ECOS3012 Midterm Exam Solutions S1, 2021

24 March 2022

Total: 32 points

1. (3 points) Find all pure and mixed Nash equilibria of the following game.

	Left	Right
Up	3, 4	**9**, **5**
Down	**6**, **2**	2, 1

Answer

Version 1: (D, L), (U, R), and a mixed NE with the following probabilities: Let $p = \Pr(U)$ and $q = \Pr(L)$

$$3q + 9(1-q) = 6q + 2(1-q) \Rightarrow q = 0.7$$

$$4p + 2(1-p) = 5p + (1-p) \Rightarrow p = 0.5$$

2. (4 points) Find all pure and mixed Nash equilibria of the following game.

	L	R
U	5, 4	2, 8
M	3, 10	3, 12
D	2, 7	6, 1

Answer:

M is strictly dominated by a mixed strategy: U with probability 1/2 and D with probability 1/2. This is because $EU(M) < \frac{1}{2}EU(U) + \frac{1}{2}EU(D)$ for any $0 \leq \Pr(L) \leq 1$.

Therefore, we can simply delete the middle row and solve the following games instead:

	L	R
U	**5**, 4	2, **8**
D	2, **7**	**6**, 1

- There isn't any pure NE.

- Let p denote $\Pr(U)$ and q denote $\Pr(L)$. Use the following equations, or draw best-response diagrams, to find the mixed NE strategies $(p^*, 1-p^*)$ for player 1 and $(q^*, 1-q^*)$ for player 2.

$$EU_1(U) = EU_1(D)$$
$$EU_2(L) = EU_2(R)$$

$$\begin{aligned} 5q + 2(1-q) &= 2q + 6(1-q) \\ 4p + 7(1-p) &= 8p + (1-p) \\ q^* &= \frac{4}{7} \\ p^* &= \frac{3}{5} \end{aligned}$$

- Marking standard:
- +1 mark for showing: M is strictly dominated by a mixed strategy of U and D (need to show workings).
- +1 mark for drawing the conclusion that there is no pure-strategy NE.
- +1 mark for p^* (need to show working), partial credit is given for wrong answers but correct equations
- +1 mark for q^* (need to show working), partial credit is given for wrong answers but correct equations

3. (8 points) A game with positive externality: n people are choosing the frequency of mask-wearing.

   Suppose that there are n (version 1: $n = 10$, version 2: $n = 5$) residents in a small community. Each person chooses how often they wear facial masks in public. Let $x_i \in [0, 1]$ denote the frequency that resident i wears a mask. $x_i = 0$ means that resident i never wears a mask in public; $x_i = 1$ means that resident i always wears a mask in public.

   All residents think it is uncomfortable to wear a mask. This discomfort is described by an increasing "total cost" function for mask-wearing:

   $C(x_i) = 2x_i^2$

   All residents also agree that masks protect them. The benefit of this protection that resident i gets depends on (i) how often i wears a mask themselves, and (ii) how often the other residents wear masks. In particular, for $i = 1, 2, 3, ..., n$ if resident i chooses to wear a mask with frequency x_i, then the total benefit for resident i is equal to

   $B_i(x_1, x_2, ..., x_n) = (x_1 + x_2 + x_3 + \cdots + x_n)^{1/2}$

   All residents want to maximise their total net benefit, i.e., total benefit - total cost.

   (a) (3 points) Suppose that in the socially optimal, symmetric scenario, every player chooses frequency x. Calculate the value of x.

   (b) (2 points) If resident 1 through $n-1$ are playing the socially optimal strategy x, what is resident n's best response?

(c) (3 points) What is the Nash equilibrium strategy for each resident?

(d) (1 point) Is the NE higher, lower, or the same compared to the SO?

Answer:

(a) The socially optimal frequency x solves

$$\max_x B(x) - C(x)$$

$$\max_x (10x)^{1/2} - 2x^2$$

Take the derivative of this objective function and set it equal to 0 to find the peak of this strictly concave function:

$$\begin{aligned} \frac{1}{2}(10x)^{-1/2} \times 10 - 4x &= 0 \\ \frac{5}{(10x)^{1/2}} &= 4x \\ x^{3/2} &= \frac{5}{4 \times 10^{1/2}} \\ x &= \left(\frac{5}{4 \times 10^{1/2}}\right)^{\frac{2}{3}} \\ &\approx 0.5386 \end{aligned}$$

(b) Take $x_1, ..., x_{n-1}$ as given, resident n chooses x_n to solve:

$$\max_{x_n} B(x_1, ..., x_n) - C(x_n)$$

$$\max_{x_n} (x_1 + x_2 + x_3 + \cdots + x_{10})^{1/2} - 2x_{10}^2$$

Plug in $x_1 = x_2 = ... = x_9 \approx 0.5386$. Take the derivative of this objective function with respect to x_{10} and set it equal to 0 to find the peak of this strictly concave function:

$$\frac{1}{2}(9 \times 0.5386 + x_{10})^{-1/2} - 4x_{10} = 0$$

Use the help of a software (e.g., Mathematica) or www.wolframalpha.com, the root of this function is

$$x_{10} \approx 0.0564.$$

If the other residents are all choosing the socially optimal frequency, the last resident wants to deviate to a lower frequency.

(c) In the Nash equilibrium, each resident takes the frequency of other residents as given, and:

- for each i, x_i solves

$$
\begin{aligned}
\frac{1}{2}\left(\sum_{j\neq i} x_j + x_i\right)^{-1/2} - 4x_i &= 0 \\
\left(\sum_{j\neq i} x_j + x_i\right)^{-1/2} &= 8x_i
\end{aligned}
$$

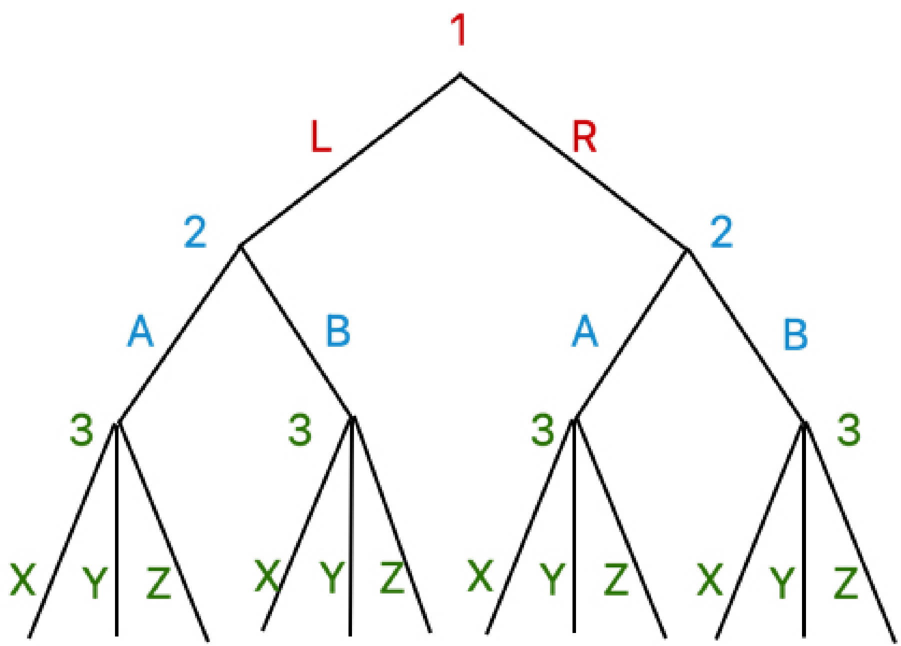
By symmetry, we expect residents to have identical solution in a symemtric equilibrium, so let's plug in $x_1 = x_2 = ... = x_{10} = x_{NE}$:

$$
\begin{aligned}
(10x_{NE})^{-1/2} &= 8x_{NE} \\
x_{NE}^{3/2} &= \frac{1}{10^{1/2} \times 8} \\
x_{NE} &= \left(\frac{1}{10^{1/2} \times 8}\right)^{2/3} \\
&\approx 0.11604
\end{aligned}
$$

(d) $x_{NE} < x_{SO}$. This is because "wearing a mask" is an activity that creates positive externality. In general, for any activity that creates positive externality, people don't do enough of it in equilibrium.

- Marking standard:
- Part (a): 1 mark for correct equation for value maximisation, 2 marks for calculation of x.
- Part (b): 1 mark for correct equation for value maximisation, 1 mark for calculation of x. (Full credit is given for carrying errors due to wrong answer in part a.)
- Part (c): 1 mark for correct equation for value maximisation, 1 mark for specifying "identical solutions in a symmetric equilibrium", 1 mark for calculation of x.
- Part (d): No partial mark.

4. (3 points)



(a) How many subgames does this game have?

Answer 7

(b) How many pure strategies does player 2 have?

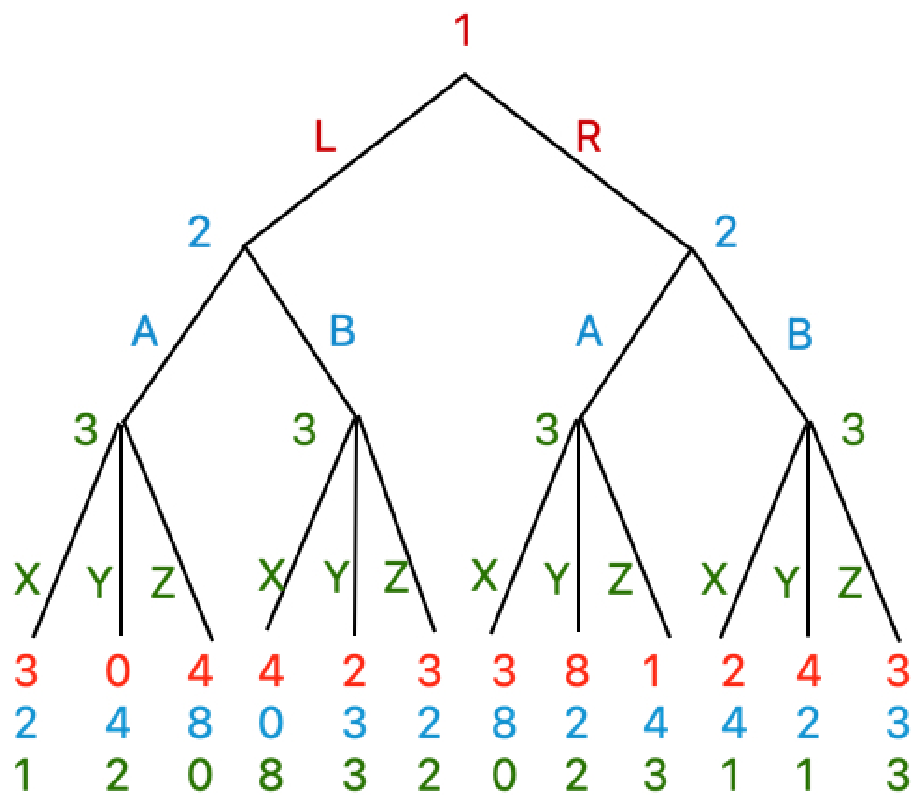
Answer $2 \times 2 = 4$.

(c) How many pure strategies does player 3 have?

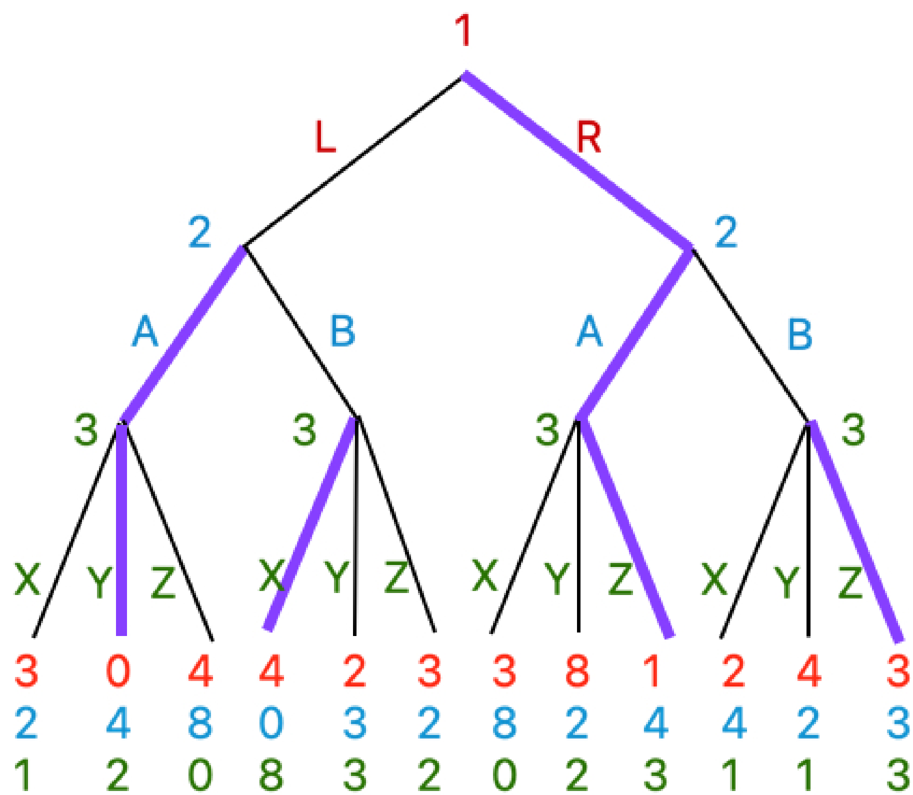
Answer:

There are 4 possible combinations of (P1's action, P2's action). Following each of these combinations, P3 has three possible choices. Therefore, the total number of pure strategies for P3 is $3 \times 3 \times 3 \times 3 = 81$.

5. (5 points)



(a) Find the subgame perfect equilibrium and correctly write down the full equilibrium strategies. (2 points)



SPE: (R, (A, A), (Y, X, Z, Z)).

- Marking standard: -1.5 if answer does not give a complete strategy that describes what to do for all subgames (for example, if the answers says "(R, A, Z)").

(b) (3 points) Is there a pure-strategy Nash equilibrium in which:

Player 3 plays Y in the equilibrium outcome.

If so, please write down the full equilibrium strategies for all three players that yield this outcome. If not, please explain why it is impossible.

- Marking standard:
- No points are given if the student simply said "yes" or "no".
- -2~3 if student incorrectly answered "yes" because of wrong equilibrium analysis.
- -2.5 if student answered no but the reasoning is wrong. For example, some students used incorrect payoff matrices. Player 1 has 2 pure strategies, player 2 has 4 pure strategies and player 3 has 3^4 = 81 strategies.

- -1~2 if student answered "no" but the explanation is incomplete (e.g., discussed only some but not all possible cases, or lacked details in the analysis, or did not describe what a profitable deviation looks like, or misinterpreted the question as "Player 3 always plays Y/Z in all subgames", which is not true).

Answer

No.

In order for Y to be a best response for P3, P1 must have chosen L and P2 must have chosen A (because otherwise P3 would have a profitable deviation to either X or Z). Let's focus on the path (L, A, Y). This can be a Nash equilibrium outcome if and only if neither P1 nor P2 has a unilateral deviation.

Player 2: Given that P1 chose L and that P3 will choose Y if P2 chooses A (but P3 is free to choose any action if P2 deviates to B), P2 gets payoff 4 if she chooses A, and a payoff of 3 or lower if she deviates to B. Therefore, P2 does not have a unilateral deviation.

Player 1: If he stays on this path (L, A, Y), his payoff is 0. However, if he deviates to R, then regardless of what P2 and P3 will choose, P1 can guarantee a payoff of 1 or higher. Therefore, P1 always has a profitable unilateral deviation.

Therefore, any path that leads to Y for P3 will either have a profitable deviation for P3 or a profitable deviation for P1. This cannot be a Nash equilibrium outcome.

6. (8 points) The following game is repeated infinitely many times. Suppose that the discount factor δ can be arbitrarily close to one (but not equal to 1).

	Left	Right
Up	4, 3	10, 6
Down	7, 1	3, 0

(a) (2 points) What is the highest average payoff that player 1 can get in any subgame perfect equilibrium?

(b) (2 points) What is the highest average payoff that player 2 can get in any subgame perfect equilibrium?

(c) (2 points) What is the infimum (greatest lower bound) of the average payoff that player 1 can get in any subgame perfect equilibrium?

(d) (2 points) What is the infimum (greatest lower bound) of the average payoff that player 2 can get in any subgame perfect equilibrium?

Answer: This is a question about the Folk Theorem. (a) and (b) ask for the highest feasible payoff, which is equal to the player's highest payoff number in the 2x2 payoff matrix; (c) and (d) ask for the minmax values.

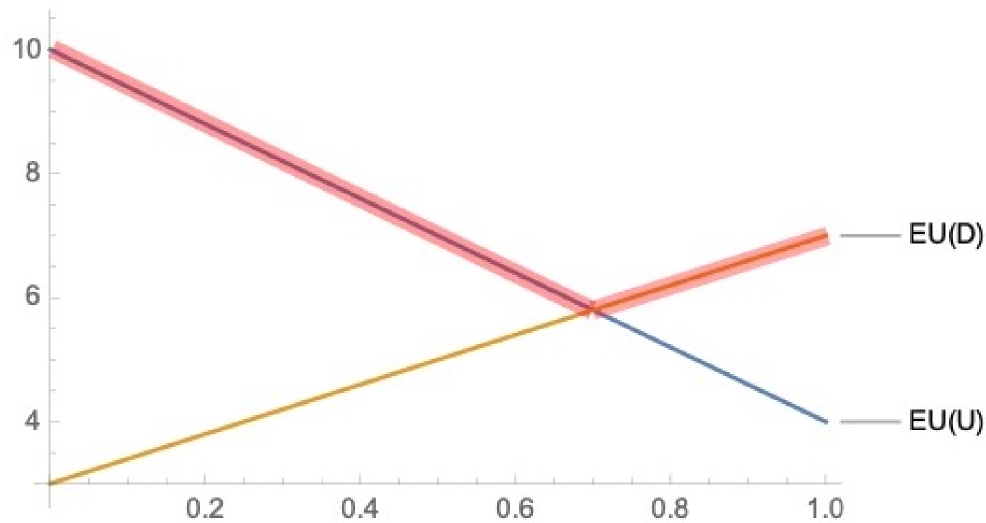
(a) 10. (b) 6. (c) 5.8 (d) 1

Let $p = \Pr(Up)$ and $q = \Pr(Left)$.

Calculate minmax for player 1:

$$EU(U) = 4q + 10(1 - q) = 10 - 6q$$

$$EU(D) = 7q + 3(1 - q) = 4q + 3$$

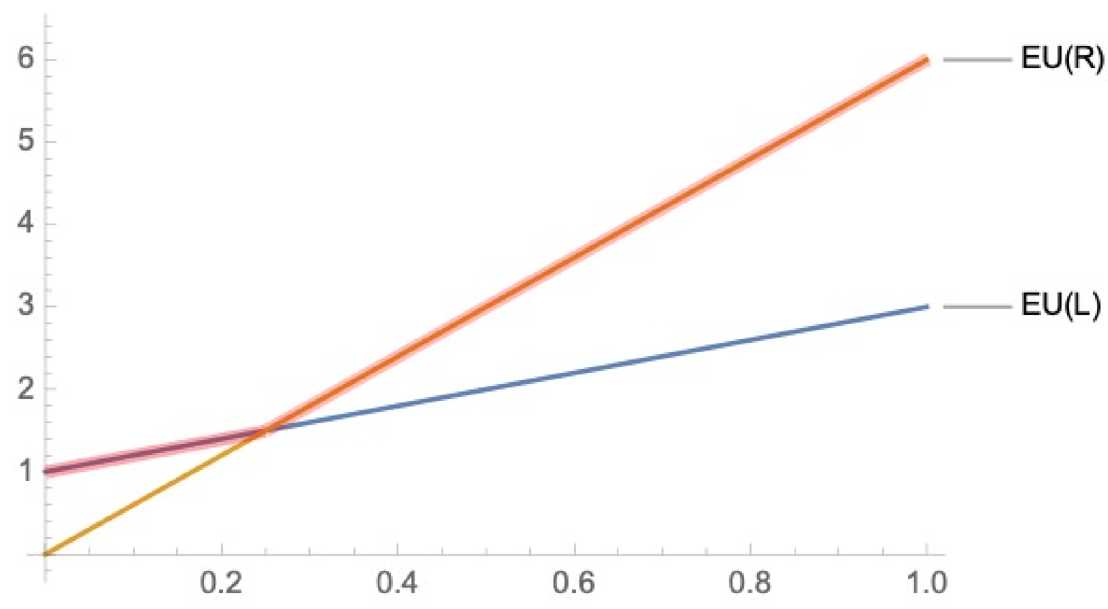


The minmax value for player 1, i.e., the lowest value on the red-highlighted best-response function, is achieved when $EU(U) = EU(D) = 5.8$.

Calculate minmax for player 2:

$$EU(L) = 3p + (1 - p) = 2p + 1$$

$$EU(R) = 6p$$



The minmax value for player 2 is equal to 1.

- Marking standard:
- -1 point each if answered (a), (b) using method for (c), (d)
- -0.5~1 point each if answered (a), (b) with present value and not average payoff
- -0.5~1 point each if answered (c), (d) with correct approach but the final conclusion on infimum is not provided
- -0.5~1 point each if student assumed that minmax for P2 is at EU(L) = EU(R) without explanation. -0.5 if correct graph is drawn but answer lacks explanation.
- Correct mentioning of the Folk Theorem gets 0.5-1 point.