Bayesian Network Practice - Solutions

Week 13

Question 1, Part 1, a

$$
\begin{aligned}
P(t) &= \sum_{r,l,q,s} P(R_r, L_l, t, S_s, Q_q) \\
&= \sum_{r,l,q,s} P(R_r)P(L_l|R_r)P(Q_q|L_l)P(t|Q_q, R_r)P(S_s|Q_q, t) \\
&= \sum_{r} P(R_r) \sum_{l} P(L_l|R_r) \sum_{q} P(Q_q|L_l)P(t|Q_q, R_r) \sum_{s} P(S_s|Q_q, t) \\
& \qquad \text{(Try to push the summations as far right as you can)} \\
&= \sum_{r} P(R_r) \sum_{l} P(L_l|R_r) \sum_{q} P(Q_q|L_l)P(t|Q_q, R_r) \times 1 \qquad \text{(See notes section)} \\
&= \sum_{r} P(R_r) \sum_{l} P(L_l|R_r) \sum_{q} P(Q_q|L_l)P(t|Q_q, R_r) \\
&= \sum_{r} P(R_r) \sum_{l} P(L_l|R_r)\Big(P(q|L_l)P(t|q, R_r) + P(\neg q|L_l)P(t|\neg q, R_r)\Big) \\
&= \sum_{r} P(R_r)\Bigg(P(l|R_r)\Big(\underbrace{P(q|l)}_{0.9}P(t|q, R_r) + \underbrace{P(\neg q|l)}_{0.1}P(t|\neg q, R_r)\Big) \\
& \qquad + P(\neg l|R_r)\Big(\underbrace{P(q|\neg l)}_{0.7}P(t|q, R_r) + \underbrace{P(\neg q|\neg l)}_{0.3}P(t|\neg q, R_r)\Big)\Bigg) \\
&= \underbrace{P(r)}_{0.2}\Bigg(\underbrace{P(l|r)}_{0.8}\Big(0.9 \times \underbrace{P(t|q, r)}_{0.7} + 0.1 \times \underbrace{P(t|\neg q, r)}_{0.2}\Big) + \underbrace{P(\neg l|r)}_{0.2}\Big(0.7 \times \underbrace{P(t|q, r)}_{0.7} + 0.3 \times \underbrace{P(t|\neg q, r)}_{0.2}\Big)\Bigg) + \\
& \quad \underbrace{P(\neg r)}_{0.8}\Bigg(\underbrace{P(l|\neg r)}_{0.3}\Big(0.9 \times \underbrace{P(t|q, \neg r)}_{0.9} + 0.1 \times \underbrace{P(t|\neg q, \neg r)}_{0.3}\Big) + \underbrace{P(\neg l|\neg r)}_{0.7}\Big(0.7 \times \underbrace{P(t|q, \neg r)}_{0.9} + 0.3 \times \underbrace{P(t|\neg q, \neg r)}_{0.3}\Big)\Bigg) \\
&= 0.2\Big(0.8\big(0.9 \times 0.7 + 0.1 \times 0.2\big) + 0.2\big(0.7 \times 0.7 + 0.3 \times 0.2\big)\Big) + \\
& \quad 0.8\Big(0.3\big(0.9 \times 0.9 + 0.1 \times 0.3\big) + 0.7\big(0.7 \times 0.9 + 0.3 \times 0.3\big)\Big) \\
&= 0.2\Big(0.8\big(0.65\big) + 0.2\big(0.55\big)\Big) + 0.8\Big(0.3\big(0.84\big) + 0.7\big(0.72\big)\Big) \\
&= 0.2 \times 0.63 + 0.8 \times 0.756 \\
&= 0.7308 \\
\therefore P(t) &= 0.7308
\end{aligned}
$$

Notes

Note: $\sum_x P(X_x) = 1$ and also $\sum_x P(X_x|something) = 1$ because you are summing over all possible values of X. It's like calculating the probability that quokkas are happy plus the probability that they aren't, which is always just 1!

Question 1, Part 1, b

Step 1: get equation

$$\begin{aligned}
P(t) &= \sum_{r,l,q,s} P(R_r, L_l, t, S_s, Q_q) \\
&= \sum_{r,l,q,s} P(R_r)P(L_l|R_r)P(Q_q|L_l)P(t|Q_q,R_r)P(S_s|Q_q,t) \\
&= \sum_r P(R_r) \sum_l P(L_l|R_r) \sum_q P(Q_q|L_l)P(t|Q_q,R_r) \sum_s P(S_s|Q_q,t)
\end{aligned}$$

(Try to push the summations as far right as you can)

$$= \sum_r \underbrace{P(R_r)}_{f_1(R)} \sum_l \underbrace{P(L_l|R_r)}_{f_2(L,R)} \sum_q \underbrace{P(Q_q|L_l)}_{f_3(Q,L)} \underbrace{P(t|Q_q,R_r)}_{f_4(Q,R)} \sum_s \underbrace{P(S_s|Q_q,t)}_{f_5(S,Q)}$$

(Label the different parts)

Step 2: work out factors

R	$f_1(R)$
T	0.2
F	0.8

L	R	$f_2(L,R)$
T	T	0.8
T	F	0.3
F	T	0.2
F	F	0.7

Q	L	$f_3(Q,L)$
T	T	0.9
T	F	0.7
F	T	0.1
F	F	0.3

Q	R	$f_4(Q,R)$
T	T	0.7
T	F	0.9
F	T	0.2
F	F	0.3

S	Q	$f_5(S,Q)$
T	T	0.9
T	F	0.4
F	T	0.1
F	F	0.6

Step 3: sum over S

$$P(t) = \sum_r f_1(R) \sum_l f_2(L,R) \sum_q f_3(Q,L) f_4(Q,R) \underbrace{\sum_s f_5(S,Q)}_{f_6(Q)}$$

(We'll name the new factor f_6)

S	Q	$f_5(S,Q)$
T	T	0.9
T	F	0.4
F	T	0.1
F	F	0.6

→

Q	$f_6(Q)$
T	0.9 + 0.1
F	0.4 +0.6

→

Q	$f_6(Q)$
T	1
F	1

Notice that we got 1 in both cases. Here, just like in the enumeration section, we could have just immediately set $\sum_s f_5(S,Q)$ to 1

Step 4: multiply factors

$$P(t) = \sum_r f_1(R) \sum_l f_2(L,R) \sum_q \underbrace{f_3(Q,L) f_4(Q,R) f_6(Q)}_{f_7(Q,R,L)}$$

(We'll name the new factor f_7)

Q	L	$f_3(Q,L)$
T	T	0.9
T	F	0.7
F	T	0.1
F	F	0.3

Q	R	$f_4(Q,R)$
T	T	0.7
T	F	0.9
F	T	0.2
F	F	0.3

Q	$f_6(Q)$
T	1
F	1

→

Q	R	L	$f_3 \times f_4 \times f_6$
T	T	T	$0.9 \times 0.7 \times 1$
T	T	F	$0.7 \times 0.7 \times 1$
T	F	T	$0.9 \times 0.9 \times 1$
T	F	F	$0.7 \times 0.9 \times 1$
F	T	T	$0.1 \times 0.2 \times 1$
F	T	F	$0.3 \times 0.2 \times 1$
F	F	T	$0.1 \times 0.3 \times 1$
F	F	F	$0.3 \times 0.3 \times 1$

→

Q	R	L	$f_7(Q,R,L)$
T	T	T	0.63
T	T	F	0.49
T	F	T	0.81
T	F	F	0.63
F	T	T	0.02
F	T	F	0.06
F	F	T	0.03
F	F	F	0.09

Step 5: sum over Q

$$P(t) = \sum_r f_1(R) \sum_l f_2(L,R) \underbrace{\sum_q f_7(Q,R,L)}_{f_8(R,L)}$$ (We'll name the new factor f_8)

Q	R	L	$f_7(Q,R,L)$
T	T	T	0.63
T	T	F	0.49
T	F	T	0.81
T	F	F	0.63
F	T	T	0.02
F	T	F	0.06
F	F	T	0.03
F	F	F	0.09

$\rightarrow$

R	L	$f_8(R,L)$
T	T	$0.63 + 0.02$
T	F	$0.49 + 0.06$
F	T	$0.81 + 0.03$
F	F	$0.63 + 0.09$

$\rightarrow$

R	L	$f_8(R,L)$
T	T	0.65
T	F	0.55
F	T	0.84
F	F	0.72

Step 6: multiply factors

$$P(t) = \sum_r f_1(R) \sum_l \underbrace{f_2(L,R) f_8(R,L)}_{f_9(R,L)}$$ (We'll name the new factor f_9)

L	R	$f_2(L,R)$
T	T	0.8
T	F	0.3
F	T	0.2
F	F	0.7

R	L	$f_8(R,L)$
T	T	0.65
T	F	0.55
F	T	0.84
F	F	0.72

$\rightarrow$

R	L	$f_2 * f_8$
T	T	0.8×0.65
T	F	0.2×0.55
F	T	0.3×0.84
F	F	0.7×0.72

$\rightarrow$

R	L	f_9
T	T	0.52
T	F	0.11
F	T	0.252
F	F	0.504

Note: be careful about the order! In the f_2 table, L is on the left. In the other table, L is on the right!

Step 7: almost there! Sum over L

$$P(t) = \sum_r f_1(R) \underbrace{\sum_l f_9(R,L)}_{f_{10}(R)}$$ (We'll name the new factor f_{10})

R	L	f_9
T	T	0.52
T	F	0.11
F	T	0.252
F	F	0.504

$\rightarrow$

R	$f_{10}(R)$
T	$0.52 + 0.11$
F	$0.252 + 0.504$

$\rightarrow$

R	$f_{10}(R)$
T	0.63
F	0.756

Step 8: multiply one more time!

$$P(t) = \sum_r \underbrace{f_1(R) f_{10}(R)}_{f_{11}(R)}$$ (We'll name the new factor f_{11})

R	$f_1(R)$
T	0.2
F	0.8

R	$f_{10}(R)$
T	0.63
F	0.756

$\rightarrow$

R	$f_1 \times f_{10}$
T	0.2×0.63
F	0.8×0.756

$\rightarrow$

R	$f_1 \times f_{10}$
T	0.126
F	0.6048

Step 9: finally, sum over R

$$\begin{aligned} P(t) &= \sum_r f_{11}(R) \\ &= 0.126 + 0.6048 \\ &= 0.7308 && \text{(Yay! :D)} \end{aligned}$$

Question 1, Part 2

Step 1: Use Bayes rule

$$P(q \mid s, \neg r) = \frac{P(q, s, \neg r)}{P(s, \neg r)}$$

Before going on

From here, if we wanted to, we could work out the numerator and denominator individually by following the same process as in the last question. That's a lot of work, though! It would be better if we did something a bit sneaky instead!

What is that, you ask? Well, let's think about this for a second. We also know:

$$P(\neg q \mid s, \neg r) = \frac{P(\neg q, s, \neg r)}{P(s, \neg r)}$$

Now, if we knew $P(q, s, \neg r)$ AND $P(\neg q, s, \neg r)$, we could work out the denominator by adding them together. Then, we could divide $P(q, s, \neg r)$ by the denominator to get our answer. In other words we could get $\mathbf{P}(Q, s, \neg r)$ and normalise this to get our answer. And that's what we're going to do!

Step 2: get equation

$$\begin{aligned} \mathbf{P}(Q, s, \neg r) &= \sum_{t,l} P(Q, T_t, L_l, \neg r, s,) \\ &= \sum_{t,l} P(\neg r) P(L_l|\neg r) P(Q|L_l) P(T_t|Q, \neg r) P(s|Q, T_t) \\ &= P(\neg r) \sum_t P(T_t|Q, \neg r) P(s|Q, T_t) \sum_l P(L_l|\neg r) P(Q|L_l) \end{aligned}$$

(Note: there is more than one way to do this! Eg. t and l swapped)

Step 3: factors

$$\mathbf{P}(Q, s, \neg r) = \underbrace{P(\neg r)}_{0.8} \sum_t \underbrace{P(T_t|Q, \neg r)}_{f_1(T,Q)} \underbrace{P(s|Q, T_t)}_{f_2(Q,T)} \sum_l \underbrace{P(L_l|\neg r)}_{f_3(L)} \underbrace{P(Q|L_l)}_{f_4(Q,L)}$$

T	Q	$f_1(Q,T)$
T	T	0.9
T	F	0.3
F	T	0.1
F	F	0.7

Q	T	$f_2(Q,T)$
T	T	0.9
T	F	0.6
F	T	0.4
F	F	0.1

L	$f_3(L)$
T	0.3
F	0.7

Q	L	$f_4(Q,L)$
T	T	0.9
T	F	0.7
F	T	0.1
F	F	0.3

Step 4: multiply

$$\mathbf{P}(Q, s, \neg r) = 0.8 \sum_t f_1(T, Q) f_2(Q, T) \sum_l \underbrace{f_3(L) f_4(Q, L)}_{f_5(Q,L)}$$

Q	L	$f_5(Q, L)$
T	T	$0.9 \times 0.3 = 0.27$
T	F	$0.7 \times 0.7 = 0.49$
F	T	$0.1 \times 0.3 = 0.03$
F	F	$0.3 \times 0.7 = 0.21$

Step 5: sum over L

$$\mathbf{P}(Q, s, \neg r) = 0.8 \sum_t f_1(T, Q) f_2(Q, T) \underbrace{\sum_l f_5(Q, L)}_{f_6(Q)}$$

Q	$f_6(Q)$
T	$0.27 + 0.49 = 0.76$
F	$0.03 + 0.21 = 0.24$

Step 6: multiply

$$\mathbf{P}(Q, s, \neg r) = 0.8 \sum_t \underbrace{f_1(T, Q) f_2(Q, T) f_6(Q)}_{f_7(Q,T)}$$

Q	T	$f_7(Q, T)$
T	T	$0.9 \times 0.9 \times 0.76 = 0.6156$
T	F	$0.1 \times 0.6 \times 0.76 = 0.0456$
F	T	$0.3 \times 0.4 \times 0.24 = 0.0288$
F	F	$0.7 \times 0.1 \times 0.24 = 0.0168$

Step 7: sum over T

$$\mathbf{P}(Q, s, \neg r) = 0.8 \underbrace{\sum_t f_7(Q, T)}_{f_8(Q)}$$

Q	$f_8(Q)$
T	$0.6156 + 0.0456 = 0.6612$
F	$0.0288 + 0.0168 = 0.0456$

Step 8: normalise

All right, we've got:

$$\begin{aligned}\mathbf{P}(Q, s, \neg r) &= 0.8 \times < 0.6612, 0.0456 > \\ &= < 0.8 \times 0.6612, 0.8 \times 0.0456 > \\ &= < 0.52896, 0.03648 >\end{aligned}$$

(Note: this 0.8 doesn't really matter, because we're going to normalise later anyway)

Now, we're going to normalise by dividing each term by their sum (ie. $0.52896 + 0.03648$), and then we can get our answer!

$$\begin{aligned}\mathbf{P}(Q \mid s, \neg r) &= \frac{< 0.52896, 0.03648 >}{0.52896 + 0.03648} \\ &= \frac{< 0.52896, 0.03648 >}{0.56544} \\ &= \frac{< 0.52896, 0.03648 >}{0.56544} \\ &\approx < 0.935, 0.065 >\end{aligned}$$

($\approx$ means "approximately")

$$\therefore P(q \mid s, \neg r) \approx 0.935$$

($\therefore$ means "therefore")

So, the probability of quokkas being happy given that people are taking lots of quokka selfies and it is not raining is about 0.935!

Question 1, Part 3

Well, in the previous question we got a probability of 0.935 that the quokkas are happy given the evidence. So, since this is more than 0.5, we will predict that they are happy. Yay! :D (I guess you could say we BAYESically had the answer already... :D)

Question 1, Parts 4 and 5

$$\begin{aligned}P(r \mid \neg l, s) &= 0.0619 \\ P(l \mid q, t, s) &= 0.44218\end{aligned}$$

Question 2

$$
\begin{aligned}
P(U,O) &= \sum_{q,w,a} P(Q_q, U, O, K_k, A_a) \\
&= \sum_{q,k,a} P(U)P(O)P(Q_q|U,O)P(K_k|Q)P(A_a|Q) \\
&= P(U)P(O)\sum_{q} P(Q_q|U,O)\sum_{k} P(K_k|Q)\sum_{a} P(A_a|Q) \\
&= P(U)P(O)\sum_{q} P(Q_q,U,O)\sum_{k} P(K_k|Q) \times 1 \\
&= P(U)P(O)\sum_{q} P(Q_q|U,O)\sum_{k} P(K_k|Q) \\
&= P(U)P(O)\sum_{q} P(Q_q|U,O) \times 1 \\
&= P(U)P(O)\sum_{q} P(Q_q|U,O) \\
&= P(U)P(O) \times 1 \\
&= P(U)P(O)
\end{aligned}
$$

Perfect! This is just what we wanted :). We've just shown U and O are independent!!!