The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates

Author(s): Stephen J. Brown and Philip H. Dybvig

Source: *The Journal of Finance*, Jul., 1986, Vol. 41, No. 3, Papers and Proceedings of the Forty-Fourth Annual Meeting of the America Finance Association, New York, New York, December 28-30, 1985 (Jul., 1986), pp. 617-630

Published by: Wiley for the American Finance Association

Stable URL: https://www.jstor.org/stable/2328491

THE JOURNAL OF FINANCE • VOL. XLI, NO. 3 • JULY 1986

The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates

STEPHEN J. BROWN and PHILIP H. DYBVIG*

ABSTRACT

The one-factor version of the Cox, Ingersoll, and Ross model of the term structure is estimated using monthly quotes on U.S. Treasury issues trading from 1952 through 1983. Using data from a single yield curve, it is possible to estimate implied short and long term zero coupon rates and the implied variance of changes in short rates. Analysis of residuals points to a probable neglected tax effect.

THE TERM STRUCTURE of interest rates is important to economists because the relationship among the yields on default free securities that differ in their term to maturity reflects the information available to the market about the future course of events. The Expectations Hypothesis, the Liquidity Preference Hypothesis (Hicks [10]) and the Market Segmentation Hypothesis (Culbertson [6]) are theories of the term structure that predict little more than that the implied forward rate is either equal to or not equal to the expectation of future spot rates. Cox, Ingersoll and Ross [5] (CIR) model the term structure of interest rates in a competitive equilibrium context. Their model has elements in common with the earlier hypotheses of the term structure. However, the CIR model has a rich class of empirical implications, not only for the pricing of default free securities, but also for the pricing of bond options, callable bonds and other types of financial claims.

This paper examines the extent to which the model (in its simplest one–factor form) is descriptive of the prices of U.S. Treasury Bills, Bonds and Notes traded from 1952 to 1983. Section I describes the model, and the parsimonious representation of bond prices which it implies. Section II describes the data and Section III outlines the preliminary results obtained by fitting the model to observed data. The final section outlines directions for future research.

I. The Model

The simplest form of the CIR model is based on a single factor model of interest rates. For completeness, we include a simple intuitive derivation of the model.[1] The dynamics of the interest rate process are given by

* Both authors are from Yale University. We wish to acknowledge the helpful comments of Jon Ingersoll, Terry Marsh, Steve Ross and workshop participants at Princeton University. All errors are our own.

[1] This exposition of the CIR model is similar to one found in an early precursor to CIR, Ingersoll [12].

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\,dz \tag{1}$$

where $\kappa(\theta - r)$ is the instantaneous rate if drift (a mean reversion if $0 < \kappa$) and dz is a standard Gauss–Wiener process. The variance of the interest rate process is proportional to the level of interest rates.[2] Given the instantaneous interest rate r at period t, let $P[r, t, T]$ represent the price of a riskless pure discount bond maturing at period T. From Ito's Lemma, the instantaneous rate of return on the bond is

$$dP/P = (P_r dr + 1/2 P_{rr}(dr)^2 + P_t dt)/P \tag{2}$$

$$= [\kappa(\theta - r)P_r/P + P_t/P + 1/2\sigma^2 r P_{rr}/P]dt + \sigma\sqrt{r}P_r/P\,dz, \tag{3}$$

where we substitute from (1) for dr.

In perfect markets, the instantaneous expected rate of return for any asset can be written as the instantaneous risk free return, r, plus a risk premium. In the present single factor model, the returns on all assets are locally perfectly correlated to the extent they are all correlated with the short interest rate, the only source of noise in this economy. If we write the instantaneous interest rate on the zero coupon bond as

$$dP/P = \mu(r, t, T)dt + v(r, t, T)dz, \tag{4}$$

the absence of arbitrage in this economy implies that

$$\mu(r, t, T) = r + \lambda^*\,(r, t)v(r, t, T) \tag{5}$$

Assuming the risk premium factor, λ^*, to be of the form $\lambda\sqrt{r}/\sigma$, substituting the expression for the expected return (5) into (3) yields

$$rP + \lambda r P_r = P_r\kappa(\theta - r) + P_t + 1/2 P_{rr}\sigma^2 r \tag{6}$$

This is the fundamental equation for the price of any asset which has a value that depends solely on the instantaneous rate, r, and the time to maturity, $T - t$ (CIR, eq. 22).

With the boundary condition that

$$P(r, T, T) = 1.0 \tag{7}$$

the solution of (6) is of the form

$$P[r, t, T] = A[t, T]\cdot e^{-B[t,\ T]r} \tag{8}$$

where for $$\tau = T - t, \tag{9}$$

[2] This model for the interest rates is discussed in CIR [4]. Marsh and Rosenfeld [14] discuss at length the empirical evidence in favor of the model, and suggest alternative models for changes in short term interest rates. CIR [5] also consider models expressed in terms of real interest rates, where inflation uncertainty is a second factor, and Brennan and Schwartz [2] consider an alternative two factor model. The argument for considering a single factor model first is its simplicity and empirical tractability. Whether a multiple factor model will represent a significant improvement is an open empirical question, particularly since we cannot identify all the parameters of interest in the single factor model given prices of the set of bonds trading at a given point in time.

$$A[t, T] \equiv \left\{\frac{\phi_1 \exp(\phi_2 \tau)}{\phi_2[\exp(\phi_1 \tau) - 1] + \phi_1}\right\} \tag{10}$$

$$B[t, T] \equiv \frac{\exp(\phi_1 \tau) - 1}{\phi_2[\exp(\phi_1 \tau) - 1] + \phi_1} \tag{11}$$

where

$$\phi_1 \equiv \{(\kappa + \lambda)^2 + 2\sigma^2\}^{1/2} \tag{12}$$

$$\phi_2 \equiv (\kappa + \lambda + \phi_1)/2 \tag{13}$$

$$\phi_3 \equiv 2\kappa\theta/\sigma^2 \tag{14}$$

Equations (8) through (14) define the basic CIR model we study in this paper. We estimate the parameters ϕ_1, ϕ_2, ϕ_3 and r on the basis of data on the prices of U.S. Treasury issues trading at a point in time, and therefore obtain a time series of estimates of ϕ_1, ϕ_2, ϕ_3 and r.

To gain some intuition for the model, observe that discount bond prices are a function of the instantaneous interest rate, r (the only state variable), the time to maturity, τ, and the parameters ϕ_1, ϕ_2, and ϕ_3 which are in turn related to the risk premium parameter λ and interest rate process parameters σ, κ and θ.

If we look at the price of the discount bond as a function of time to maturity, $\tau = T - t$ (leaving t, r, ϕ_1, ϕ_2, and ϕ_3 fixed), we are essentially looking at a single yield curve for such bonds trading as of period t. Specifically, the yield to maturity τ, ytm, is given by

$$\text{ytm} = -\log(P)/\tau \tag{15}$$

For small time to maturity τ, $P \approx \exp[-r\tau]$ and ytm $\approx r$. For τ large, P is of the order $\exp[-(\phi_1 - \phi_2)\phi_3\tau]$ and r_L, the discount rate on such long discount bonds is asymptotically given by[3]

$$r_L = (\phi_1 - \phi_2)\phi_3 \tag{16}$$

Of the U.S. Treasury issues trading on a given date, only Treasury Bills are pure discount issues priced by (8). Other Treasury issues are coupon bonds. Such bonds can be priced by the model if we ignore tax effects and regard each as a portfolio of discount issues, one for each coupon payment and one for the terminal payment on the bond. Consider a coupon bond that entitles the holder to the vector of remaining payments, $\boldsymbol{c}$, to be received on the vector of dates, $\boldsymbol{d}$. The value of such a bond at period t is given by

$$V^*(t, \boldsymbol{c}, \boldsymbol{d}) = \textstyle\sum_{d_i > t} c_i P(r, t, d_i) \tag{17}$$

In terms of this notation, a pure discount bond such as a Treasury Bill can be represented as a bond with a single payment to be received at the time the Bill matures.

[3] This is the result (expressed in terms of the definitions (10) through (14) above) given by CIR as their Equation (26). See Dybvig, Ingersoll and Ross [7] for an analysis of the properties of long interest rates.

To estimate the parameters of the model, we make the further assumption that the bond price quoted in period t, $V(t, \boldsymbol{c}, \boldsymbol{d})$, deviates from the model price $V^*(t, \boldsymbol{c}, \boldsymbol{d})$ by a zero-mean error, $\epsilon_{t,T}$:

$$V(t, \boldsymbol{c}, \boldsymbol{d}) = V^*(t, \boldsymbol{c}, \boldsymbol{d}) + \epsilon_{t,T} \tag{18}$$

The error in (18) is assumed to be independent and identically distributed as Normal in the cross section of bonds that cover the maturities traded at that point of time.[4]

Given the model for the observations (18), it is possible to obtain maximum likelihood estimates of the parameters ϕ_1, ϕ_2, ϕ_3 and the instantaneous interest rate, r, using nonlinear least squares procedures applied to data on the prices of bonds of different maturities trading at a given point in time.[5] From these estimates, it is possible to identify the parameter σ using

$$\sigma^2 = 2[\phi_1\phi_2 - \phi_2^2] \tag{19}$$

and the implied long rate r_L, using Equation (16), although it is not possible to separately identify the parameters θ, κ and λ.[6]

The advantage of the CIR model is that it provides for a parsimonious representation of the yield curve within the context of a relatively flexible functional form. If correct, it would simplify the comparison of bonds of different maturities. However, it represents a highly nonlinear function of the parameters to be estimated, and data from the cross section of securities representing the yield curve will suffice to identify only certain of the parameters of interest. Furthermore, little is known of the statistical properties of the data by which the model is to be estimated.

II. The Data

Data on U.S. Treasury security prices was taken from the CRSP Bond files for the period from December, 1952 through December, 1983. For each month we used data for every Treasury Bill, Note and Bond trading as of that quote date, excluding from the sample callable bonds, bonds not federally taxed, flower bonds

[4] This stochastic specification is motivated by the necessity to use the mean of bid and ask price quotations instead of prices that represent actual trade data. These price quotations are themselves subject to measurement errors. The assumption that the errors are i.i.d. is relatively strong. We would expect that since bonds of differing maturities trade with different frequencies the variance arising from quotation errors need not be constant across maturities. If we were to assume instead that the errors arise from possible misspecification of the pricing relation, the errors would be associated with the discount bond component $P(r, t, T)$, which would imply that the variance of the errors should increase with maturity and be correlated across bonds of different maturities. In addition, it might be reasonable to suppose that such pricing errors would be proportional rather than additive. For these reasons, it is crucial to examine carefully the specification of the error process in light of the observed data.

[5] Marsh [13] (especially pp. 427–431) proposes (but does not actually implement) an entirely different approach to obtaining maximum likelihood estimates of the model using time series of ratios of discount bond prices.

[6] This observation is exactly analogous to the observation that one can identify the variance but not the mean of the process generating stock returns from observing the prices of options trading on the stock, if the Black-Scholes option pricing formula (Black and Scholes [1]) is correct.

and bonds which had limited negotiability because of prohibitions and restrictions on commercial bank ownership. These prices were either the trading prices or the mean of bid and ask price quotations where trading prices were otherwise not available, plus the accumulated interest as of that date. For each price, the corresponding time to maturity as well as (in the case of coupon bonds) the coupon payments, the number of payments remaining and the time to next payment were computed using information available on the CRSP tape. These data were ordered by quote date and then by time to maturity, providing for 373 cross sections of default-free coupon bond prices, each of which was used to estimate the parameters of the CIR model.

III. Results

Table I reports the time series means and standard deviations of maximum likelihood estimates of three identifiable parameters of the model, for the overall period and for subperiods chosen to conform with those used by CIR [4].[7] Panel A reports estimates of the underlying process variance, σ^2. Panel B reports the model estimates of the instantaneous rate of interest. This is compared to the average of the short rates of interest obtained as the average yield of Treasury Bills with up to 14 days to maturity for each bond quote date. Panel C gives the mean estimates of the implied return on a long term discount bond based on Equation (16). This is compared to the average yield to maturity of the 14 bonds with the longest term to maturity as of each quote date. These numbers are not strictly comparable, since the average yield to maturity was computed on the basis of coupon rather than discount bonds, and therefore we should expect the observed difference between the returns to be positive (negative) when the yield curve is upward (downward) sloping.

The results reported in Panel A indicate that estimates of the implied process variance differ from the variance parameters of the process estimated by CIR [4] on the basis of a time series of week by week interest on Treasury Bills with 13 weeks to maturity. However, the numbers are of a similar order of magnitude, and the implied process variance appears more stable through time than the time series estimates. The differences may simply reflect the relative lack of precision of the time series based estimates.

In Figure 1 we report the implied standard deviation of changes in short rates given as $s = \hat{\sigma}\sqrt{r}$ (breaks in the graph represent months for which the estimate of the implied variance, $\hat{\sigma}^2$, was zero or was (slightly) negative). Month by month changes in the implied standard deviation of the short rate are quite dramatic. These are consistent with the sampling error in these estimates. On the same figure we report the annualized standard deviation of month by month changes in observed short rates estimated for each year of our sample period.[8] The fact

[7] We should note that the asymtotic standard errors of the individual estimates are large since the likelihood functions proved relatively flat in the region of the MLE of the parameters ϕ_1, ϕ_2, and ϕ_3.

[8] Inspection of Figure 1 would seem to suggest a regime shift in the interest rate process bracketed by the period from late 1979 through late 1982. Such a regime shift has been documented by others (Huizinga and Mishkin [11]) who associate the shift with changes in Federal Reserve operating procedures in October 1979 and October 1982.

Table I

Comparison of Estimates Based on Crossection of Bonds Trading at a Point in Time with Estimates Based on Time Series of Bond Returns[a]

A. Implied process variance compared to time series estimates

	$\bar{\sigma}^2$	Standard Deviation of σ^2	Time Series Estimate of σ^2 [b]
1952/12–1983/12	0.2121	0.3401	NA
1967/01–1976/12	0.2041	0.2140	0.316
1967/01–1968/12	0.3178	0.3358	0.158
1969/01–1970/12	0.2651	0.2608	0.162
1971/01–1972/12	0.1212	0.0760	0.400
1973/01–1974/12	0.2266	0.2518	1.712
1975/01–1976/12	0.1758	0.1578	0.524
1977/01–1978/12	0.2102	0.4717	NA
1979/01–1980/12	0.4740	0.6572	NA
1981/01–1982/12	0.5396	0.4510	NA
1983/01–1983/12	0.2054	0.1097	NA
1952/12–1966/12	0.1196	0.2323	NA
1967/01–1970/12	0.2770	0.2742	0.172
1970/01–1976/12	0.1641	0.1612	0.412
1977/01–1983/12	0.3790	0.5118	NA

B. Implied instantaneous interest rate compared to short term Treasury Bill rate[c]

	$\bar{\hat{r}}$	$\overline{\hat{r} - r_f}$	$\hat{\sigma}(\hat{r} - r_f)$	t value
1952/12–1983/12	0.0534	0.0045	0.0089	9.77
1967/01–1976/12	0.0579	0.0043	0.0057	8.26
1967/01–1968/12	0.0497	0.0048	0.0049	4.80
1969/01–1970/12	0.0666	0.0057	0.0052	5.37
1971/01–1972/12	0.0425	0.0030	0.0033	4.45
1973/01–1974/12	0.0764	0.0040	0.0087	2.25
1975/01–1976/12	0.0543	0.0039	0.0049	3.90
1977/01–1978/12	0.0678	0.0058	0.0050	5.68
1979/01–1980/12	0.1163	0.0182	0.0154	5.79
1981/01–1982/12	0.1252	0.0110	0.0200	2.69
1983/01–1983/12	0.0805	−0.0039	0.0042	−3.22
1952/12–1966/12	0.0272	0.0023	0.0042	7.12
1967/01–1970/12	0.0582	0.0052	0.0050	7.21
1970/01–1976/12	0.0585	0.0036	0.0057	5.79
1977/01–1983/12	0.0999	0.0094	0.0155	5.56

[a] Time series means of CIR parameters were estimated on the basis of US Treasury Issue prices from 1952/12 to 1983/12. σ^2 represents the time series mean of estimates of the interest rate variance parameter, σ^2 (excluding from the computation those months for which the implied variance was estimated to be zero or negative).

[b] These numbers were taken from Cox, Ingersoll and Ross [4], Table 1, expressed on an annualized basis.

[c] $\bar{\hat{r}}$ represents the time series mean of estimates of the implied instantaneous interest rate, and $\overline{\hat{r} - r_f}$ is the mean difference between these estimates and the short term rate given as the mean yield on US Treasury Bills with at most 14 days to maturity as of each quote date.

that there is a correspondence between the two sets of numbers is remarkable given that the solid lines represent standard deviations implied by the shape of the yield curve alone at a single point in time, whereas the stars give estimates

Table I—*continued*

C. Implied long term interest rate compared to yield on long term Treasury Bond issues[d]

	$\bar{\hat{r}}_L$	$\overline{\hat{r}_L - r_{fL}}$	$\hat{\sigma}(\hat{r}_L - r_{fL})$	t value
1952/12–1983/12	0.0666	0.0059	0.0439	2.60
1967/01–1976/12	0.0755	0.0107	0.0741	1.58
1967/01–1968/12	0.1068	0.0558	0.1538	1.78
1969/01–1970/12	0.0659	−0.0030	0.0120	−1.22
1971/01–1972/12	0.0661	0.0073	0.0084	4.26
1973/01–1974/12	0.0576	−0.0136	0.0255	−2.61
1975/01–1976/12	0.0791	0.0057	0.0054	5.17
1977/01–1978/12	0.0771	0.0006	0.0062	0.47
1979/01–1980/12	0.1003	−0.0015	0.0037	−1.99
1981/01–1982/12	0.1266	−0.0013	0.0039	−1.63
1983/01–1983/12	0.1137	0.0037	0.0020	6.41
1952/12–1966/12	0.0390	0.0056	0.0146	4.99
1967/01–1970/12	0.0864	0.0264	0.1119	1.63
1970/01–1976/12	0.0685	−0.0001	0.0171	−0.05
1977/01–1983/12	0.1031	−0.0001	0.0047	−0.20

[d] $\bar{\hat{r}}_L$ represents the time series mean of estimates of the implied long term rate, and $\overline{\hat{r} - r_{fL}}$ is the mean difference between these estimates and the long term rate given as the mean yield to maturity of the 14 Treasury Bonds quoted as of the same quote date. Note that these numbers are not strictly comparable, as $\bar{\hat{r}}_L$ applies to *discount bonds* and r_{fL} is estimated from long term *coupon bonds*. With a rising yield curve, we would expect the difference to be positive.

based on the time series of short rates.[9] If we consider the annual average of the implied standard deviations, $\bar{s}_t$, as a predictor of the time series estimate of the standard deviation of changes in the short rate, $\hat{s}_t$, we cannot reject the hypothesis that it is an unbiased predictor:

$$\underset{}{\hat{s}_t} = \underset{(.01484)}{-.00707} + \underset{(\ .12971)}{.87635}\ \bar{s}_t \qquad R^2 = .61 \tag{20}$$

(Standard errors in parentheses)

with an intercept not significantly different from zero and a slope coefficient not significantly different from unity. At this level, it would appear that the model is quite well specified. Furthermore, the model may predict the variance of interest rate changes as well as or better than historical time series based estimates.

As indicated in Panel B of Table I, the model appears to systematically overestimate the implied short rates of return. The degree of overestimation is significant in all but the subperiod of the data from January to December 1983. It is interesting that this apparent misspecification matches a similar misspecification reported by Fama and MacBeth [8] in their study of equilibrium in the equity securities markets. However, it would be premature to conclude that this represents compelling evidence that the CIR model is misspecified; it may merely

[9] Stephen Ross has pointed out to us that, of course, if one were solely interested in using the CIR model to estimate the standard deviation parameter σ implied by the term structure, a simpler and probably more efficient alternative to full maximum likelihood estimation would be to linearize Equations (9) through (14) around σ.

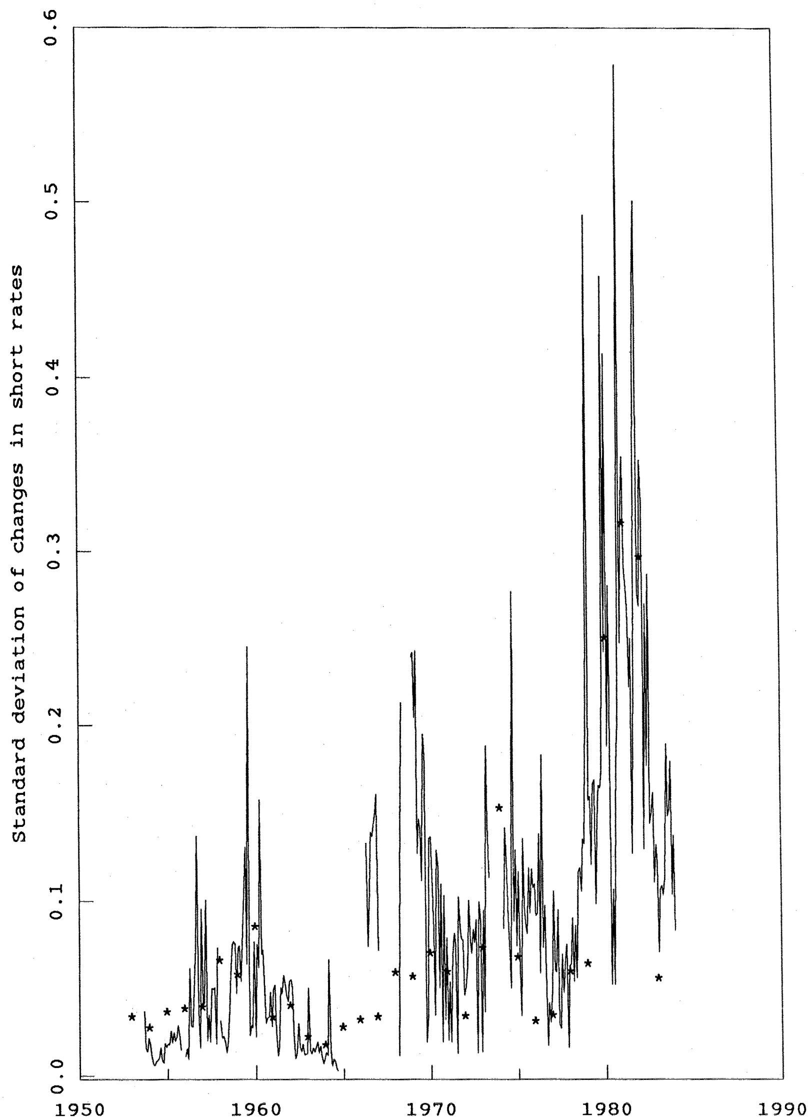

Figure 1. Implied and time series estimates of standard deviation.

indicate problems with the stochastic specification and procedures used to estimate the implied short rate.[10]

Turning to the long rates reported in Panel C of Table I, we find that while the implied long rate is greater than the yield to maturity on long term bonds for

[10] Computing the average term premia given as the difference between the average of forward rates implied by the monthly estimates of the model parameters for a range of terms to maturity, and the average observed short rates of return, yielded results consistent with those reported by Fama [9]. These results (available from the authors) are noteworthy in view of the fact that the CIR model is a parsimonious representation estimated on the basis of a wider class of Treasury issues than those considered by Fama.

the entire sample, the result is mixed for subperiods of the data. This result is not surprising. For the overall period,the yield curve is positively sloped ($\bar{\bar{r}} < \bar{\bar{r}}_L$) whereas in the subperiods where the implied long rate is significantly less than the yield to maturity, the yield curve is negatively sloped ($\bar{\bar{r}} > \bar{\bar{r}}_L$). In these circumstances, we would expect precisely the relationship we observe between the implied long rate and yields to maturity on long coupon bonds.

To examine in some greater depth the issue of how well specified the model is, Figure 2 reports boxplots of the standardized residuals from fitting the model using the method of maximum likelihood. These residuals are classified according to

1. whether the bond in question is a Treasury Bill or other Treasury issue,
2. whether the bond is priced to trade at a premium or at a discount (note that the latter category accounts for all Treasury Bills), and
3. whether the bond is long term or short term to maturity, where "short term" is defined as the period for which capital gains are taxed as ordinary income (six months prior to 1977 and one year subsequent to that date, except in the case of Treasury Bills the discount for which is treated as ordinary income in every year of our sample, and were thus considered "short term" issues).

To interpret these boxplots, note that 95 percent of the data falls within the extent defined by the "whiskers", while the box gives the interquartile range. The median is the dashed line within the box and the "notches" on either side of the box give the 95 percent confidence interval for the median. We see from these plots that the dispersion of the errors associated with Treasury Bills is much less pronounced than for the other Treasury issues. We see also some evidence that the distribution of errors differs across bonds trading at a premium and bonds trading at a discount in that the residuals in each case appear skewed but skewed in opposite directions. Furthermore, there is some evidence that the model appears to overpredict prices of bonds trading at a premium, to the extent that the median residual for such bonds is significantly negative.

This difference between bonds trading at a premium and bonds trading at a discount is highlighted by comparing the residuals prior to 1979 (Figure 3) with those subsequent to 1979 (Figure 4). While the general appearance of these figures is similar to Figure 2, the model appears to be seriously misspecified not only with respect to the distinction between Treasury Bills and other Treasury issues, but also between premium and discount bonds. This can be seen most clearly in the post-1979 results of Figure 4. Not only do premium and discount issues differ according to the direction of their relative skewness, but discount bond prices are very significantly *underestimated* by the model, and premium prices are as significantly *overestimated.*

Treasury Bills are traded in a more active secondary market than are other Treasury issues, and one would expect quoted prices to be much closer to trade prices for the Bills. It is not surprising that the model seems to fit better to the quoted prices for Treasury Bills than to the prices quoted for short term discount bonds which are otherwise indistinguishable from the Bills. This argument does not explain the apparent difference between discount and premium bonds. Since the two categories of bonds differ according to their United States income tax

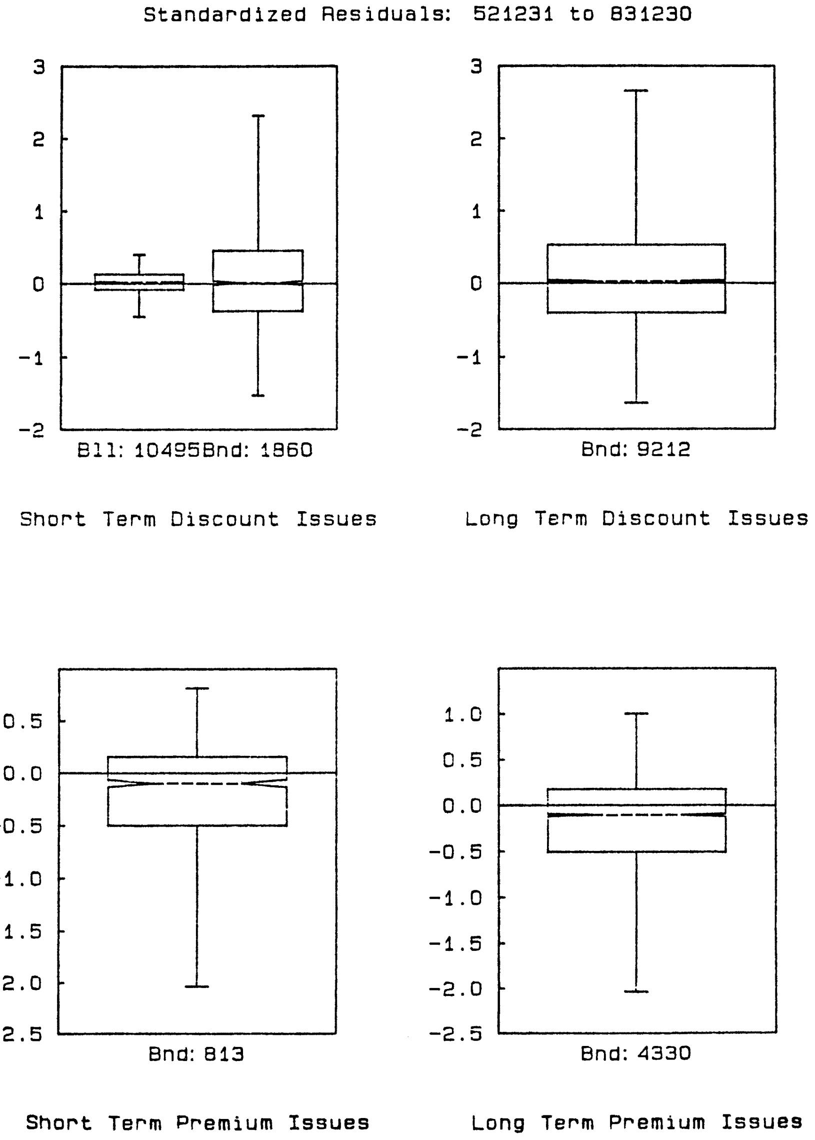

Figure 2. These boxplots (described in text) give the median, interquartile range and 95 percent confidence intervals for the residuals from the fitted model (actual price less estimated model price) standardized by the standard deviation of the residuals estimated for each month in the sample. The residuals are classified in each month by whether the bonds in question are trading at a premium or discount, and whether the remaining time to maturity would qualify any holding period gain for long term capital gains tax treatment. The numbers under each panel give the number of residuals from T Bills (Bll) and from T Bonds and Notes (Bnd) falling within each classification.

treatment, it is reasonable to explore possible tax explanations of the specification errors associated with these bonds.

The CIR model was derived without consideration of taxes, and to this point we have ignored the tax implications associated with purchase or sale of default

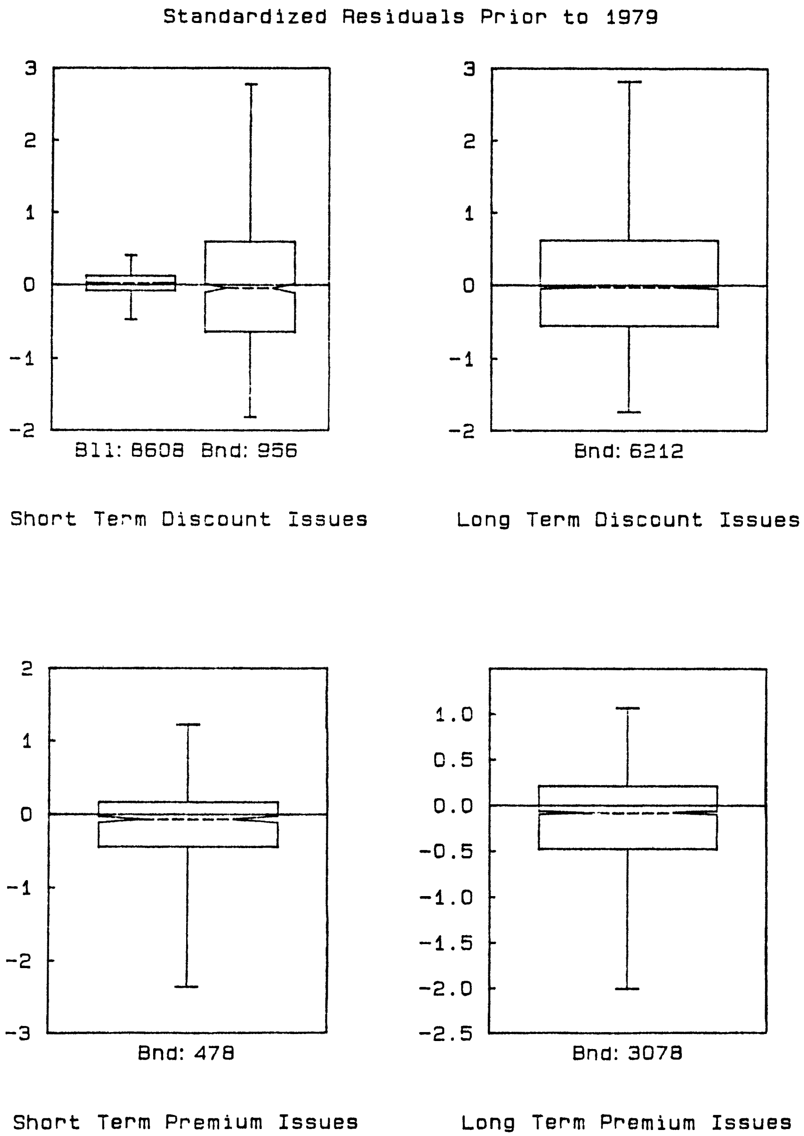

Figure 3. Boxplots of standardized residuals for the period 1952/12 through 1978/12.

free bonds. To see how the model might be adjusted to consider tax effects, note that the United States tax code treats premia and discounts on long term bonds held to maturity in an asymmetric fashion: the bondholder can amortize the premium over the life of the bond to reduce taxable income, whereas the discount is taxed at favorable capital gains rates when the bond is sold or redeemed. This is not the complete story, of course. There is an option effect introduced when we consider that changes in interest rates can transform a premium bond into a

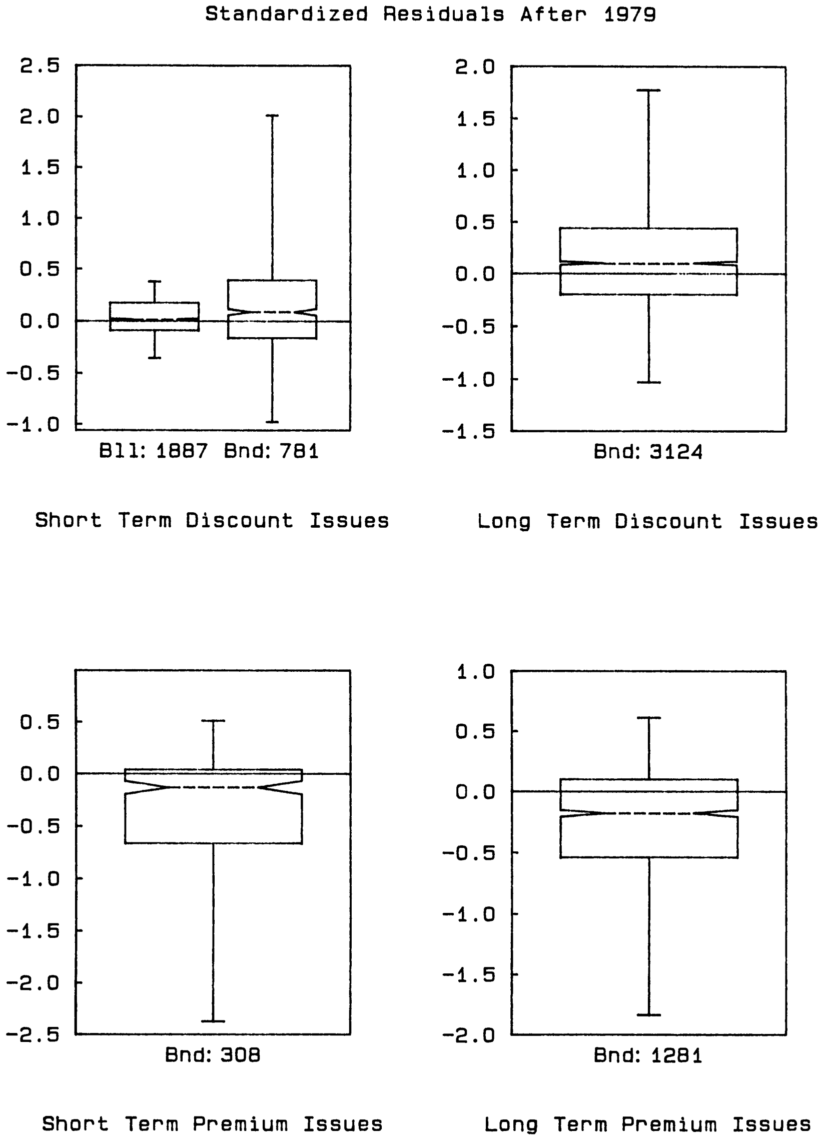

Figure 4. Boxplots of standardized residuals for the period 1979/1 through 1983/12.

discount bond.[11] However, other things equal, investors should prefer bonds trading at a discount to bonds trading at a premium.[12]

The model misspecification apparent in the comparison of premium and discount bonds presented in Figure 4 is consistent with such a tax hypothesis.

[11] See Constantinedes and Ingersoll [3] for an analysis of the option effect and numerical results that indicate that the effect is significant.

[12] We should note that recent changes in US tax law may serve to minimize the importance of these tax related effects.

This argument is made more compelling when we consider that the degree of model misspecification increased subsequent to 1979, a year in which there were major changes in the tax law relating the the taxable status of capital gains relative to other income. The simplest version of the tax hypothesis would predict that the degree of misspecification associated with the long term bonds should differ from that of the short term bonds. However, the degree of misspecification appears to be as severe for both classes of bonds. Clearly, more work needs to be done at both the theoretical and empirical levels to account for this phenomenon.

IV. Conclusions

The CIR model, at least in its simplest form, is readily estimable on the basis of prices of United States Treasury issues covering the maturity spectrum that are quoted at a point in time. Using data from the yield curve alone, it is possible to estimate *both* the instantaneous default free interest rate and the variance of changes in that rate. It is possible to compare such estimates implied by the prices of a cross section of bonds trading at a point of time with estimates obtained from studying the time series of short term interest rates.

While the variance of the default free return implied by the prices of different bonds trading at a point in time seems to correspond quite well to the time series variance of short interest rates, the model systematically overestimates short interest rates. Furthermore, studying the residuals from the model we find further evidence that the model is misspecified in the context of these data. The model appears to fit Treasury Bills better than it does other Treasury issues. This violates the assumption that errors in pricing are identically distributed across Treasury issues. In addition the model significantly overprices premium issues and underprices discount issues, partially consistent with a neglected tax hypothesis. Further work needs to be done to revise the specification of the model to account for these issues.

REFERENCES

1. F. Black and M. Scholes. "The pricing of options and corporate liabilities." *Journal of Political Economy* 81 (May/June, 1973), 637–659.
2. M. Brennan and E. Schwartz. "A Continuous Time Approach to the Pricing of Bonds." *Journal of Banking and Finance* 3 (July, 1979), 133–155.
3. G. Constantinedes and J. Ingersoll. "Optimal Bond Trading with Personal Taxes." *Journal of Financial Economics* 13 (September, 1984), 229–335.
4. J. Cox, J. Ingersoll and S. Ross. "Forward Rates and Expected Spot Rates: The Effect of Uncertainty." Unpublished working paper: University of Chicago, 1977.
5. ———. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53 (March, 1985), 385–407.
6. J. Culbertson. "The Term Structure of Interest Rates." *Quarterly Journal of Economics* 71 (November, 1957), 485–517.
7. P. Dybvig, J. Ingersoll and S. Ross. "Do Interest Rates Converge?" Unpublished working paper: Yale University, 1985.
8. E. Fama. "Term Premiums in Bond Returns." *Journal of Financial Economics* 13 (December, 1984), 529–546.
9. ——— and J. MacBeth. "Risk, Return and Equilibrium: Empirical Tests." *Journal of Political Economy* 81 (May/June, 1973), 607–636.

10. J. Hicks. *Value and Capital.* London: Oxford University Press, 1946.
11. J. Huizinga and F. Mishkin. "Monetary Policy Regime Shifts and the Unusual Behavior of Real Interest Rates." Unpublished working paper: National Bureau of Economic Research, Cambridge, 1985.
12. J. Ingersoll. "Interest Rate Dynamics, the Term Structure, and the Valuation of Contingent Claims." Unpublished working paper: U. of Chicago, 1976.
13. T. Marsh. "Equilibrium Term Structure Models: Test Methodology." *Journal of Finance* 35 (May, 1980), 421–434.
14. T. Marsh and E. Rosenfeld. "Stochastic Processes for Interest Rates and Equilibrium Bond Prices." *Journal of Finance* 38 (May, 1983) 635–645.

DISCUSSION

WAYNE E. FERSON*: Brown and Dybvig [2] examine the single state variable term structure model of Cox, Ingersoll and Ross (CIR) using price data for U.S. Treasury securities of various maturities. Although the work they present is admittedly preliminary, it seems to have been thoughtfully executed and the approach is appealing for several reasons. First, given the possibility of substitution across maturities it seems reasonable that discount bonds with similar maturities should have similar yields; that is, the term structure should be "smooth." Alternative approaches to imposing smoothness in term structure estimation—such as the use of splines—do so in a relatively ad hoc way. The present approach imposes smoothness across the term structure by using a functional form for bond prices that derives from an economic model and which depends on a "small" number of parameters. A second attractive feature is that such an approach has the potential to produce estimates of parameters that may be of broader interest. As CIR and Brown and Dybvig point out, the term structure in this model embodies the information currently available to the market about the future course of events. Expected market risk premiums for example, are reflected in the term structure. Recent empirical work of Campbell [3], Keim and Stambaugh [5] and others suggests that ex ante yield curve information may have some predictive power for future rates of return on other securities. It seems to me that a confluence of these two strains might prove profitable in future research, potentially providing both guidance on functional form for predictive empirical models as well as interesting tests of pricing theory based on conditional moments.

Brown and Dybvig do not exploit an opportunity to address empirically the classical question of the term structure, namely the relation of the yield curve to expectations about future interest rates. The CIR model as estimated here does not seem to allow separate identification of term premiums; that is, the difference between forward rates and expected future spot rates. It does seem to be possible to obtain, at each date, estimates of the parameters needed to forecast the spot rate, except for the speed of adjustment coefficient. Given an estimate of these parameters (say, from a previous period time series) then ex ante forecasts of future interest rates could be formed using term structure information as of the current date. An important caveat here is that nondeterministic variation in such

* University of Chicago.