ECOS3012 mid-semester exam: answers to 13 practice questions

March 24, 2022

Q1 Consider the game:

		Player 2	
		L	R
	U	10, 1	2, 2
Player 1	M	2, 3	10, 3
	D	5, 3	5, 5

(a) Is there any strictly dominated strategy? If so, please find all strictly dominated strategies. If not, please explain.

Answer: D is strictly dominated by the mixed strategy "play U with probability $\frac{1}{2}$ and M with probability $\frac{1}{2}$". (You can construct alternative mixed strategy that dominates D. This is just one example.)

To see why, first let the probability that player 2 plays L be q. We need to show that for all $q \in [0,1]$, the expected utility from playing this mixed strategy is higher than the expected utility from playing D.

Start by calculating the expected utilities for the mixed strategy and D.

$$EU(\frac{1}{2}U, \frac{1}{2}M) = \frac{1}{2}[10q + 2(1-q)] + \frac{1}{2}[2q + 10(1-q)] = 6$$

$$EU(D) = 5q + 5(1-q) = 5$$

Since $EU(\frac{1}{2}U, \frac{1}{2}M) > EU(D)$ for all $q \in [0,1]$, D is strictly dominated by the mixed strategy "play U with probability $\frac{1}{2}$ and M with probability $\frac{1}{2}$".

This is the only strictly dominated strategy of the game. (L is weakly dominated by R but it is not strictly dominated.)

(b) Find all Nash equilibria (pure and mixed).

Answer: There are infinitely many Nash equilibria. Let q be the probability that player 2 plays L, then any strategy profile $(M, (q, 1-q))$ with $q \leq \frac{1}{2}$ is a Nash equilibrium.

Reasoning:

Since D is strictly dominated, delete it from the game and get

			Player 2	
			L	R
			q	1-q
Player 1	U	p	**10**, 1	2, **2**
	M	1-p	2, **3**	**10, 3**

There is one pure-strategy equilibrium: (M, R).

To find out if this game has any mix-strategy Nash equilibrium, let p be the probability that player 1 plays U, and q be the probability that player 2 plays L.

Calculate each player's expected utility for each pure strategy.

$$EU_1(U) = 10q + 2(1-q)$$
$$EU_1(M) = 2q + 10(1-q)$$
$$EU_2(L) = p + 3(1-p)$$
$$EU_2(R) = 2p + 3(1-p)$$

If, in a mixed-strategy Nash equilibrium, player 2 plays both L and R with positive probability, then he must be indifferent between L and R. Since L is weakly dominated by R, $EU_2(L) = EU_2(R)$ if and only if $p = 0$. That is, player 2 is willing to randomize if and only if player 1 plays M only.

But when is $p = 0$ (play M only) a best response for player 1? Note that,

$$EU_1(M) \geq EU_1(U) \Rightarrow q \leq \frac{1}{2}.$$

In other words, playing M is a best response for player 1 if $q \leq \frac{1}{2}$, and playing a mixed strategy with $q \leq \frac{1}{2}$ is a best response for player 2 when player 1 is playing M.

Therefore, any strategy profile $(M, (q, 1-q))$ with $q \leq \frac{1}{2}$ is a Nash equilibrium.

There doesn't exist any Nash equilibrium in which player 1 is playing both U and M with positive probability. To see why, note that if that is the case, then player 1 must be indifferent between U and M:

$$EU_1(U) = EU_1(M) \Rightarrow q = \frac{1}{2}.$$

But $q = \frac{1}{2}$ is never a best response for player 2 if player 1 is playing both U and M with positive probability. This is because L is weakly dominated by R. Therefore, if player 1 plays U with positive probability, player 2 strictly prefers to play R ($q = 0$).

Note: Alternately, you can solve (b) by drawing the players' best-response correspondence graphs. The two graphs intersect at infinitely many points, which implies that there are infinitely many NE.

Q2 Consider the following static game.

		Player 2		
		A	B	C
	A	1, 1	9, 0	2, 0
Player 1	B	0, 10	8, 8	1, 3
	C	0, 2	4, 0	5, 5

(a) Does this game have strictly dominated strategies? If so, state all strictly dominated strategies. If not, please explain.

(b) Find all Nash equilibria of this game.

(c) Suppose that the players act sequentially. Player 2 chooses her action after observing the choice of player 1. Find the subgame perfect equilibrium(/ia) of this sequential game. State each player's equilibrium strategy.

A: (a) Yes. For player 1, row A strictly dominates row B. For player 2, column A strictly dominates column B, too. Therefore, eliminate row B and column B to simplify the game into

		Player 2	
		A	C
Player 1	A	**1, 1**	2, 0
	C	0, 2	**5, 5**

A: (b) There are three Nash equilibria. (A, A), (C, C), and a mixed-strategy NE in which both players play A with probability $\frac{3}{4}$ and C with probability $\frac{1}{4}$.

Verify the mixed NE: let p be the prob. that player 1 plays A; let q be the prob. that player 2 plays A.

When player 1 is indifferent between A and C: $q + 2(1-q) = 5(1-q) \Rightarrow q = \frac{3}{4}$.

When player 2 is indifferent between A and C: $p + 2(1-p) = 5(1-p) \Rightarrow p = \frac{3}{4}$.

A: (c) Player 2's best response is

1. Choose **A** if P1 chooses A
2. Choose **A** if P1 chooses B
3. Choose **C** if P1 chooses C

Given P2's best response, P1 should choose C to maximize payoff.

The subgame perfect equilibrium is (C, (A, A, C)). (Note that your answer should specify P2's action in all three possible scenarios.)

(The subgame perfect equilibrium **outcome** is (C, C) with payoff (5, 5).)

Q3 The products from firm 1 and firm 2 are imperfect substitutes. If the price of firm 1's product is p_1 and the price of firm 2's product is p_2, then the market demand for firm 1 is $q_1 = 100 - 2p_1 + p_2$ and the market demand for firm 2 is $q_2 = 100 - p_2 + p_1$. The marginal production cost is 10 for both firms.

(a) Express firm 1's optimal price as a function of p_2 and express firm 2's optimal price as a function of p_1.

(b) What prices will these firms charge in equilibrium? Which firm earns a higher profit?

The owners of the two firms are considering a merge. If they merge, the firms become a monopoly. The new monopoly will charge a single price p_m, and the market demand is $q_m = 200 - p_m$. The two owners split the monopolistic profits evenly. The two firms will merge if and only if both owners agree to merge.

(c) Will the firms merge to become a monopoly?

Answer:

(a) $\pi_1 = (p_1 - 10)(100 - 2p_1 + p_2)$

FOC: $100 - 4p_1 + p_2 + 20 = 0$.

$$p_1^* = \frac{120 + p_2}{4} \tag{1}$$

$\pi_2 = (p_2 - 10)(100 - p_2 + p_1)$

FOC: $100 - 2p_2 + p_1 + 10 = 0$.

$$p_2^* = \frac{110 + p_1}{2} \tag{2}$$

(b) (1) and (2):

$$p_2 = \frac{110 + \frac{120+p_2}{4}}{2}$$
$$8p_2 = 440 + 120 + p_2$$
$$p_2^* = \frac{560}{7} = 80.$$
$$p_1^* = \frac{120 + 80}{4} = 50.$$
$$\pi_1^* = (50 - 10)(100 - 2 \cdot 50 + 80) = 3200.$$
$$\pi_2^* = (80 - 10)(100 - 80 + 50) = 4900.$$

Firm 2 earns a higher profit because its market demand is more inelastic in its own price.

(c) If the two firms merge, the monopoly maximizes $\pi_m = (p_m - 10)(200 - p_m)$.

FOC: $200 - 2p_m + 10 = 0$.

$p_m^* = 105$. $\pi_m^* = (105 - 10)(200 - 105) = 9025$.

Each owner gets $\frac{9025}{2} = 4512.5$. This is lower than π_2^*. The owner of firm 2 vetoes the proposal. The firms do not merge.

Q4 Three oligopolists operate in a market with inverse demand given by $P(Q) = 50 - Q$, where $Q = q_1 + q_2 + q_3$ and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, $c = 5$, and no fixed cost. First, firm 1 and 2 simultaneously chooses $q_1 \geq 0$ and $q_2 \geq 0$. Then, after observing q_1 and q_2, firm 3 chooses q_3.

(a) What quantities do firms choose in the subgame perfect equilibrium of this game?

(b) How much profit does firm 3 make?

(c) Suppose instead that firm 3 cannot observe q_1 or q_2 before choosing its own quantity. Does it produce at a higher or lower quantity compared to your answer in (a)? Does it make more or less profit compared to your answer in (b)? For full credit, please support your answer with calculation.

Answer:

(a) Solve the game backwards.

Given q_1 and q_2, firm 3 solves the following maximization problem:

$$\max_{q_3} \pi_3 = (50 - q_1 - q_2 - q_3) q_3 - 5q_3$$

$$\pi_3^{'}(q_3) = 45 - q_1 - q_2 - 2q_3$$

Setting $\pi_3^{'}(q_3) = 0$ yields firm 3's best-response function

$$q_3^*(q_1, q_2) = \frac{45 - q_1 - q_2}{2}.$$

Now, knowing firm 3's best-response function, firms 1 and 2 solve the following problems simultaneously:

$$\max_{q_1} \pi_1 = \left(50 - q_1 - q_2 - \frac{45 - q_1 - q_2}{2}\right) q_1 - 5q_1$$

$$\max_{q_2} \pi_2 = \left(50 - q_1 - q_2 - \frac{45 - q_1 - q_2}{2}\right) q_2 - 5q_2$$

$$\begin{aligned} \pi_1^{'}(q_1) &= 50 - 2q_1 - q_2 - \frac{45}{2} + q_1 + \frac{q_2}{2} - 5 \\ &= \frac{45 - q_2}{2} - q_1 \end{aligned}$$

$$\begin{aligned} \pi_2^{'}(q_2) &= 50 - q_1 - 2q_2 - \frac{45}{2} + \frac{q_1}{2} + q_2 - 5 \\ &= \frac{45 - q_1}{2} - q_2 \end{aligned}$$

Setting $\pi_1^{'}(q_1) = 0$ and $\pi_1^{'}(q_1) = 0$ yields firms 1 and 2's best-response functions:

$$\begin{aligned} q_1^*(q_2) &= \frac{45 - q_2}{2} \\ q_2^*(q_1) &= \frac{45 - q_1}{2} \end{aligned}$$

Solving these two equations simultaneously yields

$$q_1^* = q_2^* = 15$$

Plug q_1^* and q_2^* back into the best-response function of firm 3 to get

$$q_3^* = \frac{15}{2} = 7.5.$$

(b) When $q_1^* = q_2^* = 15$ and $q_3^* = \frac{15}{2}$, $\pi_3^* = \left(50 - 15 - 15 - \frac{15}{2}\right)\frac{15}{2} - 5 \times \frac{15}{2} = \frac{225}{4} = 56.25$.

(c) When firm 3 cannot observe q_1 and q_2, the game becomes a Cournot game with three firms. Each firm i solves the following problem:

$$\max_{q_i} \pi_i = \left(50 - q_i - q_j - q_k\right) q_i - 5q_i$$

Setting

$$\pi_i^{'}(q_i) = 45 - 2q_i - q_j - q_k = 0$$

yields the best-response function for firm i:

$$q_i^*\left(q_j, q_k\right) = \frac{45 - q_j - q_k}{2}.$$

Since the game's set-up is symmetric, we expect that, in equilibrium, $q_1^* = q_2^* = q_3^* = q$. If this is the case, then q must satisfy

$$q = \frac{45 - q - q}{2} \Rightarrow q = \frac{45}{4} = 11.25.$$

You can verify that $q_1^* = q_2^* = q_3^* = 11.25$ indeed satisfy the best-response function for each firm.

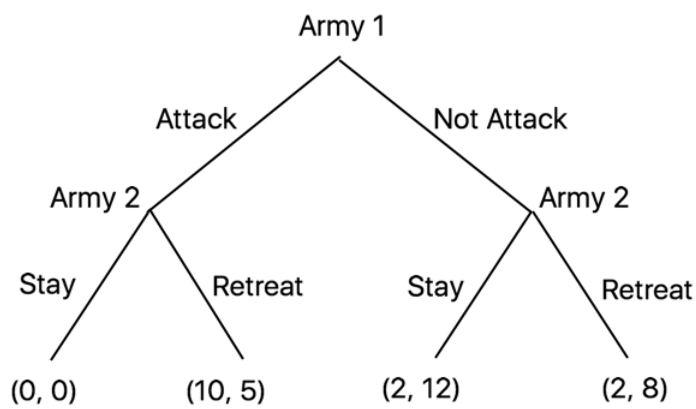
Therefore, in equilibrium, firm 3 chooses to produce 11.25, which is higher than the quantity 7.5 in (a).

Moreover,

$$\pi_3 = \left(50 - 3 \times \frac{45}{4}\right)\frac{45}{4} - 5 \times \frac{45}{4} = \frac{2025}{16} = 126.563 > 56.25.$$

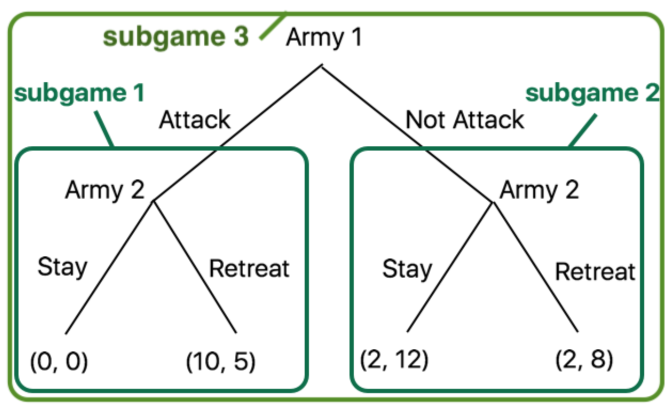
Firm 3 earns more when it cannot observe q_1 or q_2.

Q5 Army 1 from country 1 must decide whether to attack army 2, of country 2, which is occupying an island between the two countries. In the event of an attack, army 2 may stay on the island and fight, or retreat over a bridge to its mainland. Each army prefers to occupy the island than not to occupy it; a fight is the worst outcome for both armies.



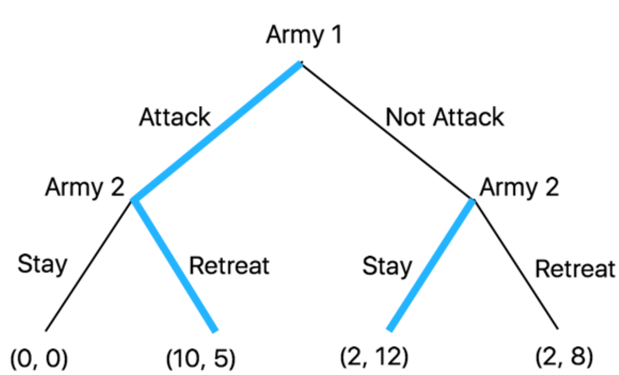
(a) How many subgames does this game have? What are they?

Answer: There are three subgames.



(b) Find the subgame perfect equilibrium. Remember, you should specify players' complete strategies following every possible history.

Answer: Use backward induction to find the SPE. The thickened blue paths in the graph below marks Army 2's best responses at each subgame and, consequently, Army 1's best response given that it knows Army 2's.



As illustrated by the graph, Army 2's strategy is to retreat if Army 1 attacks, and to stay if Army 1 does not attack. Given Army 2's strategy, Army 1 chooses to attack.

We use the notation (Attack, (Retreat, Stay)) to represent this subgame perfect equilibrium.

(c) Convert the game to its normal-form representation (i.e. draw the payoff matrix) and find all Nash equilibria.

Answer:

		Army 2			
		(R, R)	(R, S)	(S, R)	(S, S)
Army 1	A	**10**, **5**	**10**, **5**	0, 0	0, 0
	NA	2, 8	2, **12**	**2**, 8	**2**, **12**

While this game possibly has many mixed-strategy Nash equilibria, for grading purpose, full credit will be given to part (c) and (d) as long as you correctly discuss all pure-strategy Nash equilibria.

The game has three pure-strategy Nash equilibria: (Attack, (Retreat, Retreat)), (Attack, (Retreat, Stay)), and (Not Attack, (Stay, Stay)).

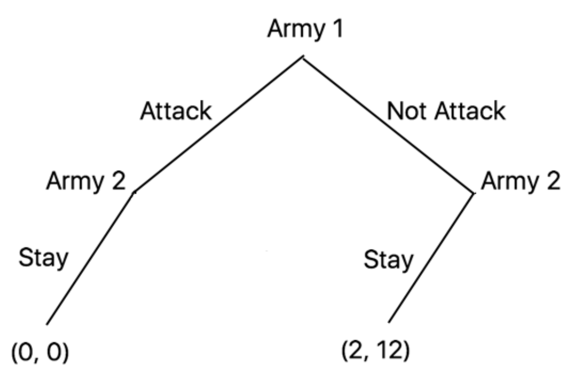
(d) For each Nash equilibrium that is not subgame perfect, identify the subgame in which the test of subgame perfection fails.

Answer: (Attack, (Retreat, Retreat)) is not subgame perfect because Retreat is not player 2's best response in subgame 2.

(Not Attack, (Stay, Stay)) is not subgame perfect because Stay is not player 2's best response in subgame 1.

(e) Show that army 2 can increase its subgame perfect equilibrium payoff (and reduce army 1's payoff) by burning the bridge to its mainland (assume this act entails no cost), eliminating its option to retreat if attacked.

Answer:



If the option to retreat is eliminated, Army 2 always chooses to Stay. In this case, it's best for Army 1 not to attack. The outcome is (Not attack, Stay) with payoffs (2, 12).

Compare with the subgame perfect equilibrium outcome of the original game (Attack, Retreat), Army 1's payoff decreased and Army 2's payoff increased.

Q6 Ann and Bob must choose among three movies, a, b, and c, for their movie night. Ann's payoff function is $U_A(a) = 3$, $U_A(b) = 2$, and $U_A(c) = 1$. Bob's payoff function is $U_B(a) = 1$, $U_B(b) = 2$, and $U_B(c) = 3$. The rules are that Ann moves first and can veto one of the three alternatives. Then Bob chooses one of the remaining two alternatives.

(a) Draw an extensive-form game tree for this game.

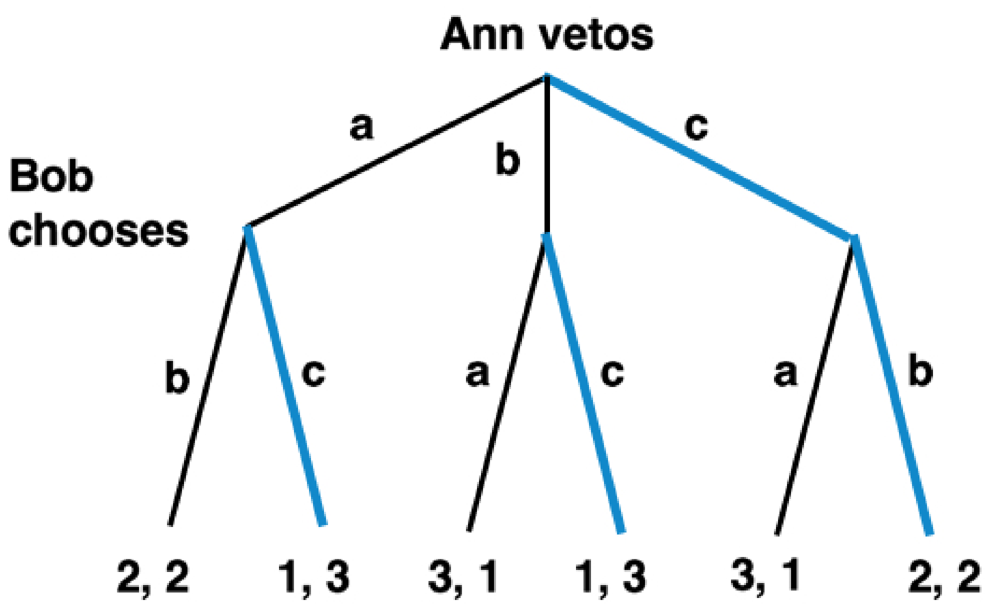
(b) Find the subgame-perfect equilibrium.

(c) How many pure strategies does Ann have? How many pure strategies does Bob have?

(d) Find a pure-strategy Nash equilibrium that is not a subgame-perfect equilibrium.

Answer:

(a) See graph below.



(b) Bob chooses c if Ann vetos a or b and he chooses b if Ann vetos c. By backward induction, Ann vetos c. Therefore, the SPE is $(c, (c, c, b))$.

(c) Ann has 3 pure strategies and Bob has 8 pure strategies. Let (x, y, z) denote a pure strategy from Bob. x is his choice when Ann vetos a; y is his choice when Ann vetos b; z is his choice when Ann vetos c. Then, the normal-form representation of the game is

		Bob							
		(b,a,a)	(c,a,a)	(b,c,a)	(c,c,a)	(b,a,b)	(c,a,b)	(b,c,b)	(c,c,b)
	veto a	2, 2	1, **3**	2, 2	1, **3**	2, 2	1, **3**	**2**, 2	1, **3**
Ann	veto b	**3**, 1	**3**, 1	1, **3**	1, **3**	**3**, 1	**3**, 1	1, **3**	1, **3**
	veto c	**3**, 1	**3**, 1	**3**, 1	**3**, 1	2, **2**	2, **2**	**2, 2**	**2, 2**

(d) As shown in the game matrix above, the only pure-strategy non-subgame-perfect NE is $(c, (b, c, b))$.

Q7 The following game "5" is a simplified version of Blackjack ("21").

Two players A and B take turns making choices. Player A starts off by choosing either 1 or 2. Player B observes this choice then increases the count by adding 1 or 2. Then player A observes B's choice and add 1 or 2. The game continues with the players taking turns, incrementing the count by 1 or 2. The player who reaches 5 wins. If the current count is already 4, the next player must choose 1 (and win the game).

Here is an example.

Round 1: A chooses 1

Round 2: B chooses 2

Round 3: A chooses 2 and wins the game

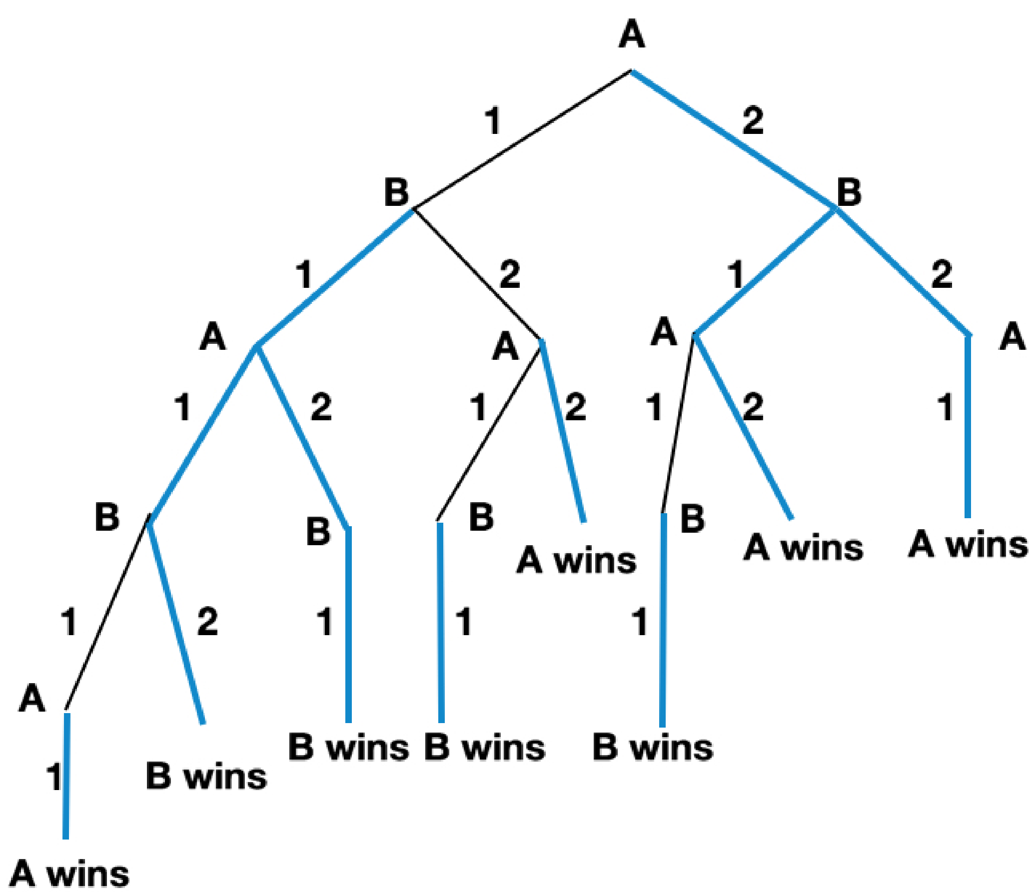
(a) Draw a game tree that describes this game.

(b) Find the subgame perfect equilibrium outcome(s).

(c) Explain whether this statement is correct: "The player that moves first always wins."

(d) How many pure strategies does B have?

Answer:



Subgame perfect equilibrium outcomes: (2, 1, 2) and (2, 2, 1). A wins in both cases.

The first mover always wins.

(d) 8 pure strategies. Player B's strategy is a protocol on how to act at each situation. Let's count how many cases B can confront. Let's say with a loose notation that [1 1 2] means A plays 1, then B adds 1, and then A adds 2. Then all the cases B can meet are [1], [2], [1 1 1], [1 1 2], [1 2 1], [2 1 1], [1 2 2], [2 1 2], [2 2 1]. Each of B's strategy will specify how to act for each of the cases. If B meets a case with total number 3 of less, which would be one of [1], [2], [1 1 1], then B has two options to choose, 1 or 2, for each of the cases. If B meets a case with total number 4, then B has only one option to choose, 1. If B meets a case with total number 5, then it means that B already lost the game and there is not any option for B to choose. That is, B has two options to choose only when she gets total number as 1,2, or 3, and there are 3 different cases that has total number as 1,2, or 3. Then, since B's strategy says how to act for each case, there are $2^3 = 8$ different strategies that B has. Note that the number of cases with total 1,2, or 3 is important.

Still confused? Let's take a step back and ask a more general question:

What is a strategy?

Suppose that, in a generic game, there are N possible scenarios in which you have **two or more** choices. A strategy is a complete plan that tells you which choice you should pick in each of these N scenarios.

For example, suppose that, in some game, there are 4 scenarios in which you have two or more choices.

In scenario 1, you have 2 choices.

In scenario 2, you have 4 choices.

In scenario 3, you have 7 choices.

In scenario 4, you have 5 choices.

Then, a strategy needs to tell you which choice you should pick in each of the four scenarios: (choice in scenario 1, choice in scenario 2, choice in scenario 3, choice in scenario 4). The total number of pure strategies is equal to (# of choices in scenario 1) x (# of choices in scenario 2) x (# of choices in scenario 3) x (# of choices in scenario 4) = 2 x 4 x 7 x 5 = 320.

Now, let's get back to Q7. In this game tree, there are **three** nodes at which B has **two** choices. (In the other nodes, B either doesn't have a turn or must choose 1.) Therefore, a strategy of B specifics which number he should choose at each of these three nodes. The total number of pure strategies = 2 x 2 x 2 = 8.

Q8 The Battle of the Exes

Suppose that Ann and Bob face the game below:

		Bob: Rock concert	Bob: Pop concert
Ann	Rock concert	1, 1	6, 2
Ann	Pop concert	2, 6	0, 0

Ann and Bob met at a rock concert, and fell in, and then out of love (and had a rough break-up). Both of them prefer a rock concert to pop concert, but really would prefer not to run into each other. They first sought a love doctor (a sociologist) to help them with their woes, and eventually turn to you, a practitioner of the dismal arts with the following questions:

a. If my Ex is still bitter about our break up and is doing whatever that minimizes my payoff, what is the highest payoff I can get in the stage game?

Answer: This question asks for each player's minmax value. Since the game is symmetric, it suffices to find only Ann's minmax value.

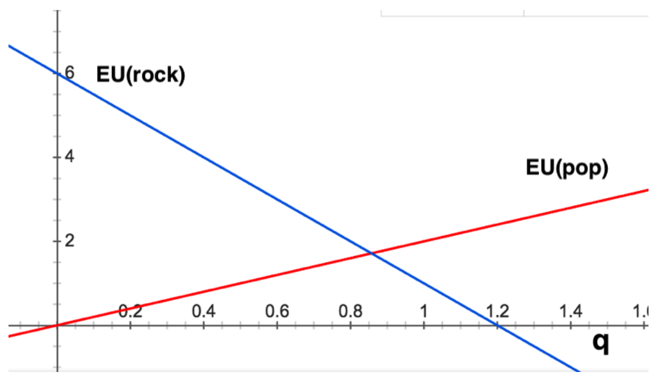
To do this, we first derive Ann's best response as a function of Bob's strategy.

Suppose that Bob chooses rock with probability q and pop with probability $1-q$. Then, Ann's expected payoffs for rock and pop as functions of q are

$$EU_A(rock) = q + 6(1-q)$$

$$EU_A(pop) = 2q$$

Plot these functions in the graph below. The horizontal axis is q and the vertical axis is the expected payoff. The blue downward sloping line is $EU_A(rock)$ and the red upward sloping line is $EU_A(pop)$.



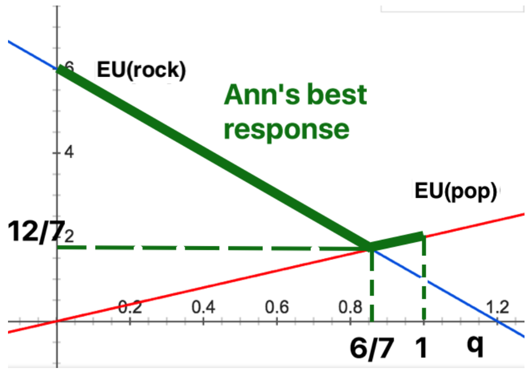
Note that

$$\begin{cases} EU_A(rock) > EU_A(pop) & if\ q < \frac{6}{7} \\ EU_A(rock) = EU_A(pop) & if\ q = \frac{6}{7} \\ EU_A(rock) < EU_A(pop) & if\ q > \frac{6}{7} \end{cases}$$

Therefore, Ann's optimal choice is

$$\begin{cases} rock & if\ q < \frac{6}{7} \\ \text{indifferent} & if\ q = \frac{6}{7} \\ pop & if\ q > \frac{6}{7} \end{cases},$$

which is labeled as the thick green curve in the graph below.



Bob is bitter about the break up and wants to choose the q that minimizes Ann's optimized payoff. Knowing Ann's best response function, Bob chooses the lowest point on the thick green curve, which corresponds to $q = \frac{6}{7}$ and $EU_A = \frac{12}{7}$. Therefore, Ann's minmax value is $\frac{12}{7}$. This is the highest payoff she can get when Bob is always minimizing her payoff.

b. If we repeat this game twice and our total payoff is the undiscounted sum of the stage payoffs, please help us find a strategy profile so that (i) it is a subgame perfect equilibrium, and (ii) in this equilibrium, we never go to the same concert.

Answer: There are many possible answers. Here's one example.

Consider the following strategy profile:

In the first period, play (rock, pop).

In the second period, play (rock, pop) again regardless of the first-period outcome.

This is a subgame perfect equilibrium because:

(i) The second-period strategy (rock, pop) is a Nash equilibrium of the stage game.

(ii) Following this strategy profile, since the second-period payoff is always (6, 2), we can update the total payoff as a function of the first-period outcome:

	rock	pop
rock	7, 3	**12, 4**
pop	**8, 8**	6, 2

We need the first-stage strategy pair (rock, pop) to be a NE of this updated game. From the matrix above we see that this is indeed the case.

c. If we repeat this game twice and our total payoff is the undiscounted sum of the stage payoffs, please tell us: is it possible to have a subgame perfect equilibrium in which we both go to the rock concert in the first period? Why? (Hint: for full credit, your answer should contain the word "mixed".)

Answer: It is not possible to have (rock, rock) as the first-period strategy in any subgame perfect equilibrium.

To see why, first note that the stage game has 3 NE.

Two pure NE: (rock, pop), (pop, rock) with payoffs (6, 2), and (2, 6).

One mixed NE: ((6/7, 1/7), (6/7, 1/7)) with payoffs (12/7, 12/7).

In a SPE, the second-period outcome must be one of these three.

The table below describes players' two-stage total payoff. x_i and y_i represent the corresponding payoff in stage 2 following a particular outcome in stage 1.

		Bob: rock	Bob: pop
Ann	rock	$1+x_1, 1+y_1$	$6+x_2, 2+y_2$
	pop	$2+x_3, 6+y_3$	$0+x_4, 0+y_4$

Suppose that the players play (rock, rock) in the first stage.

If Ann does not have a profitable deviation from rock in the first stage, we must have

$$1+x_1 \geq 2+x_3 \Rightarrow x_1 \geq 1+x_3.$$

The only way to satisfy this inequality, given that players must play a NE in stage 2, is to let $(x_1, y_1) = (6, 2)$ and $(x_3, y_3) = (2, 6)$ or $(12/7, 12/7)$.

If Bob does not have a profitable deviation from rock in the first stage, we must have

$$1+y_1 \geq 2+y_2 \Rightarrow y_1 \geq 1+y_2.$$

However, because $(x_1, y_1) = (6, 2)$ and y_2 must be equal to 12/7, 2, or 6, it is impossible to satisfy this inequality. This means that if Ann does not have a profitable deviation from (rock, rock) in stage 1, Bob must have a profitable deviation.

Therefore, no matter how we design the players' strategies, (rock, rock) can never be a SPE outcome in the first period.

Q9 Consider the following game.

		Player 2: A	Player 2: B	Player 2: C
Player 1	A	1, 1	9, 0	2, 0
	B	0, 10	8, 8	1, 3
	C	0, 2	4, 0	5, 5

(a) Suppose that the players repeat this stage game twice. In each stage, they simultaneously choose their actions, and their total payoff from the twice-repeated game is the (undiscounted) sum of the stage payoffs. Find a subgame perfect equilibrium in which the players play (B, B) in the first stage. State the players' strategies in this equilibrium and explain why they constitute a subgame perfect equilibrium.

(b) Suppose that the players repeat this stage game infinitely many times. In each stage, they simultaneously choose their actions, and their total payoff from the infinitely-repeated game is the discounted sun of the stage payoffs with some discount factor $\delta \in (0, 1)$. State the lowest possible payoffs player 1 and player 2 can get in any subgame perfect equilibrium. Explain your answer.

A: (a) A correct answer must specify each player's conditional plan "I'll play X in stage 2 if the outcome in stage 1 is (Y, Z)" for all possible stage-1 outcomes (Y, Z). For the proposed strategy profile to constitute a subgame perfect equilibrium, it should satisfy the following conditions:

1. 2nd-stage outcome is a NE of the stage game. I.e., players must play (A, A), (C, C), or the mixed NE $\left(\left(\frac{3}{4}, \frac{1}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right)$ in stage 2.
2. Write down a matrix that represents each player's total payoffs in both stages as a function of the 1st-stage outcome. Then, (B, B) must be a NE of this new matrix.

Here is one example that satisfy these conditions and there are possibly other correct answers. The general idea is that, to give players an incentive to play (B, B), their conditional plan in stage 2 should have the flavor "we will play a good NE like (C, C) if we played B in stage 1; otherwise, we'll play a bad NE like (A, A)."

Example: Each player follows the strategy below

- Play B in stage 1.
- In stage 2, play C if the outcome in stage 1 is (B, B). Otherwise, play A.

Based on this strategy, the players' total payoffs as functions of the first-stage outcomes are

		Player 2 A	Player 2 B	Player 2 C
Player 1	A	1+1, 1+1	9+1, 0+1	2+1, 0+1
	B	0+1, 10+1	8+5, 8+5	1+1, 3+1
	C	0+1, 2+1	4+1, 0+1	5+1, 5+1

which is equivalent to

		Player 2 A	Player 2 B	Player 2 C
Player 1	A	**2, 2**	10, 1	3, 1
	B	1, 11	**13, 13**	2, 4
	C	1, 3	5, 1	**6, 6**

(B, B) is a NE of this new matrix, as required.

A: (b) By the Folk Theorem, the lowest average payoff a player can get in any subgame perfect equilibrium = the minmax value for this player in the stage game.

		Player 2 A	Player 2 B	Player 2 C
Player 1	A	1, 1	9, 0	2, 0
	B	0, 10	8, 8	1, 3
	C	0, 2	4, 0	5, 5

Conditions for player i's minmax value:

1. Given the opponent's strategy, player i always chooses a best response.

2. Given that player i is always best responding, the opponent chooses a strategy to minimize player i's payoff.
3. The minimized "best-response" payoff is player i's minmax value.

- For player 1, B is never a best response, so let's simplify player 1's relevant payoffs into the following table:

		Player 2		
		A	B	C
Player 1	A	1	9	2
	C	0	4	5

  – (This part can be skipped.) Let q_1 be the prob. that P2 chooses A, q_2 be the prob. that P2 chooses B, and $1-q_1-q_2$ be the prob. that P2 chooses C. P1's expected utility when choosing A is $q_1+9q_2+2(1-q_1-q_2)=2-q_1+7q_2$. P1's expected utility when choosing C is $4q_2+5(1-q_1-q_2)=5-5q_1-q_2$. Therefore, P1's best response is to choose A when $2-q_1+7q_2>5-5q_1-q_2$ and C when $2-q_1+7q_2<5-5q_1-q_2$.
  – Given that P1 always best responds to P2's strategy, to minimize P1's payoff, P2 must play A with probability 1. In this case, **P1's minmax payoff is 1**. If P2 plays B or C with positive probability, then P1's payoff must be strictly higher than 1.

- Similarly, for player 2, B is never a best response, so let's simplify player 2's relevant payoffs into the following table:

		Player 2	
		A	C
Player 1	A	1	0
	B	10	3
	C	2	5

  – Given that P2 always best responds to P1's strategy, to minimize P2's payoff, P1 must play A with probability 1. In this case, **P2's minmax payoff is also 1**. If P1 plays B or C with positive probability, then P2's payoff must be strictly higher than 1.

- **Therefore, the lowest possible average payoff player 1 or player 2 can get in any subgame perfect equilibrium is 1.**

Q10 Suppose that the following prisoners' dilemma is repeated infinitely with discount factor $\delta \in (0,1)$.

		Prisoner 2	
		Confess	Not Confess
Prisoner 1	Confess	1, 1	5, 0
	Not Confess	0, 5	4, 4

Consider a "tit-for-tat" strategy profile in which both players play NC in the first period. In later periods, each player mimics the action that the opponent picked in the previous period. (For example, if the outcome in period t is (C, NC) then players play (NC, C) in period $t+1$.)

a) If both prisoners play the tit-for-tat strategy, what is the outcome path? What is each prisoner's average payoff?

Answer: the outcome path is (NC, NC) forever. Each prisoner's average payoff is 4.

b) Suppose that prisoner 1 plays C forever while prisoner 2 follows tit-for-tat. What is prisoner 1's average payoff from this infinite-shot deviation?

Answer: In this case, the outcome path is (C, NC), (C, C), (C, C), (C, C), ...

Prisoner 1 gets payoff 5 in the first period, and 1 in all other periods. The average payoff is

$$(1-\delta)\left(5+\delta+\delta^2+\delta^3+...\right) = (1-\delta)\left(5+\frac{\delta}{1-\delta}\right) = 5-4\delta$$

c) Consider the one-shot deviation from the equilibrium path: suppose that prisoner 1 chooses to deviate only at the first period, but then follows tit-for-tat forever after. What is his average payoff? Under what condition on δ is this not a profitable one-shot deviation?

Answer: the outcome path in this case is (C, NC), (NC, C), (C, NC), (NC, C), (C, NC), (NC, C), ...

The average payoff for player 1 is

$$\begin{aligned}
&(1-\delta)(5+0\delta+5\delta^2+0\delta^3+5\delta^4+0\delta^5+5\delta^6...)\\
=&(1-\delta)(5+5\delta^2+5\delta^4+5\delta^6+...)\\
=&(1-\delta)\frac{5}{1-\delta^2}\\
=&\frac{5}{1+\delta}
\end{aligned}$$

Recall that if no player deviates from the tit-for-tat strategy, then the average payoff is 4. Therefore, one-shot deviation from the beginning of the game is not profitable if and only if

$$\frac{5}{1+\delta} \leq 4$$
$$\delta \geq \frac{1}{4}.$$

d) Can tit-for-tat be a subgame perfect equilibrium? If so, find the condition on δ. If not, please explain why.

Answer: Tit-for-tat constitutes a subgame perfect equilibrium if and only if $\delta = \frac{1}{4}$.

There are 4 types of subgames, corresponding to 4 types of histories

(i) Either the outcome in the last period is (NC, NC) or the current period is the first period of the game

If both players play tit-for-tat, the outcome path is (NC, NC) forever and the average payoff is 4.

According to c), there is no one-shot deviation in this subgame if and only if $\delta \geq \frac{1}{4}$.

(ii) The outcome in the last period is (C, NC).

If both players play tit-for-tat, the outcome path is (NC, C), (C, NC), (NC, C), (C, NC), ...

Player 1's average payoff is

$$\begin{aligned}
&(1-\delta)(0+5\delta+0\delta^2+5\delta^3+0\delta^4+5\delta^5+0\delta^6...)\\
=&(1-\delta)(5\delta+5\delta^3+5\delta^5+5\delta^7+...)\\
=&(1-\delta)\frac{5\delta}{1-\delta^2}\\
=&\frac{5\delta}{1+\delta}
\end{aligned}$$

Player 2's average payoff is

$$\begin{aligned}
&(1-\delta)(5+0\delta+5\delta^2+0\delta^3+5\delta^4+0\delta^5+5\delta^6...)\\
=&(1-\delta)(5+5\delta^2+5\delta^4+5\delta^6+...)\\
=&(1-\delta)\frac{5}{1-\delta^2}\\
=&\frac{5}{1+\delta}
\end{aligned}$$

Suppose that player 1 deviates to C in the first period and then revert back to tit-for-tat. This leads to an outcome path of (C, C) forever and an average payoff of 1. Therefore, in this subgame, player 1 does not have a profitable deviation if and only if

$$\frac{5\delta}{1+\delta} \geq 1$$

$$\delta \geq \frac{1}{4}$$

Suppose that player 2 deviates to NC in the first period and then revert back to tit-for-tat. This leads to an outcome path of (NC, NC) forever and an average payoff of 4. Therefore, in this subgame, player 2 does not have a profitable deviation if and only if

$$\frac{5}{1+\delta} \geq 4$$

$$\delta \leq \frac{1}{4}$$

Therefore, in this subgame, no player has any profitable deviation if and only if

$$\delta = \frac{1}{4}$$

(iii) The outcome in the last period is (NC, C).

This is symmetric to the subgame in (ii). Again, there is no profitable deviation if and only if $\delta = \frac{1}{4}$.

(iv) The outcome in the last period is (C, C).

If both players play tit-for-tat, the outcome path is (C, C), (C, C), (C, C), (C, C), ... and the average payoff for both players is 1.

If player 1 deviates to NC for one period and reverts back to tit-for-tat, the outcome path is (NC, C), (C, NC), (NC, C), (C, NC), ...

Player 1's average payoff for this deviation is

$$
\begin{aligned}
&(1-\delta)(0+5\delta+0\delta^2+5\delta^3+0\delta^4+5\delta^5+0\delta^6...)\\
=&(1-\delta)(5\delta+5\delta^3+5\delta^5+5\delta^7+...)\\
=&(1-\delta)\frac{5\delta}{1-\delta^2}\\
=&\frac{5\delta}{1+\delta}
\end{aligned}
$$

This deviation is not profitable for player 1 if and only if

$$\frac{5\delta}{1+\delta} \leq 1,\ \delta \leq \frac{1}{4}$$

Similarly, a deviation to NC for one period is not profitable for player 2 if and only if $\delta \leq \frac{1}{4}$.

(i)+(ii)+(iii)+(iv) => tit-for-tat is SPE if and only if $\delta = \frac{1}{4}$.

Q11 Suppose the following game is repeated infinitely with a discount factor close to 1.

		Player 2 L	Player 2 R
Player 1	U	0, 2	4, 1
	D	1, 4	2, 0

Can the following payoff pairs be achieved as the average payoff in some subgame perfect equilibrium of this repeated game? Please explain your answer in details.

(a) (2.5, 3)

(b) (1.5, 3)

(c) (1.5, 2)

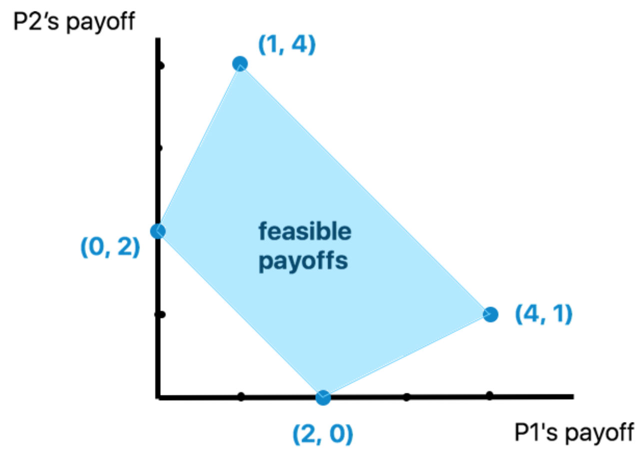
(d) (0.5, 2)

Answer: Only (b) can.

The folk theorem states that, as δ is sufficiently close to 1, any feasible payoff *strictly* above the minmax value can be supported as the average payoff in some subgame perfect equilibrium. Therefore, to answer this question, we need to determine the set of feasible payoffs as well as the minmax value for each player.

Feasible payoffs:

The set of feasible payoffs in this game is the convex set defined by the four payoff pairs, as illustrated in the blue region in the graph below.



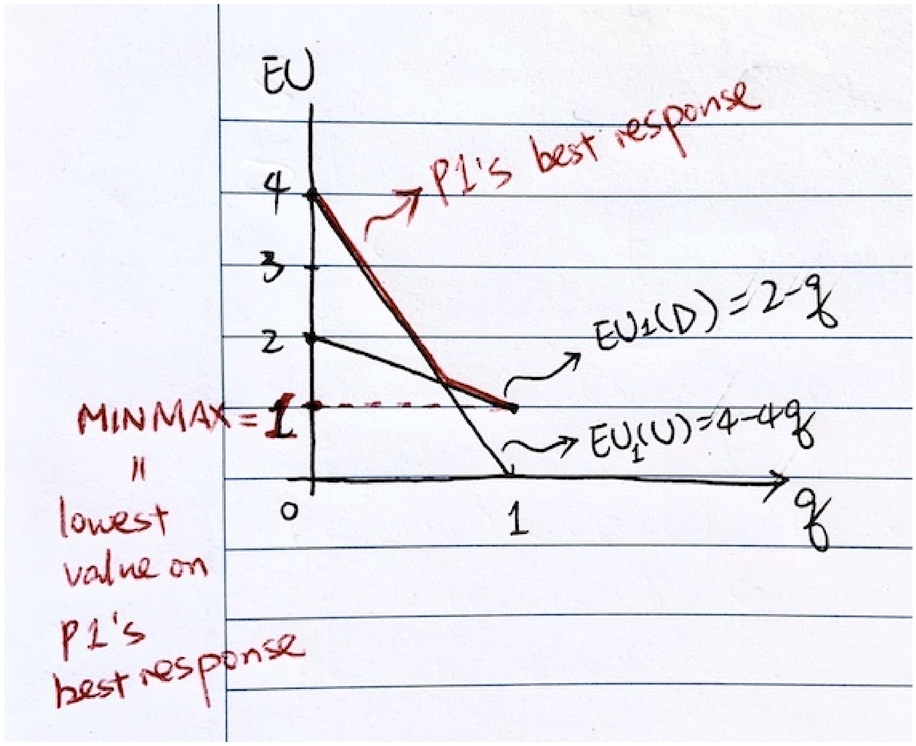
Minmax value:

In this game, the minmax value for player 1 is 1. To see why, let q be the probability that player 2 plays L. Then player 1's expected utility is

$$EU_1(U) = 4(1-q) = 4-4q$$

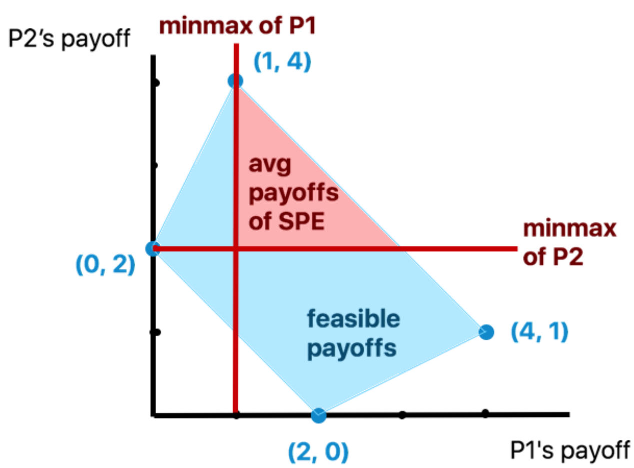
$$EU_2(D) = q + 2(1-q) = 2-q$$

Player 1's best response is to play U when $q < \frac{2}{3}$, U or D when $q = \frac{2}{3}$, and D when $q > \frac{2}{3}$. Player 1's payoff is minimized when player 2 chooses $q = 1$: in this case player 1 chooses D and gets a payoff of 1, which is his minmax value.



The minmax value for player 2 is 2. To see why, note that since L strictly dominates R, player 2's best response is to play L always and his payoff is minimized when player 1 chooses U: in this case player 2's minmax value is 2.

Therefore, any feasible payoff larger than players' minmax value can be supported as the average payoff in some SPE. Graphically, this is the triangular region defined by (1, 2), (1, 4), (3, 2).



Answers (b) is correct because (1.5, 3) belongs in the interior of this red region, which means that it is a feasible payoff pair that is *strictly* higher than the minmax values, as required by the theorem.

However, answer (c) (1.5, 2) is *incorrect* because while 1.5 is strictly higher than player 1's minmax, 2 is *not strictly higher* than player 2's minmax. To see why this payoff pair cannot be achieved in any SPE, first note that if (1.5, 2) is the average payoff of a repeated game, then both players must be randomizing over time. In particular, player 2 must sometimes play R; otherwise player 1's average payoff cannot be 1.5. But since L strictly dominates R, player 2 has an incentive to deviate from the mixed strategy to the pure strategy of playing L always. Can we prevent such a deviation by punishing it? The answer is no, because the worst punishment for player 2 merely keeps his payoff at the minmax value forever. But player 2 was already at his minmax value, which means that no punishment can make him worse off in the long run. Therefore, since it's profitable in the short run to deviate to L, and there is no negative consequence in the long run following this deviation, it is strictly profitable for player 2 to deviate.

Q12 Suppose that the following game is repeated infinitely with a discount factor δ.

		Player 2 L	Player 2 R
Player 1	U	5, 0	1, 1
	D	3, 3	1, 4

Can the payoff pair (1.5, 1.5) be achieved as the average payoff in some subgame perfect equilibrium of this repeated game if δ is sufficiently close to 1? Please explain your answer in details.

Answer:

Calculate the minmax values for each player. Let $p = \Pr(U)$ and $q = \Pr(L)$.

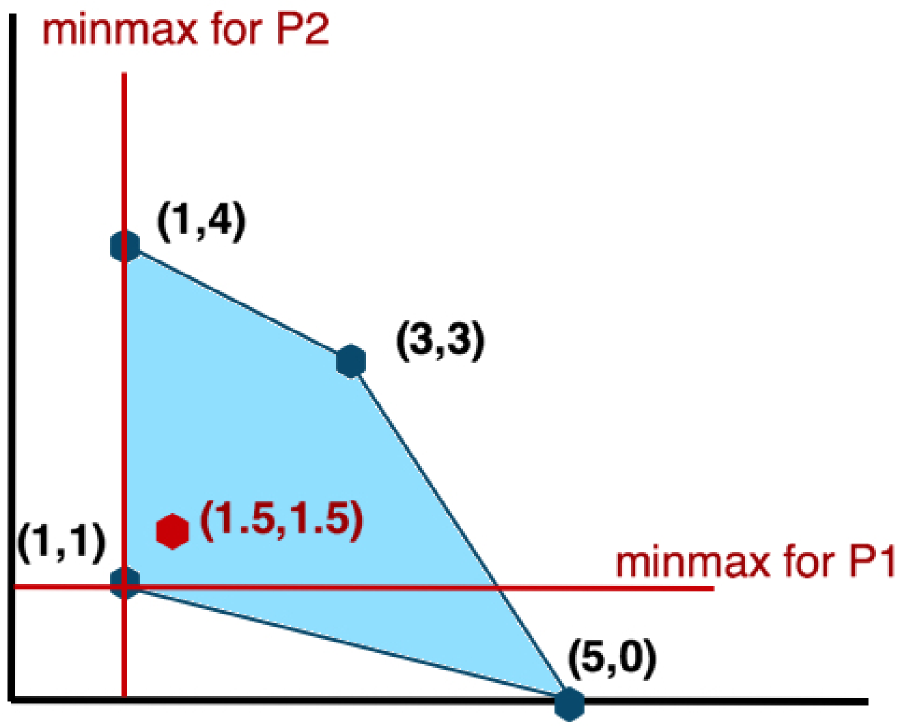
For player 1,

$EU_1(U) = 5q + 1 - q = 4q + 1$. $EU_1(D) = 3q + 1 - q = 2q + 1$. $\max\{EU_1(U), EU_1(D)\} = 4q + 1$, which is minimized when $q = 0$. Therefore, player 1's minmax value is 1.

For player 2,

R strictly dominates L. Therefore, player 2's best response is to play R for all $p \in [0, 1]$. $EU_1(R) = p + 4 - 4p = 4 - 3p$, which is minimized when $p = 1$. Hence, player 2's minmax value is 1.

As shown in the plot below, (1.5, 1.5) is feasible and strictly higher than each player's minmax value. Therefore, it can be achieved as the average payoff in some SPE of the repeated game when δ is sufficiently high.

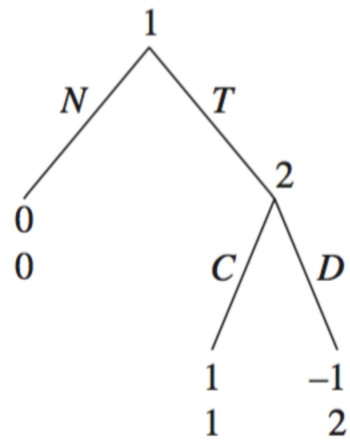


Q13 Consider the following "trust game".

Player 1 first chooses whether to ask for the services of player 2. He can trust player 2 (T) or not trust him (N).

If player 1 plays T, then player 2 can chooses to cooperate (C), which represents offering player 1 some fair level of service, or he can defect (D), which is better for player 2 at the expense of player 1.

The players prefer engaging in a cooperative exchange over not interacting at all (no trust). This game captures many real-life exchanges in which one party must trust another in order to achieve some gains from trade, but the second party can abuse that trust.



(a) Suppose that this trust game is played only once. Find the subgame-perfect equilibrium outcome.

(b) Suppose that this trust game is repeated infinitely with a discount factor of $\delta = 0.6$. Can you construct a subgame-perfect equilibrium in which player 1 always trusts player 2 and player 2 always cooperates? Prove your answer.

A: (a) Player 1 chooses N and the game ends. (Player 2 would choose D if player 1 had chosen T.)

(b) Yes. Here's an example (trigger strategy).

> Player 1: I trust player 2 in period 1 and as long as there were no deviations from (T, C) in any period. If a deviation occurs then I will not trust him forever after.
>
> Player 2: I will cooperate in period 1 and as long as there were no deviations from (T, C) in any period. If a deviation occurs then I will defect forever after.

This trigger strategy separates the game into two phases (two types of subgames):

1. Cooperation phase (no deviation in the past)

   In the cooperation phase, without deviation, the outcome path is (T, C) forever with average payoff of 1 for each player.

   Player 1 never finds it optimal to deviate because the payoff from playing N is 0, which is lower than the current payoff.

   Suppose player 2 deviates from C to D for one period and then reverts back to the trigger strategy. Then the outcome path is (T, D), (N, D), (N, D), (N, D), ... and the average payoff is $2(1-\delta)$. Therefore, there is no profitable deviation from the cooperation phase if and only if

$$1 \geq 2(1-\delta)$$

   or

$$\delta \geq \frac{1}{2}.$$

2. Punishment phase (someone deviated in the past)

   In the punishment phase, without deviation, the outcome path is (N, D) forever with average payoff of 0 for each player. Given that player 2 is playing D, player 1 has no incentive to deviate to T because it simply results in a lower payoff (-1). Given that player 1 is playing N, player 2 has no strictly incentive to play C instead because the payoff is 0 either way. Therefore, there is no profitable deviation from the punishment phase.

1+2: Even though the unique SPE of the stage game is (N, D), we can construct the trigger strategy so that (T, C) is played in every period as long as $\delta \geq \frac{1}{2}$. Since $\delta = 0.6$ in this question, the answer is yes.