Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation

BOND PRICING AND THE TERM STRUCTURE OF INTEREST RATES: A NEW METHODOLOGY FOR CONTINGENT CLAIMS VALUATION[1]

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This paper presents a unifying theory for valuing contingent claims under a stochastic term structure of interest rates. The methodology, based on the equivalent martingale measure technique, takes as given an initial forward rate curve and a family of potential stochastic processes for its subsequent movements. A no arbitrage condition restricts this family of processes yielding valuation formulae for interest rate sensitive contingent claims which do not explicitly depend on the market prices of risk. Examples are provided to illustrate the key results.

Keywords: Term structure of interest rates, interest rate options, contingent claims, martingale measures.

1. INTRODUCTION

In relation to the term structure of interest rates, arbitrage pricing theory has two purposes. The first, is to price all zero coupon (default free) bonds of varying maturities from a finite number of economic fundamentals, called state variables. The second, is to price all interest rate sensitive contingent claims, taking as given the prices of the zero coupon bonds. This paper presents a general theory and a unifying framework for understanding arbitrage pricing theory in this context, of which all existing arbitrage pricing models are special cases (in particular, Vasicek (1977), Brennan and Schwartz (1979), Langetieg (1980), Ball and Torous (1983), Ho and Lee (1986), Schaefer and Schwartz (1987), and Artzner and Delbaen (1988)). The primary contribution of this paper, however, is a new methodology for solving the second problem, i.e., the pricing of interest rate sensitive contingent claims given the prices of all zero coupon bonds.

The methodology is new because (i) it imposes its stochastic structure directly on the evolution of the forward rate curve, (ii) it does not require an "inversion of the term structure" to eliminate the market prices of risk from contingent claim values, and (iii) it has a stochastic spot rate process with multiple stochastic factors influencing the term structure. The model can be used to consistently price (and hedge) all contingent claims (American or European) on the term structure, and it is derived from necessary and (more importantly) *sufficient* conditions for the absence of arbitrage.

The arbitrage pricing models of Vasicek (1977), Brennan and Schwartz (1979), Langetieg (1980), and Artzner and Delbaen (1988) all require an

"inversion of the term structure" to remove the market prices of risk when pricing contingent claims. This inversion is required due to the two-step procedure utilized in these papers to price contingent claims. The first step is to price the zero coupon bonds from a finite number of state variables. Given these derived prices, the second step is to value contingent claims. It is the first step in this procedure that introduces the explicit dependence on the market prices of risk in the valuation formulae. The equilibrium model of Cox, Ingersoll, and Ross (1985), when used to value contingent claims, also follows this same two-step procedure. To remove this dependence, for parameterized forms of the market prices for risk, it is possible to invert the bond pricing formula after step one, to obtain the market prices for risk as functions of the zero coupon bond prices.

This "inversion of the term structure" removes the market prices for risk from contingent claim values, but it is problematic. First, it is computationally difficult since the bond pricing formulae are highly nonlinear. Secondly, as will be shown later, the spot rate and bond price processes parameters are not independent of the market prices for risk. Hence, arbitrarily specifying a parameterized form of the market prices for risk as a function of the state variables can lead to an inconsistent model, i.e., one which admits arbitrage opportunities. This possibility was originally noted by Cox, Ingersoll, and Ross (1985, p. 398).

A second class of arbitrage pricing models, illustrated by Ball and Torous (1983) and Shaefer and Schwartz (1987) avoids this two-step procedure by taking a finite number of initial bond prices and bond price processes as exogenously given. Unfortunately, Schaefer and Schwartz's model requires a constant spot rate process, and as shown by Cheng (1987), Ball and Torous' model is inconsistent with stochastic spot rate processes and the absence of arbitrage.

The model of Ho and Lee (1986) also avoids the two-step procedure by taking the initial bond prices and bond price processes as exogenously given. Unlike all the previous models, however, they utilize a discrete trading economy. In this economy, the zero coupon bond price curve, in contrast to a finite number of bond prices, is assumed to fluctuate randomly over time according to a binomial process. Unfortunately, it is only a single factor model, so bonds of all maturities are perfectly correlated. Furthermore, to implement their model, they estimate the parameters of the discrete time binomial process including the risk neutral probability. For large step sizes, as shown by Heath, Jarrow, and Morton (1990), the parameters are not independent. This makes estimation problematic, as the dependence is not explicitly taken into account. The continuous time version of this model, which is studied below as a special case, is not subject to this same estimation difficulty.

We generalize the Ho and Lee model to a continuous time economy with multiple factors. Unlike the Ho and Lee model, however, we impose the exogenous stochastic structure upon forward rates, and not the zero coupon

bond prices. This change in perspective facilitates the mathematical analysis and it should also facilitate the empirical estimation of the model. Indeed, since zero coupon bond prices are a fixed amount at maturity, their "volatilities" must change over time. In contrast, constant forward rate volatilities are consistent with a fixed value for a zero coupon bond at maturity.

The model in this paper takes as given the initial forward rate curve. We then specify a general continuous time stochastic process for its evolution across time. To ensure that the process is consistent with an arbitrage free economy (and hence with some equilibrium), we use the insights of Harrison and Kreps (1979) to characterize the conditions on the forward rate process such that there exists a unique, equivalent martingale probability measure. Under these conditions, markets are complete and contingent claim valuation is then a straightforward application of the methods in Harrison and Pliska (1981). We illustrate this approach with several examples.

An outline of this paper is as follows: Section 2 presents the terminology and notation. Section 3 presents the forward rate process. Section 4 characterizes arbitrage free forward rate processes. Section 5 extends the model to price interest rate dependent contingent claims. Sections 6 and 7 provide examples. Section 8 relates the arbitrage pricing approach to the equilibrium pricing approach, while Section 9 summarizes the paper and discusses generalizations.

2. TERMINOLOGY AND NOTATION

We consider a continuous trading economy with a trading interval $[0, \tau]$ for a fixed $\tau > 0$. The uncertainty in the economy is characterized by the probability space (Ω, F, Q) where Ω is the state space, F is the σ-algebra representing measurable events, and Q is a probability measure. Information evolves over the trading interval according to the augmented, right continuous, complete filtration $\{F_t: t \in [0, \tau]\}$ generated by $n \geqslant 1$ independent Brownian motions $\{W_1(t), W_2(t), \ldots, W_n(t): t \in [0, \tau]\}$ initialized at zero. We let $E(\cdot)$ denote expectation with respect to the probability measure Q.

A continuum of default free discount bonds trade with differing maturities, one for each trading date $T \in [0, \tau]$. Let $P(t, T)$ denote the time t price of the T maturity bond for all $T \in [0, \tau]$ and $t \in [0, T]$. We require that $P(T, T) = 1$ for all $T \in [0, \tau]$, $P(t, T) > 0$ for all $T \in [0, \tau]$ and $t \in [0, T]$, and that $\partial \log P(t, T)/\partial T$ exists for all $T \in [0, \tau]$ and $t \in [0, T]$. The first condition normalizes the bond's payoff to be a certain dollar at' maturity. The second condition excludes the trivial arbitrage opportunity where a certain dollar can be obtained for free. The last condition guarantees that forward rates are well-defined.

The instantaneous *forward rate* at time t for date $T > t$, $f(t, T)$, is defined by

$$(1) \qquad f(t, T) = -\partial \log P(t, T)/\partial T \quad \text{for all} \quad T \in [0, \tau], \quad t \in [0, T].$$

It corresponds to the rate that one can contract for at time t, on a riskless loan

that begins at date T and is returned an instant later. Solving the differential equation of expression (1) yields

$$(2) \qquad P(t,T) = \exp\left(-\int_t^T f(t,s)\,ds\right) \quad \text{for all} \quad T \in [0,\tau], \quad t \in [0,T].$$

The *spot rate*[3] at time $t, r(t)$, is the instantaneous forward rate at time t for date t, i.e.,

$$(3) \qquad r(t) = f(t,t) \quad \text{for all} \quad t \in [0,\tau].$$

3. TERM STRUCTURE MOVEMENTS

This section of the paper presents the family of stochastic processes representing forward rate movements, condition (C.1). This condition describes the evolution of forward rates, and thus uniquely specifies the spot rate process and the bond price process. Additional boundedness conditions, (C.2) and (C.3), are required to guarantee that the spot rate and the bond price process are well-behaved.

C.1—A FAMILY OF FORWARD RATE PROCESSES: *For fixed, but arbitrary* $T \in [0,\tau]$, $f(t,T)$ *satisfies the following equation*:

$$(4) \qquad f(t,T) - f(0,T) = \int_0^t \alpha(v,T,\omega)\,dv$$

$$+ \sum_{i=1}^n \int_0^t \sigma_i(v,T,\omega)\,dW_i(v) \quad \text{for all} \quad 0 \leqslant t \leqslant T$$

where: (i) $\{f(0,T): T \in [0,\tau]\}$ *is a fixed, nonrandom initial forward rate curve which is measurable as a mapping* $f(0,\cdot): ([0,\tau], B[0,\tau]) \to (R, B)$ *where* $B[0,\tau]$ *is the Borel σ-algebra restricted to* $[0,\tau]$; (ii) $\alpha: \{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times \Omega \to R$ *is jointly measurable from* $B\{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times F \to B$, *adapted, with*

$$\int_0^T |\alpha(t,T,\omega)|\,dt < +\infty \quad \text{a.e.} \ Q, \quad \text{and}$$

(iii) *the volatilities* σ_i: $\{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times \Omega \to R$ *are jointly measurable from* $B\{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times F \to B$, *adapted, and satisfy*

$$\int_0^T \sigma_i^2(t,T,\omega)\,dt < +\infty \quad \text{a.e.} \ Q \quad \text{for} \quad i = 1,\ldots,n.$$

In this stochastic process n independent Brownian motions determine the stochastic fluctuation of the *entire* forward rate curve starting from a fixed initial

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[3]This is equivalent to $r(t) = \lim_{h \to 0} [1 - P(t,t+h)]/P(t,t+h)h = f(t,t)$.

curve $\{f(0,T): T \in [0,\tau]\}$. The sensitivity of a particular maturity forward rate's change to each Brownian motion is reflected by differing volatility coefficients. The volatility coefficients $\{\sigma_i(t,T,\omega): T \in [0,\tau]\}$ for $i = 1,\ldots,n$ are left unspecified, except for mild measurability and integrability conditions, and can depend on the entire past of the Brownian motions. Different specifications for these volatility coefficients generate significantly different qualitative characteristics of the forward rate process. The family of drift functions $\{\alpha(\cdot,T): T \in [0,\tau]\}$ is also unrestricted (at this point), except for mild measurability and integrability conditions.

It is important to emphasize that the only substantive economic restrictions imposed on the forward rate processes are that they have continuous sample paths and that they depend on only a finite number of random shocks (across the entire forward rate curve).

Given condition (C.1), we can determine the dynamics of the spot rate process:

$$(5) \qquad r(t) = f(0,t) + \int_0^t \alpha(v,t,\omega)\, dv$$

$$+ \sum_{i=1}^n \int_0^t \sigma_i(v,t,\omega)\, dW_i(v) \quad \text{for all} \quad t \in [0,T].$$

The spot rate process is similar to the forward rate process, except that both the time and maturity arguments vary simultaneously.

For the subsequent analysis, it is convenient to define an accumulation factor, $B(t)$, corresponding to the price of a money market account (rolling over at $r(t)$) initialized at time 0 with a dollar investment, i.e.,

$$(6) \qquad B(t) = \exp\left(\int_0^t r(y)\, dy\right) \quad \text{for all} \quad t \in [0,\tau].$$

Given the dynamics of the spot rate process, we need to ensure that the value of the money market account satisfies

$$(7) \qquad 0 < B(t,\omega) < +\infty \quad \text{a.e. } Q \quad \text{for all} \quad t \in [0,\tau].$$

This is guaranteed by condition (C.2).

C.2—REGULARITY OF THE MONEY MARKET ACCOUNT:

$$\int_0^\tau |f(0,v)|\, dv < +\infty \quad and \quad \int_0^\tau \left\{\int_0^t |\alpha(v,t,\omega)|\, dv\right\} dt < +\infty \quad \text{a.e. } Q.$$

Next, we are interested in the dynamics of the bond price process. The following condition imposes sufficient regularity conditions so that the bond price process is well-behaved.

C.3—REGULARITY OF THE BOND PRICE PROCESS:

$$\int_0^t \left[\int_v^t \sigma_i(v, y, \omega) \, dy \right]^2 dv < +\infty \quad \text{a.e. } Q$$

for all $t \in [0, \tau]$ *and* $i = 1, \ldots, n;$

$$\int_0^t \left[\int_t^T \sigma_i(v, y, \omega) \, dy \right]^2 dv < +\infty \quad \text{a.e. } Q$$

for all $t \in [0, T], \quad T \in [0, \tau], \quad i = 1, \ldots, n;$

and

$$t \to \int_t^T \left[\int_0^t \sigma_i(v, y, \omega) \, dW_i(v) \right] dy \quad \text{is continuous a.e. } Q$$

for all $T \in [0, \tau]$ *and* $i = 1, \ldots, n.$

It is shown in the Appendix that under conditions C.2–C.3, the dynamics of the bond price process (suppressing the notational dependence on ω) are

$$(8) \qquad \ln P(t, T) = \ln P(0, T) + \int_0^t [r(v) + b(v, T)] \, dv$$

$$- (1/2) \sum_{i=1}^n \int_0^t a_i(v, T)^2 \, dv$$

$$+ \sum_{i=1}^n \int_0^t a_i(v, T) \, dW_i(v) \quad \text{a.e. } Q$$

where

$$a_i(t, T, \omega) \equiv - \int_t^T \sigma_i(t, v, \omega) \, dv \quad \text{for} \quad i = 1, \ldots, n \quad \text{and}$$

$$b(t, T, \omega) \equiv - \int_t^T \alpha(t, v, \omega) \, dv + (1/2) \sum_{i=1}^n a_i(t, T, \omega)^2.$$

A straightforward application of Ito's lemma to expression (8) yields $P(t, T)$ as the strong solution to the following stochastic differential equation:

$$(9) \qquad dP(t, T) = [r(t) + b(t, T)] P(t, T) \, dt$$

$$+ \sum_{i=1}^n a_i(t, T) P(t, T) \, dW_i(t) \quad \text{a.e. } Q.$$

In general, the bond price process is non-Markov since the drift term $(r(t, \omega) + b(t, T, \omega))$ and the volatility coefficients $a_i(t, T, \omega)$ for $i = 1, \ldots, n$ can depend on the history of the Brownian motions. The form of the bond price process as given in expression (9) is similar to, but more general than, that appearing in the existing literature (see, for example, Brennan and Schwartz (1979) or Langetieg (1980)), because it requires less regularity assumptions and it need not be Markov.

We define the relative bond price for a T-maturity bond as $Z(t, T) = P(t, T)/B(t)$ for $T \in [0, \tau]$ and $t \in [0, T]$. This is the bond's value expressed in units of the accumulation factor, not dollars. This transformation removes the portion of the bond's drift due to the spot rate process. As such, it is particularly useful for analysis. Applying Ito's lemma to the definition of $Z(t, T)$ yields

$$(10) \qquad \ln Z(t, T) = \ln Z(0, T) + \int_0^t b(v, T)\, dv - (1/2) \sum_{i=1}^n \int_0^t a_i(v, T)^2\, dv$$

$$+ \sum_{i=1}^n \int_0^t a_i(v, T)\, dW_i(v) \quad \text{a.e. } Q.$$

Again, the relative bond price at date t may depend on the path of the Brownian motion through the cumulative forward rate drifts and volatilities. In general, it cannot be written as a function of only the current values of the Brownian motions.

4. ARBITRAGE FREE BOND PRICING AND TERM STRUCTURE MOVEMENTS

Given conditions C.1–C.3, this section characterizes necessary and sufficient conditions on the forward rate process such that their exists a unique, equivalent martingale probability measure.

C.4—EXISTENCE OF THE MARKET PRICES FOR RISK: *Fix* $S_1, \ldots, S_n \in [0, \tau]$ *such that* $0 < S_1 < S_2 < \ldots < S_n \leqslant \tau$. *Assume there exists solutions*

$$\gamma_i(\cdot, \cdot\, ; S_1, \ldots, S_n)\colon \Omega \times [0, S_1] \to R \quad \text{for} \quad i = 1, \ldots, n \qquad \text{a.e. } Q \times \lambda$$

to the following system of equations:

$$(11) \quad \begin{bmatrix} b(t, S_1) \\ \vdots \\ b(t, S_n) \end{bmatrix} + \begin{bmatrix} a_1(t, S_1) \ldots a_n(t, S_1) \\ \vdots \\ a_1(t, S_n) \ldots a_n(t, S_n) \end{bmatrix} \begin{bmatrix} \gamma_1(t; S_1, \ldots, S_n) \\ \vdots \\ \gamma_n(t; S_1, \ldots, S_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

which satisfy

(12.a) $\int_0^{S_1} \gamma_i(v; S_1, \ldots, S_n)^2 \, dv < +\infty$ a.e. Q *for* $i = 1, \ldots, n$,

(12.b) $E \left(\exp \left\{ \sum_{i=1}^n \int_0^{S_1} \gamma_i(v; S_1, \ldots, S_n) \, dW_i(v) \right. \right.$

$$\left. \left. - (1/2) \sum_{i=1}^n \int_0^{S_1} \gamma_i(v; S_1, \ldots, S_n)^2 \, dv \right\} \right) = 1,$$

(12.c) $E \left(\exp \left\{ \sum_{i=1}^n \int_0^{S_1} [a_i(v, y) + \gamma_i(v; S_1, \ldots, S_n)] \, dW_i(v) \right. \right.$

$$\left. \left. - (1/2) \sum_{i=1}^n \int_0^{S_1} [a_i(v, y) + \gamma_i(v; S_1, \ldots, S_n)]^2 \, dv \right\} \right) = 1$$

$$\textit{for} \quad y \in \{S_1, \ldots, S_n\}$$

where λ is Lebesgue measure.

The system of equations in expression (11) gives $\gamma_i(t; S_1, \ldots, S_n)$ for $i = 1, \ldots, n$ the interpretation of being the *market prices for risk* associated with the random factors $W_i(t)$ for $i = 1, \ldots, n$, respectively. Indeed, to see this, we can rewrite expression (11) for the T-maturity bond as

(13) $b(t, T) = \sum_{i=1}^n a_i(t, T)(-\gamma_i(t; S_1, \ldots, S_n))$.

The left side of expression (13) is the instantaneous excess expected return on the T-maturity bond *above* the risk free rate. The right side is the sum of (minus) the "market price of risk for factor i" times the instantaneous covariance between the T-maturity bond's return and the ith random factor for $i = 1$ to n. It is important to emphasize that the solutions to expression (11) depend, in general, on the vector of bonds $\{S_1, \ldots, S_n\}$ chosen.

The following proposition shows that condition C.4 guarantees the existence of an equivalent martingale probability measure.

PROPOSITION 1—Existence of an Equivalent Martingale Probability Measure: *Fix $S_1, \ldots, S_n \in [0, \tau]$ such that $0 < S_1 < S_2 < \ldots < S_n \leqslant \tau$. Given a vector of forward rate drifts $\{\alpha(\cdot, S_1), \ldots, \alpha(\cdot, S_n)\}$ and volatilities $\{\sigma_i(\cdot, S_1), \ldots, \sigma_i(\cdot, S_n)\}$ for $i = 1, \ldots, n$ satisfying conditions C.1–C.3, then condition C.4 holds if and only if there exists an equivalent probability measure $\tilde{Q}_{S_1, \ldots, S_n}$ such that $(Z(t, S_1), \ldots, Z(t, S_n))$ are martingales with respect to $\{F_t : t \in [0, S_1]\}$.*

PROOF: In the Appendix.

This proposition asserts that under conditions C.1–C.3, condition C.4 is both necessary and sufficient for the existence of an equivalent martingale probability measure $\tilde{Q}_{S_1,\ldots,S_n}$. The key argument in the proof is Girsanov's Theorem, and it identifies this probability measure as

$$(14) \qquad d\tilde{Q}_{S_1,\ldots,S_n}/dQ = \exp\left\{ \sum_{i=1}^{n} \int_0^{S_1} \gamma_i(v; S_1,\ldots,S_n)\, dW_i(v) \right.$$

$$\left. -(1/2) \sum_{i=1}^{n} \int_0^{S_1} \gamma_i(v; S_1,\ldots,S_n)^2\, dv \right\}.$$

Furthermore, it can also be shown that

$$(15) \qquad \tilde{W}_i^{S_1,\ldots,S_n}(t) = W_i(t) - \int_0^t \gamma_i(v; S_1,\ldots,S_n)\, dv \quad \text{for} \quad i = 1,\ldots,n$$

are independent Brownian motions on $\{(\Omega, \tilde{Q}_{S_1,\ldots,S_n}, F), \{F_t\colon t \in [0, S_1]\}$.

Although condition C.4 guarantees the existence of an equivalent martingale probability measure, it does not guarantee that it is unique. To obtain uniqueness, we impose the following condition:

C.5—UNIQUENESS OF THE EQUIVALENT MARTINGALE PROBABILITY MEASURE: *Fix $S_1,\ldots,S_n \in [0,\tau]$ such that $0 < S_1 < S_2 < \ldots < S_n \leqslant \tau$. Assume that*

$$\begin{bmatrix} a_1(t, S_1) & \ldots & a_n(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_n) & \ldots & a_n(t, S_n) \end{bmatrix} \quad \textit{is nonsingular a.e. } Q \times \lambda.$$

The following proposition demonstrates that condition C.5 is both necessary and sufficient for the uniqueness of the equivalent martingale measure.[4]

PROPOSITION 2—Characterization of Uniqueness of the Equivalent Martingale Probability Measure: *Fix $S_1,\ldots,S_n \in [0,\tau]$ such that $0 < S_1 < S_2 < \ldots < S_n \leqslant \tau$. Given a vector of forward rate drifts $\{\alpha(\cdot, S_1),\ldots,\alpha(\cdot, S_n)\}$ and volatilities $\{\sigma_i(\cdot, S_1),\ldots,\sigma_i(\cdot, S_n)\}$ for $i = 1,\ldots,n$ satisfying conditions C.1–C.4, then condition C.5 holds if and only if the martingale measure is unique.*

PROOF: In the Appendix.

Conditions C.1–C.5, through the functions $\gamma_i(t; S_1,\ldots,S_n)$ for $i = 1,\ldots,n$ impose restrictions upon the drifts for the forward rate processes

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[4]For the case of a single Brownian motion, condition C.5 simplifies to the statement that $\sigma_1(t, S_1) > 0$ a.e. $Q \times \lambda$.

$\{\alpha(\cdot, S_1), \ldots, \alpha(\cdot, S_n)\}$. It imposes just enough restrictions so that there is a unique equivalent martingale probability measure for the bonds $(Z(t, S_1), \ldots, Z(t, S_n))$ with $0 < S_1 < \ldots < S_n \leqslant \tau$. Both the market prices for risk and the martingale measure, however, depend on the particular bonds $\{S_1, \ldots, S_n\}$ chosen. To guarantee that there exists a unique equivalent martingale measure simultaneously making *all* relative bond prices martingales, we prove the following proposition.

PROPOSITION 3—Uniqueness of the Martingale Measure Across All Bonds: *Given a family of forward rate drifts $\{\alpha(\cdot, T): T \in [0, \tau]\}$ and a family of volatilities $\{\sigma_i(\cdot, T): T \in [0, \tau]\}$ for $i = 1, \ldots, n$ satisfying conditions C.1–C.5, the following are equivalent*:

(16) *$[\tilde{Q}$ defined by $\tilde{Q} = \tilde{Q}_{S_1, \ldots, S_n}$ for any $S_1, \ldots, S_n \in (0, \tau]$ is the unique equivalent probability measure such that $Z(t, T)$ is a martingale for all $T \in [0, \tau]$ and $t \in [0, S_1]]$;*

(17) *$[\gamma_i(t; S_1, \ldots, S_n) = \gamma_i(t, T_1, \ldots, T_n)$ for $i = 1, \ldots, n$ and all $S_1, \ldots, S_n, T_1, \ldots, T_n \in [0, \tau]$, $t \in [0, \tau]$ such that $0 \leqslant t < S_1 < \ldots < S_n \leqslant \tau$ and $0 \leqslant t < T_1 < \ldots < T_n \leqslant \tau]$;*

(18) *$[\alpha(t, T) = -\sum_{i=1}^{n} \sigma_i(t, T)(\phi_i(t) - \int_t^T \sigma_i(t, v) \, dv)$ for all $T \in [0, \tau]$ and $t \in [0, T]$ where for $i = 1, \ldots, n$, $\phi_i(t) = \gamma_i(t; S_1, \ldots, S_n)$ for any $S_1, \ldots, S_n \in (t, \tau]$ and $t \in [0, S_1]]$.*

PROOF: From Proposition 2, for each vector $(S_1, \ldots, S_n)$ with $S_1 < S_2 < \ldots < S_n \leqslant \tau$, $\tilde{Q}_{S_1, \ldots, S_n}$ is the unique equivalent probability measure making $Z(t, S_i)$ a martingale over $t \leqslant S_1$ for $i = 1, \ldots, n$. These measures are all equal to $\tilde{Q}$ if and only if $\gamma_i(t; S_1, \ldots, S_n) = \gamma_i(t, T_1, \ldots, T_n)$ for $i = 1, \ldots, n$ and all $S_1, \ldots, S_n, T_1, \ldots, T_n \in [0, \tau]$ and $t \in [0, \tau]$ such that $0 \leqslant t < S_1 < \ldots < S_n \leqslant \tau$ and $0 \leqslant t < T_1 < \ldots < T_n \leqslant \tau$. To obtain the third condition, by expression (13) and the fact that $(\phi_1(t), \ldots, \phi_n(t))$ is independent of T, one obtains $b(t, T) = -\sum_{i=1}^{n} a_i(t, T)\phi_i(t)$. Substitution for $b(t, T), a_i(t, T)$ for $i = 1, \ldots, n$ and taking the partial derivative with respect to T gives (18).   Q.E.D.

This proposition asserts that the existence of a unique equivalent probability measure, $\tilde{Q}$, making relative bond prices martingales (condition (16)) is equivalent to the condition that the market prices for risk are independent of the vector of bonds $\{S_1, \ldots, S_n\}$ chosen (condition (17)). Furthermore, condition (17) is also equivalent to a restriction on the drift of the forward rate process (condition (18)). We discuss each of these conditions in turn.

The martingale condition (16) implies that

$$(19) \qquad P(t,T) = B(t)E\left(\exp\left\{\sum_{i=1}^{n}\int_{0}^{T}\phi_i(t)\,dW_i(t)\right.\right.$$

$$\left.\left. -(1/2)\sum_{i=1}^{n}\int_{0}^{T}\phi_i(t)^2\,dt\right\}\middle/ B(T)\middle| F_t\right).$$

Expression (19) demonstrates that the bond's price depends on the forward rate drifts $\{\alpha(\cdot,T)\colon T \in [0,\tau]\}$, the initial forward rate curve $\{f(0,T)\colon T \in [0,\tau]\}$, and the forward rate volatilities $\{\sigma_i(\cdot,T)\colon T \in [0,\tau]\}$ for $i = 1,\ldots,n$. All of these parameters enter into expression (19) implicitly through $\phi_i(t)$ for $i = 1,\ldots,n$, the market prices for risk and $B(T)$, the money market account.

Condition (17) of Proposition 3 is called the *standard finance condition* for arbitrage free pricing. This is the necessary condition for the absence of arbitrage used in the existing literature to derive the fundamental partial differential equation for pricing contingent claims (see Brennan and Schwartz (1979) or Langetieg (1980)).

Last, for purposes of contingent claim valuation, the final condition contained in expression (18) will be most useful. It is called the *forward rate drift restriction*. It shows the restriction needed on the family of drift processes $\{\alpha(\cdot,T)\colon T \in [0,\tau]\}$ in order to guarantee the existence of a unique equivalent martingale probability measure. As seen below, not all potential forward rate processes satisfy this restriction.

5. CONTINGENT CLAIM VALUATION

This section demonstrates how to value contingent claims in the preceding economy. As this analysis is a slight extension of the ideas contained in Harrison and Kreps (1979) and Harrison and Pliska (1981), the presentation will be brief. More importantly, it also provides the unifying framework for categorizing the various arbitrage pricing theories in the literature (i.e., Vasichek (1977), Brennan and Schwartz (1979), Langetieg (1980), Ball and Torous (1983), Ho and Lee (1986), Schaefer and Schwartz (1987), Artzner and Delbaen (1988)) in relation to our own.

Let conditions C.1–C.5 hold. Fix any vector of bonds $\{S_1,\ldots,S_n\} \in [0,\tau]$ where $0 < S_1 < S_2 < \ldots < S_n \leqslant \tau$. By Proposition 2, there exists a unique $\tilde{Q}_{S_1,\ldots,S_n}$ making all $Z(t,S_i)$ martingales for $i = 1,\ldots,n$. The uniqueness of $\tilde{Q}_{S_1,\ldots,S_n}$ implies that the market is *complete* (Harrison and Pliska (1981; Corollary 3.36, p. 241)), i.e., given any random variable $X\colon \Omega \to R$ which is nonnegative, F_{S_1} measurable with $\tilde{E}_{S_1,\ldots,S_n}(X/B(S_1)) < +\infty$ where $\tilde{E}(\cdot)_{S_1,\ldots,S_n}$ denotes expectation with respect to $\tilde{Q}_{S_1,\ldots,S_n}$, there exists an admissible[5] self-financing trading strategy $\{N_0(t), N_{S_1}(t),\ldots,N_{S_n}(t)\colon t \in [0,S_1]\}$ such that the

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[5]For the definition of an admissible self-financing trading strategy, see Harrison and Pliska (1981).

value of the portfolio satisfies

$$(20) \qquad N_0(S_1)B(S_1) + \sum_{i=1}^{n} N_{S_i}(S_1)P(S_1, S_i) = X \quad \text{a.e. } Q.$$

The random variable X is interpreted as the payout to a *contingent claim* at time S_1. Harrison and Pliska (1981) define an arbitrage opportunity and show, in the absence of arbitrage, that the time t price of the contingent claim to X at time S_1 must be given by

$$(21) \qquad \tilde{E}_{S_1,\ldots,S_n}(X/B(S_1)|F_t)B(t).$$

Substituting expression (20) into (21) yields

$$(22) \qquad \tilde{E}_{S_1,\ldots,S_n}\left(N_0(S_1) + N_{S_1}(S_1)/B(S_1) + \sum_{i=2}^{n} N_{S_i}(S_1)Z(S_1, S_i)|F_t \right)B(t).$$

To value this contingent claim, expression (22) demonstrates that we need to know the dynamics for $r(t)$ and $Z(t, S_i)$ for $i = 1,\ldots, n$, all under the martingale measure; that is,

$$(23) \qquad r(t) = f(0, t) + \int_0^t \alpha(v, t)\, dv + \sum_{i=1}^{n} \int_0^t \sigma_i(v, t)\, d\tilde{W}_i^{S_1,\ldots,S_n}(v)$$

$$+ \sum_{i=1}^{n} \int_0^t \gamma_i(v; S_1,\ldots, S_n)\sigma_i(v, t)\, dv \quad \text{a.e. } Q$$

and

$$(24) \qquad Z(t, u) = Z(0, u)\exp\left\{ -(1/2)\sum_{i=1}^{n}\int_0^t a_i(v, u)^2\, dv \right.$$

$$\left. + \sum_{i=1}^{n}\int_0^t a_i(v, u)\, d\tilde{W}_i^{S_1,\ldots,S_n}(v) \right\} \quad \text{a.e. } Q$$

for $u \in \{S_1,\ldots, S_n\}$. Therefore, we need to know $\gamma_i(v; S_1,\ldots, S_n)$ for $i = 1,\ldots, n$, the market prices for risk. These enter through the dynamics of the spot rate process in expression (23). This is true even though the evaluation proceeds in the risk neutral economy under the martingale measure.

All other bonds of differing maturities $u \in [0, \tau]$ are assumed to have values at time S_1, through expression (8), which are F_{S_1} measurable. Since $\tilde{E}_{S_1,\ldots,S_n}(P(S_1, u)/B(S_1)) < +\infty$ for all $u \in [0, \tau]$ and the market is complete, every other bond can be duplicated with an admissible self-financing trading strategy involving only the n bonds $\{S_1,\ldots, S_n\}$ and the money market account. Thus, one can price *all* the remaining bonds and all contingent claims. These are the two purposes for the arbitrage pricing methodology as stated in the introduction.

As expressions (23) and (24) make clear, the dynamics for the bond price process, spot rate process, and the market prices for risk cannot be chosen

independently. Independently specifying these processes will in general lead to inconsistent pricing models. This is the logic underlying the criticism of the arbitrage pricing methodology presented in Cox, Ingersoll, and Ross (1985, p. 398).

The model, as presented above, captures the essence of all the existing arbitrage pricing models. To see this, let us first consider Vasicek (1977), Brennan and Schwartz (1979), Langetieg (1980), and Artzner and Delbaen (1988). Since all four models are similar, we focus upon that of Brennan and Schwartz. Brennan and Schwartz's model has $n = 2$. Instead of specifying the two bond processes for $\{S_1, S_2\}$ directly as in expression (24), they derive these expressions from other assumptions. First, they exogenously specify a long rate process and a spot rate process. Second, they assume that *all* bond prices at time t can be written as twice-continuously differentiable functions of the current values of these long and short rates. In conjunction, these assumptions (by Ito's lemma) imply condition (24). The analysis could then proceed as above, yielding contingent claim values dependent on the market prices for risk.[6,7]

Along with the framework for categorizing the various models, an additional contribution of our approach is to extend the above analysis to eliminate the market prices for risk from the valuation formulas. Intuitively speaking, this is done by utilizing the remaining information contained in the bond price processes to "substitute out" the market prices for risk. For this purpose, we add the following condition:

C.6—COMMON EQUIVALENT MARTINGALE MEASURES: *Given conditions C.1–C.3, let C.4 and C.5 hold for all bonds* $\{S_1, \ldots, S_n\} \in [0, \tau]$ *with* $0 < S_1 < \ldots < S_n \leqslant \tau$. *Further, let* $\bar{Q} = \bar{Q}_{S_1, \ldots, S_n}$ *(on their common domain).*

To remove the market prices for risk from expression (23), we assume condition C.6. Proposition 3, the no arbitrage condition (expression (18)) gives

$$(25) \qquad \int_0^t \alpha(v, t)\, dv = -\sum_{i=1}^n \int_0^t \sigma_i(v, t)\phi_i(v)\, dv$$

$$+ \sum_{i=1}^n \sigma_i(v, t) \int_v^t \sigma_i(v, y)\, dy\, dv.$$

[6]Brennan and Schwartz (1979), however, didn't use this martingale approach. Instead, they priced based on the necessary conditions given by the partial differential equation satisfied by a contingent claim's value under condition (17). Artzner and Delbaen (1988) use the martingale approach.

[7]Ball and Torous (1983) and Schaefer and Schwartz (1987) exogenously specify two bond price processes $\{P(t, S_1), P(t, S_2)\}$ directly. They price contingent claims based on necessary, but not sufficient conditions, for the absence of arbitrage. Unfortunately, both the Ball and Torous model (as shown by Cheng (1987)) and the Schaefer and Schwartz model can be shown to be inconsistent with stochastic spot rate processes and the absence of arbitrage.

Substitution of this expression into expression (23) for the spot rate yields

$$(26) \qquad r(t) = f(0,t) + \sum_{i=1}^{n} \int_0^t \sigma_i(v,t) \int_v^t \sigma_i(v,y)\, dy\, dv$$

$$+ \sum_{i=1}^{n} \int_0^t \sigma_i(v,t)\, d\tilde{W}_i(v).$$

The market prices for risk *drop out* of expression (23) and they are replaced with an expression involving the volatilities across different maturities of the forward rates, i.e., a "term structure of volatilities." Thus, contingent claim values can be calculated independently of the market prices for risk. We further illustrate this abstract procedure with concrete examples in the next two sections.

6. EXAMPLES

This section presents two examples to illustrate and to clarify the analysis in Section 5. One example, a continuous time limit of Ho and Lee's (1986) model (see Heath, Jarrow, and Morton (1988)), may prove useful in practical applications due to its computational simplicity.[8]

We assume that forward rates satisfy the stochastic process from condition C.1 with a single Brownian motion and the volatility $\sigma_1(t,T,\omega) \equiv \sigma > 0$, a positive constant. We let the initial forward rate curve $\{f(0,T): T \in [0,\tau]\}$ be measurable and absolutely integrable (as in condition C.2). Given a particular, but arbitrary stochastic process for the market price of risk, $\phi: [0,\tau] \times \Omega \to R$ which is predictable and bounded, we also assume that the forward rate drift condition (18) is satisfied:

$$(27) \qquad \alpha(t,T) = -\sigma\phi(t) + \sigma^2(T - t)$$

$$\text{for all} \quad T \in [0,\tau] \quad \text{and} \quad t \in [0,T].$$

It is easy to verify that conditions C.1–C.6 are satisfied. This implies, therefore, that contingent claim valuation can proceed as in Section 5. Before that, however, we analyze the forward rate, spot rate, and bond price processes in more detail.

Under the equivalent martingale measure, and in terms of its Brownian motion (see expression (15)), the stochastic process for the forward rate is

$$(28) \qquad f(t,T) = f(0,T) + \sigma^2 t(T - t/2) + \sigma\tilde{W}(t).$$

Under condition (28), forward rates can be negative with positive probability.

The stochastic spot rate process under the equivalent martingale measure is

$$(29) \qquad r(t) = f(0,t) + \sigma\tilde{W}(t) + \sigma^2 t^2/2.$$

Spot rates can also be negative with positive probability.

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[8]The example in this section is similar to a model independently obtained by Jamshidian (1989).

The dynamics of the bond price process over time is given by substituting expression (28) into expression (2):

$$(30) \qquad P(t,T) = [P(0,T)/P(0,t)]e^{-(\sigma^2/2)Tt(T-t)-\sigma(T-t)\tilde{W}(t)}.$$

Next, consider a European call option on the bond $P(t,T)$ with an exercise price of K and a maturity date t^* where $0 \leqslant t \leqslant t^* \leqslant T$. Let $C(t)$ denote the value of this call option at time t. The cash flow to the call option at maturity is

$$(31) \qquad C(t^*) = \max[P(t^*,T) - K,0].$$

By Section 5, the time t value of the call is

$$(32) \qquad C(t) = \bar{E}\big(\max[P(t^*,T) - K,0]B(t)/B(t^*)|F_t\big).$$

An explicit calculation,[9] using normal random variables, shows that expression (32) simplifies to

$$(33) \qquad C(t) = P(t,T)\Phi(h) - KP(t,t^*)\Phi\big(h - \sigma(T-t^*)\sqrt{(t^*-t)}\big)$$

where

$$(34) \qquad h = \Big[\log\big(P(t,T)/KP(t,t^*)\big) + (1/2)\sigma(T-t^*)^2(t^*-t)\Big]\Big/$$
$$\sigma(T-t^*)\sqrt{(t^*-t)}$$

and $\Phi(\cdot)$ is the cumulative normal distribution.

The value of the bond option is given by a modified Black-Scholes formula. The parameter, $\sigma(T-t^*)$, is not equal to the variance of the instantaneous return on the T-maturity bond, but it is equivalent to the variance of the instantaneous return on the forward price (at time t^*) of a T-maturity bond, $(P(t,T)/P(t,t^*))$.

For the second example, assume that forward rates satisfy condition C.1 with the volatilities $\sigma_1(t,T,\omega) \equiv \sigma_1 > 0$ and $\sigma_2(t,T,\omega) \equiv \sigma_2 e^{-(\lambda/2)(T-t)} > 0$ where $\sigma_1,\sigma_2,\lambda$ are strictly positive constants, i.e.,

$$(35) \qquad df(t,T) = \alpha(t,T)\,dt + \sigma_1\,dW_1(t) + \sigma_2 e^{-(\lambda/2)(T-t)}\,dW_2(t)$$
$$\text{for all} \quad T \in [0,\tau] \quad \text{and} \quad t \in [0,T].$$

Here, the instantaneous changes in forward rates are caused by two sources of randomness. The first, $\{W_1(t): t \in [0,\tau]\}$, can be interpreted as a "long-run factor" since it uniformly shifts all maturity forward rates equally. The second, $\{W_2(t): t \in [0,\tau]\}$, affects the short maturity forward rates more than it does long term rates and can be interpreted as a spread between a "short" and "long term factor."

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[9]This calculation and the one in the next section can be found in Brenner and Jarrow (1992).

The volatility functions are strictly positive and bounded. Furthermore, the matrix

$$(36) \quad \begin{bmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{bmatrix} = \begin{bmatrix} -\sigma_1(S-t) + 2\sigma_2(e^{-(\lambda/2)(S-t)} - 1)/\lambda \\ -\sigma_1(T-t) + 2\sigma_2(e^{-(\lambda/2)(T-t)} - 1)/\lambda \end{bmatrix}$$

is nonsingular for all $t, S, T \in [0, \tau]$ such that $t \leqslant S < T$.

We arbitrarily fix two bounded, predictable processes for the market prices of risk, $\phi_i \colon [0, \tau] \times \Omega \to R$ for $i = 1, 2$. To ensure the process is arbitrage free, we set

$$(37) \quad \alpha(t,T) = -\sigma_1 \phi_1(t) - \sigma_2 e^{-(\lambda/2)(T-t)} \phi_2(t) + \sigma_1^2(T-t)$$
$$- 2(\sigma_2^2/\lambda) e^{-(\lambda/2)(T-t)} (e^{-(\lambda/2)(T-t)} - 1).$$

The above forward rate process satisfies conditions C.1–C.6. Under the martingale measure $\tilde{Q}$ and its Brownian motions $\{\tilde{W}_1(t), \tilde{W}_2(t) \colon t \in [0, \tau]\}$, the forward rate process is

$$(38) \quad f(t,T) = f(0,T) + \sigma_1^2 t(T - t/2)$$
$$- 2(\sigma_2/\lambda)^2 [e^{-\lambda T}(e^{\lambda t} - 1) - 2e^{-(\lambda/2)T}(e^{(\lambda/2)t} - 1)]$$
$$+ \sigma_1 \tilde{W}_1(t) + \sigma_2 \int_0^t e^{-(\lambda/2)(T-v)} d\tilde{W}_2(v).$$

This expression shows that forward rates can be negative with positive probability. The spot rate follows the simpler process:

$$(39) \quad r(t) = f(0,t) + \sigma_1^2 t^2/2 - 2(\sigma_2/\lambda)^2 [(1 - e^{-\lambda t}) - 2(1 - e^{-(\lambda/2)t})]$$
$$+ \sigma_1 \tilde{W}_1(t) + \sigma_2 \int_0^t e^{-(\lambda/2)(t-v)} d\tilde{W}_2(v).$$

As before, we can calculate the value of a European call option on the bond $P(t,T)$ with an exercise price of K and a maturity date t^* where $0 \leqslant t \leqslant t^* \leqslant T$. Let $C(t)$ denote the value of this call option at time t. By Section 5, the call's value is

$$(40) \quad C(t) = P(t,T)\Phi(h) - KP(t,t^*)\Phi(h-q)$$

where

$$(41) \quad h = [\log(P(t,T)/KP(t,t^*)) + (1/2)q^2]/q,$$
$$q^2 = \sigma_1^2(T - t^*)^2(t^* - t)$$
$$+ (4\sigma_2^2/\lambda^3)(e^{-(\lambda/2)T} - e^{-(\lambda/2)t^*})^2(e^{\lambda t^*} - e^{\lambda t}).$$

7. A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS

The previous section provides examples of forward rate processes satisfying conditions C.1–C.6. These processes have deterministic volatilities which are independent of the state $\omega \in \Omega$. This section provides a class of processes with

the volatilities dependent on $\omega \in \Omega$. This class of processes can be described as the solutions (if they exist) to the following stochastic integral equation with restricted drift:

$$(42) \qquad f(t,T) - f(0,T) = \int_0^t \alpha(v,T,\omega)\, dv + \sum_{i=1}^n \int_0^t \sigma_i(v,T,f(v,T))\, dW_i(v)$$

$$\text{for all} \quad 0 \leqslant t \leqslant T$$

where

$$\alpha(v,T,\omega) \equiv -\sum_{i=1}^n \sigma_i(v,T,f(v,T)) \left[\phi_i(v) - \int_v^T \sigma_i(v,y,f(t,y))\, dy \right]$$

$$\text{for all} \quad T \in [0,\tau],$$

$\sigma_i \colon \{(t,S) : 0 \leqslant t \leqslant S \leqslant T\} \times R \to R$ is jointly measurable and satisfies

$\int_0^T \sigma_i(t,T,f(t,T))^2\, dt < +\infty$ a.e. Q for $i = 1,\ldots,n$ and

$\phi_i \colon \Omega \times [0,\tau] \to R$ is a bounded predictable process for $i = 1,\ldots,n$.

We now study sufficient conditions on the volatility functions such that strong solutions to this class of stochastic differential equations exist. The continuous time analogue of Ho and Lee's (1986) model as given in expression (28) is a special case of this theorem. The example of a proportional volatility function is also provided below to show that additional hypotheses are needed.

A key step in proving the existence theorem is the following lemma, which asserts that the existence of a class of forward rate processes in the initial economy is guaranteed if and only if it can be guaranteed in an "equivalent risk neutral economy."

LEMMA 1—Existence in an Equivalent Risk Neutral Economy:

$$(43) \qquad \begin{array}{l}[\textit{The processes } \{f(t,T) \colon T \in [0,\tau]\} \textit{ satisfy } (42) \textit{ with} \\ \gamma_i(t; S_1,\ldots,S_n) = \phi_i(t) \textit{ for all } 0 \leqslant t < S < \ldots < S_n \leqslant \tau \quad \textit{and} \\ i = 1,\ldots,n]\end{array}$$

if and only if

$$(44) \qquad \begin{array}{l}[\textit{The process } \{\tilde{\alpha}(\cdot,T) \colon T \in [0,\tau]\} \textit{ defined by } \tilde{\alpha}(t,T) = \\ \sum_{i=1}^n \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,v,f(t,v))\, dv \textit{ for all } T \in [0,\tau] \\ \textit{satisfies } (42) \textit{ with } \tilde{\alpha}(t,T) \textit{ replacing } \alpha(t,T), \textit{ } \tilde{W}_i(t) \textit{ replacing} \\ W_i(t) \textit{ where } \tilde{W}_i(t) \equiv W_i(t) - \int_0^t \phi_i(y)\, dy \textit{ is a Brownian mo-} \\ \textit{tion with respect to } [(\Omega, F, \tilde{Q}), \{F_t \colon t \in [0,\tau]\}, \textit{ and } \tilde{Q} \textit{ replac-} \\ \textit{ing } Q \textit{ where } d\tilde{Q}/dQ = \exp\{\sum_{i=1}^n \int_0^T \phi_i(t)\, dW_i(t) - \\ (1/2)\sum_{i=1}^n \int_0^T \phi_i(t)^2\, dt\}].\end{array}$$

PROOF: A straightforward application of Girsanov's Theorem.   *Q.E.D.*

Combined with this, the next lemma generates our existence theorem given in Proposition 4.

LEMMA 2—*Existence of Forward Rate Processes: Let* σ_i: $\{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times R \to R$ *for* $i = 1, \ldots, n$ *be Lipschitz continuous in the last argument, nonnegative, and bounded. Let* $(\Omega, F, \tilde{Q})$ *be any equivalent probability space with* $\{\tilde{W}_1(t), \ldots, \tilde{W}_n(t): t \in [0, \tau]\}$ *independent Brownian motions; then, there exists a jointly continuous* $f(\cdot, \cdot)$ *satisfying* (42) *with* $\tilde{W}_i(t)$ *replacing* $W_i(t)$ *and*

$$\tilde{\alpha}(t,T) = \sum_{i=1}^n \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,v,f(t,v)) \, dv$$

for all $T \in [0, \tau]$

replacing $\alpha(t,T)$.

The proof of this lemma is contained in Morton (1988). The hypotheses of Lemma 2 differ from the standard hypotheses guaranteeing the existence of strong solutions to stochastic differential equations due to the boundedness condition on the volatility functions.

PROPOSITION 4—*Existence of Arbitrage-Free Forward Rate Drift Processes: Let* ϕ_i: $[0, \tau] \times \Omega \to R$ *be bounded predictable processes for* $i = 1, \ldots, n$. *Let* σ_i: $\{(t,s): 0 \leqslant t \leqslant s \leqslant T\} \times R \to R$ *for* $i = 1, \ldots, n$ *be Lipschitz continuous in the last argument, nonnegative, and bounded; then, there exists a jointly continuous forward rate process satisfying condition* (42).

By appending the nonsingularity condition C.5, this proposition provides sufficient conditions guaranteeing the existence of a class of forward rate processes satisfying conditions C.1–C.6. This set of sufficient conditions is easily verified in applications.

To show that the boundedness condition in Proposition 4 cannot be substantially weakened, we consider the special case of a single Brownian motion where $\sigma_1(t, T, f(t, T)) \equiv \sigma \cdot f(t, T)$ for a fixed constant $\sigma > 0$. This volatility function is positive and Lipschitz continuous, but not bounded.

For this volatility function, the no arbitrage condition of Proposition 4 with $\phi_i(t) \equiv 0$ implies that the forward rate process must satisfy

$$(45) \qquad f(t,T) = f(0,T) \exp\left\{ \int_0^t \int_u^T f(u,v) \, dv \, du \right\} \exp\{-\sigma^2 t/2 + \sigma W(t)\}$$

for all $T \in [0, \tau]$ and $t \in [0, T]$.

Unfortunately, it can be shown (see Morton (1988)) that there is no finite valued solution to expression (45). In fact, it can be shown that under (45), in finite time, forward rates explode with positive probability for the martingale measure, and hence for any equivalent probability measure. Infinite forward rates generate zero bond prices and hence arbitrage opportunities.

The forward rate process given in (45) is in some ways the simplest model consistent with nonnegative forward rates. The incompatibility of this process with arbitrage free bond prices raises the issue as to the general existence of a drift process $\{\alpha(\cdot, T): T \in [0, \tau]\}$ satisfying conditions C.1–C.6, *and* with non-negative forward rates. This existence issue is resolved through an example.

This example can be thought of as a combination of the two previous examples. When forward rates are "small" the process has a proportional volatility, and when forward rates are "large" it has a constant volatility. Intuitively, as shown below, rates cannot fall below zero nor explode. Formally, consider a single Brownian motion process with $\sigma_1(t, T, f(t, T)) = \sigma \min(f(t, T), \lambda)$ for $\sigma, \lambda > 0$ positive constants. This volatility function is positive, Lipschitz continuous, and bounded; thus, for an arbitrary initial forward rate curve Proposition 4 guarantees the existence of a jointly continuous $f(t, T)$ which solves

$$(46) \qquad df(t, T) = \sigma \min(f(t, T), \lambda) \left(\int_t^T \sigma \min(f(t, s), \lambda) \, ds \right) dt$$

$$+ \sigma \min(f(t, T), \lambda) \, dW(t).$$

The following proposition guarantees that this forward rate process remains positive for any strictly positive initial forward rate curve.

PROPOSITION 5—A Nonnegative Forward Rate Process: *Given $f(t, T)$ solves expression (46) and given an arbitrary initial forward rate curve $f(0, t) = I(t) > 0$ for all $t \in [0, \tau]$, then with probability one, $f(t, T) \geqslant 0$ for all $T \in [0, \tau]$ and $t \in [0, \tau]$.*

PROOF: In the Appendix.

Since the forward rate process is a mixture of the constant volatility and proportional volatility models, it is easy to see (using expression (46)) that the forward rate drifts $\{\alpha(\cdot, T): T \in [0, \tau]\}$ will be dependent upon the path of the Brownian motion. Another forward rate process consistent with nonnegative forward rates is provided in the next section.

8. THE EQUILIBRIUM PRICING VERSUS THE ARBITRAGE PRICING METHODOLOGY

The crucial difference between our methodology for pricing contingent claims on the term structure of interest rates and that of Cox, Ingersoll, and Ross (1985) (CIR) is the difference between the arbitrage free pricing methodology and that of equilibrium pricing, respectively. To clarify the relationship between these approaches, we illustrate how to describe (or model) the equilibrium determined CIR square root model in our framework. The CIR model is based on a single state variable, represented by the spot interest rate $r(t)$ for $t \in [0, \tau]$.

The spot rate is assumed to follow a square root process

$$(47) \qquad dr(t) = K(\theta(t) - r(t)) \, dt + \sigma\sqrt{r(t)} \, dW(t)$$

where $r(0), K, \sigma$ are strictly positive constants, $\theta: [0, \tau] \to (0, +\infty)$ is a continuous function of time, $\{W(t): t \in [0, \tau]\}$ is a standard Wiener process initialized at zero, and $2K\theta(t) \geqslant \sigma^2$ for all $t \in [0, \tau]$.

The condition that $2K\theta(t) \geqslant \sigma^2$ for all $t \in [0, \tau]$ guarantees that zero is an inaccessible boundary for spot rates. Although this stochastic differential equation has a solution (see Feller (1951)), an explicit representation is unavailable. In equilibrium, CIR show that the equilibrium bond dynamics are:

$$(48) \qquad dP(t, T) = r(t)\big[1 - \lambda \bar{B}(t, T)\big] P(t, T) \, dt$$
$$- \bar{B}(t, T) P(t, T) \sigma\sqrt{r(t)} \, dW(t)$$

where λ is a constant,

$$\bar{B}(t, T) = 2(e^{\gamma(T-t)} - 1) / \big[(\gamma + K + \lambda)(e^{\gamma(T-t)} - 1) + 2\gamma\big],$$
$$\text{and} \quad \gamma = \big((K + \lambda)^2 + 2\sigma^2\big)^{1/2}.$$

The parameter λ is related to the market price of risk, $\phi(t) = -\lambda\sqrt{r(t)}/\sigma$. The market price of risk is restricted in equilibrium to be of this particular functional form. CIR solve for the bond price process, and from this one can deduce the forward rate process:

$$(49) \qquad f(t, T) = r(t)\big(\partial\bar{B}(t, T)/\partial T\big) + K\int_t^T \theta(s)\big(\partial\bar{B}(s, T)/\partial T\big) \, ds.$$

Given its parameters, CIR's model has a predetermined functional form for the forward rate process at time 0 given by expression (49). To match any arbitrary, but given initial forward rate curve, CIR suggest that one "inverts" expression (49) when $t = 0$ for $\{\theta(t): t \in [0, \tau]\}$ to make the spot rate process's parameters implicitly determined by the initial forward rate curve; see CIR (p. 395).

CIR never prove that such an inversion is possible, i.e., that a "solution" $\{\theta(t): t \in [0, \tau]\}$ exists to expression (49) with $t = 0$. In fact, if $\{\partial f(0, T)/\partial T: T \in [0, \tau]\}$ exists and is continuous, then there is a *unique* continuous solution.[10] Using standard procedures, one can show that the solution $\{\theta(s): s \in [0, \tau]\}$ to

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[10] Let $\{f(0, T): T \in [0, \tau]\}$ be twice continuously differentiable. Note that $\bar{B}_T(t, T) \equiv \partial\bar{B}(t, T)/\partial T$ and $\bar{B}_{TT}(t, T) \equiv \partial^2\bar{B}_T(t, T)/\partial T$ are continuous on $0 \leqslant t \leqslant T \leqslant \tau$ with $\bar{B}(t, t) = 0$ and $\bar{B}_T(t, T) = 1$. Expression (49) with $t = 0$ is

$$f(0, T) = r(0)\bar{B}_T(0, T) + K\int_0^T \theta(s)\bar{B}_T(s, T) \, ds.$$

Differentiating with respect to T yields

$$\big[\partial f(0, T)/\partial T - r(0)\bar{B}_{TT}(0, T)\big]/K = \theta(T) + \int_0^T \theta(s)\big(K\bar{B}_{TT}(s, T)\big) \, ds.$$

This is a Volterra integral equation of the second kind with a unique continuous solution $\theta(\cdot)$ on $[0, \tau]$; see Taylor and Lay (1980, p. 200).   Q.E.D.

equation (49) with $t = 0$ can be approximated to any order of accuracy desired (see Taylor and Lay (1980, pp. 196–201)). Nonetheless, the CIR model is not consistent with all initial forward rate curves. This is due to the requirement that $2K\theta(t) \geqslant \sigma^2$ for all $t \in [0, \tau]$. Indeed, consider expression (49) initialized at $t = 0$. Substitution of the inaccessible boundary condition into it, and simplification yields

$$(50) \qquad f(0, T) \geqslant r(0)\partial\bar{B}(0, T)/\partial T + \sigma^2\bar{B}(0, T)/2.$$

Not all initial forward rate curves will satisfy this expression.

Hence, in our framework we have that CIR's term structure model can be written as

$$(51) \qquad df(t, T) = r(t)K(\partial^2\bar{B}(t, T)/\partial t\partial T - \partial\bar{B}(t, T)/\partial T)\, dt$$
$$+ (\partial\bar{B}(t, T)/\partial T)\sigma\sqrt{r(t)}\, dW(t)$$

where

$$r(t) = \left[f(t, T) - K\int_t^T \theta(s)(\partial\bar{B}(s, T)/\partial T)\, ds \right] \Big/ (\partial\bar{B}(t, T)/\partial T),$$

$\{f(0, T): T \in [0, \tau]\}$ is a continuously differentiable, fixed, initial forward rate curve, and $\theta: [0, \tau] \to (0, +\infty)$ is the unique continuous solution to expression (49) with $t = 0$.

To apply our analysis based on expressio (51), we need to guarantee that conditions C.1–C.6 are satisfied. Recall that conditions C.1–C.3 guarantee that the bond price process satisfies expression (8). Next, given expression (8), conditions C.4 and C.5 guarantee that for any vector of bonds $\{S_1, \ldots, S_n\}$ an equivalent martingale measure exists and is unique. Finally, condition C.6 ensures that the martingale measure is identical across all vectors of bonds. These conditions are sufficient to price all contingent claims when starting from forward rates.

Alternatively, CIR exogenously specify the spot rate process. Consequently, using different methods, they are able to guarantee that the bond price process satisfies expression (8). Hence, we don't need to check sufficient conditions C.1–C.3, since expression (8) is the starting point of our analysis. Next, given that the bond prices are generated by an equilibrium with a single Brownian motion, conditions C.4, C.5, and C.6 are easily verified. In fact, to check condition C.6 one can easily verify that expression (18) is satisfied.

Given the form of the CIR model as in expression (51), we can now proceed directly as in Section 5 to price contingent claims. This analysis will generate the *identical* contingent claim values as in CIR subject to the determination of $\{\theta(s): s \in [0, \tau]\}$. Note that the forward rate's quadratic variation

$$\langle f(t, T)\rangle_t = \int_0^t \left[(\partial\bar{B}(s, T)/\partial T)\sigma\sqrt{r(s)} \right]^2 ds$$

depends on the parameters $\lambda, \sigma, K, r(0)$, and $\{f(0, T): T \in [0, \tau]\}$. The parameter λ, however, is functionally related to the market price of risk. This makes

contingent claim valuation explicitly dependent on this parameter as well (e.g., see CIR (expression (32), p. 396)).

With this analysis behind us, we can now discuss some differences between the two pricing approaches. First, CIR's model fixed a particular market price for risk and endogenously derived the stochastic process for forward rates. In contrast, our approach takes the stochastic process for forward rates as a given (it could be from an equilibrium model) and prices contingent claims from it.

9. SUMMARY

This paper presents a new methodology for pricing contingent claims on the term structure of interest rates. Given an initial forward rate curve and a mechanism which describes how it fluctuates, we develop an arbitrage pricing model which yields contingent claim valuations which do not explicitly depend on the market prices for risk.

For practical applications, we specialize our abstract economy and study particular examples. For these examples, closed form solutions are obtained for bond options depending only upon observables and the forward rate volatilities. These models are testable and their empirical verification awaits subsequent research.

The paper can be generalized by imbedding our term structure model into the larger economy of Harrison and Pliska (1981), which includes trading in alternative risky assets (e.g., stocks) generated by additional (perhaps distinct) independent Brownian motions. Our model provides a consistent structure for the interest rate process employed therein. This merging of the two analyses can be found in Amin and Jarrow (1989).

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APPENDIX

PROOF OF EXPRESSION (8): Before proving expression (8), we need to state a generalized form of Fubini's theorem for stochastic integrals. This proof of this theorem follows Ikeda and Watanabe (1981, p. 116) very closely, and is available from the authors upon request.

LEMMA 0.1: *Let (Ω, F, Q) be a probability space. Let (F_t) be a reference family satisfying the usual conditions and generated by a Brownian motion $\{W(t): t \in [0, \tau]\}$.*

Let $\{\Phi(t, a, \omega): (t, a) \in [0, \tau] \times [0, \tau]\}$ *be a family of real random variables such that*:

(i) $\quad\quad ((t, \omega), a) \in \{([0, \tau] \times \Omega) \times [0, \tau]\} \to \Phi(t, a, \omega)$

is $L \times B[0, \tau]$ *measurable where L is the predictable σ-field*;

(ii) $\quad\quad \int_0^t \Phi^2(s, a, \omega)\, ds < +\infty \quad a.e. \quad for\ all \quad t \in [0, \tau]$;

(iii) $\quad\quad \int_0^t \left\{ \int_0^\tau \Phi(s, a, \omega)\, da \right\}^2 ds < +\infty \quad a.e. \quad for\ all \quad t \in [0, \tau]$.

If $t \to \int_0^\tau \{ \int_0^t \Phi(s, a, \omega) dW_s \}\, da$ *is continuous a.e., then*

$$\int_0^t \left\{ \int_0^\tau \Phi(s, a, \omega)\, da \right\} dW_s = \int_0^\tau \left\{ \int_0^t \Phi(s, a, \omega)\, dW_s \right\} da \quad for\ all \quad t \in [0, \tau].$$

COROLLARY 1: *Let the hypotheses of Lemma 0.1 hold. Define*

$$\Phi(s, a, \omega) = \begin{cases} 0 & if\ (s, a) \notin [0, t] \times [t, \tau], \\ \sigma(s, a, \omega) & if\ (s, a) \in [0, t] \times [t, \tau]. \end{cases}$$

Then

$$\int_0^y \left\{ \int_t^T \sigma(s, a, \omega)\, da \right\} dW(s) = \int_t^T \left\{ \int_0^y \sigma(s, a, \omega)\, dW(s) \right\} da \quad for\ all \quad y \in [0, t].$$

COROLLARY 2: *Let the hypotheses of Lemma 0.1 hold. Define*

$$\Phi(s, a, \omega) = \begin{cases} 0 & if\ (s, a) \notin [0, t] \times [0, t], \\ \sigma(s, a, \omega) 1_{s \leqslant a} & if\ (s, a) \in [0, t] \times [0, t]. \end{cases}$$

Then

$$\int_0^y \left[\int_s^t \sigma(s, a, \omega)\, da \right] dW(s) = \int_0^t \left[\int_0^{a \wedge y} \sigma(s, a, \omega)\, dW(s) \right] da \quad for\ all \quad y \in [0, t].$$

Now we can proceed with the proof of expression (8).

$$\ln P(t, T) = -\int_t^T f(0, y)\, dy - \int_t^T \left[\int_0^T \alpha(v, y)\, dv \right] dy - \sum_{i=1}^n \int_t^T \left[\int_0^t \sigma_i(v, y)\, dW_i(v) \right] dy.$$

Note that the integrals are well-defined by conditions C.1, C.2.

By condition C.2, we can apply the standard Fubini's theorem. By conditions C.1–C.3 we can apply Corollary 1 with $y = t$ to get

$$\ln P(t, T) = -\int_t^T f(0, y)\, dy - \int_0^t \left[\int_t^T \alpha(v, y)\, dv \right] dy - \sum_{i=1}^n \int_0^t \left[\int_t^T \sigma_i(v, y)\, dy \right] dW_i(v).$$

Adding and subtracting the same terms yields

$$= -\int_0^T f(0, y)\, dy - \int_0^t \left[\int_v^T \alpha(v, y)\, dy \right] dv - \sum_{i=1}^n \int_0^t \left[\int_v^T \sigma_i(v, y)\, dy \right] dW_i(v)$$

$$+ \int_0^t f(0, y)\, dy + \int_0^t \left[\int_v^t \alpha(v, y)\, dy \right] dv + \sum_{i=1}^n \int_0^t \left[\int_v^t \sigma_i(v, y)\, dy \right] dW_i(v).$$

But, expression (5) yields with Corollary 2 (by C.1–C.3) for $y = t$:

$$\ln P(t, T) = \ln P(0, T) + \int_0^t r(y) \, dy - \int_0^t \left[\int_v^T \alpha(v, y) \, dv \right] dy$$

$$- \sum_{i=1}^n \int_0^t \left[\int_v^T \sigma_i(v, y) \, dy \right] dW_i(v).$$

This completes the proof.   Q.E.D.

PROOF OF PROPOSITION 1: This proposition is proved through the following two lemmas. The straightforward proofs of these lemmas are omitted.

LEMMA 1.1: *Assume C.1–C.3 hold for fixed* $\{S_1, \ldots, S_n\} \in [0, \tau]$ *such that* $0 < S_1 < \ldots < S_n \leqslant \tau$. *Define*

$$X(t, y) = \int_0^t b(v, y) \, dv + \sum_{i=1}^n \int_0^t a_i(v, y) \, dW_i(v)$$

$$\text{for all} \quad t \in [0, y] \quad \text{and} \quad y \in \{S_1, \ldots, S_n\}.$$

Then $\gamma_i \colon \Omega \times [0, \tau] \to R$ *for* $i = 1, \ldots, n$ *satisfies*:

(i) $\begin{bmatrix} b(t, S_1) \\ \vdots \\ b(t, S_n) \end{bmatrix} + \begin{bmatrix} a_1(t, S_1) & \ldots & a_n(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_n) & \ldots & a_n(t, S_n) \end{bmatrix} \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ a.e. $\lambda \times Q$;

(ii) $\int_0^{S_1} \gamma_i(v)^2 \, dv < +\infty$ a.e. Q for $i = 1, \ldots, n$;

(iii) $E\left(\exp\left\{ \sum_{i=1}^n \int_0^{S_1} \gamma_i(v) \, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^{S_1} \gamma_i(v)^2 \, dv \right\} \right) = 1$;   and

(iv) $E\left(\exp\left\{ \sum_{i=1}^n \int_0^{S_1} [a_i(v, y) + \gamma_i(v)] \, dW_i(v) \right. \right.$

$$\left. \left. - (1/2) \sum_{i=1}^n \int_0^{S_1} [a_i(v, y) + \gamma_i(v)]^2 \, dv \right\} \right) = 1$$

for $y \in S_1, \ldots, S_n$ *if and only if: there exists a probability measure* $\tilde{Q}_{S_1 \ldots S_n}$ *such that*

(a) $d\tilde{Q}_{S_1 \ldots S_n}/dQ = \exp\left\{ \sum_{i=1}^n \int_0^{S_1} \gamma_i(v) \, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^{S_1} \gamma_i(v)^2 \, dv \right\}$;

(b) $\tilde{W}_i^{S_1 \ldots S_n}(t) = W_i(t) - \int_0^t \gamma_i(v) \, dv$ *are Brownian motions on*

$$\left\{ (\Omega, F, \tilde{Q}_{S_1 \ldots S_n}), \{F_t \colon t \in [0, S_1]\} \right\} \quad \text{for} \quad i = 1, \ldots, n;$$

(c) $\begin{pmatrix} dX(t, S_1) \\ \vdots \\ dX(t, S_n) \end{pmatrix} = \begin{pmatrix} a_1(t, S_1) & \ldots & a_n(t, S_1) \\ \vdots & & \vdots \\ a_1(t, S_n) & \ldots & a_n(t, S_n) \end{pmatrix} \begin{pmatrix} d\tilde{W}_1^{S_1 \ldots S_n}(t) \\ \vdots \\ d\tilde{W}_n^{S_1 \ldots S_n}(t) \end{pmatrix}$ *for* $t \in [0, S_1]$;

and

(d) $Z(t, S_i)$ *are martingales on* $\left\{ (\Omega, F, \tilde{Q}_{S_1 \ldots S_n}), \{F_t \colon t \in [0, S_1]\} \right\}$ *for* $i = 1, \ldots, n$.

LEMMA 1.2: *Assume* C.1–C.3 *hold for fixed* $\{S_1 \ldots S_n\} \in [0, \tau]$ *such that* $0 < S_1 < \ldots < S_n \leqslant \tau$. *Define*

$$X(t, y) = \int_0^t b(v, y)\, dv + \sum_{i=1}^n \int_0^t a_i(v, y)\, dW_i(v)$$

for all $t \in [0, y]$ *and* $y \in \{S_1 \ldots S_n\}$.

There exists a probability measure $\overline{Q}$ *equivalent to* Q *such that* $Z(t, S_i)$ *are martingales on* $\{(\Omega, F, \overline{Q}), \{F_i: t \in [0, S_1]\}\}$ *for all* $i = 1, \ldots, n$ *if and only if there exists* $\gamma_i: \Omega \times [0, \tau] \to R$ *for* $i = 1, \ldots, n$ *and a probability measure* $\overline{Q}_{S_1 \ldots S_n}$ *such that* (a), (b), (c), *and* (d) *of Lemma 1.1 hold.*

PROOF OF PROPOSITION 2: The proof of this proposition requires the following two lemmas.

LEMMA 2.1: *Fix* $S < \tau$. *Let* $\beta_i: \Omega \times [0, \tau] \to R$ *for* $i = 1, \ldots, n$ *be such that* $\int_0^S \beta_i^2(v)\, dv < +\infty$ *a.e.* Q. *Define*

$$T_m \equiv \inf \left\{ t \in [0, S]: E\left(\exp\left\{ (1/2) \sum_{i=1}^n \int_0^t \beta_i(v)^2\, dv \right\} \right) \geqslant m \right\},$$

$$M^m(t) \equiv \exp\left\{ \sum_{i=1}^n \int_0^{\min(T_m, t)} \beta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^{\min(T_m, t)} \beta_i(v)^2\, dv \right\}.$$

Then

$$E\left(\exp\left\{ \sum_{i=1}^n \int_0^S \beta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^S \beta_i(v)^2\, dv \right\} \right) = 1$$

if and only if $\{M^m(S)\}_{m=1}^\infty$ *are uniformly integrable.*

PROOF: Define $\beta_i^m(v) \equiv \beta_i(v) 1_{\{v \leqslant T_m\}}$; then by Elliott (1982, p. 165),

$$M^m(t) = \exp\left\{ \sum_{i=1}^n \int_0^S \beta_i^m(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^S \beta_i^m(v)^2\, dv \right\}$$

is a supermartingale. Since

$$E\left(\exp\left\{ (1/2) \sum_{i=1}^n \int_0^{T^m} \beta_i(v)^2\, dv \right\} \right) = E\left(\exp\left\{ (1/2) \sum_{i=1}^n \int_0^S \beta_i^m(v)^2\, dv \right\} \right) \leqslant m,$$

by Elliott (1982, p. 178) $E(M^m(S)) = 1$. Hence, $M^m(t)$ is a martingale. Note

$$\lim_{m \to \infty} M^m(S) = \exp\left\{ \sum_{i=1}^n \int_0^S \beta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^S \beta_i^2(v)\, dv \right\}$$

with probability one since $T_m \to S$ with probability one. Observe that $\{M^m(S)\}_{m=1}^\infty$ is a martingale with respect to $m = 1, 2, \ldots$ because $\sup E(M^m(S)) = 1 < +\infty$ and $E(M^{m+1}(S)|F_{\min(S, T_m)} = M^{m+1}(\min(S, T_m))$ by the Optional Stopping Theorem (since $T_m \leqslant S$, see Elliott (1982, p. 17)) $= M^m(S)$ by the definition of M^m.

Step 1: Suppose $\{M^m(S)\}_{m=1}^\infty$ are uniformly integrable; then

$$\lim_{m \to \infty} M^m(S) = \exp\left\{ \sum_{i=1}^n \int_0^S \beta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^S \beta_i(v)^2\, dv \right\}$$

in L^1 (see Elliott (1982, p. 22)), and thus

$$E\left(\exp\left\{ \sum_{i=1}^n \int_0^S \beta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^n \int_0^S \beta_i(v)^2\, dv \right\} \right) = \lim_{m \to \infty} E(M^m(S)).$$

But, $E(M^m(S)) = 1$. This completes the proof in one direction.

Step 2: Conversely, suppose

$$E\left(\exp\left\{\sum_{i=1}^{n}\int_{0}^{S}\beta_i(v)\,dW_i(v)-(1/2)\sum_{i=1}^{n}\int_{0}^{S}\beta_i(v)^2\,dv\right\}\right)=1.$$

We know

$$E\left(\exp\left\{\sum_{i=1}^{n}\int_{0}^{S}\beta_i(v)\,dW_i(v)-(1/2)\sum_{i=1}^{n}\int_{0}^{S}\beta_i(v)^2\,dv\right\}\middle|F_{T_n}\right)=M^m(S),$$

hence $M^m(S)$ is uniformly integrable.   Q.E.D.

LEMMA 2.2: *Assume conditions C.1–C.3 hold for fixed $S_1,\ldots,S_n\in[0,\tau]$ such that $0<S_1<\ldots<S_n\leq\tau$. Suppose conditions (i), (ii), (iii), and (iv) of Lemma 1.1 hold; then $\gamma_i(t)$ for $i=1,\ldots,n$ satisfying (i), (ii), (iii), and (iv) are unique (up to $\lambda\times Q$ equivalence) if and only if*

$$A(t)\equiv\begin{pmatrix}a_1(t,S_1)&\ldots&a_n(t,S_1)\\&\vdots&\\a_1(t,S_n)&\ldots&a_n(t,S_n)\end{pmatrix}$$

is singular with $(\lambda\times Q)$ measure zero.

PROOF: Suppose $A(t)$ is singular with $(\lambda\times Q)$ measure zero. Then, by condition (i) of Lemma 1.1, $\gamma_i(t)$ for $i=1,\ldots,n$ are unique (up to $\lambda\times Q$ equivalence).

Conversely, suppose $\Sigma\equiv\{t\times\omega\in[0,S]\times\Omega:A(t)$ is singular$\}$ has $(\lambda\times Q)(\Sigma)>0$. We want to show that the functions satisfying conditions (i), (ii), (iii), and (iv) are not unique. First, by hypothesis, we are given a vector of functions $(\gamma_1(t),\ldots,\gamma_n(t))$ satisfying (i), (ii), (iii), and (iv).

Step 1: Show that there exists a bounded, adapted, measurable vector of functions $(\delta_1(t),\ldots,\delta_n(t))$ nonzero on Σ such that

$$A(t)\begin{bmatrix}\delta_1(t)\\\vdots\\\delta_n(t)\end{bmatrix}=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}\quad\text{and}$$

$$g(t)\equiv\exp\left\{\sum_{i=1}^{n}\int_{0}^{t}\delta_i(v)\,dW_i(v)-\sum_{i=1}^{n}\left(\int_{0}^{t}\delta_i(v)\gamma_i(v)\,dv\right)-(1/2)\sum_{i=1}^{n}\int_{0}^{t}\delta_i^2(v)\,dv\right\}$$

is bounded a.e. Q. Let $\Sigma_i=\{(t,w):A(t)$ has rank $i\}$. Σ_i is a measurable set. Then $\Sigma=\bigcup_{i=1}^{n-1}\Sigma_i$ and $\Sigma_i\cap\Sigma_j=\varnothing$ for $i\neq j$. Fix $\eta>0$. On each set Σ_i, set $\delta_i^\eta(t)$ for $i=1,\ldots,n$ equal to a solution to

$$A(t)\begin{bmatrix}\delta_1^\eta(t)\\\vdots\\\delta_n^\eta(t)\end{bmatrix}=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$$

such that $\delta_i^\eta(t)$ are bounded by $\min(\eta,1/\gamma_i(t)$ for $i=1,\ldots,n)$. Finally, let $\delta_i^\eta(t)$ be zero on Σ^c for $i=1,\ldots,n$. Note that we shall always interpret superscripts on δ as the upper bound on the process, and not as an exponent.

By construction, $\delta_i^\eta(t)$ are adapted, measurable, bounded by η, and

$$\left|\sum_{i=1}^{n}\int_{0}^{t}\delta_i^\eta(v)\gamma_i(v)\,dv+(1/2)\sum_{i=1}^{n}\int_{0}^{t}\delta_i^\eta(v)^2\,dv\right|\leq[2+\eta^2]\tau\quad\text{a.e. }Q.$$

Let $\alpha=\inf\{j\in\{1,2,3,\ldots\}:(1/2)^{2j}S<1\}$. Define inductively the stopping times:

$$\tau_1=\inf\left\{t\in[0,S]:\sum_{i=1}^{n}\int_{0}^{t}\delta_i^{(1/2)^\alpha}(v)\,dW_i(v)\geq(1/2)\right\},$$

$$\tau_j=\inf\left\{t\in[0,S]:\sum_{i=1}^{n}\int_{\tau_{j-1}}^{t}\delta_i^{(1/2)^{2j+\alpha}}(v)\,dW_i(v)\geq(1/2)^j\right\}\quad\text{for}\quad j=2,3,4,\ldots.$$

We claim that $Q(\lim_{j \to \infty} \tau_j = S) = 1$. Indeed,

$$Q(\tau_j < S | F_{\tau_{j-1}}) \leq Q\left(\left| \sum_{i=1}^{n} \int_{\tau_{j-1}}^{S} \delta_i^{(1/2)^{2j+\alpha}}(v) \, dW_i(v) \right| \geq (1/2)^j | F_{\tau_{j-1}} \right)$$

$$\leq \frac{1}{[1/2]^{2j}} \int_{\tau_{j-1}}^{S} \left(\delta_i^{(1/2)^{2j+\alpha}}(v) \right)^2 dv \quad \text{by Chebyshev's inequality,}$$

$$\leq \frac{1}{[1/2]^{2j}} \left([1/2]^{2j+\alpha} \right)^2 S < (1/2)^{2j} \quad \text{by choice of } \alpha.$$

Hence $E[Q(\tau_j < S | F_{\tau_{j-1}})] = Q\ (\tau_j < S) < (1/2)^{2j}$. Since

$$Q\left(\lim_{j \to \infty} \tau_j = S \right) = 1 - Q\left(\lim_{j \to \infty} \tau_j < S \right) \quad \text{and}$$

$$Q\left(\lim_{j \to \infty} \tau_j < S \right) < Q\left(\bigcap_{j=1}^{\infty} (\tau_j < S) \right) \leq \inf\left\{ Q(\tau_j < S): \quad j = 1, 2, 3, \ldots \right\} = 0,$$

this proves the claim.

Set

$$\delta_i(t) = \sum_{j=0}^{\infty} 1_{[\tau_j, \tau_{j+1}]}^{(t)} \delta_i^{(1/2)^{2j+\alpha}}(t) \quad \text{for} \quad i = 1, \ldots, n.$$

$\delta_i(t)$ is bounded, adapted, and measurable and satisfies

$$A(t) \begin{bmatrix} \delta_1(t) \\ \vdots \\ \delta_n(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{a.e.} \quad \lambda \times Q.$$

Note that for all $t \in [0, S]$,

$$\left| \sum_{i=1}^{n} \int_0^t \delta_i(t) \, dW_i(t) \right| \leq \sum_{j=0}^{\infty} (1/2)^j = 2,$$

so

$$\exp\left\{ \sum_{i=1}^{n} \int_0^t \delta_i(t) \, dW_i(t) - \sum_{i=1}^{n} \int_0^t \delta_i(v) \gamma_i(v) \, dv - (1/2) \sum_{i=1}^{n} \int_0^t \delta_i^2(v) \, dv \right\}$$

is bounded a.e. $\lambda \times Q$. This completes Step 1.

Step 2: Show that $(\gamma_1(t) + \delta_1(t), \ldots, \gamma_n(t) + \delta_n(t))$ satisfies conditions (i), (ii), (iii), and (iv) of Lemma 1.1. This step will complete the proof.

Conditions (i) and (ii) are obvious. To obtain condition (iii), define

$$T_m = \inf\left\{ t \in [0, T]: E\left(\exp\left\{ (1/2) \sum_{i=1}^{n} \int_0^t (\gamma_i(t) + \delta_i(t))^2 \, dv \right\} \geq m \right) \right\},$$

$$M^m(t) = \exp\left\{ \sum_{i=1}^{n} \int_0^{\min(T_m, t)} [\gamma_i(v) + \delta_i(v)] \, dW_i(v) \right.$$

$$\left. - (1/2) \sum_{i=1}^{n} \int_0^{\min(T_m, t)} [\gamma_i(v) + \delta_i(v)]^2 \, dv \right\}.$$

By Lemma 2.1, we need to show that $M^m(S)$ is uniformly integrable. But

$$
M^m(s) = \exp\left\{ \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \gamma_i(v)\, dW_i(v) \right.
$$

$$
\left. - (1/2) \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \gamma_i(v)^2\, dv \right\} \exp\left\{ \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \delta_i(v)\, dW_i(v) \right.
$$

$$
\left. - (1/2) \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \left[2\gamma_i(v)\delta_i(v) + \delta_i(v)^2 \right] dv \right\}.
$$

Since

$$
\exp\left\{ \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \delta_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \left[2\gamma_i(v)\delta_i(v) + \delta_i(v)^2 \right] dv \right\}
$$

is bounded,

$$
0 \leqslant M^m(S) \leqslant K \exp\left\{ \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \gamma_i(v)\, dW_i(v) - (1/2) \sum_{i=1}^{n} \int_{0}^{\min(S,T_m)} \gamma_i(v)^2\, dv \right\}
$$

$$
\text{for some} \quad K > 0.
$$

By Lemma 2.1 since $\gamma_i(t)$ satisfies (iii), the right hand side is uniformly integrable. By Kopp (1984, p. 29), it can be shown that $M^m(S)$ is uniformly integrable. Finally, an analogous argument used to prove (iii) shows (iv) holds as well.   Q.E.D.

PROOF OF PROPOSITION 5: Fix a T_0. Consider

$$
\eta(t) \equiv -\sigma \min\left(f(t,T_0), \lambda \right) \int_{t}^{T_0} \sigma \min\left(f(t,s), \lambda \right) ds / \sigma \min\left(f(t,T_0), \lambda \right)
$$

$$
= -\int_{t}^{T_0} \sigma \min\left(f(t,s), \lambda \right) ds.
$$

Since $\sigma \min(f(t,s), \lambda)$ is bounded, $\eta(t)$ is bounded. Hence,

$$
E\left(\exp\left\{ (1/2) \int_{0}^{T_0} \eta(t)^2\, dt \right\} \right) < +\infty.
$$

By Girsanov's theorem, there exists an equivalent probability measure $\overline{Q}$ and a Brownian motion $\overline{W}(t)$ such that $df(t,T_0) = \sigma \min(f(t,T_0), \lambda)\, d\overline{W}(t)$. Define $t_0 = \inf\{t \in [0,T_0]: f(t,T_0) = 0\}$. By Karlin and Taylor (1981, Lemma 15.6.2), zero is an unattainable boundary, i.e., $\overline{Q}\{t_0 \leqslant T_0\} = Q\{t_0 \leqslant T_0\} = 0$. Since $f(t,T_0)$ has continuous sample paths, $f(t,T_0) > 0$ a.e. Let $\{T_i: i = 1, 2, 3, \ldots\}$ be the rationals in $[0, \tau]$:

$$
Q\{ f(t,T_i) = 0 \text{ for some } T_i \text{ and some } t \in [0, T_i] \}
$$

$$
= Q\left\{ \bigcup_{i=1}^{\infty} \{ f(t,T_i) = 0 \text{ for some } t\ [0, T_i] \} \right\}
$$

$$
\leqslant \sum_{i=1}^{\infty} Q\{ f(t,T_i) = 0 \text{ for some } t \in [0, T_1] \} = 0.
$$

By the joint continuity of $f(t,T)$, $Q\{ f(t,T) \geqslant 0 \text{ for all } T \in [0,\tau] \text{ and all } t \in [0,T] \} = 1$.   Q.E.D.

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