A Computational Exploration of the Chaotic Lorenz System

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1. Abstract

With the construction of his famous system of differential equations, meteorologist Edward Lorenz sparked decades of study into the mathematical concept of "chaos" and processes that behave noperiodically. Lorenz's study was inspired by a desire to understand the unpredictability of weather, but he found that the system, while somewhat constrained in its outcome, would toggle between two situations with little consistency. In the sixty years since, much has been discovered about both the general field of chaos theory and the Lorenz system in particular, but there is still much that is not understood or cannot be easily analysed. Here, several methods for solving differential equations are used to explore the system and examine the ways it behaves both chaotically and periodically.

2. Introduction

In 1963, Edward Lorenze published an article titled "Deterministice Nonperiodic Flow" in the *Journal of the Atmospheric Sciences* [3]. While the topic was meteorological in origin, this paper proved hugely influential in mathematics and helped to pioneer the field of chaos theory. Wishing to investigate the apparent unpredictability of weather phenomena, Lorenz studied a deterministic model of fluid convection and simplified it to focus on its nonperiodic behaviour. He discovered that the resulting system was indeed unpredictable, but constrained to a certain space of results, later called the Lorenz Attractor. His analysis of the system includes some defining characteristics of chaotic processes, including nonperiodic orbits, sensitivity to initial conditions and the existence of unstable periodic solutions.

In the years since, chaos theory has been greatly expanded upon, with many more theoretical and practical examples of chaotic systems. But the Lorenz system remains notable for the complexity of the problem, being a set of differential equations that requires numerical methods to effectively examine, and for its implications that real-world processes like weather cannot be deterministically predicted without extremely precise knowledge of the factors at play. The aim of this paper is to use a few basic to mid-level iterative methods to model the Lorenz system and assess the behavior of the attractor. Using only the classic incarnation of the system, several observations will be compiled regarding the trajectories under different conditions.

3. Mathematical Background

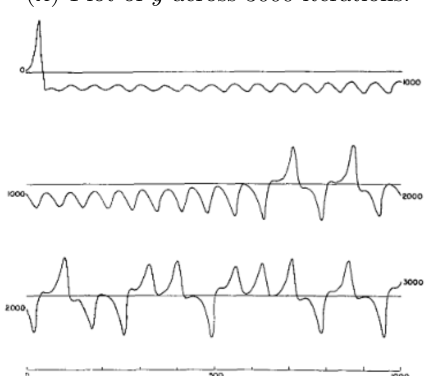
The Lorenz system is a three-dimensional set of differential equations which provided an early example of nonperiodic behavior in deterministic systems. The general form of the system is as follows:

$$\frac{dx}{dt} = -\sigma x + \sigma y$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = xy - \beta z$$

The system was adapted from a model for fluid convection developed by Barry Saltzman. While the variables x, y, and z each have their own physical implications within the context of that study, they were isolated by Lorenz for further investigation of their nonperiodic nature. The constants σ, ρ, and β are likewise somewhat divorced from their original meaning, but classically are set with $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. Under these parameters, there exist two steady-state solutions, $x = y = \pm 6\sqrt{2}, \quad z = 27$, along with the trivial $x = y = z = 0$, where $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0$. The orbit of other points, however, is not stable, and numerical methods have been required to determine their trajectory.

Choosing an initial condition $\mathbf{x}_0 = (0, 1, 0)$ to be near the trivial fixed point, Lorenz found that the system would make an arc towards and begin to oscillate around the steady-state solution of $C' = (-6\sqrt{2}, -6\sqrt{2}, 27)$. However, the magnitude of this oscillation would grow, and after a sufficient duration, it would change course to curve around the solution $C = (6\sqrt{2}, 6\sqrt{2}, 27)$ before returning to its oscillation about C'. Following this, the orbit would switch in proximity between the two solutions, with an inconsistent number of oscillations around one before reverting to the other.

(A) Plot of y across 3000 iterations.



(B) Parametric plot of y and z from iterations 1400-1900

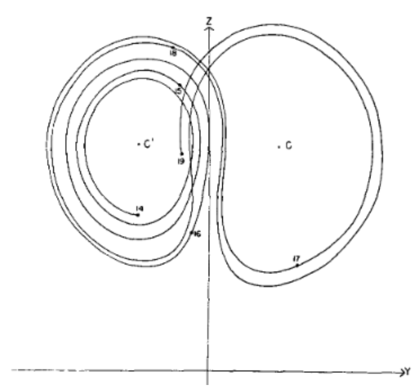


Figure 1. Sketches from Lorenz's paper showing the chaotic behavior of the system.

In subsequent analysis, Lorenz used the local maxima of z to quantify the scale of each oscillation and what effect that had on which solution was proximate. What he found is that each successive maximum would be larger, the curve arcing further outward, until passing some threshold that would cause it to switch to the other solution, usually shrinking back down to a smaller oscillation. Comparing pairs of successive maxima, Lorenz likened the system to the Tent Map and postulated that periodic orbits for the system may exist.

4. Implementation

Applied here is a suite of seven methods for numerically solving differential equations. Euler's method, the "Euler's Improved" method (also known as RK2), and Runge and Kutta's RK4 are each fixed-step methods derived from Taylor expansions and have error terms of order 1, 2, and 4, respectively. Seperating the time window into a number of discrete steps, each method iteratively calculates the vector between one step and the next. In addition to their explicit versions, using the derivative at the start of each sub-interval, Euler's and Improved Euler's methods were implimented to implicitly use the derivate at the end of the sub-intervals for stabler performance on stiff differential equations. Lastly, the adaptive-step methods RKF45 from Runge, Kutta, and Fehlberg and DP45 from Dormand and Prince were constructed with the intent of preserving computational power when a small step size is unnesscessary while ensuring precision can be maintained where needed.

Each method was implimented in GNU Octave according to [1] and [2], taking a function, a starting time, an initial condition, and an ending time. The fixed-step methods have the final parameter h for the step size, while the adaptive methods take four additional arguments instead - minimum and maximum supposed error tolerance, and minimum and maximum step size. Note that by the nature of the adaptive methods, the error tolerance only serves in adhering the component fourth-order method to its fifth-order counterpart, and so, as we shall see, it may still diverge from the exact function values. Each program is written to handle the dependent variable as a vector in addition to a scalar, so as long as the input function and initial condition are defined correctly, we can use them to solve a multi-element system simultaneously. A main driver was then created containing the Lorenz system (See Appendix), which can be edited to choose the method for a trial and adjust parameters and conditions of the system, and which generates a few predetermined visualisations of the generated approximation. By default the system is run for a window $0 \leq t \leq 50$, sufficient to show several oscillations and demonstrate chaotic behavior.

5. Assessment of Methods

The first step was to gauge the practical performance of each of the concerned methods on the Lorenz system. This was achieved through investigation of the x value over time. By the nature of the system, x oscillates chaotically between two states, the positive and negative cycles. Several preliminary trials were run starting at $\mathbf{x}_0 = (-8, 8, 27)$ with the RKF45 algorithm and varying error tolerance, with the results for x plotted and visually compared. It was found that with smaller and smaller tolerance, the behavior of x - and of the system as a whole - was more and more stable, staying consistent with other trials for larger ranges of t (Figure 2). Stability through $t = 10$ was chosen as a baseline, with corresponding error

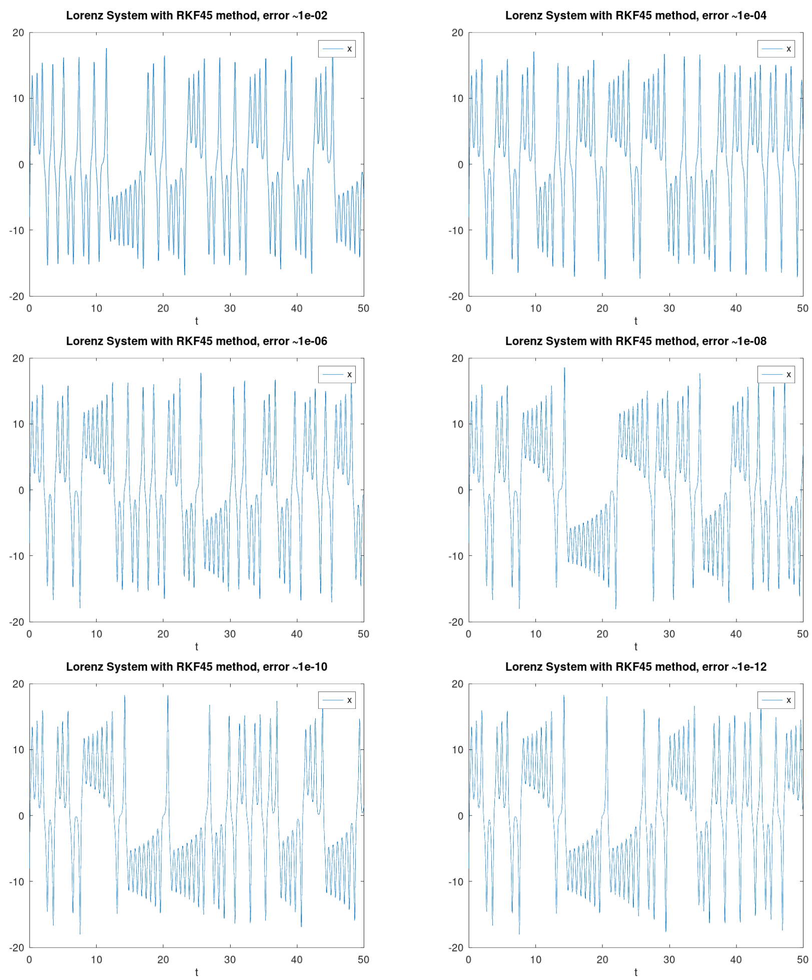


FIGURE 2. Plots of x generated by RKF45, with error tolerances an order of magnitude above and below the stated value. Note that the duration of agreement between successive attempts increases as the tolerance is lowered.

tolerance on the order of 10^{-6}.

With this in mind, values of h were chosen for the fixed-step methods to generate results for a similar expected error, and trials were run of each method at this level. In addition to the value of x, all three variables were plotted against t as well as in different parametric combinations to better visualise the system. The desired error was then steadily lowered to assess the performance of each method at different scales. Of course, with the varying efficiency of the methods, some would take multiple minutes to run at the specified precision. These were subsequently dropped from further trials to keep the data collection time manageable.

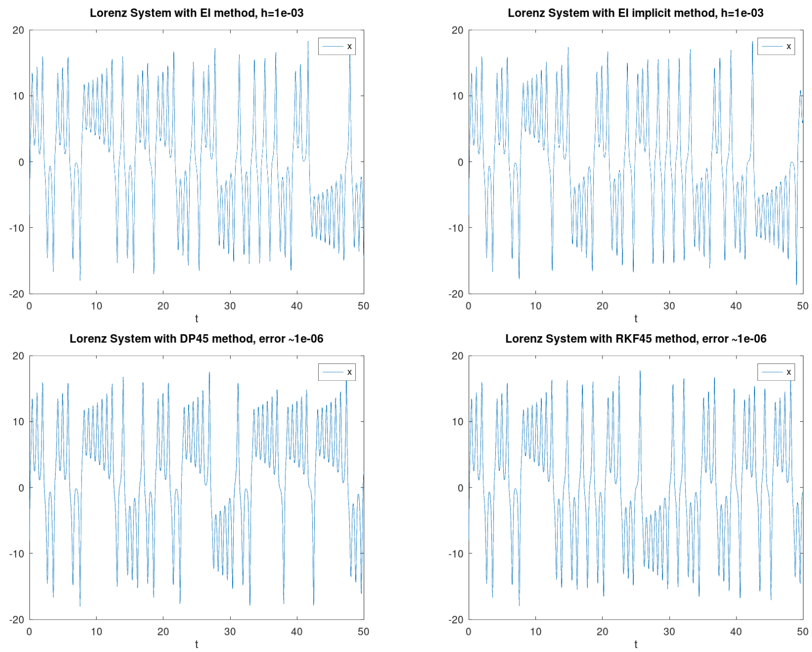


FIGURE 3. Plots of x across various methods, with expected error on the order of 10^{-6}. Note approximate agreement through $t = 10$.

For both versions of the Improved Euler's method and both adaptive methods, there was unanimity where expected, and while the Improved - particularly in its implicit form - is computationally expensive with $\mathcal{O}(n^2)$, it exhibits quite reasonable accuracy. Unsurprisingly, having only $\mathcal{O}(n)$, both versions of the mundane Euler's method are too intensive to run at this precision, and they appear somewhat behind the curve in comparative accuracy (Figure 4). Also behind the curve, however, was the RK4 method; with $\mathcal{O}(h^4)$, it would be expected that h could be chosen as the fourth root as the general error target, but it consistently required a half order of magnitude smaller to produce outputs consistent with the others (Figure 5).

As the error tolerance was lowered, these findings of limited efficiency have eliminated all but the adaptive methods, which both manage to run well at a tolerance of 10^{-14}. At this stage, computations take somewhat over a minute to run, and because of this, along with concerns of floating-point limitations, further precision was not pursued. At least under these conditions, this seems to cap off consistent results at around $t = 35$, which is no small feat but still a testament to the system's chaotic nature. As for finalising the optimal method for these calculations, RKF45 and DP45 have previously been found to produce rather comparable results, but it generally seems to be a trend that the latter will be slightly more accurate at a cost of slightly longer computation time. Interestingly, DP45 here takes slightly less time to run, and while there is no exact function to compare true errors, close inspection of the output plots suggests that it does indeed remain consistent slightly longer

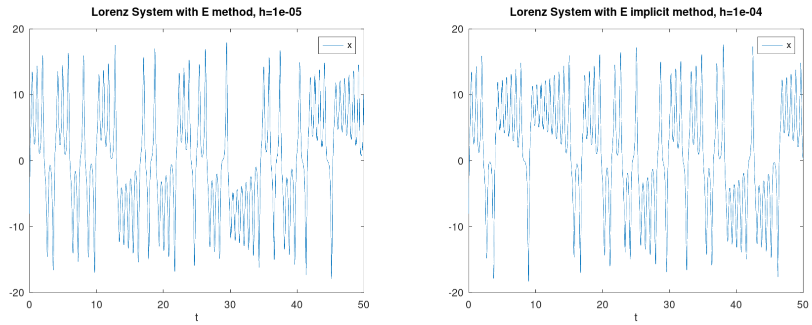


FIGURE 4. Plots of x for explicit and implicit Euler's method, at the smallest value of h that could be run in reasonable time.

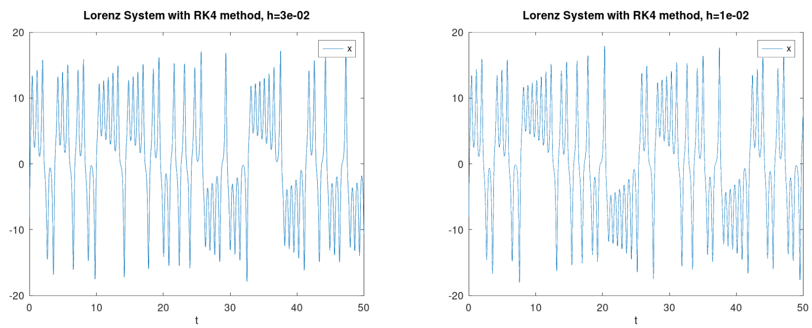


FIGURE 5. Plots of x for RK4, showing how a somewhat tighter step than expected is needed to match the Improved and adaptive methods.

with what is subsequently established as the correct trace (Figure 6). Because reliability is at a premium, and the computation time even at this scale manageable, DP45 will thus be used with a tolerance of 10^{-14} for all future calculations.

6. Analysis of Lorenz System

The initial condition $\mathbf{x}_0 = (-8, 8, 27)$ used for method comparison is equidistant from both steady-state solutions, and whether by coincidence or not, it appears to start behaving chaotically much more quickly than Lornez found with $\mathbf{x}_0 = (0, 1, 0)$. Looking at Figure 7, however, it is clear that this is much the same two-lobed attracter that he had encountered in his pioneering studies. And consistent with his findings, we find that oscillation around one solution only becomes more and more exaggerated until reaching some critical extreme, whereby it switches to the other solution for at least one cycle around.

Of course, a defining feature of chaotic systems is their sensitivity to changes in initial conditions, and so truly observing chaos requires more than one trial. Adjusting each dimension by only 10^{-6}, we can see in Figure 8 that this has a very noticeable effect on the resulting orbit, even if it still constrained to the shape of the attracter. The traces of each actually stays in agreement until around $t = 15$,

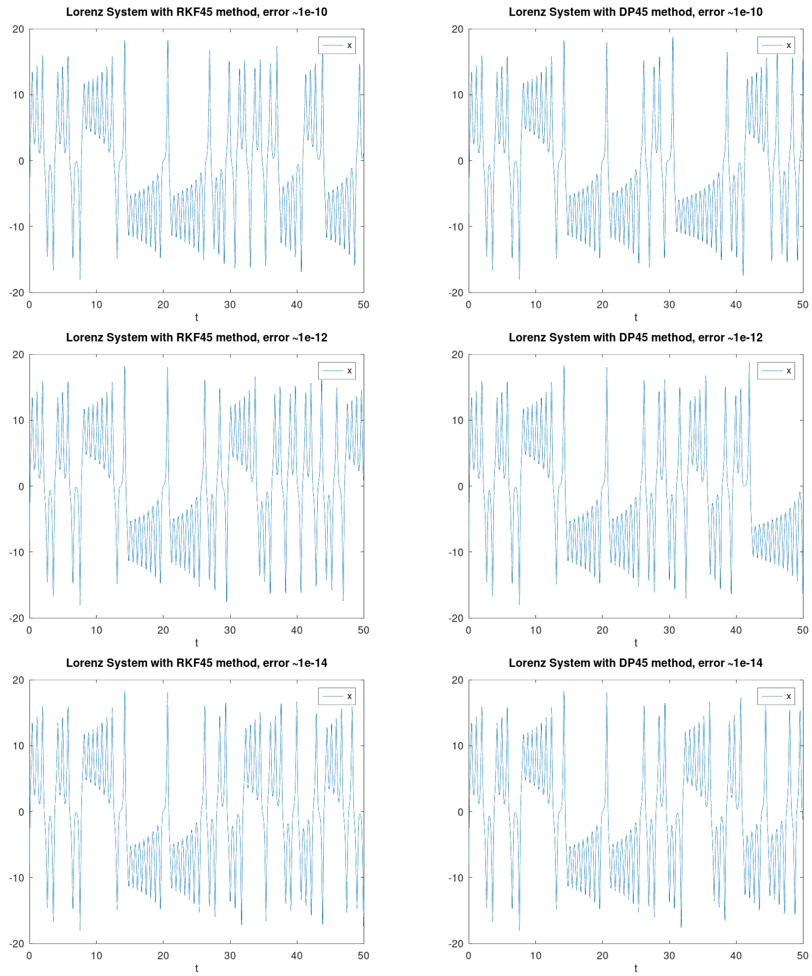


FIGURE 6. Plots of x from both adaptive methods for smaller error tolerances.

at which point all positively-adjusted trials diverge to the other lobe while the negatively-adjusted trials remain in place for one more cycle around. It doesn't take too long after that for the trajectories to finishing splitting up, and their separation is apparent. While one could accuse this variation as being the product of the methods, the earlier tests suggest that at least with these parameters, results can be trusted through to $t = 35$. And after all, again, the fact that each test trial would eventually beging to deviate dramatically is only further testament to chaos.

Another common characteristic of chaotic systems is that large-scale differences, while quite consequential, may result in orbits which are still rather indistinguishable in the long term. Setting $\mathbf{x}_0 = (0, 1, 0)$ Lorenz's own results were reproduced, and as suggested earlier they eventually follow a trace with no special difference despite the difference in initial conditions (Figure 9). But this orignal case is actually quite interesting when one considers the behavior of the system and its similarity to the Tent Map. It appears that $\mathbf{x}_0$ can be considered part of the "positive" lobe

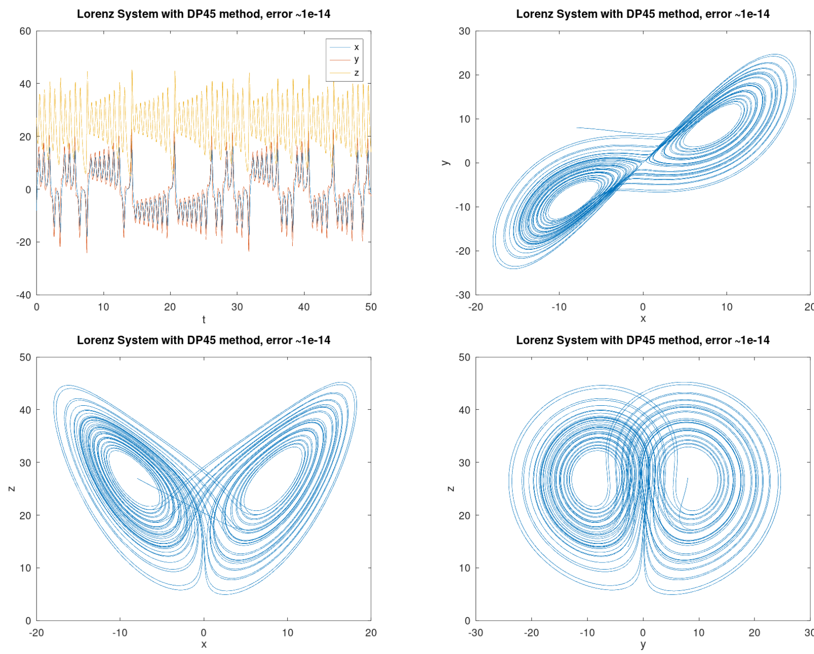


FIGURE 7. Plot of all three variables and parametric plots of each pair, $\mathbf{x}_0 = (-8, 8, 27)$.

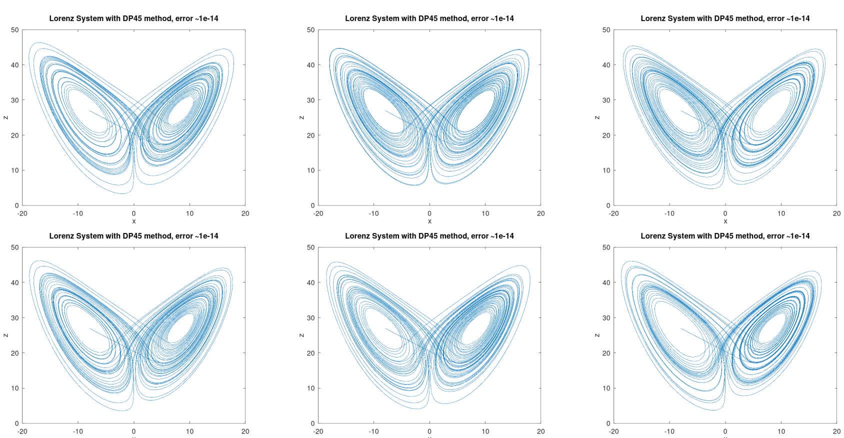


FIGURE 8. x-z parametric plots starting from slight adjustments of $\mathbf{x}_0 = (-8, 8, 27)$. Left, middle, and right columns have x_0, y_0, and z_0 adjusted, respectively; top row has $+10^{-6}$, bottom row has -10^6.

of the attractor, yet it is far enough out to not only immediately switch to the other lobe but arc in much closer to the corresponding solution than has been seen yet. This results in the slow-growing oscillation Lorenz saw for the first half of his

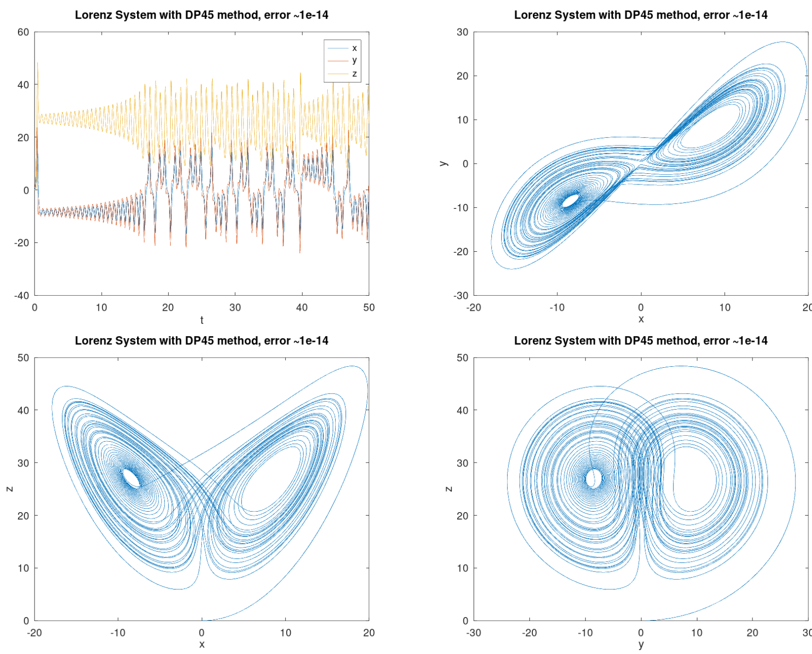


FIGURE 9. Plot of all three variables and parametric plots of each pair, $\mathbf{x}_0 = (0, 1, 0)$.

simulation, which, once it grows large enough, begins the familiar range of orbit and remains in that scope.

Another quality of chaotic systems is the existence of periodic orbits, infinitely dense but repelling nearby trajectories. Already it was noted by Lorenz that three fixed points exists, and running the system with any as an initial condition confirms this (Figure 10). Making any slight adjustment to C or C' will accordingly cause

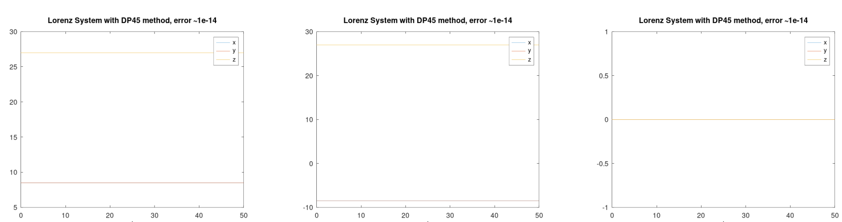


FIGURE 10. Values of x, y, and z for each fixed point. Values of $\mathbf{x}_0$ from left to right: $C = (6\sqrt{2}, 6\sqrt{2}, 27)$, $C' = (-6\sqrt{2}, -6\sqrt{2}, 27)$, and the trivial $(0, 0, 0)$.

divergence, although starting at quite a slow rate. This fact is consistent with the previous examination of Lorenz's pick of $\mathbf{x}_0$, where, after reaching rather close to C', it took much longer for the orbit to spiral back out to the regular scope of the attractor. However, the fixed point at the origin is not actually wholly repelling, as a quick examination of the governing equations will reveal. For any point $(0, 0, z_0)$,

$\frac{dx}{dt} = \frac{dy}{dt} = 0$, but as β is positive, $\frac{dz}{dt}$ will point towards the origin, and so any orbit starting at such a value will be attracted to $(0, 0, 0)$ exponentially.

Much more interesting, and harder to find, are non-fixed periodic orbits. Some such trajectories have been found with great effort, but Alexander N. Pchelintsev [4] was able to analytically derive an alternate method for numerically finding cycles in the Lorenz system. While the method is still highly involved, even for a low-level cycle, Pchelintsev was able to use it to confirm that $\mathbf{x_0} \approx (-2.147367631, 2.078048211, 27)$ results in a cycle of period ~ 1.55865221. Plotting it here, unfortunately, still results in divergence, but only after $t = 20$, suggesting that this is once again a symptom of machine limitations in numerical methods. Examining the parametric plots in Figure 11, one can see the cycle as the thicker-appearing part of the curve, where several slight deviations were stacked before the error was too great and the system started behaving chaotically. Notable is that in each view, that part corresponding to the cycle appears rather symmetric, rotationally in the x-y plot and bilaterally for x-z and y-z. In contrast, the nonperiodic behavior tends to result in the two lobes looking more lopsided, both in largest arc and general density and distribution. Given previous speculations on the nature of the system, it seems reasonable that this orbit remains stable (or would, if the methods were more precise) because it has a magnitude which is neither low enough to keep oscillating at one lobe nor high enough to end up at such a low magnitude once passed to the other lobe. Instead it hits some perfect value which allows it to retain its magnitude through every half-cycle, thus mapping on to itself. It seems unlikely then that any other cycles exist with this structure, but longer periods may likewise exhibit these symmetries, either in themselves or with an opposite chiral pair.

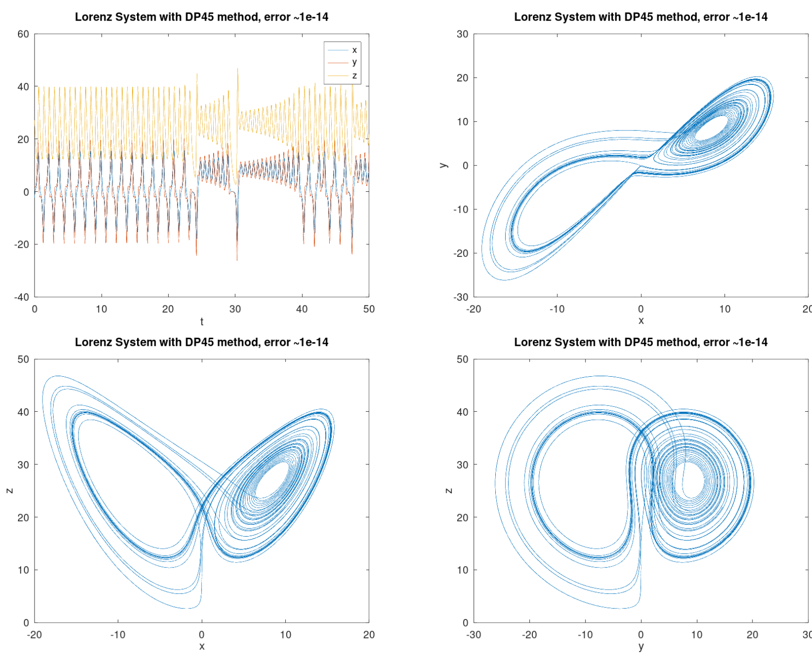


FIGURE 11. Plot of all three variables and parametric plots of each pair, $\mathbf{x}_0 = (-2.147367631, 2.078048211, 27)$.

7. Conclusion

There may be implications of the Lorenz system that will never be discovered, and meteorological processes may never be predictable with full accuracy. But with the development of stronger and more efficient modes of approximation, there is much that has been and will continue to be learned about chaos both manufactured in theory and observed in the world. With a greater understanding of the logic that governs this system and what form periodicity takes in it, it may be possible to predict not what exactly will occur, but when stability is to be expected or change imminent. And with peridoic and nonperiodic behavior seeming to be governed by the same rules, only separated by precise imbalance, more research could find a better way to quantify the system as a whole.

References

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