Financial Econometrics Part 2 - Project - Serge Nyawa

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Contents

1 Introduction

The main goal of our project was to implement and estimate the multivariate GARCH models we have seen as part of the course ”Financial Econometrics”- part 2. We transfer ourselves into the position of a portfolio manager analysing her portfolio dynamics and corresponding extreme events through Value-at-Risk (VaR) metric. We choose a portfolio of 3 stocks with different market Betas, taking 15 years of data from 2010 until the end of 2024. We begin our analysis by performing some exploratory data analysis. We plot prices, perform stationarity tests, and take log first differences of the time series that are not stationary (in our case all 3). We then subsequently de-mean returns by calculating the conditional mean for each of the series for a respective time. To simplify our analysis we do not analyse nor model possible cointegration relationships between the time series. Although accounting for possible cointegration relationships can enrich models' predictions, it entails additionally computational complexity. Furthermore, if present, cointegration models longer term relationship whilst we aim to model more immediate short term dynamics.

We proceed our analysis of the portfolio's multivariate volatility by estimating the moving average (MA) covariance taking a monthly and quarterly window. Comparing the resulting VaR forecasts to to a exponentially weighted moving average (EWMA) covariance estimation, we find better forecasting performance of the latter when using a decay factor of $\lambda = 0.998$.

We additionally estimate our VaR using a observable factor covariance model. We estimate our factors by using additional data, first, running a principle component analysis (PCA) on returns of companies listed in the S&P 500 index, conditioning on data availability and quality; second, using Fama-French daily factors. Generally, using the components of the PCA run on the S&P 500 outperforms the model using Fama-French factors. When choosing a 252 day window, we do not reject unconditional coverage for the observable factor covariance model using PCA, whilst still rejecting independence and conditional coverage of violations.

Finding that overall both models showcase poor predictive capability, we expand our model selection to a multivariate GARCH set-up. We analyse multivariate GARCH models such as BEKK that assume symmetry and positive definiteness in the variance-covariance matrix for computational efficiency. We find significantly better model results using a general GARCH/BEKK model compared to a more parsimonious GARCH model assuming constant conditional correlation. When estimating VaR we find that we do not reject the number of violations being significantly different from 5% using the

Kupiec's and Christoffersen's "proportion of failures" coverage test where violations are assumed to be Bernoulli, representing a good model fit.

2 Data

2.1 Data Description

We use daily stock data from the Yahoo finance API. The observation period of the collected data ranges from 2010 to end of 2024 with 3000 data points per time series. As all stocks within the portfolio are listed in the US, taking closing prices, all stocks are synchronised. Table 1 shows summary statistics for the selected stocks after the data were cleaned of missing values. The data were selected to provide a reasonable amount of heterogeneity - comparing tech, pharmaceutical, and a retail company - that would allow an interesting comparison. Comparing the standard deviations, it is already notable that stocks are similar but not equal in terms of deviations from the zero mean. Within our portfolio, Johnson and Johnson (JNJ) and Walmart (WMT) show a lower STD than tech-stocks like Apple (AAPL). When looking at de-meaned log differenced returns (Table 2) we see the (excess) kurtoses of the variables already indicates higher risks of extreme returns. Interestingly, following this comparison, it is WMT that shows the heaviest tail and thus the highest probability of large shocks to returns.

Table 1: Summary Statistics of Stock Prices

Ticker	AAPL	JNJ	WMT
Mean	71.2939	102.5481	31.7191
Std	67.5314	40.1896	17.3109
Skew	0.9674	−0.0560	1.4004
Kurtosis	−0.4614	−1.3122	2.1784

Table 2: Summary Statistics of De-meaned Log Returns

Ticker	AAPL	JNJ	WMT
Mean	−0.0001	−0.0001	0.0000
Std	0.0175	0.0106	0.0122
Skew	−0.2448	−0.2707	−0.2277
Kurtosis	5.5696	9.2730	16.0018

Table 3: ADF Test Results for Raw Prices and Log Returns

Ticker	Raw Prices p-value	Raw Prices Result	Log Returns p-value	Log Returns Result
AAPL	0.9955	Non-stationary	0.0000	Stationary
JNJ	0.7756	Non-stationary	0.0000	Stationary
WMT	1.0000	Non-stationary	0.0000	Stationary

2.2 Data Preparation and Preliminary Testing

To model the multivariate volatility of the portfolio stationarity of the time series are required. We perform visual inspections and Augmented Dickey Fuller (ADF) tests to test for stationary and the absence of a unit root. Not rejecting the null hypothesis of a unit root being present, we take the log first difference of prices to get returns, and conditionally de-mean the data. The first value of return time series - naturally a missing value - was furthermore dropped. Our results of performing the ADF test on the differenced series is shown in in Table 3. Rejecting the null hypothesis of unit root behaviour, we find that our time series are now stationary. For the sake of simplicity we do not analyse possible Co-Integration relationships between the individual time series which could indicate a longer term relationship.

2.3 Distribution Visualisations: Histograms and QQ-Plots

As a final preliminary check, we used Q-Q Plots to understand the distributional characteristics of our portfolio. Figures 9 to 11 show both the stock distribution (right) as well as the deviation of observed quantiles of the relative frequencies (y-axis) to the quantiles theoretic standard normal probability (x-axis). If the distribution would indeed follow a normal distribution, the scatter points as well as the red regression line would show a nearly perfect 45° line. However, as already noted to the kurtosis values, the stock market data comes with much heavier tails than the normal distribution assumed in many models.

3 Univariate Volatility and Rolling Volatility Time Series

As a preliminary analysis, we computed the 5-day rolling standard deviation of log returns to approximate stock-specific volatility. Figure 7 and 8 show the time series of respective log returns as well as calculated rolling volatility (rolling standard variation). We added an exemplary stress date such as the initial COVID-19 sell-off in red. Due to the large number of data points (3648) chart techniques are not applicable in these plots. Nevertheless, one can see that volatility and log returns are seemingly stationary with the COVID-19 Pandemic being the greatest outlier. For visual clarity, Figure 5 and figure 6 show explanatory time series plots for demeaned log returns and standard deviations of a single stock - Apple (AAPL), respectively. In line with the standard theory, demeaned log returns are stationary but volatility is not white noise, with clustering and persistency appearing.

4 Multivariate Volatility Analysis

We perform three multivariate models to analyse the co-volatility relationship of AAPL, JNJ, and WMT stock returns. First, we perform Moving Average Covariance and Exponentially Weighted Moving Average Covariance Models. Next, we perform a Observable Factor Covariance model. Finally, we estimate a Multivariate GARCH model including BEKK and Diagonal GARCH extensions.

Value-at-Risk (VaR) Forecasting Value-at-Risk quantifies the maximum expected loss of a portfolio over a given time horizon at a specified confidence level. Formally, for a loss random variable L_{t+1} and confidence level α, the VaR at level α is defined as the smallest number ℓ such that

$$\Pr\big(L_{t+1} \leq \ell\big) \geq \alpha,$$

or equivalently

$$\mathrm{VaR}_{t+1|\,t}(\alpha) = \inf\big\{\ell : F_{L_{t+1}}(\ell) \geq \alpha\big\},$$

where $F_{L_{t+1}}$ is the cumulative distribution function of losses. Under the assumption of conditionally Gaussian portfolio returns with mean zero and variance $\sigma^2_{p,t+1|t}$, the one-day ahead VaR at confidence level $1-\alpha$ can be written as

$$\mathrm{VaR}_{t+1|t}(\alpha) = z_\alpha\, \sigma_{p,t+1|t},$$

where $z_\alpha = \Phi^{-1}(\alpha)$ is the α-quantile of the standard normal distribution.

In our application, we form an equally-weighted portfolio $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})'$ of the three selected stocks. Using an estimated multivariate volatility model , the one-day ahead conditional portfolio variance is

$$\sigma^2_{p,t+1|t} = w' \Sigma_t\, w,$$

and thus the 95% VaR forecast is

$$\mathrm{VaR}_{t+1|t} = z_{0.05}\sqrt{w'\,\Sigma_t\,w},$$

with $z_{0.05} = \Phi^{-1}(0.05) \approx -1.645$.

4.1 Moving Average Covariance Model Estimation and Results

To estimate our one-day ahead VaR forecast, we first estimate the conditional covariance matrix via an n-period moving average covariance estimation:

$$\Sigma_t \;=\; \frac{1}{n}\sum_{i=1}^{n} \tilde{r}_{t-i}\,\tilde{r}'_{t-i}.$$

Out of possible window-sizes $n \in \{22, 66, 252\}$ the optimal window was chosen as $n = 66$ (quarterly) to minimise in-sample VaR violations. Figure 1 shows the forecasted VaR plotted against realised portfolio returns. Using $n = 66$, the MA estimate for the daily covariance matrix on December 30, 2024 is as follows:

Table 4: Moving Average Covariance on 2024-12-30

	AAPL	JNJ	WMT
AAPL	2.0172×10^{-4}	-8.0685×10^{-6}	9.1508×10^{-6}
JNJ	-8.0685×10^{-6}	8.9481×10^{-5}	1.5314×10^{-5}
WMT	9.1508×10^{-6}	1.5314×10^{-5}	1.2087×10^{-4}

We back-test the 95% VaR forecasts using Kupiec's unconditional coverage (UC), independence (IND) and conditional coverage (CC) tests. Over 3 435 out-of-sample observations there were 209 violations. Results at the 5% level are:

Table 5: Kupiec and Christoffersen's Test Results for MA Model (95% VaR)

Test	LR Statistic	p-value	Decision
Unconditional Coverage (UC)	1.5722	0.2099	Accept
Independence (IND)	9.8902	0.0017	Reject
Conditional Coverage (CC)	11.5750	0.0031	Reject

The UC test does not reject, indicating the overall violation rate is consistent with 5%. However, both IND and CC are rejected at 5%, suggesting that the model systematically overlooks the VaRs in case of the extreme events (clustering of the excess losses), which is expected given the model simplicity.

4.2 Exponentially Weighted Moving Average Covariance Models

The EWMA model estimates the conditional covariance matrix by exponentially weighting past return innovations, placing more emphasis on recent observations. Formally, for asset return vector $\tilde{r}_t$ and decay factor $\lambda \in (0,1)$, the EWMA recursion is

$$\Sigma_t^{\mathrm{EWMA}} = (1-\lambda)\,\tilde{r}_{t-1}\,\tilde{r}_{t-1}' + \lambda\,\Sigma_{t-1}^{\mathrm{EWMA}}.$$

Unfolding the recursion yields the equivalent infinite-lag representation

$$\Sigma_t^{\mathrm{EWMA}} = (1-\lambda)\sum_{i=1}^{\infty}\lambda^{\,i-1}\,\tilde{r}_{t-i}\,\tilde{r}_{t-i}',$$

which ensures positive definiteness and provides a one-parameter trade-off between noise reduction (larger λ) and responsiveness (smaller λ). The effective memory of the process is often characterised by the half-life h satisfying $\lambda^h = 0.5$.

We select the decay parameter $\lambda = 0.998$ by minimising out-of-sample VaR violations over the 2010-2024 period. A decay factor as high implies that the EWMA covariance estimator retains very long-memory—approximately $1/(1-\lambda) \approx 500$ periods—so that past return shocks continue to influence the covariance matrix for an extended duration. While this induces smoother, lower-variance estimates and reduces the noise inherent in high-frequency returns, it also slows the model's responsiveness to volatility shifts, delaying the upward adjustment of VaR following market stress. In practice, such sluggish adaptation can lead to clusters of VaR breaches immediately after a volatility spike, as reflected in the rejected independence and conditional coverage tests in Table (7), and suggests a trade-off between noise-reduction and timely risk reactivity when selecting λ.

The EWMA estimate of the covariance matrix on December 30, 2024, is shown in Table 6.

Table 6: EWMA Covariance on 2024-12-30

	AAPL	JNJ	WMT
AAPL	2.7574×10^{-4}	3.5308×10^{-5}	5.0086×10^{-5}
JNJ	3.5308×10^{-5}	1.1210×10^{-4}	3.5347×10^{-5}
WMT	5.0086×10^{-5}	3.5347×10^{-5}	1.5567×10^{-4}

Table 7: Kupiec and Christoffersen's Test Results for EWMA Model ($\lambda = 0.998$, 95% VaR)

Test	LR Statistic	p-value	Decision
Unconditional Coverage (UC)	0.00052	0.9818	Accept
Independence (IND)	9.5187	0.0020	Reject
Conditional Coverage (CC)	9.6217	0.0081	Reject

Figure 1 compares the one-day ahead 95% VaR forecasts from the MA model (window $n = 252$) and the EWMA model (decay factor $\lambda = 0.998$). Both series exhibit smooth trajectories and under-react to the abrupt volatility spike, causing VaR violations to cluster immediately after the stress event. This clustering of breaches highlights the models' sluggish adaptation and violates the independence assumption for VaR exceedances.

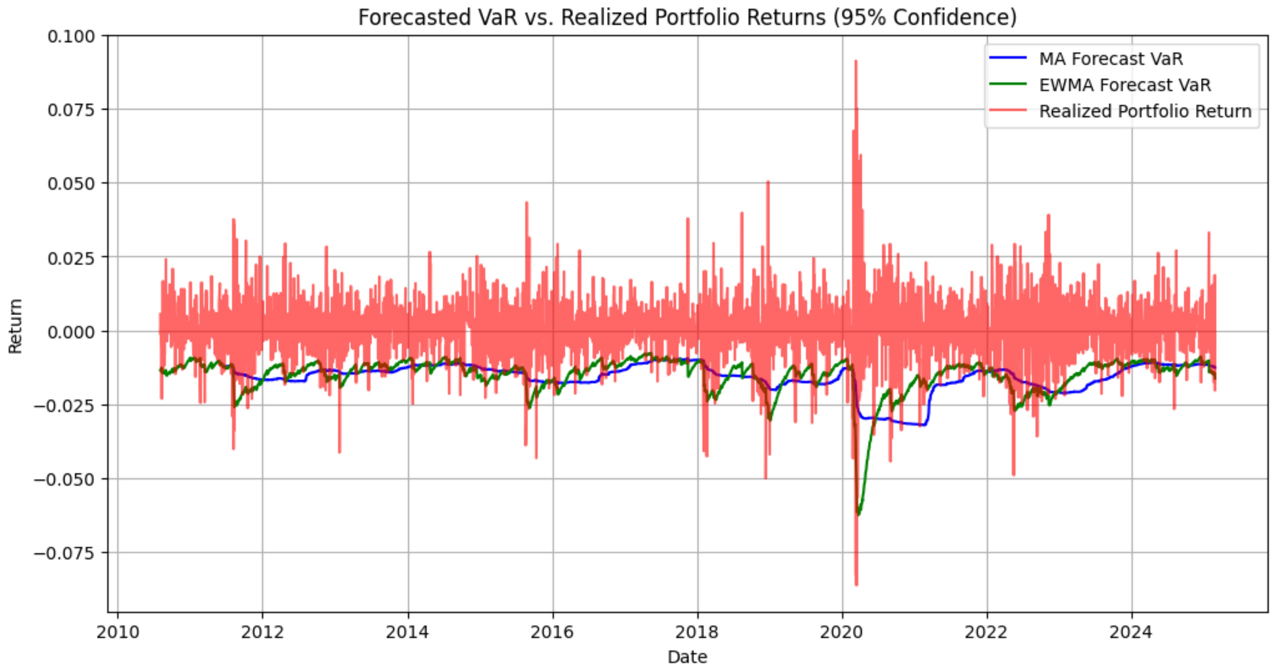

Figure 1: Value-at-Risk forecasted with moving average model of window 252 vs. forecasted with exponentially weighted moving average model with decay parameter $\lambda = 0.998$ vs. Realised Portfolio Returns (95% Confidence).

4.3 Observable Factor Covariance Model with Fama–French Factors

In the observable factor covariance model, asset returns are decomposed into a systematic component driven by a set of observable factors and an idiosyncratic component. Formally, if $\tilde{r}_t$ is the k–vector of demeaned asset returns at time t, and f_t is the p–vector of factor returns, then

$$\tilde{r}_t \;=\; \beta\, f_t \;+\; \eta_t,$$

where β is a $k \times p$ matrix of factor loadings and η_t is a k–vector of residuals with $\mathrm{E}[\eta_t \eta_t'] = \Omega_t$ diagonal. The conditional covariance is

$$\Sigma_t \;=\; \beta\, \Sigma_{f,t}\, \beta' \;+\; \Omega_t,$$

with

$$\Sigma_{f,t} \;=\; \frac{1}{n}\sum_{i=1}^{n} f_{t-i} f_{t-i}', \quad \omega_{j,t}^2 = \frac{1}{n}\sum_{i=1}^{n} \eta_{j,t-i}^2.$$

In our implementation we use the four Fama–French factors $\{$*Mkt–RF*, *SMB*, *HML*, *CMA*$\}$ downloaded from the Kenneth French data library.

We estimate loadings and idiosyncratic variances over a rolling window of $n = 252$ (yearly). The Kupiec backtest over 3,435 observations of day-ahead VaR presented in Table (9) states that while the distribution of the extreme events does not follow the binomial distribution with parameter α, they are independent.

Table 8: Observable Factor (Fama–French) Covariance on 2024-12-30

	AAPL	JNJ	WMT
AAPL	2.0177×10^{-4}	-2.4222×10^{-6}	-3.6377×10^{-7}
JNJ	-2.4222×10^{-6}	8.9499×10^{-5}	7.5555×10^{-7}
WMT	-3.6377×10^{-7}	7.5555×10^{-7}	1.2090×10^{-4}

Table 9: Kupiec and Christoffersen's Test Results for Observable Factor Model with Fama-French factors ($n = 66$, 95% VaR)

Test	LR Statistic	p-value	Decision
Unconditional Coverage (UC)	53.3784	2.75×10^{-13}	Reject
Independence (IND)	–1.8996	1.0000	Accept
Conditional Coverage (CC)	56.5624	5.22×10^{-13}	Reject

The UC rejection indicates the model underestimates the frequency of extreme losses, while the acceptance of IND suggests no significant clustering of violations. However, the CC rejection implies that the joint hypothesis of correct coverage and independence still fails, pointing to misspecification in the conditional loss distribution despite the observable factor structure.

4.4 Observable Factor Covariance Model via PCA

Instead of relying on a fixed set of observable economic factors, we apply PCA to the entire universe of NYSE-listed stocks over our sample period to extract the principal components that drive co-movements in asset returns. Under the efficient-markets hypothesis, any macroeconomic or firm-specific shock that materially affects a subset of stocks will be reflected in their prices, and thus in the covariance structure of the cross-section. By decomposing the full high-dimensional return matrix into orthogonal eigenvectors, PCA isolates the dominant directions of shared variation—effectively capturing the "true" systematic shocks without imposing an ex ante factor specification. Projecting our three-stock portfolio onto these data-driven factors then allows us to quantify and forecast the portion of extreme-loss risk attributable to broad market dynamics, while treating the residual as idiosyncratic noise.

Given a universe of k asset returns $\tilde{r}_t \in R^k$, PCA seeks an orthogonal matrix Q and diagonal matrix Λ such that the sample covariance

$$S = \frac{1}{T}\sum_{t=1}^{T} \tilde{r}_t \tilde{r}_t' = Q \Lambda Q'$$

with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. The principal components are

$$f_t = Q' \tilde{r}_t,$$

and account for decreasing shares of total variance: $\lambda_i / \sum_j \lambda_j$. Retaining only the first m components yields a reduced factor model

$$\tilde{r}_t \approx Q_m f_{m,t} + \eta_t,$$

where Q_m comprises the first m eigenvectors and $f_{m,t} \in R^m$. The conditional covariance is then

$$\Sigma_t^{\mathrm{PCA}} = Q_m \Sigma_{f,t} Q_m' + \Omega_t,$$

with

$$\Sigma_{f,t} = \frac{1}{n}\sum_{i=1}^{n} f_{m,t-i} f_{m,t-i}', \qquad \Omega_t = \mathrm{diag}\big(\tfrac{1}{n}\sum_{i=1}^{n} \eta_{j,t-i}^2\big).$$

Although 127 components are required to explain 80% of the variance across 432 NYSE stocks (see Appendix A for the scree plot), the incremental variance explained beyond the first three eigenvalues is negligible. By the "elbow" criterion on the marginal explanatory power curve, we select $m = 3$ to balance parsimony and coverage of common movements. This yields a tractable model while capturing the dominant co-movements in the return universe.

Using a rolling window of $n = 252$ trading days, we estimate the PCA factor model and backtest the one-day ahead 95% VaR (equally-weighted portfolio $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})'$). Over 3,396 observations we observe 187 violations (Table 11). The UC acceptance indicates the overall violation rate is close to 5%, but rejection of IND and CC reveals clustering in breaches and suggests that idiosyncratic noise remains insufficiently captured by only three components.

Table 10: Observable Factor Covariance (PCA) on 2024-12-30

	AAPL	JNJ	WMT
AAPL	2.0172×10^{-4}	-1.1447×10^{-5}	7.9255×10^{-6}
JNJ	-1.1447×10^{-5}	8.9481×10^{-5}	1.8818×10^{-5}
WMT	7.9255×10^{-6}	1.8818×10^{-5}	1.2087×10^{-4}

Table 11: Kupiec and Christoffersen's Test Results for PCA Factor Model ($m = 3$, $n = 252$, 95% VaR)

Test	LR Statistic	p-value	Decision
Unconditional Coverage (UC)	1.7781	0.1824	Accept
Independence (IND)	4.3276	0.0375	Reject
Conditional Coverage (CC)	11.9042	0.0026	Reject

Figure 2 compares the 95% VaR forecasts generated by the two observable factor covariance models (Fama–French and PCA-based). Both models produce relatively smooth VaR trajectories that under-react to sudden volatility spikes, resulting in clusters of violations during periods of market stress. This clustering of breaches highlights the models' inability to capture rapid shifts in tail risk and violates the independence assumption of VaR excesses.

4.5 Multivariate GARCH Models

We expand our model selection to a multivariate GARCH set-up. We also analysed multivariate GARCH models such as BEKK 1 that assume symmetry and positive definiteness in the variance-covariance matrix for computational efficiency. These assumptions greatly reduce the amount of estimated parameters over the more general vector GARCH 4 and therefore has better asymptotic properties in smaller samples. When compared to a more parsimonious GARCH model assuming

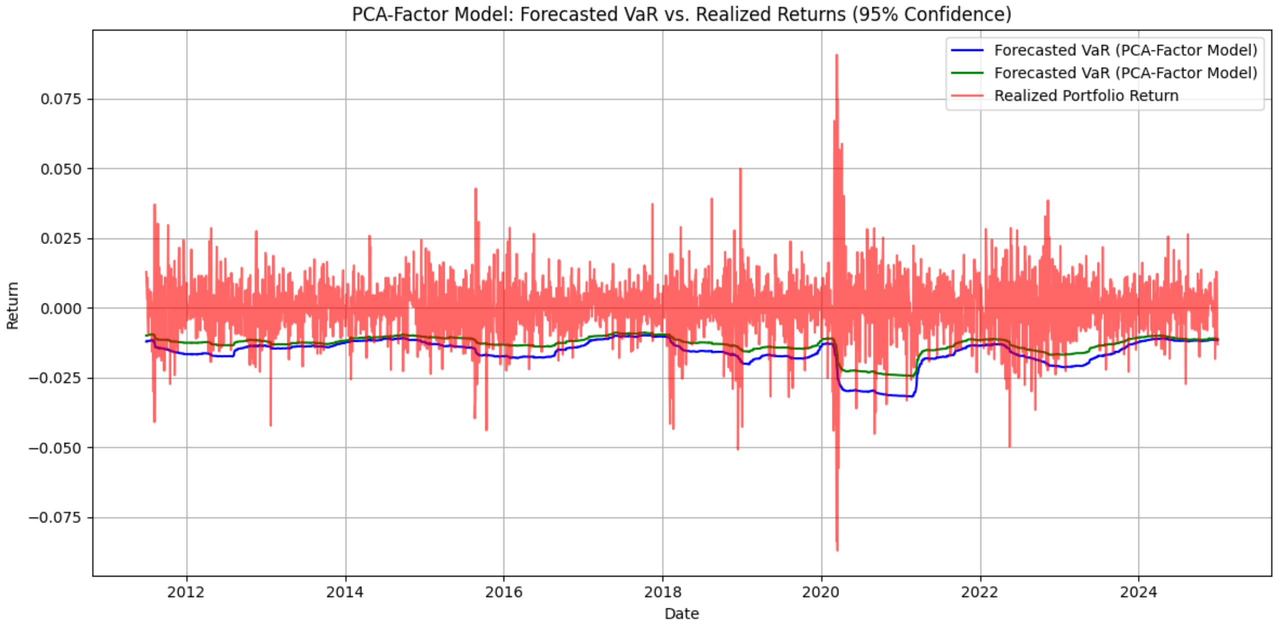

Figure 2: Value-at-Risk forecasted with observable factor covariance model on 4 Fama-French factors with rolling window of a size 252 vs. observable factor covariance model with rolling window of a size 252 on 3 PCA components constructed on 432 stocks present on NYSE over the considered period vs. Realised Portfolio Returns (95% Confidence).

constant conditional correlation between stocks, essentially estimating univariate GARCH models for every stock and then linking them together through a correlation estimation, we find significantly better model results when estimating value-at-risk.

We compare the different model estimations of 95% Value-at-Risk, performing the Christoffersen's/Kupiec's tests. We do not reject the null hypothesis of the number of unconditional violations to be equal to $1-\alpha = 5\%$ with a p-value of 0.5636 (Table 13) for UC, with IND and CC tests also not rejecting the null-hypothesis. we observe violations in 187 out of 3647 observations (Figure 3). Additionally, we estimated GARCH models with additional lags. We found that model performance did not significantly improve with the optimal model selection ultimately being the most parsimonious GARCH(1,1) model. Overall, the GARCH(1,1) model does not over- or underestimate risk on average, with no clustering of risk violations and the model being stable over time.

Table 12: Full Vector GARCH Covariance on 2024-12-30

	AAPL	JNJ	WMT
AAPL	2.8151×10^{-4}	2.8948×10^{-5}	6.1827×10^{-5}
JNJ	2.8948×10^{-5}	7.5573×10^{-5}	3.6503×10^{-5}
WMT	6.1827×10^{-5}	3.6503×10^{-5}	1.5541×10^{-4}

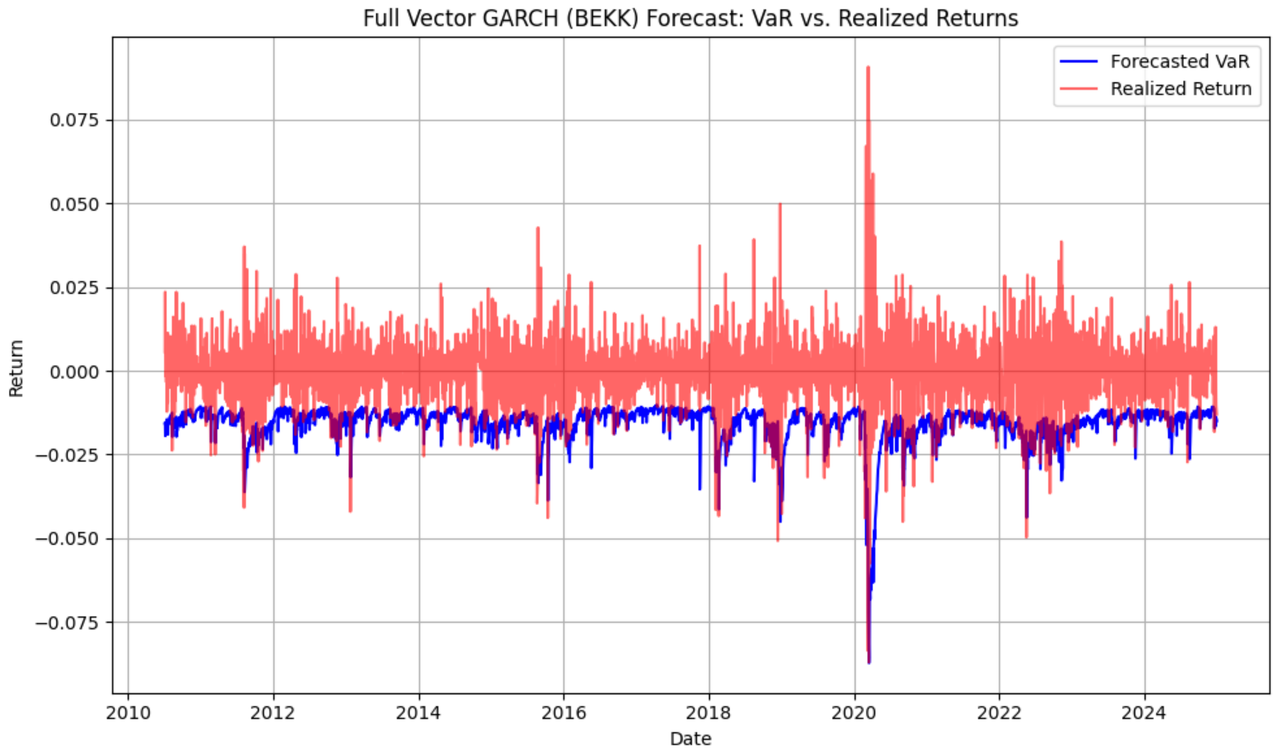

Figure 3: GARCH(1,1) model VaR forecast compared to realisations

Table 13: Kupiec and Christoffersen's Tests Results for Full Vector GARCH Model (95% VaR)

Test	LR Statistic	p-value	Decision
Unconditional Coverage (UC)	0.3334	0.5636	Accept
Independence (IND)	2.5636	0.1094	Accept
Conditional Coverage (CC)	3.0040	0.2227	Accept

5 Extension: Realised Covariance Models

The realised covariance model uses high-frequency returns to non-parametrically estimate the integrated covariance of asset prices over a fixed interval (here, one trading day). We assume that each asset price $P_t^{(i)}$ follows a continuous-time diffusion

$$dP_t^{(i)} = \mu_t^{(i)}\,dt + \sigma_t^{(i)}\,dB_t^{(i)},$$

where $B^{(i)}$ are Brownian motions with instantaneous covariance $\mathrm{Cov}(dB_t^{(i)}, dB_t^{(j)}) = \rho_{ij}(t)\,dt$. The object of interest is the daily integrated covariance

$$\int_0^T \sigma_s^{(i)}\,\sigma_s^{(j)}\,\rho_{ij}(s)\,ds,$$

which governs the variability and co-movements of returns over $[0,T]$. For our purpose, we approximate this integral with the sum of synchronous high-frequency (1-minute) log-return outer products:

$$\widehat{\Sigma}_{ij}(T) = \sum_{k=1}^{m} r_k^{(i)} r_k^{(j)},$$

where $r_k^{(i)} = \ln P_{t_k}^{(i)} - \ln P_{t_{k-1}}^{(i)}$ and $t_0 < t_1 < \cdots < t_m = T$ are the one-minute sampling times during the trading day; $m = 958$ for 16 hours that the trading is open. Under ideal conditions (no microstructure noise, perfect synchronisation), this realised covariance converges in probability to the true integrated covariance as the sampling interval tends to zero.

In our implementation, we retrieved 1-minute prices for AAPL, JNJ and WMT on 2024-12-30 from the AlphaVantage API, computed log-returns (no de-meaning). The data is perfectly synchronised due to the chosen stock liquidity, so we did not proceed with Hayashi Yoshida estimator. After application of the above summation we obtain:

Table 14: Realised Covariance Matrix for 2024-12-30 (1-Minute Returns)

	AAPL	JNJ	WMT
AAPL	0.000194	0.000036	0.000044
JNJ	0.000036	0.000161	0.000027
WMT	0.000044	0.000027	0.000243

This realised covariance estimate can then be directly compared to the one-day ahead forecasts produced by the MA, EWMA, and factor-based models to assess their accuracy in capturing intra-day risk.

5.1 Kernel-Based Realised Covariance Estimator

The kernel estimator extends the basic realised covariance by down-weighting distant lags to mitigate microstructure noise and asynchrony. Since our data is sampled with 1-minute frequency, it is contaminated with micro-structure noise, which is why the use of Kernel estimator might be preferred for the modelling of the realised volatility.

Denote the vector of d-dimensional intra-day log-returns at time j by r_j, for $j = 1, \ldots, n$. Define the

hth-lag autocovariance matrix

$$\Gamma_h = \sum_{j=h+1}^{n} r_j r'_{j-h}, \quad h = 0, 1, \ldots, H,$$

and choose a kernel weight sequence w_h satisfying $w_0 = 1$ and $w_h \to 0$ as $h \to H$. Under the Bartlett kernel,

$$w_h = 1 - \frac{h}{H+1}, \quad h = 0, 1, \ldots, H,$$

the estimator is

$$\widehat{\Sigma}_{\text{Kernel}} = \Gamma_0 + \sum_{h=1}^{H} w_h \left(\Gamma_h + \Gamma'_h\right).$$

This formulation guarantees positive semi-definiteness, consistency in the presence of noise, and accounts for non-synchronous trading.

In our empirical application for 2024-12-30, using $H = 5$, the resulting kernel-estimated realised covariance matrix is shown in Table 15.

Table 15: Kernel Estimated Realised Covariance Matrix for 2024-12-30 ($H = 5$)

	AAPL	JNJ	WMT
AAPL	0.000134	0.000024	0.000036
JNJ	0.000024	0.000112	0.000030
WMT	0.000036	0.000030	0.000120

6 Discussion and Comparison of Model Performance

Across the different multivariate volatility estimators, a clear trade-off emerges between model simplicity, responsiveness to new information, and forecasting accuracy. The simple moving-average (MA) and EWMA models both achieve acceptable unconditional coverage (UC) at the 5% level (Tables 5 and 7), but reject the independence (IND) and conditional coverage (CC) hypotheses, indicating clustered VaR breaches around volatility spikes (Figures 1). The MA model with a yearly window ($n = 252$) delivers the smoothest covariance estimates but is the slowest to adapt; the EWMA model ($\lambda = 0.998$) smooths noise effectively yet remains sluggish to regime shifts, with clustering of violations occurring.

The observable factor models exhibit mixed performance. Using Fama–French factors over a quarterly window ($n = 66$), the UC test is strongly rejected (Table 9), showing underestimation of extreme losses, though independence holds; this suggests that even well-known economic factors fail to capture idiosyncratic shocks in our small portfolio. In contrast, the PCA-based factor model with three components and a yearly window passes the UC test (Table 11) but still rejects IND and CC, implying that while data-driven factors improve average fit, they cannot fully explain clustering and the model is unstable over time.

Simple MA and EWMA estimators are easy to implement and computationally cheap, but their fixed-memory structure cannot accommodate sudden volatility shifts or heavy-tailed return distributions. Observable factor models reduce dimensionality but rely on a priori factor choices or broad PCA decompositions that may neglect non-linear dependencies. Overall, the multivariate GARCH model works best and captures time-varying correlations and volatility clustering. In general though, it can suffer from overfitting and numerical instability when performing out of sample forecasting. For high frequency data, the realised and Kernel-based realised covariance estimators exploit high-frequency data to mitigate microstructure noise, yet require careful synchronisation and bandwidth selection, and cannot directly forecast next-day risk without further modelling.

7 Conclusion

In our Project we analysed the volatility dynamics of a portfolio of 3 stocks taking daily closing price data from the past 15 years. Focussing our analysis on VaR metrics and modelling de-meaned log first differences we observe significant heterogeneity in model performance. Using simple approaches to multivariate volatility such as estimating the MA and EWMA covariance yields too static model predictions to forecast realised return volatility. Estimating the portfolio's variance-covariance matrix using observable factors improves model predictions when using factors estimated through a PCA on the S&P 500, but not when using Fama-French factors. Factor selection significantly impacts model performance. Finally, using a multivariate GARCH model we obtain significantly better model performance with violations not occurring too frequently, nor too scarcely, making the multivariate GARCH(1,1) our model of choice.

A Appendix A: PCA

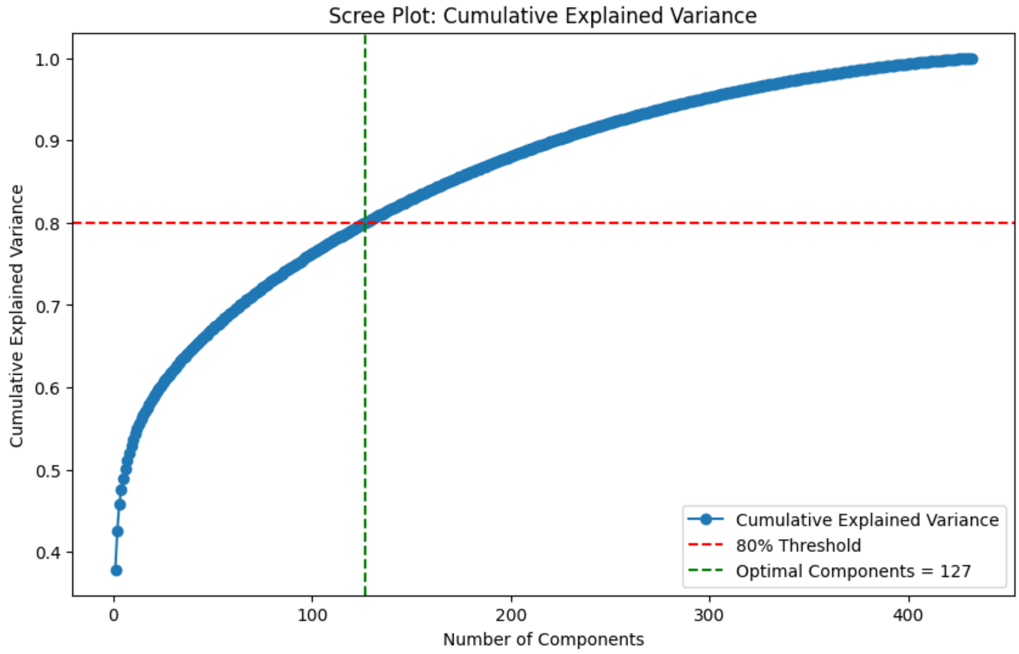

Figure 4: Scree plot of cumulative explained variance by PCA components for the NYSE universe. The red dashed line indicates the 80% variance threshold, and the green dashed line shows the optimal number of components ($m = 127$) for the threshold.

Table 16: List of NYSE Stocks Used for PCA

A	AAPL	ABT	ACGL	ACN	ADBE	ADI	ADM	ADP	ADSK
AEE	AEP	AES	AFL	AIG	AIZ	AJG	AKAM	ALB	ALGN
ALL	AMAT	AMD	AME	AMGN	AMP	AMT	AMZN	ANSS	AON
AOS	APA	APD	APH	ARE	ATO	AVB	AVGO	AVY	AWK
AXON	AXP	AZO	BA	BAC	BALL	BAX	BBY	BDX	BEN
BG	BIIB	BK	BKNG	BKR	BLDR	BLK	BMY	BR	BRO
BSX	BX	BXP	C	CAG	CAH	CAT	CB	CBOE	CBRE
CCI	CCL	CDNS	CF	CHD	CHRW	CHTR	CI	CINF	CL
CLX	CMCSA	CME	CMG	CMI	CMS	CNC	CNP	COF	COO
COP	COR	COST	CPB	CPRT	CPT	CRL	CRM	CSCO	CSGP
CSX	CTAS	CTRA	CTSH	CVS	CVX	D	DAL	DD	DE
DECK	DFS	DG	DGX	DHI	DHR	DIS	DLR	DLTR	DOC
DOV	DPZ	DRI	DTE	DUK	DVA	DVN	DXCM	EA	EBAY
ECL	ED	EFX	EG	EIX	EL	ELV	EMN	EMR	EOG
EQIX	EQR	EQT	ERIE	ES	ESS	ETN	ETR	EVRG	EW
EXC	EXPD	EXPE	EXR	F	FAST	FCX	FDS	FDX	FE
FFIV	FI	FICO	FIS	FITB	FRT	FSLR	FTNT	GD	GE
GEN	GILD	GIS	GL	GLW	GNRC	GOOG	GOOGL	GPC	GPN
GRMN	GS	GWW	HAL	HAS	HBAN	HD	HES	HIG	HOLX
HON	HPQ	HRL	HSIC	HST	HSY	HUBB	HUM	IBM	ICE
IDXX	IEX	IFF	INCY	INTC	INTU	IP	IPG	IRM	ISRG
IT	ITW	IVZ	J	JBHT	JBL	JCI	JKHY	JNJ	JNPR
JPM	K	KDP	KEY	KIM	KLAC	KMB	KMX	KO	KR
L	LDOS	LEN	LH	LHX	LII	LIN	LKQ	LLY	LMT
LNT	LOW	LRCX	LULU	LUV	LVS	LYB	LYV	MA	MAA
MAR	MAS	MCD	MCHP	MCK	MCO	MDLZ	MDT	MET	MGM
MHK	MKC	MKTX	MLM	MMC	MMM	MNST	MO	MOH	MOS
MPWR	MRK	MS	MSCI	MSFT	MSI	MTB	MTCH	MTD	MU
NDAQ	NDSN	NEE	NEM	NFLX	NI	NKE	NOC	NRG	NSC
NTAP	NTRS	NUE	NVDA	NVR	O	ODFL	OKE	OMC	ON
ORCL	ORLY	OXY	PARA	PAYX	PCAR	PCG	PEG	PEP	PFE
PFG	PG	PGR	PH	PHM	PKG	PLD	PM	PNC	PNR
PNW	PODD	POOL	PPG	PPL	PRU	PSA	PTC	PWR	QCOM
RCL	REG	REGN	RF	RJF	RL	RMD	ROK	ROL	ROP
ROST	RSG	RTX	RVTY	SBAC	SBUX	SCHW	SHW	SJM	SLB
SMCI	SNA	SNPS	SO	SPG	SPGI	SRE	STE	STLD	STT
STX	STZ	SWK	SWKS	SYK	SYY	T	TAP	TDG	TDY
TECH	TEL	TER	TFC	TGT	TJX	TKO	TMO	TMUS	TPL
TPR	TRMB	TROW	TRV	TSCO	TSLA	TSN	TT	TTWO	TXN
TXT	TYL	UAL	UDR	UHS	ULTA	UNH	UNP	UPS	URI
USB	V	VLO	VMC	VRSK	VRSN	VRTX	VTR	VTRS	VZ
WAB	WAT	WBA	WBD	WDC	WEC	WELL	WFC	WM	WMB
WMT	WRB	WSM	WST	WTW	WY	WYNN	XEL	XOM	YUM
ZBH	ZBRA								

B Appendix B: Equations

B.0.1 BEKK(1,1) Model Equation

$$H_t = CC' + A\,(r_{t-1}\,r'_{t-1})\,A' + B\,H_{t-1}\,B', \tag{1}$$

where

- H_t is the $k \times k$ conditional covariance matrix of asset returns at time t.
- r_t is a $k \times 1$ vector of asset returns.
- C is a $k \times k$ lower triangular matrix ensuring that H_t is positive definite.
- A and B are $k \times k$ parameter matrices that capture, respectively, the impact of lagged squared returns (or shocks) and lagged covariances.

B.0.2 Vector GARCH(1,1) Model Equation

Under the assumption of normality, the asset returns are modeled as conditionally normally distributed. That is,

$$r_t \mid \mathscr{F}_{t-1} \sim \mathscr{N}\big(0, H_t\big), \tag{2}$$

with the corresponding conditional density function

$$f(r_t \mid \mathscr{F}_{t-1}) = \frac{1}{(2\pi)^{k/2}|H_t|^{1/2}} \exp\left(-\frac{1}{2} r_t' H_t^{-1}\, r_t\right). \tag{3}$$

A general vector GARCH model (beyond the specific BEKK formulation) can be represented as

$$H_t = \Omega + \sum_{i=1}^{p} A_i\, \varepsilon_{t-i}\varepsilon'_{t-i} A'_i + \sum_{j=1}^{q} B_j\, H_{t-j} B'_j, \tag{4}$$

where

- Ω is a $k \times k$ constant matrix.
- α_i and β_j are parameter matrices corresponding to the effects of previous shocks and previous covariances, respectively.
- p and q denote the number of lags for the innovations and the past covariances.

C Appendix C: Figures

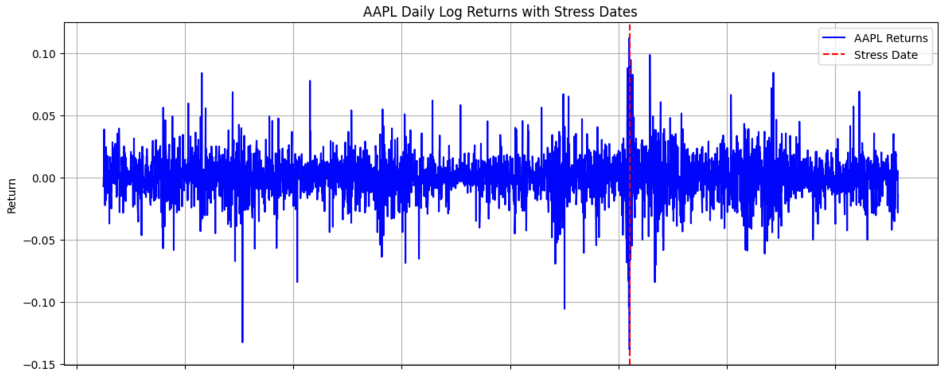

Figure 5: AAPL Log Returns

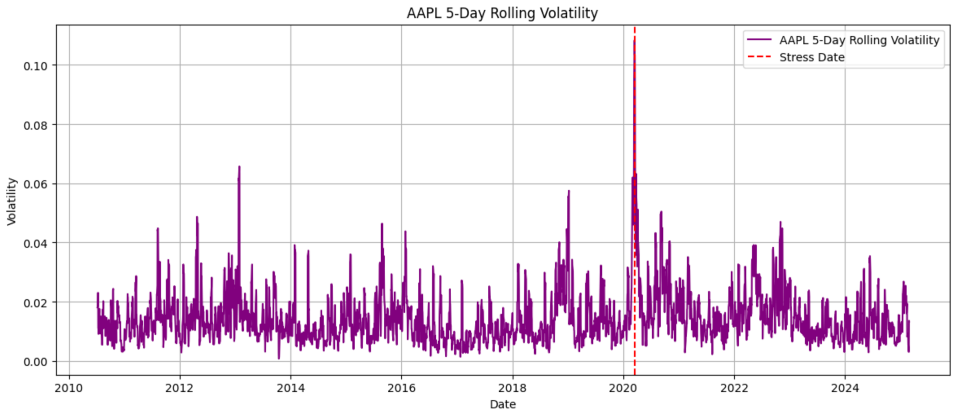

Figure 6: AAPL Volatility

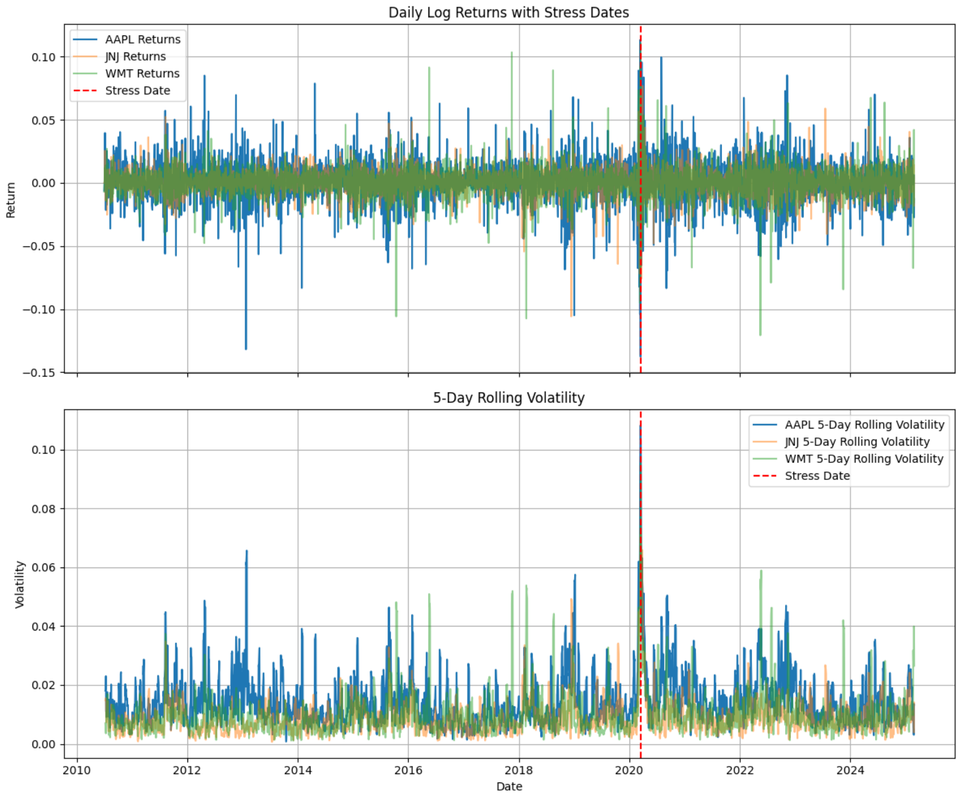

Figure 7: AAPL, JNJ, and WMT Log Returns

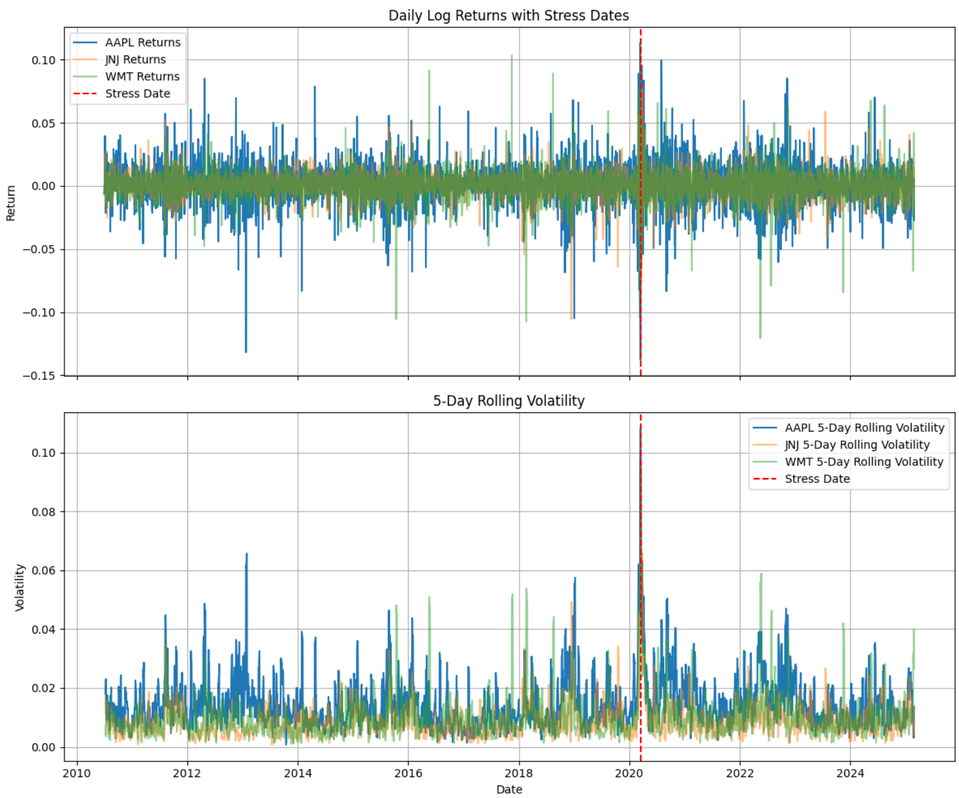

Figure 8: AAPL, JNJ, and WMT Volatility

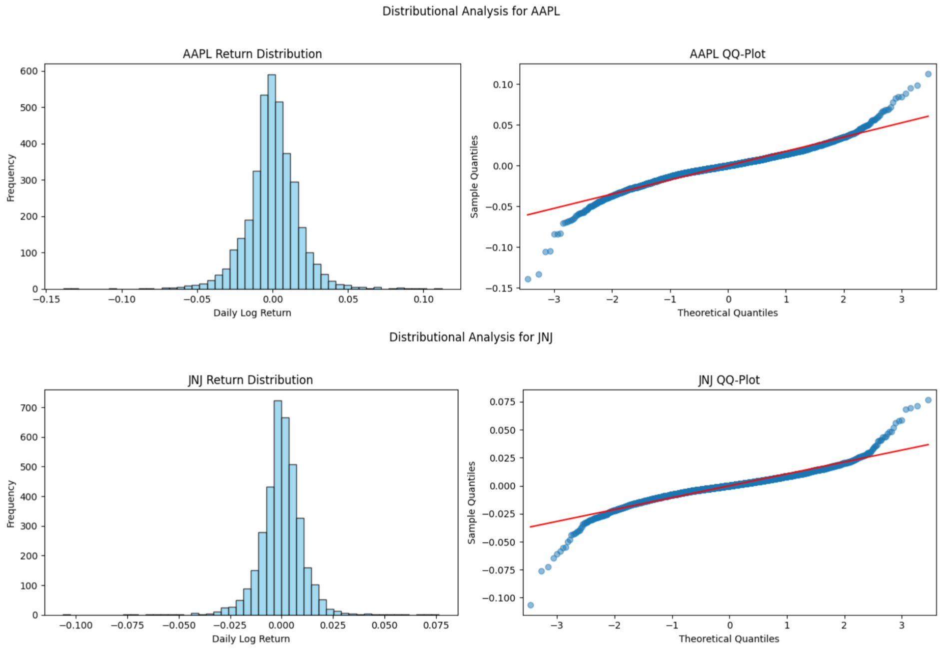

Figure 9: Distributional Pre-Analysis

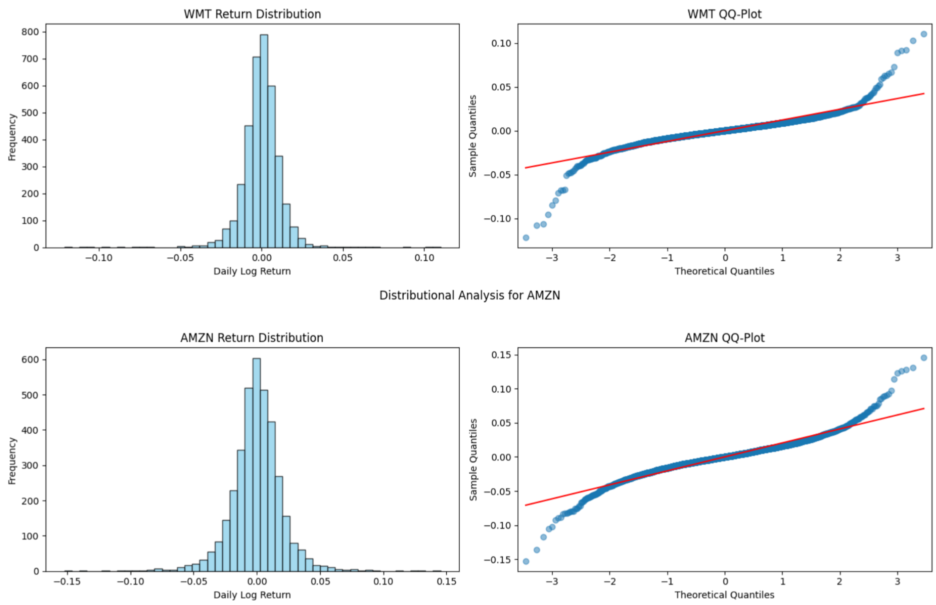

Figure 10: Distributional Pre-Analysis

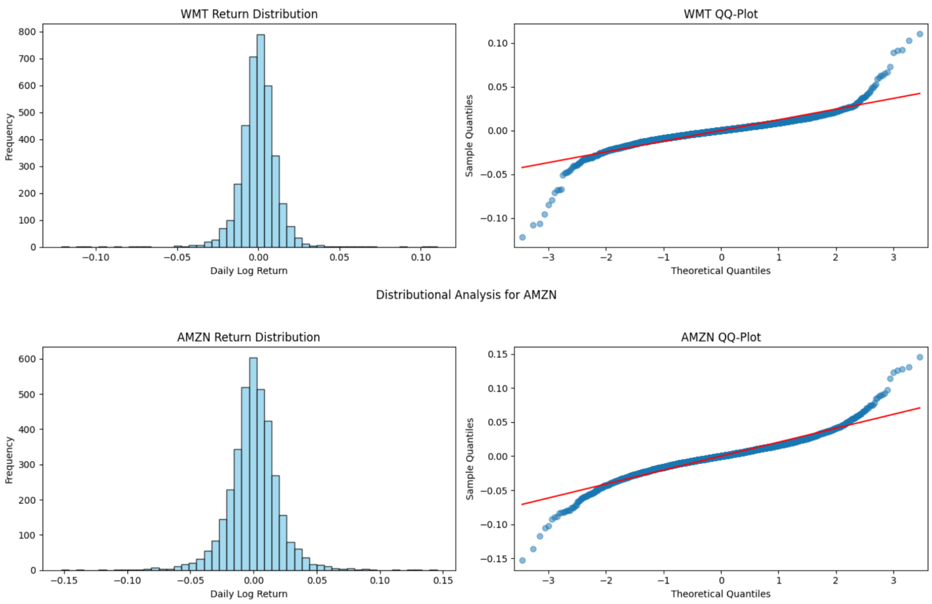

Figure 11: Distributional Pre-Analysis