}\dot{x} + \frac{x}{2} = 0.$$

(b) Is the system under-damped, critically damped or over-damped? Briefly justify your answer.

(c) Suppose we want the displacement of the spring from equilibrium to be described by $x_p(t) = \sin(t)$.
Assuming the same mass, damping constant and spring constant as in part (c), what external force should we apply to the mass to obtain this displacement?

Question 11 (12 marks)

Consider the function

$$f(x, y) = (x^2 + y^2 - 4)^2.$$

(a) Sketch the level curves $f(x, y) = c$ for $c = 0$, $c = 4$ and $c = 16$. Clearly label the curves so that a reader can tell which curve is which. Label all intercepts. (6)

(b) Sketch the x-z and y-z cross sections of the graph of $z = f(x, y)$. Label all intercepts, as well as any stationary points.

(c) Suppose, instead of the $x - z$ or $y - z$ cross-sections, we considered the cross-section of the surface $z = f(x, y)$, with the plane $y - x = 0$. Describe in words (or with a drawing) what the cross-section of the surface would look like, giving a brief reason for your answer.
Hint: It might help to think about what the whole surface looks like first.

Question 12 (5 marks)
The tangent plan to a function $f(x, y)$ of two variables at the point $(x, y) = (1, 1)$ is given by the equation

$$2x + 3y + z = 1$$

(a) Find the direction that f is increasing the most rapidly. Give your answer as a unit vector.

(b) Find the directional derivative of f at the point $(1, 1)$ going from $(1, 1)$ to $(2, 3)$.

Question 13 (12 marks)

Consider the function

$$f(x, y) = \cos(x + y) + y^2 \tag{7}$$

(a) Find all the critical points of f and determine whether they are local minimums, local maximums or saddle points.

(b) Compute the iterated integral $\int_0^{\pi} \int_0^{\pi} f(x, y)\ \mathrm{d}y\ \mathrm{d}x$, without changing the order of integration.

(c) Compute the iterated integral $\int_0^{\pi} \int_0^{\pi} f(x, y)\ \mathrm{d}x\ \mathrm{d}y$, without changing the order of integration. Compare this with the result in part b) and name the theorem which explains your findings.

End of Exam — Total Available Marks = 119

MAST10006 Calculus 2 Formulae Sheet

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$

$1 + \tan^2 x = \sec^2 x$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$ $\qquad$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\cos(2x) = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh(2x) = \cosh^2 x + \sinh^2 x$

$\cos(2x) = 2\cos^2 x - 1$ $\qquad$ $\cosh(2x) = 2\cosh^2 x - 1$

$\cos(2x) = 1 - 2\sin^2 x$ $\qquad$ $\cosh(2x) = 1 + 2\sinh^2 x$

$\sin(2x) = 2\sin x\cos x$ $\qquad$ $\sinh(2x) = 2\sinh x\cosh x$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ $\qquad$ $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$

$\sin(x+y) = \sin x\cos y + \cos x\sin y$ $\qquad$ $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1}) \qquad \operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$$

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0) \qquad \lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0) \qquad \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = \mathrm{e}^a \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R},\ a > 1)$$