Mid-semester review

ECOS3012

Multi-agent interaction

Infinitely many feasible outcomes

Which outcomes are more likely?

Games with complete information

If players act "simultaneously" only once → Look for *stable outcomes* (those that do not have any profitable one-sided deviation) → Nash Equilibrium (NE)

If there are many stages of interaction → Look for stable outcomes that do not involve non-credible threat/promise → Subgame-perfect Nash equilibrium (SPE)

Static games

Players "simultaneously" act once.

Frequently asked question:

- Find all Nash equilibria (if there is any)

Practice midterm Q1

"Find all Nash equilibria of the following game."

		Player 2	
		L	R
Player 1	U	10, 1	2, 2
	M	2, 3	10, 3
	~~D~~	~~5, 3~~	~~5, 5~~

STEP 1 Delete strictly dominated strategies

Playing U and M with equal probabilities gives a payoff of 6 on average, which strictly dominates the pure strategy D.

			Player 2 L (q)	Player 2 R (1-q)
Player 1	U	(p)	10, 1	2, 2
	M	(1-p)	2, 3	10, 3

STEP 2 Find all NE of the simplified game.

METHOD A:

First find all pure-strategy NE by highlighting best responses.
Then find all mix-strategy NE algebraically.

METHOD B:

First write down each player's best response as a function of the opponent's strategy (p or q).
Then draw the two best response functions on a diagram to find all intersections.

Dynamic games

Two types of dynamic games:

1. Only one player acts in each stage
2. Multiple players act simultaneously in each stage

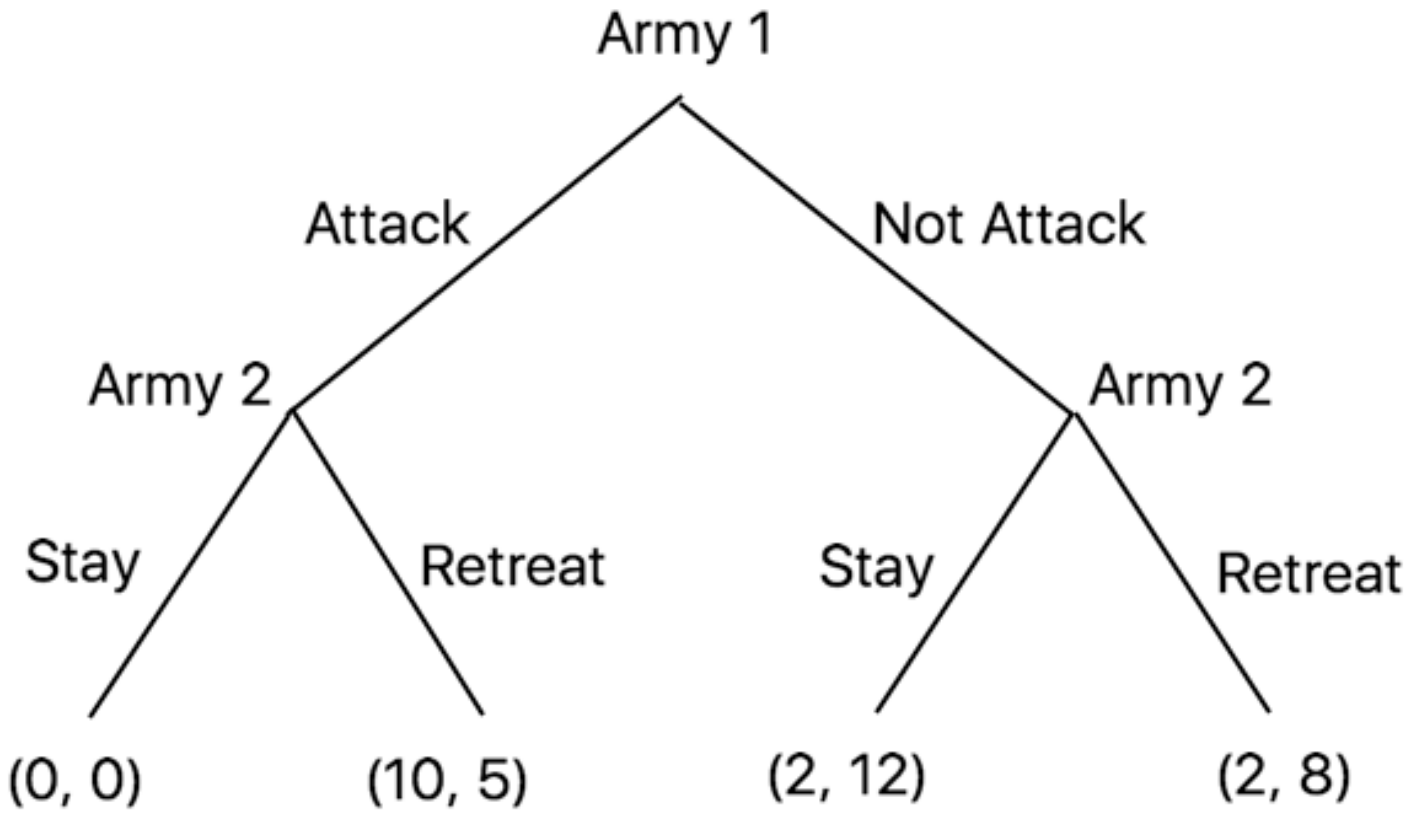
Dynamic games

Type 1: Only one player acts in each stage

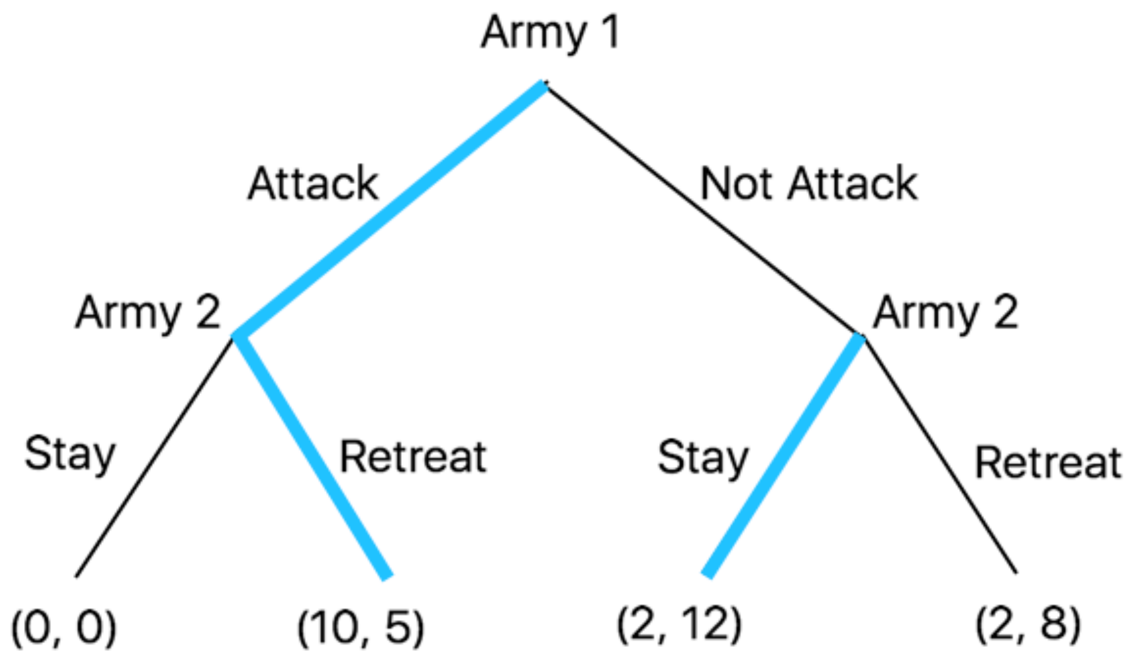
Frequently asked questions:

- Find all subgame perfect equilibria
- Find other Nash equilibria that are not subgame perfect

Practice Midterm Q5



(b) Find the subgame perfect equilibrium.



SPE is
(Attack, (Retreat, Stay))

SPE *outcome* is
(Attack, Retreat)

- Use backward induction. First mark A2's best responses in each proper subgame, then pick A1's best choice between the two marked outcomes.
- Your answer for the SPE must describe *every player's complete plan of action in every possible scenario*. I.e., you need to describe all three blue marked paths.

Remark: extensive form (game tree) vs. normal form (game matrix)

Extensive form: best for finding subgame perfect equilibria

Normal form: best for finding all Nash equilibria

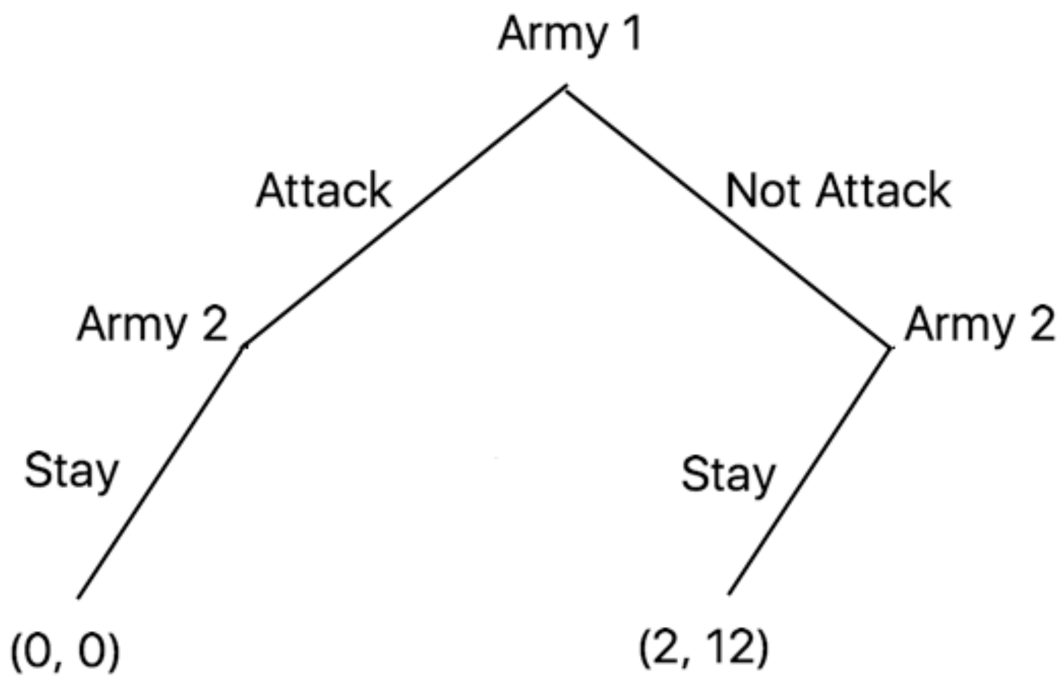
Example:

Convert the previous game to its normal-form representation and find all Nash equilibria.

Note: The key is to understand that the second player has 4 pure strategies; each strategy is a pair of actions:

(action in subgame 1, action in subgame 2).

(e) Show that army 2 can increase its subgame perfect equilibrium payoff (and reduce army 1's payoff) by burning the bridge to its mainland, eliminating its option to retreat if attacked.



Remark:

Previously, "never retreat" is not a credible threat.

By burning the bridge, it is now credible.

Dynamic games

Type 2: Multiple players act simultaneously in each stage

Frequently asked questions:

- For simple games with only one SPE (type 2.1): find the SPE
- For complicated games with many SPE (type 2.2):
  - Verify if a particular strategy profile is SPE
  - Verify whether a particular outcome can be achieved in any SPE

Type 2.1 Simple games with only one SPE

Features:

- Finite stages, or
- Unique Nash equilibrium in every stage
- Solve with backward induction

Examples:

- International tariff
- Finitely repeated prisoner’s dilemma

Type 2.2 Complicated games with many SPE

Type 2.2.A: Finite stages; multiple Nash equilibria in each stage

- Backward induction

Type 2.2.B: Infinitely repeated games

- ~~Backward induction~~
- Forward looking, history-dependent strategy: *"I'll do X in the future if the history is …"*
- Check one-shot deviation

Type 2.2.A Example

2-stage games with multiple SPE

"Is strategy profile S a subgame perfect equilibrium?"

↓

Check if players play a NE in the 2nd stage

↓

Based on S, update total payoffs in the 1st stage and check if players play a NE of the updated game matrix

Type 2.2.A Example
2-stage games with multiple SPE

"Can there be a SPE that achieves outcome X in 1st stage?"

⬇

Design proper 2nd-stage strategy:

play a "good" NE if outcome is X in 1st stage;

play a "bad" NE otherwise

("good" and "bad" depend on the specific game matrix)

Q: Is there an SPE ~~play~~ (Pop, Rock) in t = 1 ?

{ play (P, R) in t = 1 '
{ play (P, R) again in t = 2 no matter what the outcome of t = 1 is

Practice midterm Q8

		Bob	
		Rock concert	Pop concert
Ann	Rock concert	1, 1	6, 2
	Pop concert	2, 6	0, 0

NE of stage

Ann and Bob met at a rock concert, and fell in, and then out of love (and had a rough break-up). Both of them prefer a rock concert to a pop concert, but really would prefer to not run into each other.

(b) If we repeat this game twice and our total payoff is the undiscounted sum of the stage payoffs, please help us find a strategy profile so that

(i) it is a subgame perfect equilibrium, and

(ii) in this equilibrium, we never go to the same concert.

		Bob	
		Rock concert	Pop concert
Ann	Rock concert	1, 1	6, 2
	Pop concert	2, 6	0, 0

(c) If we repeat this game twice and our total payoff is the undiscounted sum of the stage payoffs, please tell us:

Is it possible to have a subgame perfect equilibrium in which we both go to the rock concert in the first period? Why?

Type 2.2.B
Infinitely repeated games

"Is strategy profile S a subgame perfect equilibrium?"

↓

subgames

Categorize all histories into a few types according to the description of S.

↓

For subgame following each type of history, compare the payoff from a one-shot deviation vs. the payoff of no deviation.

Practice midterm Q10

✓ payoffs are symmetric
✓ tit-for-tat strategy is also symmetric
↳ it's okay to just check p1's profitable dev.

Suppose that the following prisoners' dilemma is repeated infinitely with discount factor $\delta \in (0,1)$.

		Prisoner 2: Confess	Prisoner 2: Not Confess
Prisoner 1	Confess	1, 1	5, 0
	Not Confess	0, 5	4, 4

S

Consider a "tit-for-tat" strategy profile in which both players play NC in the first period.
In later periods, each player mimics the action that the opponent picked in the previous period.
(For example, if the outcome in period t is (C, NC) then players play (NC, C) in period t+1.)

S is SPE if & only if $\delta = \frac{1}{4}$

Q: Can tit-for-tat be a subgame perfect equilibrium? If so, find the condition on δ. If not, explain why.

A: $\delta \geq \frac{1}{4}$ B: $\delta \leq \frac{1}{4}$ C: $\delta \geq \frac{1}{4}$ D: $\delta \leq \frac{1}{4}$ $\Rightarrow$ Intersection of A B C D

(b) Outcome yesterday is (NC, NC) — P1, P2

no dev payoff: $4 + 4\delta + 4\delta^2 + \cdots = \frac{4}{1-\delta}$

no deviation from S: $\overset{4}{(NC, NC)}, \overset{4}{(NC, NC)}, \overset{4}{(NC, NC)}, \cdots$

One-shot dev. from S: $\overset{5}{(C, NC)}, \overset{0}{(NC, C)}, \overset{5}{(C, NC)}, \overset{0}{(NC, C)}$

return to S

dev payoff: $5 + \cancel{0}\,\delta + 5\cdot\delta^2 + \cancel{0}\,\delta^3 + 5\cdot\delta^4 + \cdots$

$5 + 5\delta^2 + 5\delta^4 + 5\delta^6 + \cdots = \frac{5}{1-\delta^2}$

$\frac{4}{1-\delta} \geq \frac{5}{1-\delta^2} = \frac{5}{(1-\delta)(1+\delta)}$

$4(1+\delta) \geq 5$

$4\delta \geq 1 \Rightarrow \boxed{\delta \geq \frac{1}{4}}$ (A)

(2) Outcome yesterday is (NC, C)

No deviation: $(C, NC), (NC, C), (C, NC), (NC, C) \cdots$

no deviation payoff $= \frac{5}{1-\delta^2}$

one-shot deviation: $(NC, NC), (NC, NC), (NC, NC) \cdots$

dev. payoff $= \frac{4}{1-\delta}$

$\frac{5}{1-\delta^2} \geq \frac{4}{1-\delta}$

$\boxed{\delta < \frac{1}{4}}$ (B)

③ Outcome yesterday is (C, NC)

No deviation : $(\overset{0}{NC}, C), (\overset{5}{C}, NC), (\overset{0}{NC}, C), (\overset{5}{C}, NC), \cdots$

$$5\delta + 5\delta^3 + 5\delta^5 + \cdots = \frac{5\delta}{1-\delta^2}$$

deviation $(\overset{1}{C}, C), (\overset{1}{C}, C), (\overset{1}{C}, C), (\overset{1}{C}, C) \cdots$

$$\frac{1}{1-\delta}$$

$$\frac{5\delta}{(1-\delta)(1+\delta)} \geq \frac{1}{1-\delta}$$

$$5\delta \geq 1+\delta$$

$$4\delta \geq 1$$

$$\Rightarrow \boxed{\delta \geq \frac{1}{4}}$$

C

④ Outcome yesterday is (C, C)

No deviation, $(\overset{1}{C}, C), (\overset{1}{C}, C), (\overset{1}{C}, C), (\overset{1}{C}, C) \cdots$

deviation $(\overset{0}{NC}, C), (\overset{5}{C}, NC), (\overset{0}{NC}, C), (\overset{5}{C}, NC) \cdots$

$$\frac{1}{1-\delta} \geq \frac{5\delta}{1-\delta^2} \Rightarrow \boxed{\delta \leq \frac{1}{4}}$$

D

Type 2.2.B
Infinitely repeated games

"Can there be a SPE that achieves average payoff $\boxed{(v_1, v_2)}$ when players are sufficiently patient?"

δ can be arbitrarily close to 1

Folk Theorem (end of L6)

Are the average payoffs feasible? Are players' average payoffs higher than their minmax values?

Player's lowest possible (expected) payoff when he is best responding

Practice Midterm Q11

Suppose the following game is repeated infinitely with a discount factor arbitrarily close to 1. Select all payoff pairs that could be achieved as the average payoff in some subgame perfect equilibrium of this repeated game.

L strictly dominates R

Minmax $_{P2}$ = 2

		Player 2	
		L always	R
Player 1	U	0, 2	4, 1
	D sometimes	1, 4 > minmax	2, 0

A. (2.5, 3) **B. (1.5, 3)** C. (1.5, 2) D. (0.5, 2)

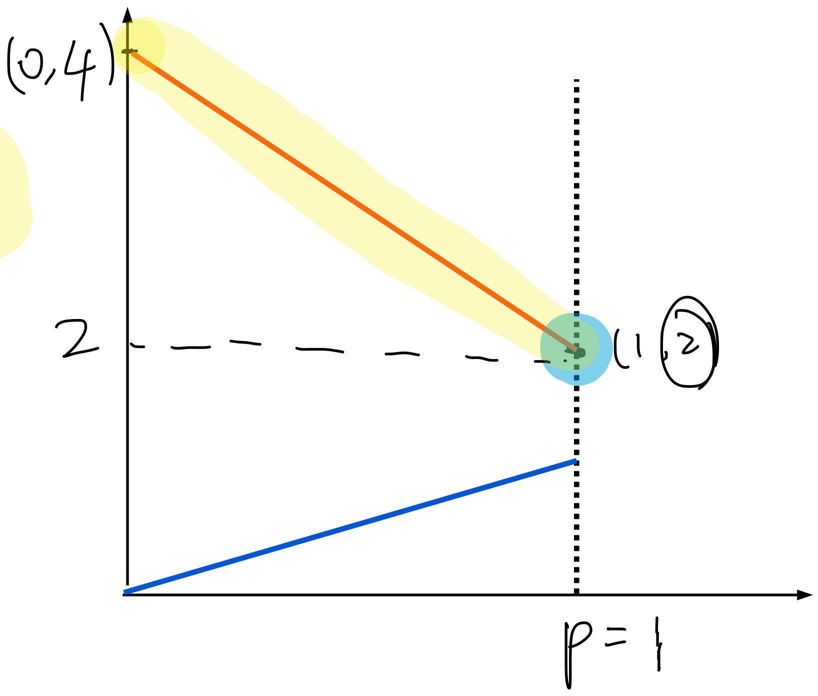
must be both P1 & P2 are playing both U&D L&R

P2 is randomising b/w L&R

MINMAX for P2 = 2

$$EU(L) = 2p + 4(1-p) = \underline{4 - 2p}$$

$$EU(R) = p + 0(1-p) = p$$

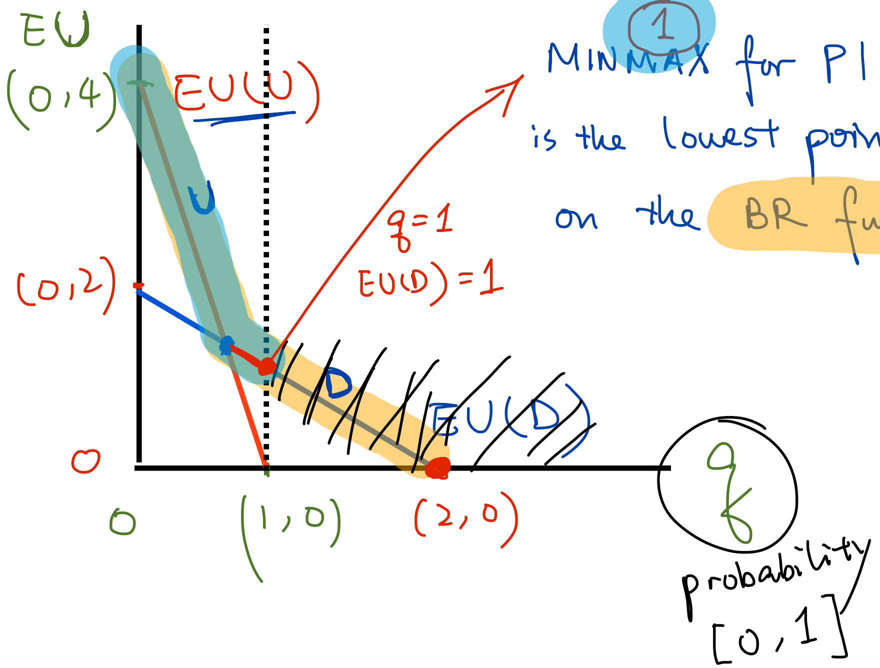


Minmax for p1.

$EU(U) = 4 - 4q$ (0, 4) , (1, 0)

$EU(D) = q + 2(1-q) = 2 - q$ $q = 1$

(2, 0) (0, 2)



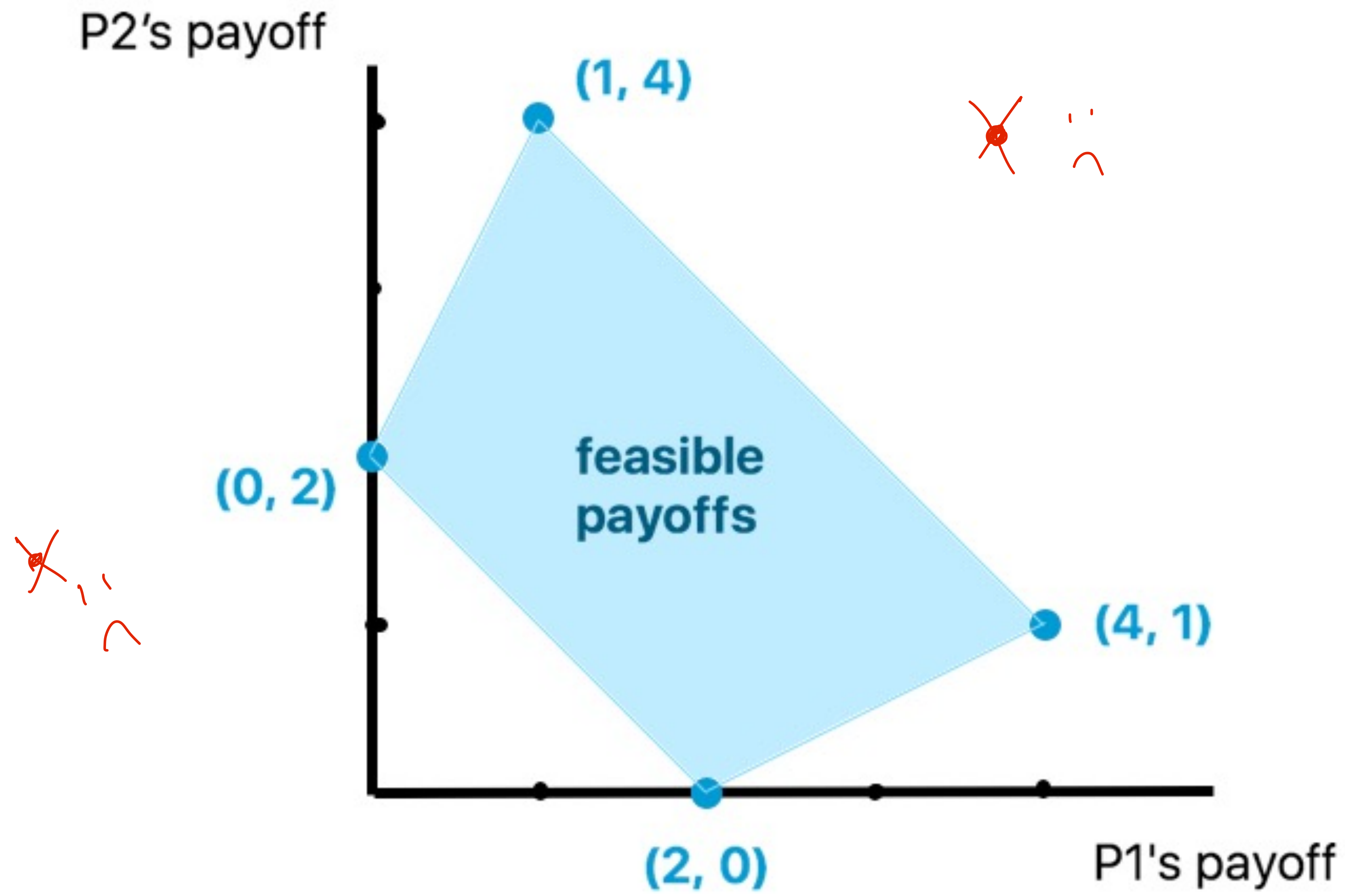
1 MINMAX for P1 is the lowest point on the BR function

Folk theorem: as δ is sufficiently close to 1, any feasible payoff *strictly* above the minmax value can be supported as the average payoff in some subgame perfect equilibrium.

STEP 1 Identify the set of feasible payoffs

STEP 2 Identify each player’s minmax payoff.

STEP 1 Identify the set of feasible payoffs

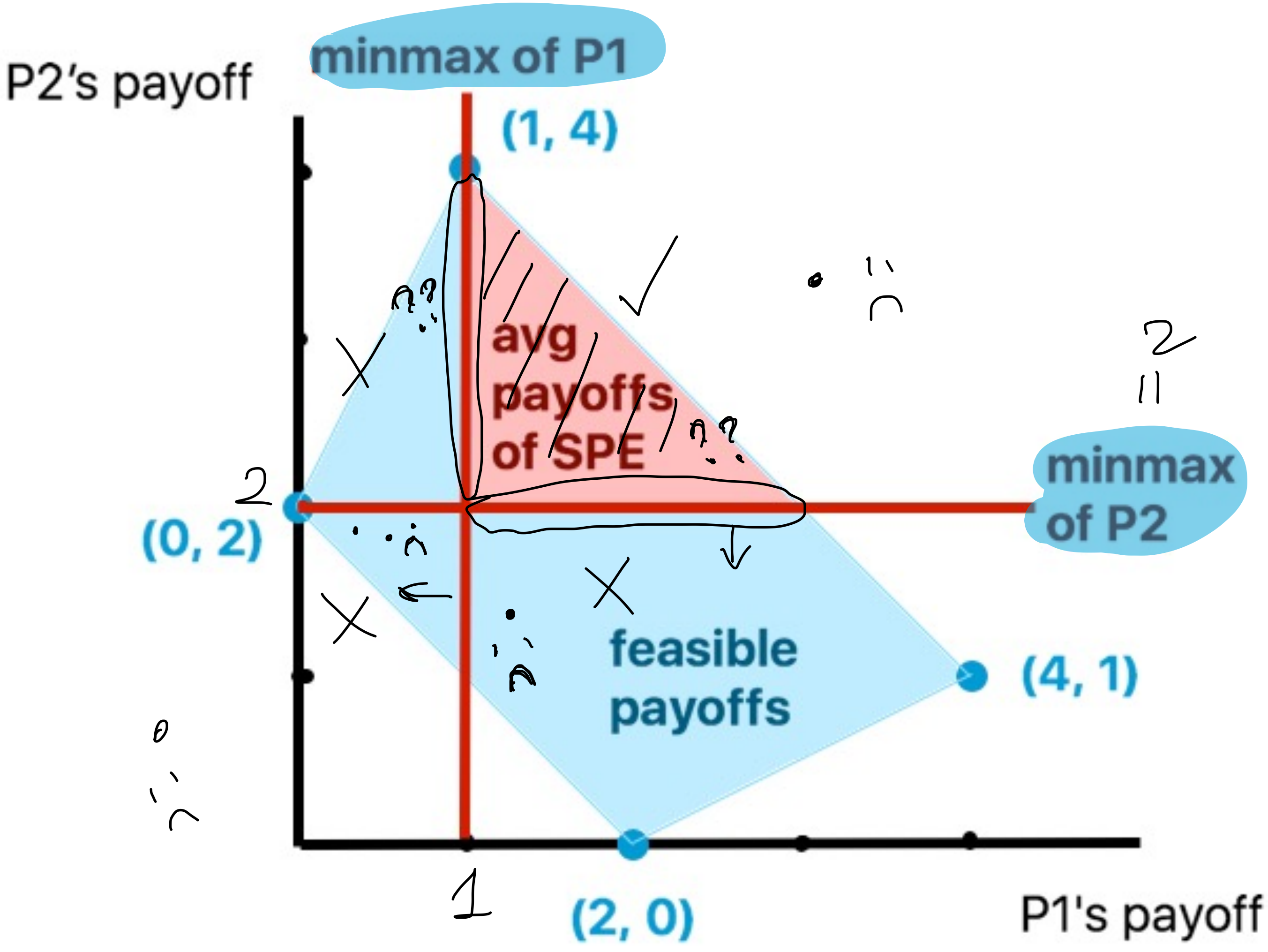


STEP 2 Identify each player's minmax payoff.

To identify player i's minmax payoff:

i. Calculate player i's expected utility of each action as a function of the opponent's strategy
ii. Derive player i's best-response payoff as a function of the opponent's strategy
iii. Find the lowest best-response payoff

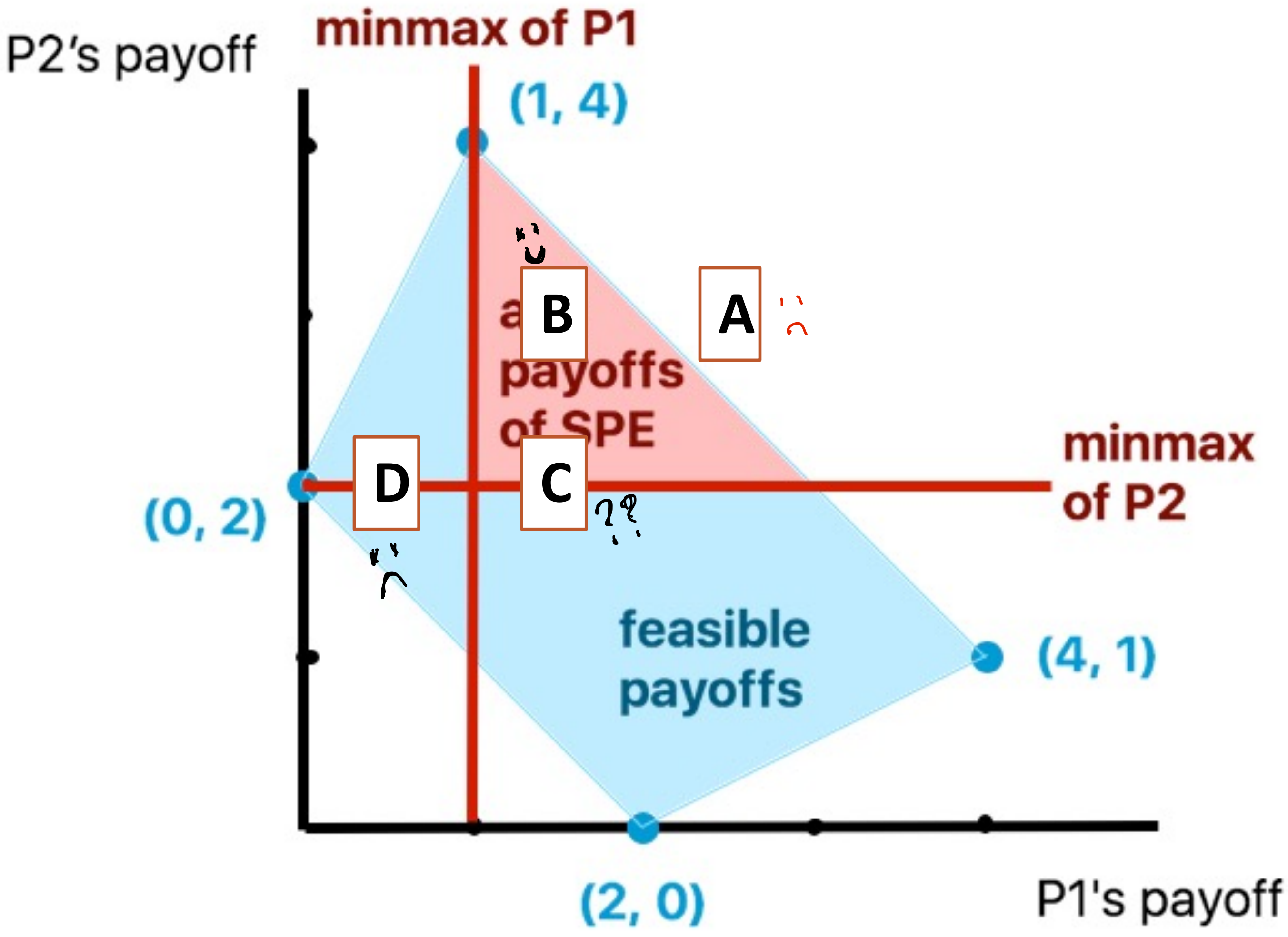
Folk Theorem: feasible payoff pairs that are strictly above minmax values ⇒ SPE ✓



- Points outside the pink region cannot be average payoff pairs of any SPE.

Points that are inside the pink region but not on the red minmax borders are average payoffs of some SPE when players are sufficiently patient (discount factor close to 1).

Points on the red minmax borders of the pink region may or may not be SPE payoffs.



Only B is correct.

A and D are outside of the pink region.

C is on the minmax border of the pink region.
In this particular game,
it cannot be the average payoff pair because it requires P2 to play a mixed strategy but P2 always has a profitable deviation.

If the question is:

"Is there a SPE that achieves ***outcome*** *X* ***with a particular*** δ*?"*

Try trigger strategy with infinite punishment:

"Play the worst equilibrium strategy if there has been any deviation from X "

↓

Check profitable one-shot deviation given the value of $\boldsymbol{\delta}$

↓

The answer is "yes" if profitable deviation doesn't exist.

Example: Suppose two prisoners repeat the following game for an indefinite number of times.

		Prisoner 2	
		Confess	Not Confess
Prisoner 1	Confess	3, 3	10, 0
	Not Confess	0, 10	4, 4

When they finish a round, the probability that they will continue to play another round of this game is 0.5. Is there a SPE where they choose not to confess in every period?

$\delta = 0.5$

(NC, NC) forever?

Answer: No. δ must be greater than 6/7 for there to be such a SPE.

propose trigger with ∞ punishment: if someone played C, then (C, C) forever

Quiz 6: Suppose two players repeat the following game for an indefinite number of times.

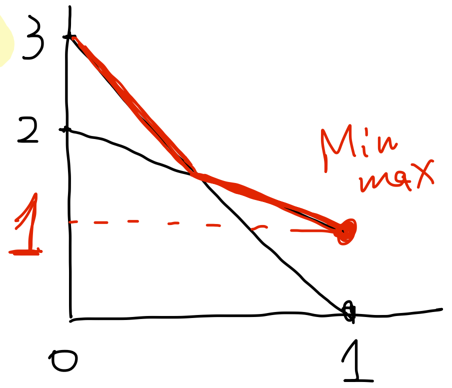
Minmax

$EU(H) = 3 - 3q$

$EU(D) = q + 2(1-q)$

$= 2 - q$

		Player 2	
		q Hawk	$1-q$ Dove
Player 1 (Minmax)	Hawk	0, 0	3, 1
	Dove	1, 3	2, 2



If players are sufficiently patient, is there a SPE in which…

(0 , 0)

1. (Hawk, Hawk) is played in every period? NO. 0 < minmax for players 1 & 2
2. (Dove, Dove) is played in every period?

Here is a wrong answer: Where is the mistake?

Yes, because I can propose the following strategy:

- Play (D, D) in period 1.
- If always played (D, D) in the past, continue to play (D, D) forever
- If someone has played H in the past, play (H, H) forever in the future
  (0, 0)

Check conditions on δ after one-shot dev.

If no deviation:

$$\overset{2}{(D, D)}, \overset{2}{(D, D)}, \overset{2}{(D . D)} \cdots\cdots$$

$$\frac{2}{1-\delta}$$

Is a SPE when $\delta \geq \frac{1}{3}$

One-shot dev.

$$\underset{3}{\overset{3}{(H, D)}}, \overset{0}{(H, H)}, \overset{0}{(H, H)}, \overset{0}{(H, H)}$$

$$\frac{2}{1-\delta} \geq 3 \Rightarrow 2 \geq 3 - 3\delta$$

$$3\delta \geq 1$$

$$\delta \geq \frac{1}{3}$$