The Term Structure of Interest Rates

Robert A. Jarrow

Johnson Graduate School of Management, Cornell University, Ithaca, New York 14853; email: raj15@cornell.edu

Annu. Rev. Financ. Econ. 2009. 1:69–96

First published online as a Review in Advance on October 22, 2009

The *Annual Review of Financial Economics* is online at financial.annualreviews.org

This article's doi:
10.1146/annurev.financial.050808.114513

1941-1367/09/1205-0069$20.00

Key Words

arbitrage-free term structures, HJM model, expectations hypothesis, LIBOR model, futures and forward contracts

Abstract

This paper reviews the term structure of interest rates literature relating to the arbitrage-free pricing and hedging of interest rate derivatives. Term structure theory is emphasized. Topics included are the HJM model, forward and futures contracts, the expectations hypothesis, and the pricing of caps/floors. Directions for future research are discussed.

1. INTRODUCTION

The term structure of interest rates literature studies the evolution of default-free interest rates of different maturities (the term structure) across time. Alternatively stated, the term structure of interest rates studies the evolution of default-free bond prices of different maturities. There are two related approaches to understanding this evolution: equilibrium and arbitrage-free pricing models.

Equilibrium models are more complex, requiring more assumptions about investor preferences and market structures. This additional structure enables equilibrium models to characterize interest rate risk premia or, equivalently, the expected returns on zero-coupon bonds of all maturities (the price evolution under the statistical probability measure). To test these models, one needs to estimate the interest rate risk premia because these differentiate the various models.

In contrast, complete market arbitrage-free pricing models study the evolution of the term structure of interest rates to price interest rate derivatives given the initial prices for a collection of zero-coupon bonds. For this reason, the arbitrage-free pricing methodology is often termed a relative pricing theory. The arbitrage-free pricing models are also useful for understanding interest rate risk management, in particular the hedging of interest rate risk. These models are valid for arbitrary interest rate risk premia; hence, they are consistent with numerous equilibrium models. To test these models, one needs to estimate only volatilities and correlations of default-free interest rates and validate (or reject) the model's hedging accuracy.

Although the term structure of interest rates literature in financial economics investigates both equilibrium and arbitrage-free pricing models, this paper concentrates on the arbitrage-free pricing approach because it is unique to financial economics, whereas equilibrium models are not. This paper focuses on the theory relating to the term structure of interest rates, which is in a relatively mature state, as evidenced by the numerous textbooks on the topic (see Baxter & Rennie 1996, Brigo & Mercurio 2001, Jarrow 2002, Musiela & Rutkowski 2004, Rebonato 2002, Schoenmakers 2005, and Zagst 2002). The empirical literature is discussed only briefly; a detailed presentation is left for a subsequent paper.

An outline of this paper is as follows. Section 2 gives a historical overview of the financial economics literature on the term structure of interest rates. Section 3 provides the preliminaries for the Heath-Jarrow-Morton (HJM) model that is presented in Section 4. Section 5 studies forward and futures contracts, a topic that is only interesting if interest rates are stochastic. Section 6 investigates the traditional expectations hypothesis, and Section 7 the prices caps and floors using the London InterBank Offer Rates (LIBOR) model. Section 8 briefly discusses the empirical evidence, and Section 9 talks about the application of term structure models to related topics. Section 10 concludes the review with a discussion on topics for future research. All significant proofs are contained in an appendix (Section 11).

2. A HISTORICAL OVERVIEW

In arbitrage-free pricing models, the term structure of interest rates literature involves the study of using derivatives to hedge (risk-manage) the financial risks generated by the evolution of default-free interest rates of different maturities. This literature best applies to two markets. The first is government debt issued by countries whose national balance sheets imply that default is highly unlikely, examples being the United States, Germany, Great Britain, and Japan. Government debt is studied in its domestic currency, excluding foreign currency exchange considerations. The second is borrowing/lending of U.S. dollars of different maturities in the Eurodollar market.[1] The Eurodollar market consists of U.S. dollars deposited into European banks that operate outside the U.S. banking system's reserve requirements. As such, Eurodollar interest rates are less subject to the supply impacts of the U.S. Treasury auction cycle and the direct impact of U.S. monetary policy. Eurodollar borrowing/lending is nearly default free due to the credit quality of the banks involved and the existence of collateral agreements between the relevant counterparties.

The study of using derivatives to hedge financial risks really began with the publication of the Black-Scholes-Merton (BSM) option pricing model (Black & Scholes 1973, Merton 1973). The BSM model assumes deterministic interest rates.[2] This assumption best applies to short-dated equity and foreign currency options where deterministic interest rates are a reasonable approximation. The discovery of the BSM technology facilitated the expansion of these option markets by enabling the hedging (risk management) of equity and foreign currency risks.

Prior to the 1970s, interest rates were relatively low and stable, so managing interest rate risk was less of a market concern. However, in the 1970s and the 1980s, due to double-digit inflation and the shift from fixed to floating currency exchange rates, interest rates became large and volatile. This increased interest rate risk generated a demand for tools and financial securities to help manage these risks. This demand led to the emergence of the over-the-counter interest rate derivatives market with the introduction of the first interest rate swap in late 1981.[3]

New interest rate risk management tools were needed because the BSM model was not well designed for this application. Assuming deterministic interest rates, the BSM model applies less well to long-dated contracts (greater than a year or two) and for short-dated contracts where the underlying asset's price process is highly correlated with interest rate movements. This, of course, is the case for interest rate derivatives, where the underlying assets are the interest rates themselves.

In response, a class of equilibrium-based interest rate pricing models was developed by Merton (1970), Vasicek (1977), Brennan & Schwartz (1979), and Cox et al. (1985); the latter is known as the Cox-Ingersoll-Ross (CIR) model. This class of models, known as spot rate models, had two limitations. First, they depended on the interest rate risk premia or, equivalently, the expected return on default-free bonds. This dependence, just as with the option pricing models pre-BSM, made their implementation problematic. Indeed, the estimation of risk premia is very difficult. The reason for this difficulty is that the empirical finance literature has documented that risk premia are nonstationary. They vary across time according to both changing tastes and changing economic fundamentals. This nonstationarity makes problematic both the modeling of risk premia and their estimation.

[1]The BSM model is the formula in the Black & Scholes (1973) paper. Merton (1973) also extended this formula to allow for a stochastic interest rate by including a single zero-coupon bond, whose maturity date matches that of the call option. This extension, however, did not introduce an entire term structure of interest rates into the valuation methodology.

[2]There is also an analog to Eurodollar markets in alternative currencies, known as Eurocurrencies, e.g., Euroyen.

[3]See Fabozzi (2000, p.585).

Indeed, at present, there is still no generally accepted model for an asset's risk premium that is consistent with historical data (see Cochrane 2001). Second, these models could not easily match the initial yield curve. This calibration is essential for the accurate pricing and hedging of interest rate derivatives because any discrepancies in yield curve matching may indicate false arbitrage opportunities in the priced derivatives.

To address these limitations, Ho & Lee (1986) applied a binomial option pricing model to interest rate derivatives with a twist. Instead of imposing an evolution on the spot rate, they had the zero-coupon bond price curve evolve in a binomial tree. This was a complete market model; that is, there were enough zero-coupon bond prices trading to hedge all interest rate risk. Risk premia are not necessary to price interest rate derivatives in a complete market setting.

Motivated by this paper, Heath et al. (1992) generalized this idea in the context of a continuous time and multifactor-complete market model, termed the HJM model. The key step in the derivation of the HJM model was determining necessary and sufficient conditions for an arbitrage-free evolution of the term structure of interest rates. To simplify the mathematics (although this is not necessary), Heath et al. focused on forward rates instead of zero-coupon bond prices. The martingale pricing technology of Harrison & Kreps (1979) and Harrison & Pliska (1981, 1983) was the tool used to obtain the desired conditions—the HJM arbitrage-free drift conditions. Given the HJM drift conditions and the fact that the interest rate market is complete, standard techniques then apply to price interest rate derivatives.

The HJM model is very general: All previous spot rate models are special cases. In fact, the labels Ho & Lee, Vasicek, extended Vasicek (or sometimes Hull & White 1990), and CIR are now exclusively used to identify subclasses of the HJM model. Subclasses are uniquely identified by a particular volatility structure for the forward rate curve's evolution. For example, the Ho & Lee (1986) model is now identified as a single factor HJM model where the forward rate volatility is a constant across maturities.

Subsequent research developed special cases of the HJM model that have nice analytic and computational properties for implementation. Perhaps the most useful class, for its analytic properties, is the affine model of Duffie & Kan (1996) and Dai & Singleton (2003). The class of models is known as affine because the spot rate can be written as an affine function of a given set of state variables. This class of term structure evolutions have known characteristic functions for the spot rate, which enables numerical computations for various interest rate derivatives (see Duffie et al. 2000). Extensions of the affine class include Filipović (2002), Chen et al. (2004), and Cheng & Scaillet (2007).

The original HJM paper showed that lognormally distributed instantaneous forward rates are inconsistent with no arbitrage. Hence, geometric Brownian motion is excluded as an acceptable forward rate process. This is unfortunate because it implies that caplets, options on forward rates, will not satisfy Black's equation (Black 1976). And historically, because of the industry's familiarity with the BSM formula (a close relative of Black's formula), Black's formula was used extensively to value caplets. This inconsistency between theory and practice led to a search for a theoretical justification for using Black's formula with caplets.

This search was ended by Sandmann et al. (1995), Miltersen et al. (1997), and Brace et al. (1997). The solution was to use a simple interest rate, compounded discretely. Of course, simple rates better match practice. Also, it was shown that the evolution of a simple interest rate can evolve as a geometric Brownian motion in an arbitrage-free setting.

This subclass of the HJM model comprises what are known as the LIBOR models, after their application to LIBOR. Subsequently, the lognormal evolution has been extended to jump diffusions (see Glasserman & Kou 2003), Levy processes (see Eberlein & Ozkan 2005), and stochastic volatilities (see Andersen & Brotherton-Ratcliffe 2001), which brings us to the current literature on interest rate term structure models.

Just as the option pricing models played a key role in enabling the expansion of the equity, foreign currency, and commodity option markets, the term structure of interest rate models played a key role in enabling the expansion of fixed income markets. This includes the over-the-counter interest rate derivative markets (swaps, caps, floors, swaptions, etc.), asset-backed security markets (mortgages—residential and commercial; loans—corporate, student, car, and credit cards; etc.), and the credit derivative markets (default swaps, collateralized default obligations, etc.). Derivative pricing models enable financial expansion because they provide the tools for risk management—the hedging of financial risks. Hedging facilitates the creation of two-sided markets via enabling a financial intermediary to synthetic-construct, the opposite side to a one-sided financial market. This expansion of financial derivative markets, through the efficient allocation of capital, has had positive and real effects on economic growth.

3. PRELIMINARIES

We consider a continuous-time, continuous-trading model with time horizon $[0, \tau]$. We are given a filtered probability space $[\Omega, \mathcal{F}, (\mathrm{F}_t)_{t\in[0,\tau]}, P]$ satisfying the usual conditions; see Protter (2005), where P is the statistical probability measure.

Trading is a collection of default-free zero-coupon bonds of all maturities, with time t price $p(t, T)$ for a sure dollar paid at time T for all $0 \leq t \leq T \leq \tau$. That is, $p(t,t) = 1$ for all t. The zero-coupon price process $p(t, T)$ is adapted to $(\mathcal{F}_t)_{t\in[0,\tau]}$ for all T. We also assume that zero-coupon bond prices are strictly positive; that is, $p(t, T) > 0$ for all t, T.

The forward rate is implicitly defined by the expression

$$p(t,T) = e^{-\int_t^T f(t,u)du}. \tag{1}$$

Taking natural logarithms and differentiating with respect to T yields the expression

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T} = -\frac{\partial p(t,T)}{\partial T}\frac{1}{p(t,T)}.$$

The forward rate corresponds to default-free borrowing and lending implicit in the zero-coupon bonds over the future time period $[T - dt, T]$. To see this, note that for small Δ,

$$f(t,T)\Delta \approx -\frac{p(t,T)-p(t,T-\Delta)}{\Delta}\frac{\Delta}{p(t,T)} = \frac{p(t,T-\Delta)}{p(t,T)} - 1.$$

The default-free spot rate is $r_t = f(t, t)$. It is the rate for immediate borrowing or lending.

We also assume that a money market account trades with initial value $B(0) = 1$ that invests in the spot rate, specifically

$$B(t) = e^{\int_0^t r_s ds}.$$

We assume that there are frictionless and competitive markets for trading in $p(t, T)$ for all T and $B(t)$. Frictionless markets mean that there are no transaction costs, taxes, or restrictions on trades (e.g., short sale constraints or margin requirements), and competitive markets means that every investor acts as a price taker, specifically, that their trades have no impact on the market price. We refer to a market constructed above an interest rate market model.

This model applies best to two markets. The first is government debt issued by countries with national balance sheets where default is highly unlikely, examples being the United States, Germany, Great Britain, Japan, and the second is borrowing/lending of U.S. dollars of different maturities in the Eurodollar market. Eurodollar borrowing/lending is nearly default free due to the existence of collateral agreements between the relevant counter parties. The frictionless and competitive markets assumptions are a good first approximation for these markets.

4. ARBITRAGE-FREE TERM STRUCTURE MODELS

The arbitrage-free interest rate term structure methodology is known as the HJM model (Heath et al. 1992). The HJM model starts by assuming an evolution for the term structure of interest rates. Although unnecessary, for simplicity of computation, they select the forward rate curve. In particular, the HJM model assumes that we are given an initial forward rate curve,

$$f(0, T) \text{ for } 0 \leq t \leq T \leq \tau,$$

and a stochastic process for the evolution of the forward rate curve,

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)dt + \sum_{i=1}^{n} \int_0^t \sigma_i(s, T)dW_i(s) \text{ for } 0 \leq t \leq T \leq \tau, \qquad (2)$$

where $W_i(t)$ for $i = 1,\ldots n$ are independent standard Brownian motions adapted to $(\mathcal{F}_t)_{t\in[0,\tau]}$, and $\alpha(t, T)$ and $\sigma_i(t, T)$ for $i = 1,\ldots n$ are $(F_t)_{t\in[0,\tau]}$ adapted, suitably measurable and integrable, so that expression 2 is well defined (see Heath et al. 1992).[4] In differential form, this is

$$df(t, T) = \alpha(t, T)dt + \sum_{i=1}^{n} \sigma_i(t, T)dW_i(t). \qquad (3)$$

This in an n – factor model for the forward rate evolution. The n – factors represent distinct economic forces affecting forward rates (e.g., inflation, unemployment, economic growth). Expression 2 represents a very general stochastic process.[5] Indeed, the evolution need not be Markov (in a finite number of state variables) and can be path dependent. In addition, given that the interest rate market model does not have currency trading (only zero-coupon bonds and a money market account), forward rates can be negative. To exclude negative forward rates, an additional restriction needs to be imposed on the evolution in expression 2.

[4]For this paper, we do not emphasize the technical conditions needed for the subsequent expressions to exist. The interested reader is referred to the original publications for these details.

[5]In fact, the only substantive economic restriction in expression 2 is that the process has continuous sample paths; however, this restriction is also unnecessary.

Under this evolution, the spot rate process is

$$r_t = f(0,t) + \int_0^t \alpha(s,t)dt + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t)dW_i(s). \tag{4}$$

Note that in this expression as time t changes, the second argument in both integrands also changes. This makes the evolution of the spot rate more complex than that of the forward rate process itself. In stochastic differential equation form, this is reflected in an additional term in the drift of the spot rate process:

$$dr_t = \left(\frac{\partial f(t,t)}{\partial T} + \alpha(t,t)\right)dt + \sum_{i=1}^{n} \sigma_i(t,t)dW_i(t). \tag{5}$$

From Equation 2, we can deduce the evolution of the zero-coupon bond price:

$$p(t,T) = p(0,T)e^{\int_0^t (r_s + b(s,T))ds - \frac{1}{2}\sum_{i=1}^{n}\int_0^t a_i(s,T)^2 ds + \sum_{i=1}^{n}\int_0^t a_i(s,T)dW_i(s)}, \tag{6}$$

where $b(s,T) = -\int_s^T \alpha(s,u)du + \frac{1}{2}\sum_{i=1}^{n} a_i(s,T)^2$ and

$a_i(s,T) = -\int_s^T \sigma_i(s,u)du$ for all $i = 1, \ldots n$.

The proof of expression 6 is contained in the appendix.

As a stochastic differential, the zero-coupon bond price evolves as

$$\frac{dp(t,T)}{p(t,T)} = [r_t + b(t,T)]dt + \sum_{i=1}^{n} a_i(t,T)dW_i(t). \tag{7}$$

The zero-coupon bond's instantaneous return consists of a drift plus n – random shocks. Within the drift, one can interpret $b(s, T)$ as the (equilibrium) risk premium on the zero-coupon bond in excess of the spot rate of interest. The bond's volatilities are $\{a_1(t, T,), \ldots, a_n(t, T)\}$.

4.1. Arbitrage-Free Conditions

From Harrison & Pliska (1981), the interest rate market model is arbitrage free if and only if there exists a probability measure Q equivalent to P such that $\frac{p(t,T)}{B(t)}$ is a $Q-$ martingale for all T. Equivalent means that both Q and P agree on zero-probability events in the set F. Heath et al. 1992 showed that for the interest rate market model, an equivalent martingale measure Q exists if and only if there exist risk premia processes

$$\phi_i(t) \text{ for } i = 1, \ldots, n,$$

suitably measurable and integrable such that

$$\alpha(t,T) = -\sum_{i=1}^{n} \sigma_i(t,T)\left[\phi_i(t) - \int_t^T \sigma_i(t,v)dv\right] \tag{8}$$

for all $0 \le t \le T \le \tau$.

This is known as the HJM arbitrage-free drift condition. The proof of the HJM arbitrage-free drift condition is contained in the appendix (Section 11). If the traded zero-coupon bonds reflect systematic risk, then the risk premia will be nonnegative; that is, $\phi_i(t) \ge 0$. This is the typical situation.

It is important to emphasize that the HJM arbitrage-free drift condition shows that not all forward rate curve evolutions are consistent with an arbitrage-free economy. In fact, as shown in the original paper by Heath et al. 1992, an evolution inconsistent with an arbitrage-free interest rate economy is geometric Brownian motion, namely

$$df(t,T) = \alpha(t,T)dt + \sigma f(t,T)dW_t, \tag{9}$$

where σ is a constant. It was shown that, under geometric Brownian motion, forward rates explode with positive probability over any finite interval, implying zero-coupon bond prices have zero value (and, hence, arbitrage exists).

In an arbitrage-free interest rate market model, $\frac{p(t,T)}{B(t)}$ being a Q – martingale implies that

$$\frac{p(s,T)}{B(s)} = \tilde{E}_t\left(\frac{p(t,T)}{B(t)}\right) \text{ for } s \leq t, \tag{10}$$

where $\tilde{E}_t(\cdot)$ denotes expectation under the martingale measure Q. From an economic perspective, this is equivalent to

$$p(t,T) = \tilde{E}_t(e^{-\int_t^T r_s ds}). \tag{11}$$

This expression states that the price of a zero-coupon bond is equal to its expected discounted payoff, where the discount rate is the spot rate of interest. Because the discount rate contains no adjustment for risk, this approach is often called risk-neutral valuation. Indeed, expression 11 is the equilibrium price that would exist in an interest rate market model where all investors are risk neutral and have beliefs represented by Q (and not P).

In essence, the risk premia are embedded in the expectations operator when computing present values, and not in the discount rate.[6] To understand this statement, it can be shown via Girsanov's theorem (see Protter 2005) that the equivalent martingale probability measure Q relates to the statistical probability measure P via the expression

$$\frac{dQ}{dP} = e^{\sum_{i=1}^{n}\int_0^\tau \phi_i(s)dW_i(s) - \frac{1}{2}\sum_{i=1}^{n}\int_0^\tau \phi_i(s)^2 ds}. \tag{12}$$

The difference between the statistical and martingale probability measures reflects the interest rate risk premia $\{\phi_1(t),\ldots, \phi_n(t)\}$. If all the risk premium were identically zero, then the two probability measures would be equal; that is, $Q = P$. We return to this issue below when we discuss the traditional expectations hypotheses.

4.2. Complete Markets

Harrison & Pliska (1981, 1983) proved that an arbitrage-free market is complete if and only if the martingale measure Q is unique. For an interest rate market model, a complete market is one where any random payoff desired at a future date, based on the information

[6]In finance textbooks, a present value formula is often written as $p(t,T) = E_t\left(e^{-\int_t^T (r_s+\theta_s)ds}\right)$, where θ_s is a risk premium adjustment to the discount rate and $E_t(\cdot)$ is the expectation using the statistical probability P. The arbitrage-free pricing methodology adjusts the expectations operator and not the discount rate. Alternatively, rewriting $p(t,T) = E_t\left(\left[1\cdot\frac{dQ}{dP}\right]e^{-\int_t^T r_s ds}\right)$, one can view the change of measure as equivalent to computing a certainty equivalent.

generated by the evolution of the term structure of interest rates, can be obtained by trading continuously in a dynamic portfolio of zero-coupon bonds. Using this result, Heath et al. (1992) showed that in an interest rate market model Q is unique if and only if there exist a set of distinct zero-coupon bonds $T_1, \ldots T_n \in [0, \tau]$ such that the matrix of zero-coupon bond price volatilities,

$$\begin{pmatrix} a_1(s, T_1) & \ldots & a_n(s, T_1) \\ \vdots & & \vdots \\ a_1(s, T_n) & \ldots & a_n(s, T_n) \end{pmatrix}, \tag{13}$$

is nonsingular for almost all s with probability one. In essence, there exists a set of zero-coupon bonds $\{T_1, \ldots, T_n\}$ that span the randomness generated by the $n-$ Brownian motions.

A complete market enables unique pricing of interest rate derivatives. Indeed, Harrison & Pliska (1981, 1983) also showed that in a complete market, any suitably measurable and integrable random payoff received at time T, denoted X_T, will have a time t value equal to

$$X_t = \tilde{E}_t(X_T e^{-\int_t^T r_s ds}). \tag{14}$$

As with expression 11, the arbitrage-free pricing methodology is termed risk-neutral valuation because the current value equals the discounted expected payoffs, where the discount rate equals the default-free spot rate of interest. The adjustment for risk is incorporated into the expectation operator and not the discount rate.

The random payoff in expression 14 could represent the cash flows from an interest rate swap, cap, floor, or any other European-style interest rate derivative. A European-style interest rate derivative's cash flows are determined by the contract's specification, not by any decisions of the holder or seller.[7] American-style interest rate derivative contract provisions require active decisions by either the holder or the seller. Using standard techniques (see Musiela & Rutkowski 2004), the arbitrage-free valuation formula in expression 14 can be extended to American-style interest rate derivatives as well.

Hedging based on expression 14 follows in the standard manner. For an n – factor model, any n zero-coupon bonds satisfying the matrix condition (Equation 13) can be used to hedge the value process X_t in expression 14. In special cases, the hedge ratios can be determined using Ito's lemma (or its generalizations, see Protter 2005). Otherwise, numerical methods can easily be employed (see Glasserman 2004). Because both pricing formulae for particular financial instruments and hedging computations are not unique to the term structure literature, we do not emphasize them in this review. Instead, we refer the interested reader to various textbooks on this topic (see Baxter & Rennie 1996, Brigo & Mercurio 2001, Jarrow 2002, Musiela & Rutkowski 2004, and Zagst 2002).

4.3. Examples

To characterize an HJM model, we just need to specify the forward rate curve evolution under the martingale measure Q. This is because in a complete market the interest rate risk premia can be arbitrarily specified. Hence, the evolution of the forward rate curve

[7]By convention, trivial decisions are usually considered European style; for example, a call option with an exercise decision on a single fixed future date is known as a European call option.

under the statistical probability measure P is not unique. However, given a particular specification of the interest rate risk premia, the HJM drift restriction (Equation 8) also uniquely identifies the evolution of the forward rate curve under the statistical probability measure. Identical under both probabilities, however, is the volatility structure of the forward rate curve's evolution. For future reference, we record the evolution of the forward rate curve, spot price, and bond price under the martingale measure:

$$f(t,T) = f(0,T) + \sum_{i=1}^{n} \left(\int_0^t \sigma_i(s,T) \int_s^T \sigma_i(s,u)du \right) ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,T) d\tilde{W}_i(s), \quad (15)$$

$$r_t = f(0,t) + \sum_{i=1}^{n} \left(\int_0^t \sigma_i(s,t) \int_s^t \sigma_i(s,u)du \right) ds + \sum_{i=1}^{n} \int_0^t \sigma_i(s,t) d\tilde{W}_i(s), \quad (16)$$

$$p(t,T) = p(0,T) e^{\int_0^t r_s ds - \frac{1}{2}\sum_{i=1}^{n} \int_0^t a_i(s,T)^2 ds + \sum_{i=1}^{n} \int_0^t a_i(s,T) d\tilde{W}_i(s)}, \quad (17)$$

where $\tilde{W}_i(t)$ for $i = 1, \ldots n$ are independent standard Brownian motions under Q. Given these evolutions, one can easily price and hedge any interest rate derivative using the standard techniques.

It is instructive to illustrate this methodology through various examples. Although these examples are simple, they all have found significant applications in industry and the academic literature. The LIBOR model, which could be logically included herein, is postponed to a subsequent section.

4.3.1. Ho & Lee. The Ho & Lee (1986) model, in the HJM framework, is represented by the forward rate curve evolution[8]

$$df(t,T) = \sigma^2(T-t)dt + \sigma d\tilde{W}(s), \quad (18)$$

where σ is a constant. This is a single-factor model. Integration yields

$$f(t,T) = f(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma\tilde{W}(t).$$

This expression shows that the forward rate curve drifts across time in a nonparallel fashion, but with parallel random shocks. The spot rate process implied by this evolution is

$$r_t = f(0,t) + \sigma^2 \frac{t^2}{2} + \sigma\tilde{W}(t).$$

This is termed an affine model for the spot rate because the spot rate is linear in the state variable process $\tilde{W}(t)$. The zero-coupon bond price evolution is

$$p(t,T) = \frac{p(0,T)}{p(0,t)} e^{-\frac{\sigma^2}{2}Tt(T-t) - \sigma(T-t)\tilde{W}(t)}.$$

The natural logarithm of the zero-coupon bond's price is also linear in the state variable process $\tilde{W}(t)$.

[8]Note that we use the HJM drift restriction (Equation 8) with $\phi_1 \equiv 0$ for all i to obtain the forward rate's drifts under the martingale measure.

4.3.2. **Vasicek.** The Vasicek (1977) model, in the HJM framework, is represented by the spot rate process[9]

$$dr_t = k(\theta - r_t)dt + \sigma d\tilde{W}(t),$$

where r_0, k, θ, and σ are positive constants. Integrating, we get

$$r_t = r_s e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right) + \sigma \int_s^t e^{-k(t-s)} d\tilde{W}(t).$$

Using expression 6, we have

$$\begin{aligned} p(t,T) &= A(T-t)e^{-C(T-t)r_t}, \text{ where} \\ A(T-t) &= e^{\left(\theta - \frac{\sigma^2}{2k^2}\right)(C(T-t)-(T-t)) - \frac{\sigma^2}{4k}C(T-t)^2}, \text{ and} \\ C(T-t) &= \frac{1}{k}\left(1 - e^{-k(T-t)}\right). \end{aligned}$$

The forward rate curve evolution implied by this process is

$$f(t,T) = \left(1 - e^{-k(T-t)}\right)\left(\theta - \frac{\sigma^2}{2k^2}\left(1 - e^{-k(T-t)}\right)\right) + e^{-k(T-t)} r_t, \text{ with}$$

$$f(0,T) = \left(1 - e^{-kT}\right)\left(\theta - \frac{\sigma^2}{2k^2}\left(1 - e^{-kT}\right)\right) + e^{-kT} r_0.$$

Not all initial forward rate curves can be fit by this model. To match any initial forward rate curve, the Vasicek model needs to be extended by making θ a function of time.

4.3.3. **Cox-Ingersoll-Ross.** The CIR model (Cox et al. 1985), in the HJM framework, is represented by the spot rate process[10]

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t} d\tilde{W}(t),$$

where r_0, θ, k, and σ are positive constants with $2k\theta > \sigma^2$. Using expression 6, we have

$$\begin{aligned} p(t,T) &= A(T-t)e^{-C(T-t)r_t}, \text{ where} \\ A(T-t) &= \left[\frac{2he^{(k+h)\frac{(T-t)}{2}}}{2h + (k+h)(e^{(T-t)h} - 1)}\right]^{\frac{2k\theta}{\sigma 2}}, \\ C(T-t) &= \frac{2\left(e^{(T-t)h} - 1\right)}{2h + (k+h)(e^{(T-t)h} - 1)}, \text{ and} \\ h &= \sqrt{k^2 + 2\sigma^2}. \end{aligned}$$

[9]See Brigo & Mercurio (2001 p. 50) for the derivation.

[10]See Brigo & Mercurio (2001, p. 56) for the derivation.

The forward rate curve evolution implied by this process is

$$f(t,T) = -\frac{d\ln A(T-t)}{dT} - \frac{dC(T-t)}{dT} r_t, \text{ with}$$

$$f(0,T) = \frac{d\ln A(T)}{dT} - \frac{dC(T)}{dT} r_0.$$

Not all initial forward rate curves can be fit by this model. To nearly match any initial forward rate curve, the CIR model needs to be extended by making θ a function of time. However, even in this circumstance, not all forward rate curves can be attained (see Heath et al. 1992).

4.3.4. Spot Rate Models.[11] The Vasicek and CIR models are special cases of the one-factor spot rate models. These models start by assuming the evolution of the spot rate under the martingale measure Q as a Markov process,

$$dr_t = \beta(r_t, t)dt + \gamma(r_t, t)d\tilde{W}(t),$$

where $\beta(r_t, t)$ and $\gamma(r_t, t)$ are suitably measurable and integrable so that this expression is well defined. Using expression 6, we have

$$p(t,T) = \tilde{E}_t(e^{-\int_t^T r_s ds}) = g(r_t, t, T)$$

for some function $g(\cdot, \cdot, \cdot)$ due to the Markov hypothesis.

Then, the forward rate process is

$$f(t,T) = -\frac{\partial \ln g(r_t, t, T)}{\partial T}, \text{ with}$$

$$f(0,T) = -\frac{\partial \ln g(r_0, 0, T)}{\partial T}.$$

Not all initial forward rate curves can be fit by this model.

4.3.5. Affine Class.[12] This section studies the affine model of Duffie & Kan (1996) and Dai & Singleton (2000). The affine models are a multifactor extension of the spot rate models. This class assumes that the spot rate and the natural logarithm of the zero-coupon bond price are affine functions of a k–dimensional vector $\mathbf{X}_t$ of state variables; specifically,

$$r_t = \rho_0 + \boldsymbol{\rho}_1 \mathbf{X}_t \text{ and}$$

$$p(t,T) = e^{A(T-t)+\mathbf{C}(T-t)\mathbf{X}_t} \text{ where}$$

$$d\mathbf{X}_t = (\mathbf{K}_0 + \mathbf{K}_1 \mathbf{X}_t)dt + \Sigma(t, \mathbf{X}_t)d\tilde{W}(t), \text{ and}$$

$$\Sigma(t, \mathbf{X}_t)\Sigma(t, \mathbf{X}_t)^T = \mathbf{H}_0 + \mathbf{H}_1 \mathbf{X}_t,$$

where ρ_0 is a constant; $\boldsymbol{\rho}_1$ is a k – vector; $\tilde{W}(t) \equiv [\tilde{W}_1(t), \ldots, \tilde{W}_n(t)]$; $\sum(t, \mathbf{X}_t)$ is a $k \times d$ matrix; $\mathbf{K}_0$ and $\mathbf{H}_0$ are k – vectors; $\mathbf{H}_1$ and $\mathbf{K}_1$ are $k \times k$ matrices; $A(T-t)$ is a scalar function; and $\mathbf{C}(T-t)$ is a k -dimensional vector function.

[11] This subsection is based on Baxter & Rennie (1996, p. 149).

[12] This subsection follows Duffie (2001, p. 149).

Duffie & Kan (1996) show that such a system exists if and only if $A(s)$, $\mathbf{C}(s)$ satisfy the system of differential equations

$$\frac{d\mathbf{C}(s)}{ds} = \boldsymbol{\rho}_1 - \mathbf{K}_1^T\mathbf{C}(s) - \frac{1}{2}\mathbf{C}(s)^T\mathbf{H}_1\mathbf{C}(s),$$
$$\frac{dA(s)}{ds} = \rho_0 - \mathbf{K}_0\mathbf{C}(s) - \frac{1}{2}\mathbf{C}(s)^T\mathbf{H}_0\mathbf{C}(s),$$

with $\mathbf{C}(T) = \mathbf{0}$ and $A(T) = 0$. The forward rate evolution implied by this process is

$$f(t,T) = -\frac{d\ln A(T-t)}{dT} - \frac{d\mathbf{C}(T-t)}{dT}\mathbf{X}_t, \text{ with}$$

$$f(0,T) = \frac{d\ln A(T)}{dT} - \frac{d\mathbf{C}(T)}{dT}\mathbf{X}_0.$$

Extensions of the affine class include Filipović (2002), Chen et al. (2004), and Cheng & Scaillet (2007).

4.4. Refinements

With respect to understanding the mathematical structure of HJM models, three questions exist. First, what structures guarantee that interest rates remain positive? Second, given an initial forward rate curve and its evolution, what is the class of forward rate curves that can be generated by all possible evolutions? Third, under what conditions is an HJM model a finite-dimensional Markov process? The first question was answered by Flesaker & Hughston (1996), Rogers (1994), and Jin & Glasserman (2001). The second was solved by Bjork & Christensen (1999) and Filipović (2001). The third was studied by Cheyette (1992), Caverhill (1994), Jeffrey (1995), Duffie & Kan (1996), and Bjork & Svensson (2001), among others.

The original HJM model had the term structure of interest rates generated by a finite number of Brownian motions. Extensions include (*a*) jump processes (see Jarrow & Madan 1995, Bjork et al. 1997, and Eberlein & Raible 1999), (*b*) stochastic volatilities (see Chiarella & Kwon 2000 and Andersen & Brotherton-Ratcliffe 2001), and (*c*) random fields (see Kennedy 1994, Goldstein 2000, and Santa-Clara & Sornette 2001).

5. FORWARD AND FUTURES CONTRACTS

Stochastic term structure of interest rate models are essential for understanding forward and futures contracts and prices because, as first shown in the classical literature by Jarrow & Oldfield (1981) and Cox et al. (1981b), there is no economic difference between these two contract types unless interest rates are stochastic.

5.1. Forward Contracts

Consider an asset with time t price $S(t)$ that is adapted to the filtration $(\mathcal{F}_t)_{t\in[0,\tau]}$. This implies that the asset's randomness is that generated by the evolution of the term structure of interest rates. For example, the asset could be a zero-coupon bond. Although this restriction can be easily relaxed, it is sufficient for our purposes.

A forward contract obligates the owner (long position) to buy the asset on the delivery date T for a predetermined price. This price is set, by market convention, such that the value of the forward contract at initiation is zero. This market clearing price is termed the forward price and denoted $K(t, T)$.

Assuming that the asset has no cash flows over the life of the forward contract, an arbitrage-free forward price must equal

$$K(t,T) = \frac{S(t)}{p(T,t)}. \tag{19}$$

Proof: The payoff to a forward contract at time T is $[S(T) - K(t, T)]$. By market convention, the forward price makes the present value of this payoff equal to zero; that is,

$$0 = \tilde{E}_t\left([S(T) - K(t,T)]\, e^{-\int_t^T r_s ds} \right),$$

$$0 = \tilde{E}_t\left(S(T) e^{-\int_t^T r_s ds} \right) - K(t,T)\tilde{E}_t\left(e^{-\int_t^T r_s ds} \right),$$

$$0 = S(t) - K(t,T)p(t,T).$$

Algebra gives the result.

This expression shows that the forward price is the future value of the asset's time t price. Indeed, if at time t one invests $S(t)$ dollars in a zero-coupon bond maturing at time T, the time T value of this investment is expression 19.

To facilitate understanding, it is convenient to introduce another equivalent probability that facilitates computation. This equivalent probability measure makes asset payoffs at some future date T martingales when discounted by the T – maturity zero-coupon bond price. The forward price martingale measure was first discovered by Jarrow (1987) and later, independently, by Geman (1989).

Fix $T \in [0, \tau]$; the forward price measure Q^T is defined by[13]

$$\frac{dQ^T}{dQ} = \frac{1}{p(0,T)B(T)}. \tag{20}$$

Using this measure, for any suitably measurable and integrable random payoff X_T received at time T, $\frac{X_t}{p(t,T)}$ is a Q^T martingale; specifically,

$$X_t = p(t,T)E_t^T(X_T), \tag{21}$$

where $E_t^T(\cdot)$ is a conditional expectation under Q^T. The proof of this expression is contained in the appendix.

Expression 21 gives a useful alternative procedure for computing present values: Take the expectation under the probability Q^T and discount the expectation using the appropriate zero-coupon bond's price. The spot rate of interest process does not explicitly appear in this expression. It is implicit, however, in the new probability Q^T. It is important to

[13]Note that $\frac{dQ_T}{dQ} = \frac{1}{p(0,T)B(T)} \geq 0$ with $\tilde{E}_0\left(\frac{dQ_T}{dQ}\right) = \tilde{E}_0\left(\frac{1}{p(0,T)B(T)}\right) = \frac{1}{p(0,T)}\tilde{E}_0\left(\frac{1}{B(T)}\right) = 1$. So, it is indeed a probability measure.

emphasize, however, that unlike the martingale probability Q, the forward price measure depends on a particular future date T corresponding to the maturity of a particular zero-coupon bond.

Applying this result to the time T price of the asset yields an equivalent expression for the forward price,[14]

$$K(t,T) = E_t^T(S(T)), \tag{22}$$

which explains the name for this probability measure. From expression 22, we see that the forward price is not an unbiased expectation of the future price of the asset under the statistical probability P unless $Q^T = P$.

The difference between these two probability measures is characterized by the interest rate risk premia $\{\phi_1(t),\ldots, \phi_n(t)\}$ and the T – maturity zero-coupon bond's volatilities $\{a_1(t, T),\ldots, a_n(t, T)\}$, as given by[15]

$$\frac{dQ^T}{dP} = e^{-\frac{1}{2}\sum_{i=1}^n \left[\int_0^T a_i(s,T)^2 ds + \int_0^\tau \phi_i(s)^2 ds\right]} \bullet e^{+\sum_{i=1}^n \int_0^T (a_i(s,T)+\phi_i(s))dW_i(t) - \sum_{i=1}^n \int_0^T a_i(s,T)\phi_i(t)dt}. \tag{23}$$

Given nonnegative interest rate risk premia, $Q^T = P$ if and only if both the interest rate risk premia and the bond's volatilities are identically zero (deterministic interest rates). Otherwise, they are distinct.

For subsequent usage, it is easy to show (see the appendix) that the forward rate equals the expected time T spot rate under the forward price measure; that is,

$$f(t,T) = E_t^T(r_T). \tag{24}$$

This implies, of course, that the forward price is not an unbiased estimate of the future spot price (under the statistical probability P).

5.2. Futures Contracts

A futures contract is similar to a forward contract. It is a financial contract, written on the asset $S(t)$, with a fixed maturity T. It represents the purchase of the asset at time T via a prearranged payment procedure. The prearranged payment procedure is known as marking to market. Marking to market obligates the purchaser (long position) to accept a continuous cash flow stream equal to the continuous changes in the futures prices for this contract.

The time t futures prices, denoted $k(t, T)$, are set (by market convention) such that newly issued futures contracts (at time t) on the same asset with the same maturity date T have zero market value. Hence, futures contracts (by construction) have zero market value at all times and a continuous cash flow stream equal to $dk(t, T)$. At maturity, the last futures price must equal the asset's price: $k(T, T) = s(T)$.

[14] $S(t) = p(t,T)E_t^T[S(T)]$. Equating this to expression 19 gives the result.

[15] A proof of this expression is contained in the appendix (Section 11).

It can be shown (see the appendix) that the arbitrage-free futures price must be

$$k(t, T) = \tilde{E}_t[S(T)]. \tag{25}$$

This expression shows that the futures price is not the expected asset's time T price, unless $Q = P$. Recall that the two probability measures are equal if and only if the interest rate risk premia are identically zero.

The relation between forward and futures prices follows directly from these expressions:

$$K(t, T) = k(t, T) + c\tilde{o}v_t\left(S(T), \frac{1}{B(T)}\right)\frac{B(t)}{p(t, T)}, \tag{26}$$

where $c\tilde{o}v_t(\cdot, \cdot)$ is the conditional covariance under the martingale probability measure Q.

Proof: Using expression 22, we get

$$\begin{aligned} K(t, T)p(t, T) &= \tilde{E}_t(S_T)\tilde{E}_t\left(\frac{1}{B(T)}\right)B(t) + c\tilde{o}v\left(S(T), \frac{1}{B(T)}\right)B(t) \\ &= \tilde{E}_t(S_T)\tilde{E}_t\left(\frac{1}{B(T)}\right)B(t) + c\tilde{o}v\left(S(T), \frac{1}{B(T)}\right)B(t). \end{aligned}$$

This completes the proof.

Expression 26 shows that forward and futures prices are equal if and only if, under the martingale measure, the covariance between the asset's spot price and the money market's value is zero. A sufficient condition for equivalence of forward and futures prices is that interest rates are deterministic (the classical result).

6. THE EXPECTATIONS HYPOTHESIS

The classical (macro) economics literature of the term structure of interest rates characterizes the possible interest rate risk premia (equilibrium term structures) according to three hypotheses about investor behavior: the market segmentation hypothesis, the liquidity preference hypothesis, and the expectations hypothesis.

The market segmentation hypothesis, normally associated with Culbertson (1957), is that the equilibrium zero-coupon bond prices of different maturities are determined in isolation of the other maturity bonds by distinct market clienteles demanding payoffs at different horizons. Hence, the relevant risk premia for each maturity bond are determined in isolation. This implies, of course, that there may be arbitrage if trading were to take place across market segments.

In contrast, Hicks's (1946) liquidity preference hypothesis is that the equilibrium zero-coupon bond prices of different maturities are jointly determined by liquidity needs across time. Also, the interest rate risk premia reflect these considerations [see also Modigliani & Sutch's (1966) preferred habitat theory]. Modern equilibrium models of the term structure of interests rates can be viewed as the representation of the liquidity preference (or preferred habitat) hypothesis in a continuous time framework (see Cox et al. 1985, Dunn & Singleton 1986, and Sundaresan 1984).

The expectations hypothesis (Fisher 1930 and Lutz 1940/1941) has as its basic premise that zero-coupon bonds of different maturities are perfect substitutes. In the literature, this hypothesis has been extended to a collection of three different possible hypotheses: the local expectations (LE) hypothesis, the return-to-maturity expectations (RME) hypothesis, and the unbiased expectations (UE) hypothesis. Each of these hypotheses can be characterized by a formula for the zero-coupon bond's price. This critique of the expectations hypothesis in the context of arbitrage-free pricing models was independently formulated by Jarrow (1981) and Cox et al. (1981a).

The LE hypothesis is that

$$p(t,T) = E_t(e^{-\int_t^T r_s ds}), \tag{27}$$

where $E_t(\cdot)$ denotes expectation under the statistical probability measure P. This hypothesis is that the price of a T – maturity zero-coupon bond is its discounted payoff using the spot rate of interest with no adjustment for risk. Equivalently, this can be written as

$$E_t\left(\frac{dp(t,T)}{p(t,T)}\right) = r_t dt.$$

Using expression 6, this implies that $Q = P$, or that the interest rate risk premia $\{\phi_1(t),\ldots, \phi_n(t)\}$ are identically zero.

The RME hypothesis is that

$$\frac{1}{p(t,T)} = E_t\left(\frac{B(T)}{B(t)}\right) = E_t\left(e^{\int_t^T r_s ds}\right). \tag{28}$$

This hypothesis is that the $(T-t)$ – period holding period return from a zero-coupon bond is equivalent to the expected return from holding the money market account. Note that the LE and RME hypotheses are mutually exclusive (unless interest rates are deterministic) because[16]

$$p(t,T)^{LE} = E_t\left(e^{-\int_t^T r_s ds}\right) > \frac{1}{E_t\left(e^{\int_t^T r_s ds}\right)} = p(t,T)^{RME}.$$

The RE hypothesis could be satisfied if the interest rate risk premia are nonnegative and they (just happen to) make the zero-coupon bond prices equal to expression 28.

The UE hypothesis is that

$$f(t,T) = E_t(r_T). \tag{29}$$

That is, forward rates provide an unbiased estimate of the future spot rate. Using expression 24, given nonnegative interest rate risk premia, this implies that $Q^T = P$ if and only if the interest rate risk premia $\{\phi_1(t),\ldots, \phi_n(t)\}$ and the T – maturity's volatilities $\{a_1(t,T),\ldots, a_n(t,T)\}$ are identically zero.

In terms of the zero-coupon bond price this is

$$p(t,T) = e^{-\int_t^T E_t(r_s)ds}. \tag{30}$$

Again the LE and UE hypotheses are mutually exclusive because[17]

[16]By Jensen's inequality, because $\frac{1}{X}$ is strictly convex, we have $E(\frac{1}{X}) > \frac{1}{E(X)}$.

[17]By Jensen's inequality, because e^X is strictly convex, we have $E(e^X) > e^{E(X)}$

$$p(t,T)^{LE} = E_t\left(e^{-\int_t^T r_s ds}\right) > e^{-\int_t^T E_t(r_s)ds} = p(t,T)^{UE}.$$

Of course, the arbitrage-free pricing of interest rate derivatives is consistent with and independent of these three expectation hypotheses.

7. PRICING CAPS AND FLOORS[18]

For this section, we consider interest rate derivatives written on Eurodollar deposit rates: caps and floors. These financial instruments are usually based on LIBOR, an index of Eurodollar borrowing rates. Caps are a portfolio of European caplets, with the same strike but with different and increasing maturity dates. The maturity dates of the included caplets are evenly spaced at fixed intervals (e.g., every 3 months) up to some final date, the maturity of the cap (e.g., 5 years). A European caplet (in the context of a continuous time setting) is a European call option with maturity T and strike k on the forward interest rate at time T; in other words, the time T payoff is $\max[f(T, T) - k, 0]$. Floors are a portfolio of floorlets with the same strike and with increasing maturity dates. The sequence of maturity dates is similar to that of a cap. A floorlet is a European put option on the forward interest rate. Using put-call parity, if one prices European caplets then the formula for European floorlets immediately follows. For this reason, this section concentrates only on pricing caplets.

Historically, given the industry's familiarity with the BSM formula for pricing equity options, the first industry model used to price caplets was Black's equation (Black 1976). Black's formula requires forward rates to evolve as geometric Brownian motion (see expression 9), but unfortunately, Heath et al. (1992) showed that this is impossible in an arbitrage-free setting. Hence, a direct application of Black's formula to price caplets (and caps) on instantaneous and continuously compounded forward rates was shown to be inappropriate.

Nonetheless, motivated by the industry's strong desire to use Black's formula, Sandmann et al. (1995), Miltersen et al. (1997), and Brace et al. (1997) found a more subtle argument to justify its usage. The solution was to use a simple interest rate, compounded discretely, instead of a continuously compounded instantaneous rate. This section describes this interest rate model, known as the LIBOR model, for pricing caps and floors. For books on the LIBOR model see Musiela & Rutkowski (2004), Rebonato (2002), and Schoenmakers (2005).

In our context, the LIBOR rate corresponds to the default-free interest rate that one can earn from investing dollars for a fixed time period, say δ units of a year (e.g., one-quarter of a year), quoted as a simple interest rate and not as the continuously compounded and instantaneous spot rate r_t.[19] Just as with the continuously compounded and instantaneous forward rates, there is an analogous forward LIBOR rate. Consider the future time interval $[T, T + \delta]$, where δ corresponds to the so-called earning interval. The forward LIBOR rate at time t for the time interval $[T, T + \delta]$ is implicitly defined by

[18]This section is based on Shreve (2004, section 4.2).

[19]Investing X dollars at time 0, a simple interest rate L at time δ is defined by the payoff of $X(1 + L\delta)$ dollars at time δ. Note that a prorated amount of the interest L is earned over the relevant earning interval. The analogous expression for the continuously and instantaneous spot rate over dt units of time is $Xe^{r_t dt}$.

$$1 + \delta L(t,T) = \frac{p(t,T)}{p(t,T+\delta)}. \tag{31}$$

The right side of this expression isolates the implicit interest embedded in the zero-coupon bonds over $[T, T+\delta]$. The spot LIBOR rate is $L(t, t)$.

Given the evolution for a 1 – factor forward rate process as in expression 15 and the definition (Equation 31) of the forward LIBOR rate involving zero-coupon bond prices, Ito's lemma enables one to compute the evolution of the LIBOR forward rate as (see Shreve 2004, p. 443)

$$dL(t,T) = \left(\frac{1+\delta L(t,T)}{\delta}\right)[a(t,T+\delta) - a(t,T)]dW^{T+\delta}(t), \tag{32}$$

where $W^{T+\delta}(t)$ is a standard independent Brownian motion under $Q^T + \delta$ (see expression 38 in the appendix). This evolution is consistent with no-arbitrage (by construction).

The LIBOR model assumes that[20]

$$\gamma(t,T) \equiv \left(\frac{1+\delta L(t,T)}{\delta L(t,T)}\right)[a(t,T+\delta) - a(t,T)] \tag{33}$$

is a deterministic function. Hence, forward LIBOR evolves as

$$dL(t,T) = \gamma(t,T)L(t,T)dW^{T+\delta}(t). \tag{34}$$

This is a 1 – factor model for the evolution of forward LIBOR, where forward LIBOR follows a (generalized) geometric Brownian motion process.

Now, let us reconsider a caplet on the forward LIBOR rate with maturity $T+\delta$ and strike k. In practice, the standard caplet pays off at time $T+\delta$, based on the LIBOR forward rate from time T; that is, the payoff is defined by $X_{T+\delta} \equiv \max\{[L(T,T) - k]\delta, 0\}$ at time $T+\delta$. Note that the caplet's payoff is proportionately adjusted for the so-called earning time interval δ.

Given that $L(T, T)$ follows a lognormal distribution under expression 34, it is straightforward to show that the caplet's value satisfies Black's formula:

$$X_0 = p(0,T+\delta)\delta[L(0,T)N(d_1) - kN(d_2)] \text{ where} \tag{35}$$

$$d_1 = \left(\frac{\log\left(\frac{L(0,T)}{k}\right) + \frac{1}{2}\int_0^T \gamma(t,T)^2 dt}{\sqrt{\int_0^T \gamma(t,T)^2 dt}}\right), \text{ and}$$

$$d_2 = d_1 - \sqrt{\int_0^T \gamma(t,T)^2 dt}.$$

Unfortunately, just as with the simple BSM model for equity options, Black's model for caplets does not capture the maturity nor strike structure well. Using implicit volatilities, smiles in volatilities result (see Jarrow et al. 2007 and Rebonato 2002). This implies that the model will not hedge properly without modification.

[20]That this assumption is nonempty is proven in Shreve (2004, section 10.4).

It is easy to generalize this 1 – factor to an n – factor model. The lognormal evolution has also been extended to jump diffusions (see Glasserman & Kou 2003), Levy processes (see Eberlein & Ozkan 2005), and stochastic volatilities (see Andersen & Brotherton-Ratcliffe 2001).

8. EMPIRICAL EVIDENCE

The empirical evidence with respect to interest rate term structure models can be decomposed into three related topics: (*a*) estimating the initial forward rate curve, (*b*) fitting the term structure evolutions, and (*c*) pricing various interest rate derivatives. The first two topics correspond to the inputs (see expression 2) and the third corresponds to the outputs (e.g., caps and floors) of the HJM model.

8.1. Forward Rate Curve Estimation

The first input to the HJM model is the initial forward rate curve $\{f(0, T)$ for $0 \leq t \leq T \leq \tau\}$. Unfortunately, this is not directly observable. Instead, one observes either (for government debt markets) the prices of a collection of government bonds, with (long maturity) and without coupons (short maturity) or (for Eurodollar markets) Eurodollar deposit rates up to one year and swaps rates thereafter. A swap rate is analogous to a coupon-bearing bond.

Conceptually, estimating the forward rate curve is a two-step procedure. Step one computes the zero-coupon bonds underlying the observed bond prices, resulting in a discrete number of zero-coupon bond price observations across maturities. Step two connects this set of discrete zero-coupon bond prices to obtain a continuous curve, from which forward rates can be computed. Of course, these two steps can be combined, and the resulting estimation is termed smoothing the forward rate curve. This is a well-studied topic in the interest rate term structure literature. Typical approaches use exponential or polynomial splines; references include Shea (1985), Nelson & Siegel (1987), Adams & van Deventer (1994), and Jarrow et al. (2004).

Forward rate curve smoothing raises an interesting issue. If one uses, for example, a polynomial spline to fit forward rates at times 0, 1, 2, and so forth, then to have a consistent HJM evolution, it should be the case that the forward rate curve evolution (Equation 2) includes in its realizations polynomial splines. Otherwise, the forward rate curve evolution is inconsistent with the smoothing procedure. This topic was studied and solved by Bjork & Christensen (1999) and Filipović (2001), who provided necessary and sufficient conditions for a consistent HJM model. As might be expected, it can be shown that not all smoothing procedures are consistent with a given forward rate curve evolution; for example, the Nelson-Siegel family of forward rate curves is inconsistent with forward rate evolutions having deterministic volatilities (see Nelson & Siegel 1987 and Filipović 1999).

8.2. Fitting Term Structure Evolutions

There is a vast literature fitting term structure evolutions. The issues relate to whether spot rates (*a*) follow a particular diffusion process (e.g., within the affine class), (*b*) follow a jump process, (*c*) or exhibit regime shifts. Related issues are the number of factors necessary in a forward rate evolution and whether markets are complete in the traded bonds. This last question also relates to the pricing and hedging of interest rate derivatives. Papers that span these issues are Ait-Sahalia (1999), Hong & Li (2005), Dai & Singleton (2000), Collin-Dufresne & Goldstein (2002), and Li & Zhao (2006). A review paper of the empirical literature is Dai & Singleton (2003).

8.3 Pricing Interest Rate Derivatives

There is also a large literature studying the pricing and hedging of interest rate derivatives. The evidence generally supports the validity of the estimated models; see Amin & Morton (1994), Flesaker (1993), Collin-Dufresne & Goldstein (2002), Li & Zhao (2006), Longstaff et al. (2001), Jarrow et al. (2007), and Gupta & Subrahmanyam (2005). A summary of the empirical evidence is contained in Rebonato (2002).

9. APPLICATIONS TO OTHER MARKETS

The HJM model is the building block for pricing and hedging all fixed-income securities, including those with credit risk. One important application has been to the risk management of mortgages, both residential and commercial. For U.S. residential mortgages guaranteed by U.S. government agencies, the default-free model applies directly. The only remaining consideration is prepayment risk (see Stanton 1995). For nongovernment guaranteed residential mortgages and commercial mortgages, default and prepayment are both important. Default requires the modeling of credit risk. Default-free interest rate term structure models are the basis for current credit risk modeling. For textbooks on the topic see Bielecki & Rutkowski (2002) and Lando (2004).

The HJM model has also been extended to include foreign currencies (see Amin & Jarrow 1991), the pricing of equities and (noninterest rate) commodities (see Amin & Jarrow 1992), to Treasury inflation protected bonds (see Jarrow & Yildirim 2003), to term structures of futures prices (see Carr & Jarrow 1995, Miltersen et al. 2006), to term structures of convenience yields (Nakajima & Maeda 2007), and to term structures of equity forward volatilities (Dupire 1992, 1996; Schweizer & Wissel 2008). In fact, it can be shown that almost all option pricing applications can be viewed as special cases of a multiple term structure HJM model (see Jarrow & Turnbull 1998). A summary of many of these applications can be found in Carmona (2007).

10. FUTURE RESEARCH DIRECTIONS

As evidenced in the text, the arbitrage-free pricing and hedging theory for the term structure of interest rates is in a mature state: Very general and abstract formulations of term structure of interest rate models are available. Yet, open questions still remain with respect to the empirical implementation of these abstract models. Which forward rate curve evolutions (HJM volatility specifications) fit markets best? The literature, for analytic convenience, has favored the affine class but with disappointing results. More general evolutions, but with more complex computational demands, need to be studied. How many factors are needed in the term structure evolution? One or two factors are commonly used, but the evidence suggests three or four are needed to accurately price exotic interest rate derivatives. And are interest rate markets complete? Equivalently, are the forward rate curve's volatilities stochastic? The answer to this question is intricately related to the form and the evolution estimated (the volatility structure), including the number of fit factors. The current evidence is based on too simplistic of models for the evolution of the term structure. As argued, although partial answers to all of these questions are available, they are not definitive. More research is needed to settle these issues.

Settling these issues is essential for understanding risk management for both corporate and financial institutions. The capital structure decision for a corporation, that is, the determination of the debt equity ratio, or the determination of economic capital for a financial institution (related to the Basel II Accord), depends crucially on the evolution of the term structure of interest rates. In this determination there are four risks to be considered: market, credit, liquidity, and operational risk. Interest rate risk is the major component of market risk. However, these term structure models still need to be extended to include credit, liquidity, and operational risk. Although the abstract model formulation for the credit risk extension is also well understood, its empirical implementation is not. This is especially true with respect to the inclusion of default correlations, that is, systemic risk. In contrast, the liquidity and operational risk extensions are still not yet adequately modeled. Although there has been some research in this direction, the models are complex and not easily estimated. Without the inclusion of these additional risks into a term structure of interest rate model, the proper determination of a firm's equity capital is impossible. There is still much to understand and to learn in this regard.

11. APPENDIX

11.1. Derivation of Expression 6

First, $p(t,T) = e^{-\int_t^T f(t,u)du}$. Substitution of expression 2 gives

$$-\ln p(t,T) = \int_t^T f(0,u)du + \int_t^T \int_0^t \alpha(s,u)dsdu + \sum_{i=1}^n \int_t^T \int_0^t \sigma_i(s,u)dW_i(s)du.$$

Fubini's theorem (see Heath et al. 1992, appendix) gives

$$\int_t^T \int_0^t \alpha(s,u)dsdu = \int_0^t \int_t^T \alpha(s,u)duds, \text{ and}$$
$$\int_t^T \int_0^t \sigma_i(s,u)dW_i(s)du = \int_0^t \left(\int_t^T \sigma_i(s,u)du\right)dW_i(s).$$

Substitution yields

$$-\ln p(t,T) = \int_t^T f(0,u)du + \int_0^t \left(\int_t^T \alpha(s,u)du\right)ds + \sum_{i=1}^n \int_0^t \int_t^T \sigma_i(s,u)dudW_i(s).$$

Adding and subtracting the same terms yields

$$\begin{aligned}-\ln p(t,T) = &\int_0^T f(0,u)du + \int_0^t \left(\int_s^T \alpha(s,u)du\right)ds + \sum_{i=1}^n \int_0^t \int_s^T \sigma_i(s,u)dudW_i(s)\\ &- \int_0^t f(0,u)du - \int_0^t \left(\int_s^t \alpha(s,u)du\right)ds - \sum_{i=1}^n \int_0^t \int_s^t \sigma_i(s,u)dudW_i(s).\end{aligned}$$

However,

$$\int_0^t \left(\int_0^u \alpha(s,u)ds\right)du = \int_0^t \left(\int_s^t \alpha(s,u)du\right)ds,$$
$$\int_0^t \int_0^u \sigma_i(s,u)dW_i(s)du = \int_0^t \int_s^t \sigma_i(s,u)dudW_i(s).$$

Noting that $-\ln p(0,T) = \int_0^T f(0,u)du$ and using expression 4 yield

$$-\ln p(t,T) = -\ln p(0,T) + \int_0^t \left(\int_s^T \alpha(s,u)du\right)ds + \sum_{i=1}^n \int_0^t \int_s^T \sigma_i(s,u)dudW_i(s) - \int_0^t r_u du.$$

Simple algebra completes the proof.

11.2. Derivation of the Heath-Jarrow-Morton Arbitrage-Free Drift Restriction

Recall Girsanov's theorem (see Shreve 2004, p. 224):

> **Theorem 1** *Let* $\frac{dQ}{dP} = e^{\sum_{i=1}^n \int_0^\tau \phi_i(s)dW_i(s) - \frac{1}{2}\sum_{i=1}^n \int_0^\tau \phi_i(s)^2 ds}$ for $\{\phi_1(t),\ldots,\phi_n(t)\}$ be such that Q is a probability measure. Then, $d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt$ for $i = 1, \ldots, n$ are standard independent Brownian motions under Q.

Given expression 6, we have

$$\begin{aligned}\frac{dp(t,T)}{p(t,T)} &= (r_t + b(t,T))dt + \sum_{i=1}^n a_i(t,T)dW_i(t)\\ &= (r_t + b(t,T))dt + \sum_{i=1}^n a_i(t,T)\left(\phi_i(t)dt + d\tilde{W}_i(t)\right)\\ &= r_t dt + \sum_{i=1}^n a_i(t,T)d\tilde{W}_i(t) + \left(b(t,T) + \sum_{i=1}^n \phi_i(t)a_i(t,T)\right)dt.\end{aligned}$$

For $\frac{p(t,T)}{B(t)}$ to be a martingale under Q, we need

$$b(t,T) + \sum_{i=1}^n \phi_i(t)a_i(t,T) = 0. \text{ Or,}$$

$$-\int_s^T \alpha(s,u)du + \frac{1}{2}\sum_{i=1}^n \left(\int_s^T \sigma_i(s,u)du\right)^2 - \sum_{i=1}^n \phi_i(t)\int_s^T \sigma_i(s,u)du = 0.$$

Differentiate in T, and simplify to obtain

$$\alpha(s,T) = -\sum_{i=1}^n \sigma_i(s,T)\left[\phi_i(t) - \left(\int_s^T \sigma_i(s,u)du\right)\right] = 0.$$

This proves the result because (dropping the arguments for simplicity)

$$d\left(\frac{p}{B}\right) = \frac{dp}{B} - \frac{p}{B^2}dB = \frac{p}{B}\left[\frac{dp}{p} - \frac{dB}{B}\right] = \frac{dp}{p} - rdt = \sum_{i=1}^n a_i(t,T)d\tilde{W}_i(t)$$

is a Q – martingale.
For future reference, note that

$$\frac{p(t,T)}{B(t)} = \frac{p(0,T)}{B(0)} e^{-\frac{1}{2}\sum_{i=1}^{n}\int_0^t a_i(s,T)^2 ds + \sum_{i=1}^{n}\int_0^t a_i(s,T) d\tilde{W}_i(s)}. \tag{36}$$

11.3. Derivation of Expression 21

First, note that $E_t^T\left(\frac{dQ^T}{dQ}\right) = E_t^T\left(\frac{1}{p(0,T)B(T)}\right) = \frac{1}{p(0,T)} E_t^T\left(\frac{1}{B(T)}\right) = \frac{p(t,T)}{p(0,T)B(t)} = e^{-\frac{1}{2}\sum_{i=1}^{n}\int_0^t a_i(s,T)^2 ds + \sum_{i=1}^{n}\int_0^t a_i(s,T) d\tilde{W}_i(s)}$. The last equality follows from expression 36. For future reference, note that this implies

$$\frac{dQ^T}{dQ} = e^{-\frac{1}{2}\sum_{i=1}^{n}\int_0^T a_i(s,T)^2 ds + \sum_{i=1}^{n}\int_0^T a_i(s,T) d\tilde{W}_i(s)}. \tag{37}$$

Then, given $X_t = \tilde{E}_t\left(X_T e^{-\int_t^T r_s ds}\right) = \tilde{E}_t\left(X_T \frac{B(t)}{B(T)}\right)$ and using Shreve (2004, lemma 5.2.2, p. 212),

$$\begin{aligned} X_t &= \frac{1}{E_t^T\left(\frac{dQ^T}{dQ}\right)} E_t^T\left(X_T \frac{B(t)}{B(T)} \frac{dQ^T}{dQ}\right) = \frac{p(t,T)}{p(0,T)B(t)} E_t^T\left(X_T \frac{B(t)}{B(T)} p(0,T)B(T)\right) \\ &= p(t,T) E_t^T(X_T) \end{aligned}$$

It follows from Equation 37 that

$$W_i^T(t) = \int_0^t a_i(s,T) ds + \tilde{W}_i(t) \text{ for } i = 1, \ldots, n \tag{38}$$

are standard independent Brownian motions under Q^T (see Shreve 2004, p. 393).

11.4. Derivation of Expression 23

$$\frac{dQ^T}{dP} = \frac{dQ^T}{dQ}\frac{dQ}{dP} = e^{-\frac{1}{2}\sum_{i=1}^{n}\int_0^T a_i(s,T)^2 ds + \sum_{i=1}^{n}\int_0^T a_i(s,T) d\tilde{W}_i(s)} e^{\sum_{i=1}^{n}\int_0^\tau \phi_i(s) dW_i(s) - \frac{1}{2}\sum_{i=1}^{n}\int_0^\tau \phi_i(s)^2 ds}.$$

This last equality follows from expressions 36 and 12.
Using $d\tilde{W}_i(t) = dW_i(t) - \phi_i(t)dt$ and algebra yields

$$\frac{dQ^T}{dP} = e^{-\frac{1}{2}\sum_{i=1}^{n}\left[\int_0^T a_i(s,T)^2 ds + \int_0^\tau \phi_i(s)^2 ds\right] + \sum_{i=1}^{n}\int_0^T (a_i(s,T)+\phi_i(s)) dW_i(t) - \sum_{i=1}^{n}\int_0^T a_i(s,T)\phi_i(t) dt}.$$

11.5. Derivation of Expression 24

Given expression 11, $p(t,T) = \tilde{E}_t\left(e^{-\int_t^T r_s ds}\right)$. Differentiating yields $-\frac{\partial p(t,T)}{\partial T} = \tilde{E}_t\left(r_T e^{-\int_t^T r_s ds}\right) = p(t,T)E_t^T(r_T)$, where the second equality follows from expression 21. Dividing by $p(t,\, T)$ completes the proof.

11.6. Derivation of Expression 25

The accumulated wealth from buying and holding a futures contract over $[t,\, T]$ and investing all proceeds into the money market account is

$$B(T)\int_t^T \frac{1}{B(s)}dk(s,T).$$

Hence, the futures price $k(t, T)$ solves

$$\tilde{E}_t\left(\frac{B(T)\int_t^T \frac{1}{B(s)}dk(s,T)}{B(T)}\right)B(t) = 0 \text{ for all } t \text{ and } k(T,T) = S(T).$$

This implies $\tilde{E}_t\left(\int_t^T \frac{1}{B(s)}dk(s,T)\right) = 0$; that is, $M_t = \int_0^t \frac{1}{B(s)}dk(s,T)$ is a $Q-$ martingale. Hence, $\int_0^t B(s)dM_s$ is a martingale [assuming $B(S)$ is suitably bounded]. However, $\int_0^t B(s)dM_s = \int_0^t B(s)\frac{1}{B(s)}dk(s,T) = k(t,T) - k(0,T)$ is a $Q-$ martingale. This finally implies $k(t,T) = \tilde{E}_t[k(T,T)] = \tilde{E}_t[S(T)]$.

DISCLOSURE STATEMENT

The author is not aware of any affiliations, memberships, funding, or financial holdings that might be perceived as affecting the objectivity of this review.

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