Financial Econometrics
Dynamic Term Structure Models

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An Empirical Example

What Does the Yield Curve Tell us About GDP Growth?

by A. Ang, M. Piazzesi, and M. Wei (2006, Journal of Econometrics)

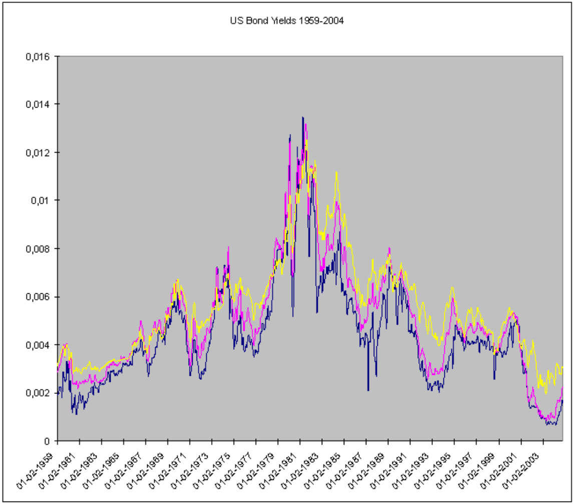


Figure 1: One-month, One-Year and Five-Year US Bond Yields

Yield Factors

Table 2: **Variation in yield changes and levels explained by the first k principal components.**

k	1	2	3	4	5
	Principal Components				
Yield Changes	79.7	91.7	96.5	97.5	98.3
Yields	96.5	99.5	99.8	99.9	100

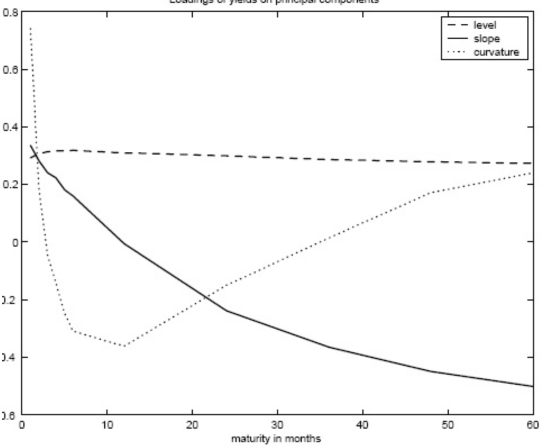


Figure 2: Level, Slope and Curvature

Motivation

How can we use information in the yield curve to forecast GDP growth?

Huge Literature ...
Harvey 1986, 1989, 1993, Laurent 1988, Stock & Watson 1989, Chen 1991, Estrella & Hardouvelis 1991, 1997, Estrella & Mishkin 1998, Hamilton & Kim 2002.

... forecasts GDP .....

1. univariate regressions on slope
2. binary choice models: $1 =$ recession
3. leading indicator by Stock & Watson

... with success: Table 1, Table 2

NBER recessions and inversions in the postwar

NBER recession	inversion	lead time
53:Q3-54:Q2		
57:Q3-58:Q2		
60:Q2-61:Q1		
- - - - - - - - - Fama-Bliss sample starts 1964:Q1 - - - - - - - - -		
	66:Q3-Q4	
69:Q4-70:Q4	68:Q2, 68:Q4, 69:Q4	6 qtrs
73:Q4-75:Q1	73:Q2-74:Q1, 74:Q4	2 qtrs
80:Q1-80:Q3	78:Q4-80:Q1	5 qtrs
81:Q3-82:Q4	80:Q3-Q4,81:Q2,82:Q1	qtrs
90:Q3-91:Q1	89:Q2	5 qtrs
01:Q1-02:Q1	00:Q3-Q4	2 qtrs

Data

real GDP, seasonally adjusted, from FRED (GDPCI)

zero-coupon yields from CRSP:
CRSP Fama riskfree rate file
Fama-Bliss discount rate file

sample: 1952:Q2 - 2001:Q4
data-quality issues in Fama & Bliss 1987: start 1964:Q1

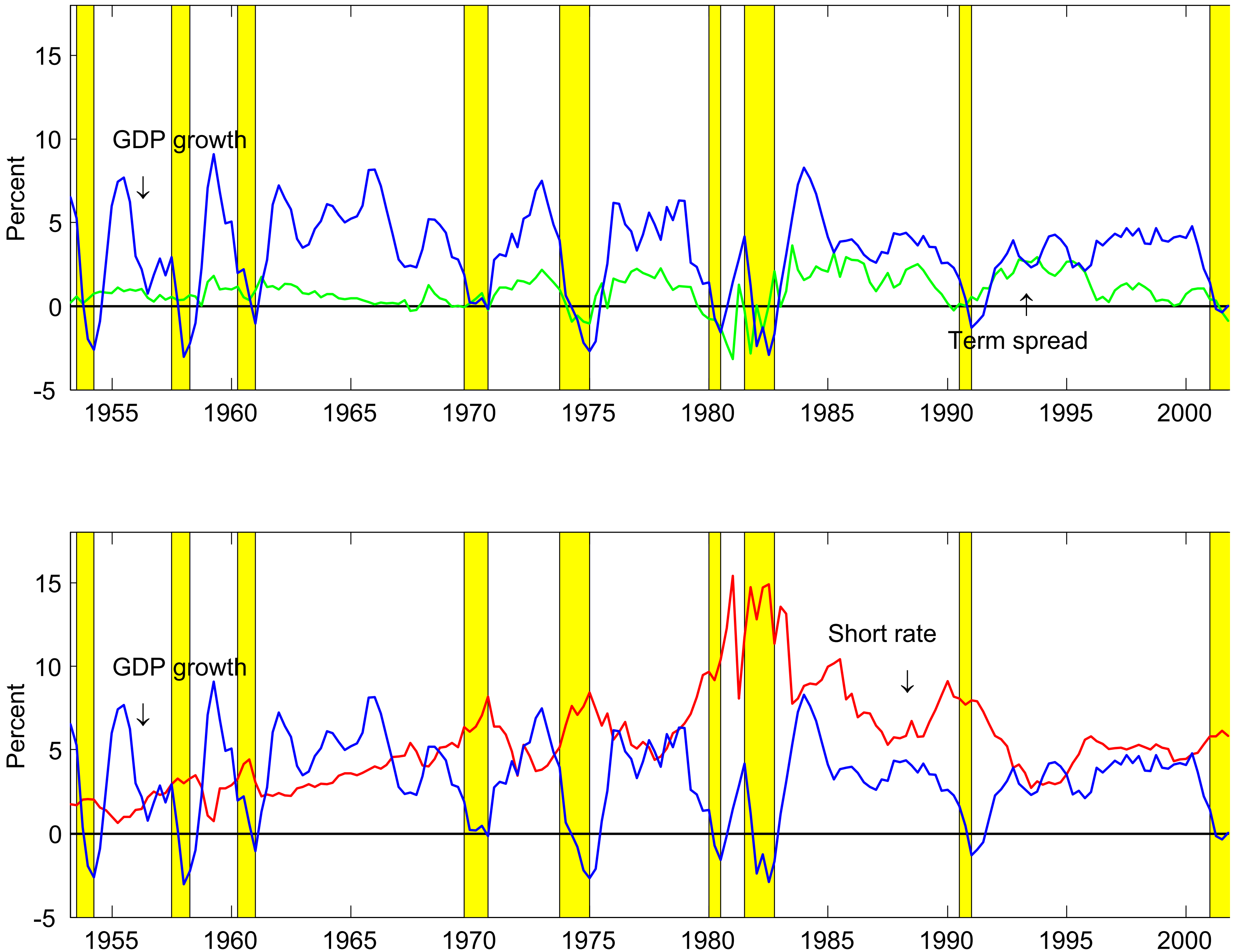
Univariate regressions

$$\begin{aligned}
g_{t\to t+k} &= \alpha + \beta\left(y_t^{(n)} - y_t^{(1)}\right) + \text{error} \\
g_{t\to t+k} &= 4/k\ (\log GDP_{t+k} - \log GDP_t) \\
&= k\text{-quarter GDP growth} \\
y_t^{(n)} &= n\text{-quarter yield}
\end{aligned}$$

overlapping periods introduce moving average,
Hodrick 1992 standard errors

Forecasts of GDP Growth from Term Spreads

Horizon	Term Spread Maturity									
	4-qtr		8-qtr		12-qtr		16-qtr		20-qtr	
k-qtrs	$\beta_k^{(4)}$	R^2	$\beta_k^{(8)}$	R^2	$\beta_k^{(12)}$	R^2	$\beta_k^{(16)}$	R^2	$\beta_k^{(20)}$	R^2
1	0.31	0.00	0.78	0.03	0.72	0.04	0.66	0.04	0.65	0.04
	(0.73)		(0.49)		(0.39)		(0.33)		(0.29)	
4	1.18	0.06	1.23	0.16	1.06	0.18	0.90	0.17	0.89	0.20
	(0.49)		(0.38)		(0.32)		(0.28)		(0.26)	
8	1.06	0.10	1.04	0.20	0.91	0.25	0.78	0.24	0.73	0.24
	(0.41)		(0.33)		(0.29)		(0.26)		(0.24)	
12	0.56	0.05	0.67	0.16	0.59	0.19	0.53	0.20	0.48	0.20
	(0.32)		(0.27)		(0.24)		(0.21)		(0.20)	



GDP growth
Term spread
Percent
GDP growth
Short rate
Percent

Questions

1. What maturity n should we pick to define slope $y_t^{(n)} - y_t^{(1)}$?

   Ex: 10 year Treasury - 3-month T-Bill

2. How about interest rate levels?

3. How about multivariate regressions using many yields and current GDP?

4. Does any of this depend on the forecasting horizon k?

Possible strategy: run MANY regressions to find out!

Methodology

Needed: model of dynamics of GDP growth and yields

Combine vectorautorgression ...

$\Longrightarrow$ explore different k in $g_{t \to t+k}$

problem with collinearity when many yields are in a large VAR

... and no-arbitrage pricing

solves collinearity problem since few factors are enough to capture yields

$\Longrightarrow$ explore different n in $y_t^{(n)} - y_t^{(1)}$

Yield-curve model

most use only latent factors
Duffie & Kan 1996, Dai & Singleton 2000, 2002

observable macro factors
Piazzesi 2001, Ang & Piazzesi 2003.

new here:

- observable yield factors:
  easy and fast to estimate, out-of-sample forecasting

- more flexible dynamics:
  conditional mean of macro variables now allowed to depend on yields

- focus on forecasting GDP

Discrete-time model

3 factors

$$X_t = \begin{pmatrix} y_t^{(1)} \\ y_t^{(20)} - y_t^{(1)} \\ g_t \end{pmatrix} \quad \begin{array}{c} \text{nominal short rate} \\ \text{5-year term spread} \\ \text{GDP growth} \end{array}$$

$X_t = \mu + \phi X_{t-1} + \Sigma \varepsilon_t$ Gaussian VAR

$\varepsilon_t \sim N(0, I)$

pricing kernel

$$m_{t+1} = \exp\left(-y_t^{(1)} - \tfrac{1}{2}\lambda_t^\intercal \lambda_t - \lambda_t^\intercal \varepsilon_{t+1}\right)$$

$\lambda_t = \lambda_0 + \lambda_1 X_t$ linear risk premia

special case: $\lambda_0 = 0, \lambda_1 = 0$ expectations hypothesis

Affine yield solutions

bond prices $p_t^{(n)}$ solve the recursion

$$p_t^{(n)} = E_t\left(m_{t+1}p_{t+1}^{(n)}\right)$$

with $p_t^{(1)} = E_t\left(m_{t+1}\right)$ since $p_{t+1}^{(0)} = 1$

find $p_t^{(n)} = \exp\left(A_n + B_n^\intercal X_t\right)$ with

$$\begin{aligned}
&A_{n+1} = A_n + B_n^\intercal\left(\mu - \boldsymbol{\Sigma}\lambda_0\right) + \tfrac{1}{2}B_n^\intercal\boldsymbol{\Sigma}\boldsymbol{\Sigma}^\intercal B_n \\
&B_{n+1} = \left(\phi - \boldsymbol{\Sigma}\lambda_1\right)^\intercal B_n - e_1 \\
&A_0 = 0, B_0 = 0
\end{aligned}$$

yields

$$\begin{aligned}
y_t^{(n)} &= -\log p_t^{(n)}/n \\
&= -\frac{A_n}{n} - \frac{B_n^\intercal}{n}X_t \\
&= a_n + b_n^\intercal X_t
\end{aligned}$$

2-step estimation procedure

1. Estimate μ, ϕ and $\mathbf{\Sigma}$ using standard SUR
   in $X_t = \mu + \phi X_{t-1} + \mathbf{\Sigma}\varepsilon_t$

2. Estimate λ_0 and λ_1 using NLS in $\lambda_t = \lambda_0 + \lambda_1 X_t$

NLS minimizes squared fitting errors

$$\min_{\lambda_0, \lambda_1} \sum_{t=1}^{T} \sum_{n=1}^{N} \left(\widehat{y}_t^{(n)} - y_t^{(n)} \right)^2$$

where $\widehat{y}_t^{(n)} = a_n + b_n^\intercal X_t$

Standard errors with GMM by stacking all moments

Advantage: speed enables out-of-sample forecasting; numerical accuracy of OLS.

Disadvantage: lose efficiency of MLE

Remarks about the model

Number of factors

$y_t^{(1)} \approx$ 1st PC
$y_t^{(20)} - y_t^{(1)} \approx$ 2nd PC
2 PCs explain 99.7% of **quarterly** yield variation

Latent vs observable factors

Appendix estimates latent 2 factor model with GDP:

1st factor $\approx y_t^{(1)}$
2nd factor $\approx y_t^{(20)} - y_t^{(1)}$

GDP forecasts almost identical

Caveats

homoskedasticity
subsample instability

Evaluating forecasts

Estimated term structure model

- get theoretical regression coefficients, such as

$$\beta = \frac{cov\left(\text{GDP growth } t \to t+k, \text{ slope at } t\right)}{var\left(\text{slope at } t\right)}$$
$$= \frac{4e_3^\intercal \phi \left(I-\phi\right)^{-1}\left(I-\phi^k\right)\Sigma_X \Sigma_X \left(b_n - b_1\right)}{k\left(b_n - b_1\right)^\intercal \Sigma_X \Sigma_X^\intercal \left(b_n - b_1\right)}$$

  where $\Sigma_X \Sigma_X^\intercal$ is the unconditional variance of X

  get coefficients for any set of RHS variables: $y_t^{(n)} - y_t^{(1)}$, $y_t^{(1)}$, g_t.

  get coefficients for all k and n!

- theoretical R^2

- rolling out-of-sample forecasts over 1990s

Results in a nutshell

1. contrary to OLS, short rate is more important
   - in univariate and multivariate predictions
   - at any horizon
   - in and out of sample
2. always use longest term spread
3. for short forecasting horizons,
   including GDP is important
4. it's inflation, not the real rate
5. term structure model beats OLS out-of-sample

Regression coefficients

	Model			OLS		
	Short	5-Year	GDP	Short	5-Year	GDP
k	Rate	Spread	Growth	Rate	Spread	Growth
4	−0.35			−0.42		
	(0.13)			(0.15)		
	−0.28	0.25	0.08	−0.28	0.60	0.11
	(0.14)	(0.31)	(0.04)	(0.15)	(0.25)	(0.06)
8	−0.28			−0.28		
	(0.11)			(0.13)		
	−0.25	0.14	0.04	−0.19	0.56	−0.02
	(0.14)	(0.25)	(0.03)	(0.13)	(0.21)	(0.03)
12	−0.24			−0.14		
	(0.10)			(0.12)		
	−0.22	0.08	0.02	−0.07	0.42	−0.02
	(0.13)	(0.20)	(0.02)	(0.11)	(0.17)	(0.02)

high short rate $\Longrightarrow$ low growth. Fed policy?

Out-of-sample forecasts

Nineties: 1990:Q1 - 2001Q:4
11 years with 2 recessions (1990-91, end of 2001)

Rolling forecasts:
re-estimate model with data up to t, forecast for $t + k$

Results

♣ Best performing models use short rates, rather than spreads

♣ Yield-curve model forecasts better than OLS

♣ Incorporating lagged GDP and the short rate produces superior forecasts than just the term spread

♣ Yield-curve model beats VAR models - useful way to reduce dimensionality

What is it - inflation or real rate?

k-qtrs	Inflation	Real Rate	Spread	GDP
1	-0.33	-0.17	0.21	0.24
	(0.14)	(0.15)	(0.34)	(0.08)
4	-0.32	-0.20	0.24	0.07
	(0.13)	(0.15)	(0.29)	(0.04)
8	-0.26	-0.18	0.18	0.03
	(0.12)	(0.13)	(0.22)	(0.02)
12	-0.21	-0.16	0.13	0.02
	(0.11)	(0.12)	(0.18)	(0.02)

Conclusion

Model guides us towards

- short rate as most important predictor

- picking highest n, which does not depend on k

- current GDP only important for short k

Higher nominal short rate $\Longrightarrow$ lower growth.

It's inflation, not the real rate.

Fed policy?