ECOS3012 Mid-semester exam solution with marking guidelines

October, 2020

Q1 (20 points) There are 213 students in ECOS3012. Each student must pick a real number x between 0 and 100 (including 0 and 100). Let y be equal to $\frac{3}{4}\times$ the average of everyone's number. A student's payoff is equal to 1 if his number x is equal to y, and 0 if it isn't.

(a) Is there a Nash equilibrium in which all students pick the same number $x = 50$? Explain why it is or isn't a Nash equilibrium. (4 points)

(b) Find all Nash equilibria of this game. Explain your answer. (8 points)

(c) Let's now change the rule. Suppose that a student's payoff is equal to 1 if his number x is equal to $y + 20$, and 0 if it isn't. Find all Nash equilibria of this game. Explain your answer. (8 points)

Answer:

(a) This is the standard "beauty contest" game that we played in Lecture 1. If all students pick $x = 50$, this is not a Nash equilibrium. Given everyone else is choosing $x = 50$, student 1's payoff is 0 if he also chooses 50 because $50 \neq \frac{3}{4} \times 50$. However, he has a strictly profitable deviation to x' that satisfies

$$x' = \frac{3}{4} \times \frac{50 \times 212 + x'}{213}$$
$$x' = \frac{10600}{283} \approx 37.4558.$$

His payoff is 1 if he chooses x' instead.

- Marking guideline:
  - Must answered the question: $x = 50$ is not a NE. (2 marks)
    - -1 mark if only explained that $x = 0$ is a NE (without directly answering the question about $x = 50$)
  - Explanation of a profitable deviation (2 marks)
    - No mark was given if only showed $y \neq$ the average
    - "Payoff = 0" is not a good explanation of why it is not a NE.

(b) By iterative elimination of strictly dominated strategies, there is a unique Nash equilibrium: everyone picks 0.

- Marking guideline:
  - Correctly identifying the NE (4 marks)
  - Explanation of iterative elimination of strictly dominated strategies (4 marks)
    * No mark was given for simply mentioning that there is no profitable deviation without explaining why.

(c) Notice that

$$\frac{3}{4} \times 0 + 20 \leq y + 20 \leq \frac{3}{4} \times 100 + 20$$

$$20 \leq y + 20 \leq 95$$

This means that no matter what numbers the other students choose, a student can always find a number x between 0 and 100 so that $x = y + 20$.

Below, I show that there is a unique NE (everyone chooses $x = 80$) by discussing two cases.

Case 1: Suppose that everyone is choosing the same number x.

Let x solve

$$x = \frac{3}{4}x + 20$$
$$x = 80$$

If everyone chooses 80, everyone's payoff is equal to 1 and no one has a profitable deviation. Therefore, this is a Nash equilibrium.

There doesn't exist any Nash equilibrium in which everyone chooses some $x^{'} \neq 80$. If this were the case, then everyone's payoff is equal to 0 and someone has a profitable deviation to some $x^{''} = y + 20$ where y is calculated using the new average after the deviation to $x^{''}$.

Case 2: Suppose that not everyone is choosing the same number.

There doesn't exist any Nash equilibrium in which some or all students are choosing different numbers. If this were the case, at least one student is receiving a payoff of 0. That particular student has a profitable deviation to some $\hat{x} = y + 20$, where y is calculated using the new average after the deviation to $\hat{x}$.

- Marking guideline:
  - 4 points for finding the correct NE.
  - 4 points for showing that this is the unique NE.
    * No mark was given for simply mentioning that there is no profitable deviation without explaining why.

Q2 (20 points) Find all Nash equilibria of the following game.

		Player 2		
		L	C	R
	U	2, 3	2, 2	9, 2
Player 1	M	5, 1	0, 2	1, 0
	D	1, 4	5, 3	0, 1

Answer: R is strictly dominated by L. Let's delete R.

		Player 2	
		L (q)	C ($1-q$)
	U	2, 3	2, 2
Player 1	M	5, 1	0, 2
	D	1, 4	5, 3

U is strictly dominated by "1/2 M, 1/2 D".

$$EU(U) = 2$$

$$EU(1/2M, 1/2D) = \frac{1}{2}(5q) + \frac{1}{2}[q + 5(1-q)] = \frac{1}{2}q + \frac{5}{2} > 2 \text{ for all } q \in [0,1]$$

So, let's delete U, too.

		Player 2	
		L (q)	C ($1-q$)
Player 1	M (p)	**5**, 1	0, **2**
	D ($1-p$)	1, **4**	**5**, 3

There isn't any pure-strategy Nash equilibrium.

$$EU(M) = 5q, \ EU(D) = 5 - 4q$$

$$EU(M) = EU(D) \Rightarrow q = \frac{5}{9}$$

$$EU(L) = 4 - 3p, \ EU(C) = 3 - p$$

$$EU(L) = EU(C) \Rightarrow p = \frac{1}{2}$$

The unique NE is a mixed-strategy equilibrium: $p = \frac{1}{2}$, $q = \frac{5}{9}$.

(Alternatively, you can draw the best-response graphs to show that they intersect at only one point: $p = \frac{1}{2}$, $q = \frac{5}{9}$.)

- Marking guideline:
  - Showing R is strictly dominated (5 marks)
  - Showing U is strictly dominated by mixed strategies (must show workings) (5 marks)

    * No workings - 0 mark
  - Identifying that there is no pure-strategy NE (3 marks)
  - Identifying the unique NE (must show workings) (7 marks)
    * Computing errors - deduct 3-4 marks
    * No workings - 0 mark

Q3 (20 points) Three countries (A, B, and C) must decide whether to declare war on a common enemy. A country cannot win the war alone, but if two or three countries declare war together, they will win for sure. Each country's payoff is described by the following rule:

Let n be the total number of countries that declare war on the enemy.

If a country *declares war (D)*, its payoff is equal to

$$\begin{cases} -10 & \text{if } n = 1 \\ 40 & \text{if } n = 2 \\ 60 & \text{if } n = 3 \end{cases}$$

If a country *does not declare war (N)*, its payoff is equal to 0.

(a) Find all pure-strategy Nash equilibria if the three countries decide simultaneously. Write down the game matrix and show your work. (8 points)

For parts (b) and (c), assume that the three countries decide sequentially: first A, then B, then C. Each country knows what the previous country(/ies) had decided.

(b) Write down the game tree and find the subgame perfect equilibrium. Clearly write down each country's strategy in the equilibrium. (7 points)

(c) Find a Nash equilibrium that is not subgame perfect. Explain why it is a Nash equilibrium. (5 points)

Answer:

(a)

		Country B	
		D	N
Country A	D	**60**, **60**, **60**	**40**, 0, **40**
	N	0, **40**, **40**	0, 0, -10

Country C chooses D

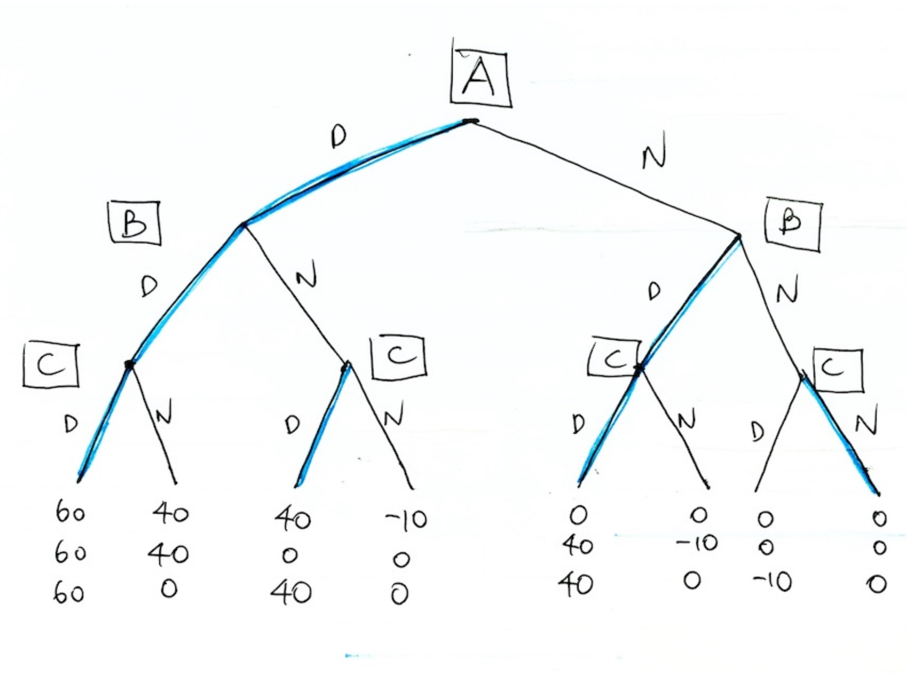
		B	
		D	N
A	D	**40**, **40**, 0	-10, 0, 0
	N	0, -10, 0	**0**, **0**, **0**

Country C chooses N

Pure NE: (D, D, D) and (N, N, N).

- Marking guideline:
  - Stated that (D, D, D) and (N, N, N) are NE without explanation. -5 marks
  - Only had two countries' payoffs in the game matrices. -3 marks
  - Wrote down the matrices correctly but did not find the equilibria. -3 marks
  - -1 mark for each wrong/missing equilibrium.

(b) SPE: Country A chooses D. Country B chooses (D, D). Country C chooses (D, D, D, N).



- Marking guideline:
  - Correct game tree. Correct and clear SPE that specifies each country's strategy in every subgame. (7 Marks)
  - Deduct 2 marks for writing down SPE = (D D D).

(c) The answer is not unique. Here are two possible answers:

Answer 1

Country A chooses N.

Country B chooses (N, N)

Country C chooses (N, N, N, N).

The equilibrium outcome is that all three countries choose N. This is a NE because, in the subgame that is reached in equilibrium, given that the other countries will always choose N in all scenarios, each country is indeed playing its best response. However, this is not a subgame perfect equilibrium, because there are profitable deviations in other subgames that were not reached in equilibrium: country B should choose D when country A chooses N, anticipating that country C will follow suit and choose D, as well.

Answer 2

Country A chooses D.

Country B chooses (D, D)

Country C chooses (D, D, D, D).

The equilibrium outcome is that all three countries choose D. This is a NE because, in the subgame that is reached in equilibrium, given that the other countries will always choose D in all scenarios, each country is indeed playing its best response by choosing D. However, this is not a subgame perfect equilibrium, because there are profitable deviations in other subgames that were not reached in equilibrium: country C should choose N when countries A and B both choose N.

- Marking guideline:
  - Correctly identifying a NE that is not a SPE (2 marks)
  - Must answer the question by explaining why it is a NE (3 marks)
    - Deduct 2 marks if student did not answer the question but explained why it is not a SPE.

Q4 (20 points) Consider the chicken game detailed below **repeated three times**. Each player's total payoff in the repeated game is the undiscounted sum of the payoffs in the three rounds.

		Player 2	
		Swerve	Keep going
Player 1	Swerve	2, 2	1, 3
	Keep going	3, 1	0, 0

Also consider the following strategy, where S stands for "Swerve" and K stands for "Keep going":

- Play (S, S) in the first round.
- Play the mixed strategy Nash equilibrium of the chicken game in the second and third rounds if (S, S) or (K, K) was played in the first round.
- Play (S, K) in the second and third rounds if (K, S) was played in the first round.
- Play (K, S) in the second and third rounds if (S, K) was played in the first round.

(a) Prove that this strategy is a subgame perfect equilibrium. (10 points)

(b) If the chicken game is repeated only **twice**, is there a subgame perfect equilibrium in which (S, S) is played in the first round? Please explain your answer. (10 points)

Answer:

(a) The mixed strategy of the chicken game:

$$2q + 1 - q = 3q \Rightarrow q = \frac{1}{2}$$

$$2p + 1 - p = 3p \Rightarrow p = \frac{1}{2}$$

Each player's expected utility from the chicken game is

$$EU(mixed) = \frac{3}{2}$$

Total 3-round payoff as a function of first-round outcome:

		Player 2	
		Swerve	Keep going
Player 1	Swerve	2+3, 2+3	1+6, 3+2
	Keep going	3+2, 1+6	0+3, 0+3

		Player 2	
		Swerve	Keep going
Player 1	Swerve	**5, 5**	**7, 5**
	Keep going	**5, 7**	3, 3

- Marking guideline:
  - +2 marks for correctly identifying the mixed-strategy equilibrium
  - +3 marks if the conclusion was correct but the explanation was wrong.
  - Deduct 3-4 marks if the conclusion was wrong because of incorrect payoff calculations.
  - Deduct 2 marks if there were minor errors in the payoff calculation which did not affect the conclusion. (For example, if incorrectly stated that $EU(mixed) = 3$.)

(b) This question is very similar to Q8(c) in practice midterm.

Suppose that the chicken game is repeated only twice and (S, S) is played in the first round. For this to be a SPE, players' payoff in the second round must be $(1, 3)$, $(3, 1)$, or $\left(\frac{3}{2}, \frac{3}{2}\right)$. The table below describes players' two-round total payoff. x_i and y_i represent the corresponding payoff in round 2 following a particular outcome in round 1.

		Player 2	
		Swerve	Keep going
Player 1	Swerve	$2 + x_1, 2 + y_1$	$1 + x_2, 3 + y_2$
	Keep going	$3 + x_3, 1 + y_3$	$0 + x_4, 0 + y_4$

If player 1 does not have any profitable deviation from (S, S) in round 1, we must have

$$2 + x_1 \geq 3 + x_3 \Rightarrow x_1 \geq 1 + x_3$$

The only way to satisfy this inequality, given that players play a NE in round 2, is to let $(x_1, y_1) = (3, 1)$ and $(x_3, y_3) = (1, 3)$ or $\left(\frac{3}{2}, \frac{3}{2}\right)$.

If player 2 does not have any profitable deviation from (S, S) in round 1, we must have

$$2 + y_1 \geq 3 + y_2 \Rightarrow y_1 \geq 1 + y_2$$

However, because $(x_1, y_1) = (3, 1)$ and y_2 must be equal to 1, $\frac{3}{2}$, or 3, it is impossible to satisfy this inequality. This means that if player 1 does not have a profitable deviation from (S, S) in round 1, player 2 must have a profitable deviation.

This proves that if the chicken game is repeated only twice, there **isn't** any subgame perfect equilibrium in which (S, S) is played in the first round.

- Marking guideline:
  - To get full marks, you must give a general proof that there isn't any SPE that supports (S, S) in round one for **any combination of strategies**. Only giving one or two examples is not enough.
    - Deduct 6 marks if you only proposed **one** strategy and showed that it's not SPE.
    - Deduct 5 marks if you only proposed **two or three** strategies and showed that they are not SPE.
    - Deduct 3 marks if you attempted to give a general proof but did not include the mixed strategy NE in your discussion.
  - Deduct 6 points if you incorrectly stated that the answer is "yes" because your payoff matrix was wrong (but the analysis after the wrong matrix was correct).
  - Deduct 8 points if you gave a wrong strategy/example.

Q5 (20 points) Consider the chicken game in Q4 **repeated infinitely many times**. Assume that the discount factor is as high as necessary (as long as it is smaller than 1), find a subgame perfect equilibrium that sustains payoffs (2, 2) in each period. In your answer, please clearly state the equilibrium strategy and the lowest discount factor for it to be a subgame perfect equilibrium.

		Player 2	
		Swerve	Keep going
Player 1	Swerve	2, 2	1, 3
	Keep going	3, 1	0, 0

Answer: There are many ways to construct a SPE that sustains (2, 2) in each period. Here I give two types of answers.

Answer 1:

Play (Swerve, Swerve) in the first period.

If players have always played (Swerve, Swerve) in all previous stages, continue to play (Swerve, Swerve) forever.

Otherwise, let "first deviation stage" refer to the first stage in which some player played "Keep going".

If "Keep going" is played by only player 1 and not player 2 in the first deviation stage, play (Swerve, Keep going) forever.

If "Keep going" is played by only player 2 and not player 1 in the first deviation stage, play (Keep going, Swerve) forever.

If both players played "Keep going" in the first deviation stage, play (Swerve, Swerve) forever.

Under this strategy, there are three types of subgames. Because the game and the proposed strategy is symmetric, it suffices to check profitable deviations for player 1.

1. “Keep going” is played by only player 1 and not player 2 in the first deviation stage.
   In this type of subgame, players should play (Swerve, Keep going) forever. Player 1’s present value, if he does not deviate, is

$$U = 1 + \delta + \delta^2 + ... = \frac{1}{1-\delta}$$

   If player 1 deviates to “Keep going”, the action sequence becomes (K, K), (S, K), (S, K), (S, K),...... and player 1’s present value is

$$U_{dev} = 0 + \delta + \delta^2 + \delta^3 + ... = \frac{\delta}{1-\delta} < \frac{1}{1-\delta}$$

   This deviation is never profitable for any $\delta < 1$.

2. “Keep going” is played by only player 2 and not player 1 in the first deviation stage.
   In this type of subgame, players should play (Keep going, Swerve) forever. Player 1’s present value, if he does not deviate, is

$$U = 3 + 3\delta + 3\delta^2 + ... = \frac{3}{1-\delta}$$

   If player 1 deviates to “Swerve”, the action sequence becomes (S, S), (K, S), (K, S), (K, S),...... and player 1’s present value is

$$U_{dev} = 2 + 3\delta + 3\delta^2 + 3\delta^3 + ... = 2 + \frac{3\delta}{1-\delta} < \frac{3}{1-\delta}$$

   This deviation is never profitable for any $\delta < 1$.

3. Neither case 1 nor case 2
   In this type of subgame, players should play (Swerve, Swerve) forever. Player 1’s present value, if he does not deviate, is

$$U = 2 + 2\delta + 2\delta^2 + ... = \frac{2}{1-\delta}$$

   If player 1 deviates to “Keep going”, the action sequence becomes (K, S), (S, K), (S, K), (S, K),...... and player 1’s present value is

$$U_{dev} = 3 + \delta + \delta^2 + \delta^3 + ... = 3 + \frac{\delta}{1-\delta}$$

   This deviation is not profitable if and only if

$$\frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta}$$

$$\delta \geq \frac{1}{2}$$

Cases 1, 2, 3: The proposed strategy is a SPE if and only if $\delta \geq \frac{1}{2}$.

Answer 2:

Play (S, S) in the first period.

If players have always played the pure strategy (S, S) in the past, continue to play (S, S) in the current period.

Otherwise, play the mixed strategy Nash equilibrium for the stage game $\left(p = \frac{1}{2},\ q = \frac{1}{2}\right)$ (which yields an expected payoff of $\frac{3}{2}$ for each player) forever.

Based on this strategy, there are two types of subgames. Because the game and the proposed strategy is symmetric, it suffices to check profitable deviations for player 1.

1. Players have always played (S, S) in the past.

   In this type of subgame, players should play (S, S) forever. Player 1's present value, if he does not deviate, is

   $$U = 2 + 2\delta + 2\delta^2 + ... = \frac{2}{1-\delta}$$

   If player 1 deviates to "Keep going", the action sequence becomes (K, S) in the first period, followed by the mixed strategy $p = q = \frac{1}{2}$ forever. Player 1's present value is

   $$U_{dev} = 3 + \frac{3}{2}\delta + \frac{3}{2}\delta^2 + \frac{3}{2}\delta^3 + ... = 3 + \frac{\frac{3}{2}\delta}{1-\delta}$$

   This deviation is not profitable if and only if

   $$\frac{2}{1-\delta} \geq 3 + \frac{\frac{3}{2}\delta}{1-\delta}$$

   $$\delta \geq \frac{2}{3}$$

2. Players have **not** always played (S, S) in the past.

   In this type of subgame, players should play the mixed-strategy NE of the stage game $\left(p = \frac{1}{2},\ q = \frac{1}{2}\right)$ in every period. Player 1's present value, if he does not deviate, is

   $$U = \frac{3}{2} + \frac{3}{2}\delta + \frac{3}{2}\delta^2 + ... = \frac{\frac{3}{2}}{1-\delta}$$

   When player 2 is playing $q = \frac{1}{2}$, player 1 is indifferent between any strategy: no matter how he deviates, he will get a payoff of $\frac{3}{2}$ today. Therefore, player 1 never strictly benefits from a one-shot deviation (he will end up getting the same present value) regardless of δ.

Cases 1 and 2: The proposed strategy is a SPE if and only if $\delta \geq \frac{2}{3}$.

- Marking guideline:
  - Deduct 11 points if you used a non-credible punishment such as "(K, K) forever" in your strategy.

– Deduct 5 points if you discussed only the subgame that starts with (S, S) but did not discuss other subgames.
– Deduct 5 points if there is no description of your strategy.
– Deduct 1 point if your strategy did not mention what would happen if both players deviated from S to K in the same round.
– Deduct 15 points if your strategy does not punish deviation from S at all. (Players still continue to play (S, S) forever after a one-shot deviation.)
– Deduct 13 points if you attempted to propose a tit-for-tat strategy but only discussed one subgame (you need to discuss 4 subgames).
– Deduct 12 points if you answered this question with Folk Theorem and the use of Folk Theorem is correct.
– Deduct 16 points if you answered this question with Folk Theorem and the use of Folk Theorem is incorrect.