Financial Econometrics
Financial Risk Management: Value-at-Risk and Expected Shortfall

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Motivation

- Volatility is a measure of risk, but there are other measures of interest.
- An important issue is big losses, i.e., one wants to have an assessment of the probability of having big losses and their amounts.
- This is particularly important from the regulatory view point or from the firm risk management perspective.
- We will focus on losses (big gains are welcome!).

Measures of Risk: Value-at-Risk and Expected Shortfall

- We are interested in computing the losses in the worst cases, i.e., the extreme cases. We have two dominant measures. For a given probability α (in practice, $\alpha = 1,\ 2$ or 5%). Let R_t be the logarithmic return and I_t the information available at time t:
  1. The Value-at-Risk (VaR) is defined as the solution to
  $$P(R_t \leq -VaR_t^{(\alpha)} \mid I_{t-1}) = \alpha.$$
  2. The Expected Shortfall (ES) or TailVar is defined by
  $$ES_t^{(\alpha)} = -E_{t-1}[R_t \mid R_t < -VaR_t^{(\alpha)}].$$
- $-VaR_t^{(\alpha)}$ is the α-quantile of the return.
- The quantities $VaR_t^{(\alpha)}$ and $ES_t^{(\alpha)}$ are positive numbers. $ES_t^{(\alpha)}$ measures the losses while $VaR_t^{(\alpha)}$ focuses on the probabilities.
- $VaR_t^{(\alpha)}$ is more popular but $ES_t^{(\alpha)}$ has better theoretical properties (coherent measure of risk).
- Observe that one can take the same definition for the arithmetic returns.

$$R_t = \mu_{t-1} + \sigma_t z_t, \quad z_t \ i.i.d. \ D(0,1).$$

- Let $F(\cdot)$ and $Q(\cdot)$ be the cumulative and quantile functions of z ($Q = F^{-1}$.)

$$\alpha = P(R_t \leq -VaR_t^{(\alpha)} \mid I_{t-1}) = P\left(\frac{R_t - \mu_{t-1}}{\sigma_t} \leq \frac{-VaR_t^{(\alpha)} - \mu_{t-1}}{\sigma_t} \mid I_{t-1}\right)$$

$$= P\left(z_t \leq \frac{-VaR_t^{(\alpha)} - \mu_{t-1}}{\sigma_t} \mid I_{t-1}\right) = F\left(\frac{-VaR_t^{(\alpha)} - \mu_{t-1}}{\sigma_t}\right).$$

- Hence,

$$-VaR_t^{(\alpha)} = \mu_{t-1} + \sigma_t Q(\alpha).$$

- Financial institutions are regulated. They have to provide (daily) VaRs, often computed with a model. The regulators have to assess whether the VaRs of the bank are violated or not.
- If we observe a time series of past ex-ante VaR forecasts and past ex-post returns, we can define the "hit sequence" of VaR violations as

$$H_t = 1, \text{ if } R_t < -VaR_t^{(\alpha)}$$

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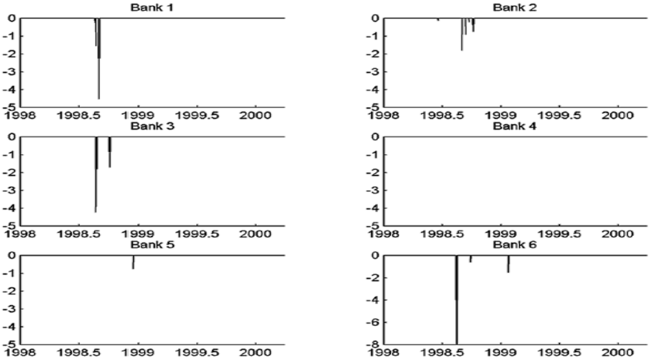
VaR Exceedences from 6 Major Commercial Banks



- Observe clustering and the magnitude of VaR violations.
- When the model is correctly specified, H_t should be a Bernoulli(α) and i.i.d.
- We have three tests:
  1. The unconditional distribution of H_t is Bernoulli(α).
  2. The H_t are i.i.d. and Bernoulli(α).
  3. Add explanatory variables to explain the violations of VaR.

First test: The unconditional distribution of H_t is Bernoulli(α)

- Likelihood:

$$L(p) = \prod_{t=1}^{T}(1-\pi)^{1-H_t}\pi^{H_t} = (1-\pi)^{T_0}\pi^{T_1},$$

where T_0 and T_1 are the numbers of zeros and ones in the sample.

- The (unconditional) MLE estimator of π is $\hat{\pi} = T_1/T$.
- The log-likelihood ratio test is given by

$$LR_{uc} = 2[\log(L(\hat{\pi})) - \log(L(\alpha))] \quad \sim \quad \chi^2(1),$$

under the null.

Second Test: The H_t are i.i.d. and Bernoulli(α)

- We need to define the alternative, i.e., what type of serial correlation we want to test against.
- A simple alternative is a homogeneous Markov chain:

$$\Pi = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix}$$

where $P[H_t = 1 \mid H_{t-1} = 1] = \pi_{11}$, $P[H_t = 1 \mid H_{t-1} = 0] = \pi_{01}$.

- The Likelihood of the Markov chain (ignoring the unconditional distribution of the first observation) is

$$L(\Pi) = (1-\pi_{01})^{T_{00}}\ \pi_{01}^{T_{01}}\ (1-\pi_{11})^{T_{10}}\ \pi_{11}^{T_{11}},$$

where T_{ij} is the number of observations with a j following an i.

- The MLE estimators of π_{01} and π_{11} are

$$\hat{\pi}_{01} = \frac{T_{01}}{T_{00}+T_{01}}, \quad \hat{\pi}_{11} = \frac{T_{11}}{T_{10}+T_{11}},$$

while

$$\hat{\pi}_{00} = 1-\hat{\pi}_{01}, \quad \hat{\pi}_{10} = 1-\hat{\pi}_{11}.$$

- Under independence, one has

$$\pi_{01} = \pi_{11} = \pi$$

and

$$\Pi_0 = \begin{bmatrix} 1-\pi & \pi \\ 1-\pi & \pi \end{bmatrix},$$

while the MLE of π is again $\hat{\pi} = T_1/T$.

- Hence, the LR test of the independence assumption is given by

$$LR_{ind} = 2[\log L(\hat{\Pi}) - \log L(\hat{\Pi}_0)] \quad \sim \quad \chi^2(1),$$

under the null, where

$$\hat{\Pi} = \begin{bmatrix} 1-\hat{\pi}_{01} & \hat{\pi}_{01} \\ 1-\hat{\pi}_{11} & \hat{\pi}_{11} \end{bmatrix} \text{ and } \hat{\Pi}_0 = \begin{bmatrix} 1-\hat{\pi} & \hat{\pi} \\ 1-\hat{\pi} & \hat{\pi} \end{bmatrix}.$$

- However, one may want to test independence and Bernoulli(α) (conditional coverage testing) , i.e. $\pi_{01} = \pi_{11} = \alpha$.
- The test is

$$LR_{cc} = 2[\log L(\hat{\Pi}) - \log L(\alpha)] = LR_{uc} + LR_{ind} \quad \sim \quad \chi^2(2),$$

under the null.

Third Test: Add explanatory variables to explain the violations of VaR

- One has

$$E[H_t \mid I_{t-1}] = \alpha \text{ and } Cov[H_t, x_{t-1}] = 0 \; \forall x_{t-1} \in I_{t-1}.$$

- Hence, consider a vector of explanatory variables x_{t-1} and do the regression

$$H_t = b_0 + x'_{t-1} b_1 + e_{t+1},$$

and test $b_0 = \alpha$ and $b_1 = 0$.

Backtesting of Expected Shortfall

- The Expected Shortfall (ES) is defined by

$$ES_t^{(\alpha)} = -E_{t-1}[R_t \mid R_t < -VaR_t^{(\alpha)}].$$

- Hence,

$$E_{t-1}[R_t + ES_t^{(\alpha)} \mid R_t < -VaR_t^{(\alpha)}] = 0.$$

- Consequently, a simple test is: consider the data violating the VaR. Then, do the regression

$$R_t + ES_t^{(\alpha)} = b_0 + x'_{t-1} b_1 + e_t$$

where x_{t-1} is a vector of explanatory variables, and test $b_0 = 0$ and $b_1 = 0$.

- A potential problem is that one could have few violations.

Additional Issues

- Backtesting of the Conditional Distribution of R_t. By assumption, the variable z_t is i.i.d., i.e.,

$$z_t = \frac{R_t - \mu_{t-1}}{\sigma_t} \;\; \text{i.i.d.} \;\sim\;\; F(\cdot).$$

  Hence,
  $F(z_t) \quad \text{i.i.d.} \;\sim\; \text{Uniform}[0,1] \;\text{ and }\; \Phi^{-1}(F(z_t)) \quad \text{i.i.d.} \;\sim\; \mathcal{N}(0,1).$

- Likewise, one could backtest a part of the distribution, like the left tail.
- One important limitation of all the previous tests is the ignorance of the parameter uncertainty. In general, one does not know μ_{t-1} and σ_t, and uses $\hat{\mu}_{t-1}$ (e.g., AR(1))and $\hat{\sigma}_t$ (e.g., GARCH(1,1)). In general, the distribution of the test will be different (use moment tests to take into account parameter uncertainty).
- Stress testing: In the simulations, add scenarios that we did not observe in the data.

Graphs from Christoffersen (IER, 1998)

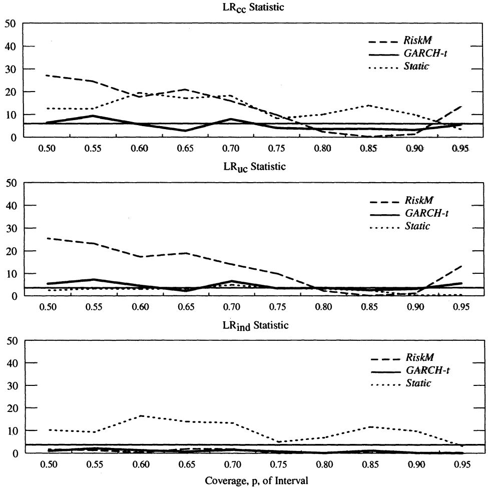


FIGURE 3

BRITISH POUND: LIKELIHOOD RATIO STATISTICS OF CONDITIONAL COVERAGE, UNCONDITIONAL COVERAGE, AND INDEPENDENCE*

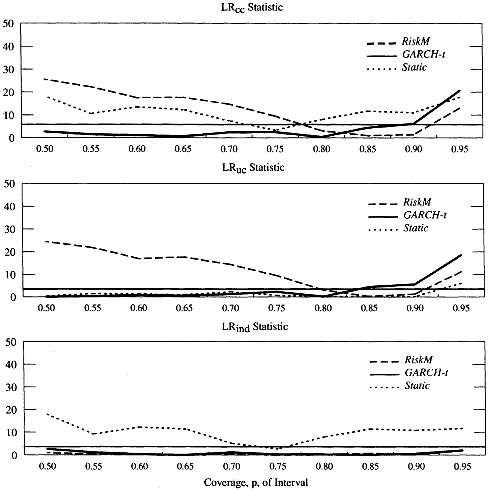


FIGURE 4

GERMAN MARK: LIKELIHOOD RATIO STATISTICS OF CONDITIONAL COVERAGE, UNCONDITIONAL COVERAGE, AND INDEPENDENCE*

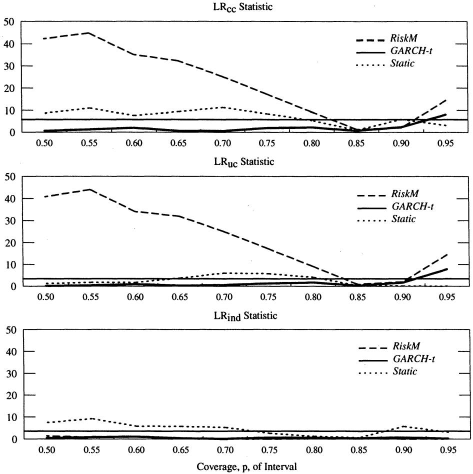


FIGURE 5

JAPANESE YEN: LIKELIHOOD RATIO STATISTICS OF CONDITIONAL COVERAGE, UNCONDITIONAL COVERAGE, AND INDEPENDENCE*

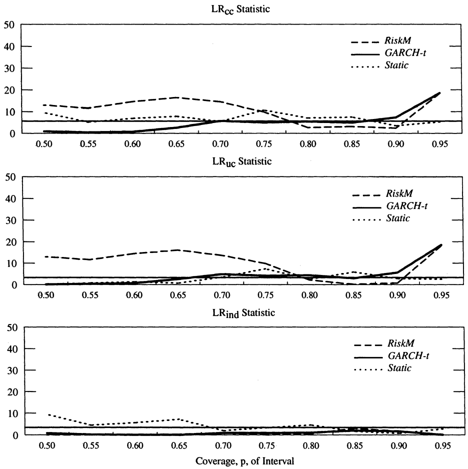


FIGURE 6

SWISS FRANC: LIKELIHOOD RATIO STATISTICS OF CONDITIONAL COVERAGE, UNCONDITIONAL COVERAGE, AND INDEPENDENCE*

TABLE 1
COVERAGE RATES AND AVERAGE WIDTHS OF INTERVAL FORECASTS*

British Pound	50.00	55.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00
RiskMetrics	55.60	60.30	64.50	69.60	73.80	78.00	81.30	84.90	89.30	93.15
Garch(1,1)-t	52.55	57.95	62.30	66.55	72.60	76.75	81.65	86.25	91.15	96.10
Static	51.75	57.00	61.90	66.90	72.25	76.75	81.55	86.20	90.30	94.70
RiskMetrics	0.90	1.01	1.12	1.24	1.38	1.53	1.71	1.92	2.19	2.61
Garch(1,1)-t	0.84	0.95	1.06	1.18	1.32	1.48	1.66	1.89	2.20	2.71
Static	0.80	0.92	1.04	1.17	1.31	1.48	1.68	1.91	2.27	2.83
German Mark	50.00	55.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00
RiskMetrics	55.50	60.15	64.45	69.40	73.80	77.90	81.55	85.35	89.25	93.30
Garch(1,1)-t	50.20	55.55	60.95	65.75	71.10	76.40	80.30	86.65	91.55	96.95
Static	50.70	56.30	61.20	66.00	71.50	75.75	80.30	85.45	90.25	96.15
RiskMetrics	0.92	1.03	1.15	1.28	1.41	1.57	1.75	1.96	2.24	2.67
Garch(1,1)-t	0.83	0.94	1.05	1.18	1.32	1.48	1.68	1.92	2.26	2.85
Static	0.79	0.90	1.02	1.16	1.33	1.48	1.67	1.99	2.35	3.12
Japanese Yen	50.00	55.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00
RiskMetrics	57.10	62.30	66.30	70.90	75.00	78.90	82.65	85.75	89.05	93.05
Garch(1,1)-t	50.40	55.70	61.00	65.50	70.70	76.05	81.15	85.45	90.85	96.30
Static	51.20	56.45	61.40	67.00	72.45	77.25	81.75	85.40	90.25	95.05
RiskMetrics	0.86	0.97	1.08	1.19	1.32	1.47	1.64	1.84	2.10	2.50
Garch(1,1)-t	0.74	0.84	0.94	1.05	1.18	1.33	1.52	1.75	2.09	2.68
Static	0.72	0.82	0.93	1.05	1.18	1.37	1.56	1.79	2.16	2.79
Swiss Franc	50.00	55.00	60.00	65.00	70.00	75.00	80.00	85.00	90.00	95.00
RiskMetrics	54.00	58.75	64.10	69.20	73.70	78.00	81.30	85.25	89.45	92.80
Garch(1,1)-t	50.20	55.50	60.85	66.65	72.20	76.90	81.85	86.35	91.55	96.95
Static	50.40	55.90	61.20	65.85	71.85	77.55	81.40	86.90	91.10	95.75
RiskMetrics	1.01	1.13	1.26	1.40	1.55	1.72	1.92	2.16	2.46	2.94
Garch(1,1)-t	0.94	1.06	1.19	1.32	1.48	1.66	1.87	2.14	2.50	3.12
Static	0.90	1.02	1.16	1.29	1.48	1.69	1.91	2.26	2.66	3.35

* For each exchange rate panel and each true coverage rate, $p = 50$ to 95 per cent, the top half of the panel shows the nominal coverage rate and the bottom half of the panel shows the average width of the interval prediction over the sample.