1 Expansion of the Sudakov form factor

In this section I work out the expansion of the quark Sudakov form factor to $\mathcal{O}(\alpha_s^2)$. The Sudakov form factor is independent of the process and thus the same expansion can be used for example for Drell-Yan and SIDIS.[1] One crucial point is that the expansion will be done in the so-called ζ-prescription introduced in Ref. [1].

The Sudakov form factor we are considering here is essentially the square of the quark evolution factor deriving from the solution of the Collins-Soper (CS) equations. As is well known, the CS equations govern the evolution of TMD distributions in two independent factorisation scales μ and ζ. Usually, μ and ζ are related to each other by assuming that $\mu = \sqrt{\zeta}$. This particular choice reduces the degrees of freedom of the problem. On the other hand, as observed in Ref. [1], this choice does not help cure large logarithms that appear in the matching functions. The mutual independence of μ and ζ can be exploited to guarantee that such large logarithms are reabsorbed in the definition of the scale ζ as a function of μ (or viceversa). This is the basic observation at the base of the ζ-prescription that achieves this goal by assuming $\zeta \equiv \zeta(\mu)$ and requiring that the total derivative of TMD distribution F with respect to μ is zero:

$$\mu^2 \frac{dF\left(x, b; \mu, \zeta(\mu)\right)}{d\mu^2} = 0. \tag{1}$$

This naturally leads to a differential equation in $\zeta(\mu)$ involving the CS anomalous dimensions and $\Gamma_{\rm cusp}$ that can be solved order by order in α_s. The exact form of the function $\zeta(\mu)$ has to be taken into account when expanding the Sudakov form factor.

The exact form of the Sudakov form factor, that we remind to be the squared evolution factor deriving from the solution of the CS equations and the evolves a pair of quark TMD distributions in b space from the scales (μ_i, ζ_i) to (μ_f, ζ_f), is:

$$\left[R(\mu_i, \zeta_i \rightarrow \mu_f, \zeta_f; b)\right]^2 = \exp\left[\int_{\mu_i^2}^{\mu_f^2} \frac{d\mu^2}{\mu^2}\left(-\gamma_V + \Gamma_{\rm cusp} \ln\left(\frac{\mu^2}{\zeta_f}\right)\right) - 2\mathcal{D} \ln\left(\frac{\zeta_f}{\zeta_i}\right)\right]. \tag{2}$$

where the following perturbative expansions hold:

$$\gamma_V = \sum_{n=1} a_s^n \gamma_V^{(n)} \qquad \Gamma_{\rm cusp} = \sum_{n=1} a_s^n \Gamma_{\rm cusp}^{(n)}\,, \tag{3}$$

and:

$$\mathcal{D} = \sum_{n=1} a_s^n \sum_{k=0}^{n} d^{(n,k)} L^k\,, \tag{4}$$

where I have defined:

$$a_s(\mu) = \frac{\alpha_s(\mu)}{4\pi} \quad \text{and} \quad L \equiv \ln\left(\frac{b^2 \mu_i^2}{4e^{-2\gamma_E}}\right)\,. \tag{5}$$

The perturbative coefficients, $\gamma_V^{(n)}$, $\Gamma_{\rm cusp}^{(n)}$, and $d^{(n,k)}$ are know up to the order necessary to implement NNLL evolution.[2] The final scales μ_f and ζ_f are usually taken to be equal to $\mu_f = \sqrt{\zeta} = \kappa Q$, where Q is, for example the virtuality of the exchanged photon in DIS and the invariant mass of the lepton pair in Drell-Yan. In addition, using the ζ-prescription, we set:

$$\zeta_i \equiv \zeta(\mu_i) = \mu_i^2 \exp\left(\sum_{n=0} a_s^n \sum_{k=0}^{n+1} \ell^{(n,k)} L^k\right) \tag{6}$$

where the coefficients $\ell^{(n,k)}$ are constants.

Before expanding the exponential we need to write its argument as a polynomial in α_s (or better a_s) computed in Q. These terms will in turn multiply powers of $\ln(Q)$. In view of the integral over

[1] Note that for gluon-initiated processes the expression will be different.

[2] In fact, with the only exception of $\Gamma_{\rm cusp}^{(4)}$, it would be possible to implement evolution up to N^{3}LL.

the impact parameter b needed to obtain cross sections differential in the transverse momentum q_T, we need to assign μ_i a dependence on b. Despite this dependence is arbitrary and needs only be dimensionally correct, the most natural choice is:

$$\mu_i = \frac{C_0}{b}\,, \quad \text{with} \quad C_0 = 2e^{-\gamma_E}\,. \tag{7}$$

This choice is such that L defined in Eq. (5) vanishes so that:

$$\mathcal{D} = \sum_{n=1} a_s^n d^{(n,0)} \tag{8}$$

and:

$$\zeta(\mu_i) = \frac{C_0^2}{b^2} \exp\left(\sum_{n=0} a_s^n \ell^{(n,0)}\right)\,. \tag{9}$$

This way the Sudakov form factor in Eq. (2) reads:

$$\begin{aligned}\left[R(Q,b)\right]^2 &= \exp\Bigg\{ -\sum_{n=1}^{2} \int_{C_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} a_s^n(\mu) \left[\gamma_V^{(n)} + \Gamma_{\text{cusp}}^{(n)} \ln\left(\frac{Q^2}{\mu^2}\right)\right] \\ &+ 2\sum_{n=1}^{2} a_s^n \left(\frac{C_0}{b}\right) d^{(n,0)} \left[-\ln\left(\frac{b^2Q^2}{C_0^2}\right) + \ell^{(0,0)} + a_s\left(\frac{C_0}{b}\right)\ell^{(1,0)}\right]\Bigg\}\,,\end{aligned} \tag{10}$$

where we limited the contributions in the exponential to $\mathcal{O}(a_s^2)$ that is the order we are interested in. Now, using the RGE at leading order:

$$\mu^2 \frac{da_s}{d\mu^2} = -\beta_0 a_s^2(\mu)\,, \tag{11}$$

whose solution is:

$$a_s(\mu) = \frac{a_s(Q)}{1 - a_s(Q)\beta_0 \ln(\mu^2/Q^2)} \simeq a_s(Q)\left[1 + a_s(Q)\beta_0 \ln(Q^2/\mu^2) + \mathcal{O}(a_s^2)\right]\,, \tag{12}$$

we write every instance of a_s appearing in Eq. (10) in terms of $a_s(Q)$ and finally retain only terms up to $a_s^2(Q)$:

$$\begin{aligned}\left[R(Q,b)\right]^2 &= \exp\Bigg\{ - a_s(Q) \int_{C_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[\gamma_V^{(1)} + \Gamma_{\text{cusp}}^{(1)} \ln\left(\frac{Q^2}{\mu^2}\right)\right] \\ &- a_s^2(Q) \int_{C_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[\gamma_V^{(2)} + \left(\Gamma_{\text{cusp}}^{(2)} + \gamma_V^{(1)}\beta_0\right) \ln\left(\frac{Q^2}{\mu^2}\right) + \Gamma_{\text{cusp}}^{(1)}\beta_0 \ln^2\left(\frac{Q^2}{\mu^2}\right)\right] \\ &+ 2a_s(Q)\left[d^{(1,0)}\ell^{(0,0)} - d^{(1,0)} \ln\left(\frac{b^2Q^2}{C_0^2}\right)\right] \\ &+ 2a_s^2(Q)\left[d^{(2,0)}\ell^{(0,0)} + d^{(1,0)}\ell^{(1,0)} + \left(-d^{(2,0)} + d^{(1,0)}\beta_0\ell^{(0,0)}\right)\ln\left(\frac{b^2Q^2}{C_0^2}\right) - d^{(1,0)}\beta_0 \ln^2\left(\frac{b^2Q^2}{C_0^2}\right)\right]\Bigg\}\,.\end{aligned} \tag{13}$$

The final step before carrying out the expansion is computing the integrals:

$$\int_{C_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \ln^k\left(\frac{Q^2}{\mu^2}\right) = \int_{\ln(C_0^2/b^2)}^{\ln Q^2} d\ln\mu^2 \ln^k\left(\frac{Q^2}{\mu^2}\right) = \int_0^{\ln(b^2Q^2/C_0^2)} dx\, x^k = \frac{1}{k+1}\ln^{k+1}\left(\frac{b^2Q^2}{C_0^2}\right)\,. \tag{14}$$

If we define:

$$\mathcal{L} \equiv \ln\left(\frac{b^2Q^2}{C_0^2}\right)\,, \tag{15}$$

the Sudakov form factor in Eq. 13 reads:

$$\begin{aligned}
[R(Q,\mathcal{L})]^2 &= \exp\left\{a_s(Q)\left[2d^{(1,0)}\ell^{(0,0)} - \left(\gamma_V^{(1)} + 2d^{(1,0)}\right)\mathcal{L} - \frac{1}{2}\Gamma_{\rm cusp}^{(1)}\mathcal{L}^2\right]\right.\\
&+ a_s^2(Q)\left[2d^{(2,0)}\ell^{(0,0)} + 2d^{(1,0)}\ell^{(1,0)} + \left(-2d^{(2,0)} + 2d^{(1,0)}\beta_0\ell^{(0,0)} - \gamma_V^{(2)}\right)\mathcal{L}\right.\\
&- \left.\left.\frac{1}{2}\left(4d^{(1,0)}\beta_0 + \Gamma_{\rm cusp}^{(2)} + \gamma_V^{(1)}\beta_0\right)\mathcal{L}^2 - \frac{1}{3}\Gamma_{\rm cusp}^{(1)}\beta_0\mathcal{L}^3\right]\right\}\,,
\end{aligned} \tag{16}$$

that can be conveniently written as:

$$[R(Q,\mathcal{L})]^2 = \exp\left\{\sum_{n=1}^{2} a_s^n(Q)\sum_{k=0}^{n+1} S^{(n,k)}\mathcal{L}^k\right\}\,, \tag{17}$$

with:

$$\begin{aligned}
&S^{(1,0)} = 2d^{(1,0)}\ell^{(0,0)} = 0\,,\quad S^{(1,1)} = -\left(\gamma_V^{(1)} + 2d^{(1,0)}\right) = 6C_F\,,\quad S^{(1,2)} = -\frac{1}{2}\Gamma_{\rm cusp}^{(1)} = -2C_F\,,\\
&S^{(2,0)} = 2d^{(2,0)}\ell^{(0,0)} + 2d^{(1,0)}\ell^{(1,0)}\,,\quad S^{(2,1)} = \left(-2d^{(2,0)} + 2d^{(1,0)}\beta_0\ell^{(0,0)} - \gamma_V^{(2)}\right)\,,\\
&S^{(2,2)} = -\frac{1}{2}\left(4d^{(1,0)}\beta_0 + \Gamma_{\rm cusp}^{(2)} + \gamma_V^{(1)}\beta_0\right)\,,\quad S^{(2,3)} = -\frac{1}{3}\Gamma_{\rm cusp}^{(1)}\beta_0\,.
\end{aligned} \tag{18}$$

The values of the coefficients of the anomalous dimensions and β-function can be read from Appendix D of Ref. [2]. The coefficients of the expansion of $\zeta(\mu)$ are instead reported in Eq. (2.29) of Ref. [1]. For the $\mathcal{O}(a_s)$ coefficients I reported the explicit values. This will help check the result against those in the literature.

Eq. (17) can be easily expanded as up to order a_s^2 as:

$$\begin{aligned}
[R(Q,\mathcal{L})]^2 &= 1 + a_s(Q)\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k + a_s^2(Q)\left[\sum_{k=0}^{3} S^{(2,k)}\mathcal{L}^k + \frac{1}{2}\left(\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k\right)^2\right] + \mathcal{O}(a_s^3)\\
&= 1 + a_s(Q)\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k + a_s^2(Q)\sum_{k=0}^{4} \widetilde{S}^{(2,k)}\mathcal{L}^k + \mathcal{O}(a_s^3)\\
&\equiv 1 + a_s(Q)R^{(1)} + a_s^2(Q)R^{(2)} + \mathcal{O}(a_s^3)\,,
\end{aligned} \tag{19}$$

with:

$$\begin{aligned}
&\widetilde{S}^{(2,0)} = S^{(2,0)} + \frac{1}{2}\left[S^{(1,0)}\right]^2\,,\\
&\widetilde{S}^{(2,1)} = S^{(2,1)} + S^{(1,0)}S^{(1,1)}\\
&\widetilde{S}^{(2,2)} = S^{(2,2)} + \frac{1}{2}\left[S^{(1,1)}\right]^2 + S^{(1,2)}S^{(1,0)}\,,\\
&\widetilde{S}^{(2,3)} = S^{(2,3)} + S^{(1,1)}S^{(1,2)}\\
&\widetilde{S}^{(2,4)} = \frac{1}{2}\left[S^{(1,2)}\right]^2\,.
\end{aligned} \tag{20}$$

Now that the expansion of the Sudakov form factor is done to $\mathcal{O}(a_s^2)$, it can be combined to the rest of the perturbative quantities entering the computation of the cross section. In the following, we will concentrate on SIDIS that involves both TMD PDFs F and TMD FFs D. Specifically, in b space the SIDIS cross section is a combination of terms having the following structure:

$$\begin{aligned}
B_{ij} &= H(Q)\, F_i(x,b;Q)\, D_j(z,b;Q) = H(Q)\, \left[R(Q,\mathcal{L})\right]^2\, F_i\left(x,b;\frac{C_0}{b}\right)\, D_j\left(z,b;\frac{C_0}{b}\right) \\
&= H(Q)\, \left[R(Q,\mathcal{L})\right]^2 \left[\sum_k \mathcal{C}_{ik}(x,\mathcal{L}) \underset{x}{\otimes} f_k\,(x;Q)\right] \left[\sum_l \mathbb{C}_{jl}(z,\mathcal{L}) \underset{z}{\otimes} d_l\,(z;Q)\right] \\
&\equiv \sum_{kl} \hat{B}_{ij,kl}(x,z;b) \underset{x}{\otimes} f_k \underset{z}{\otimes} d_l\,,
\end{aligned} \tag{21}$$

where f_k and d_l are the (non-perturbative) collinear PDFs and FFs, respectively. The other terms, including R^2, are perturbatively computable as:

$$\begin{aligned}
&H(Q) = 1 + a_s(Q)H^{(1)} + a_s^2(Q)H^{(2)} + \mathcal{O}(a_s^3)\,, \\
&\mathcal{C}_{ik}(x,\mathcal{L}) = \delta_{ik}\delta(1-x) + a_s(Q)\mathcal{C}_{ik}^{(1)}(x,\mathcal{L}) + a_s^2(Q)\mathcal{C}_{ik}^{(2)}(x,\mathcal{L}) + \mathcal{O}(a_s^3)\,, \\
&\mathbb{C}_{jl}(z,\mathcal{L}) = \delta_{jl}\delta(1-z) + a_s(Q)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L}) + a_s^2(Q)\mathbb{C}_{jl}^{(2)}(z,\mathcal{L}) + \mathcal{O}(a_s^3)\,.
\end{aligned} \tag{22}$$

We now need to put everything together and truncate to order a_s^2 in such a way that:

$$\hat{B}_{ij,kl}(x,z;b) = \hat{B}_{ij,kl}^{(0)}(x,z;b) + a_s(Q)\hat{B}_{ij,kl}^{(1)}(x,z;b) + a_s^2(Q)\hat{B}_{ij,kl}^{(2)}(x,z;b) + \mathcal{O}(a_s^3)\,, \tag{23}$$

with:

$$\begin{aligned}
\hat{B}_{ij,kl}^{(0)} &= \delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\,, \\
\hat{B}_{ij,kl}^{(1)} &= (H^{(1)} + R^{(1)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ \mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L}) \\
\hat{B}_{ij,kl}^{(2)} &= (H^{(2)} + H^{(1)}R^{(1)} + R^{(2)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ (H^{(1)} + R^{(1)})\left[\mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L})\right] \\
&+ \mathcal{C}_{ik}^{(2)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\mathbb{C}_{jl}^{(1)}(z,\mathcal{L}) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(2)}(z,\mathcal{L})
\end{aligned} \tag{24}$$

Despite we derived the expansion of the resummed cross section up to $\mathcal{O}(a_s^2)$, these are not yet the final formulas. The reason is that some of the terms of the expansion above do not depend on the impact parameter b. This means that these terms, upon Fourier transform needed to obtain the expression in the transverse moment q_T, will give rise to terms proportional to $\delta(q_T)$. In view of the matching procedure, since the fixed-order expression we are going to match to does not include any $\delta(q_T)$ terms, we need to make sure that they are not subtracted.Therefore, we need to identify and remove any term that does not depend on b. In fact, by construction, this is equivalent to leave only terms proportional to a power of $\mathcal{L}$. In order to do so, we start observing that the hard factor H does not contain any $\mathcal{L}$. In addition, the coefficients $R^{(n)}$ are polynomials in $\mathcal{L}$ and also include a constant term ($\mathcal{L}^0$). Finally, the ζ-prescription, Eq. (1), provides us with a simple recipe to compute the logarithmic terms of the matching functions $\mathcal{C}_{ik}^{(n)}$ and $\mathbb{C}_{jl}^{(n)}$. Specifically, the condition of independence of the TMD PDFs and FFs from the factorisation scale μ is such that TMDs behave like physical observable (*e.g.* deep-inelastic-scattering inclusive structure functions) and thus obey the standard scale variation rules derived, for example, in Eq. (2.17) of Ref. [3]. More in particular, one finds that the matching function coefficients have the usual logarithmic structure:

$$\mathcal{C}_{ij}^{(n)}(x,\mathcal{L}) = \sum_{k=0}^{n} \mathcal{C}_{ij}^{(n,k)}(x)\mathcal{L}^k \quad \text{and} \quad \mathbb{C}_{ij}^{(n)}(x,\mathcal{L}) = \sum_{k=0}^{n} \mathbb{C}_{ij}^{(n,k)}(x)\mathcal{L}^k\,, \tag{25}$$

where the non-logarithmic terms $\mathcal{C}_{ij}^{(n,0)}$ and $\mathbb{C}_{ij}^{(n,0)}$ have to be computed explicitly while the other terms proportional to a positive power of $\mathcal{L}$ can be expressed in terms of the non-logarithmic term of the previous orders and of the coefficients of the DGLAP splitting functions and of the QCD β function:

$$\begin{aligned}
\mathcal{C}_{ij}^{(1,1)}(x) &= -\mathcal{P}_{ij}^{(1)}(x)\,,\\
\mathcal{C}_{ij}^{(2,1)}(x) &= -\left(\mathcal{P}_{ij}^{(2)}(x) + \mathcal{C}_{ik}^{(1,0)}(x)\otimes\mathcal{P}_{kj}^{(1)}(x) - \beta_0\mathcal{C}_{ij}^{(1,0)}(x)\right)\,,\\
\mathcal{C}_{ij}^{(2,2)}(x) &= \frac{1}{2}\left(\mathcal{P}_{ik}^{(1)}(x)\otimes\mathcal{P}_{kj}^{(1)}(x) - \beta_0\mathcal{P}_{ij}^{(1)}(x)\right)\,,
\end{aligned} \tag{26}$$

and:

$$\begin{aligned}
\mathbb{C}_{ij}^{(1,1)}(x) &= -\mathbb{P}_{ij}^{(1)}(x)\,,\\
\mathbb{C}_{ij}^{(2,1)}(x) &= -\left(\mathbb{P}_{ij}^{(2)}(x) + \mathbb{C}_{ik}^{(1,0)}(x)\otimes\mathbb{P}_{kj}^{(1)}(x) - \beta_0\mathbb{C}_{ij}^{(1,0)}(x)\right)\,,\\
\mathbb{C}_{ij}^{(2,2)}(x) &= \frac{1}{2}\left(\mathbb{P}_{ik}^{(1)}(x)\otimes\mathbb{P}_{kj}^{(1)}(x) - \beta_0\mathbb{P}_{ij}^{(1)}(x)\right)\,,
\end{aligned} \tag{27}$$

where $\mathcal{P}_{ij}^{(n)}$ and $\mathbb{P}_{ij}^{(n)}$ are the coefficients of the a_s^n terms of the space- and time-like splitting functions, respectively. With this information at hand, knowing the logarithmic expansion of the coefficients $R^{(n)}$, and keeping in mind that the hard coefficients $H^{(n)}$ do not contain any logarithms, we can organize the coefficients inEq. (24) in terms of powers of $\mathcal{L}$. Specifically, we find that:

$$\hat{B}_{ij,kl}^{(n)} = \sum_{p=0}^{2n} \hat{B}_{ij,kl}^{(n,p)}\mathcal{L}^p\,, \tag{28}$$

with the $\mathcal{O}(1)$ coefficient being:

$$\hat{B}_{ij,kl}^{(0,0)} = \delta_{ik}\delta_{kl}\delta(1-x)\delta(1-z)\,, \tag{29}$$

the $\mathcal{O}(a_s)$ coefficients being:

$$\begin{aligned}
\hat{B}_{ij,kl}^{(1,0)} &= (H^{(1)}+S^{(1,0)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + \mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,0)}(z)\,,\\
\hat{B}_{ij,kl}^{(1,1)} &= S^{(1,1)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + \mathcal{C}_{ik}^{(1,1)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,1)}(z)\,,\\
\hat{B}_{ij,kl}^{(1,2)} &= S^{(1,2)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)
\end{aligned} \tag{30}$$

and the $\mathcal{O}(a_s^2)$ coefficients being:

$$\begin{aligned}
\hat{B}^{(2,0)}_{ij,kl} &= (H^{(2)} + H^{(1)}S^{(1,0)} + \widetilde{S}^{(2,0)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\,, \\
&+ (H^{(1)} + S^{(1,0)})\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,0)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,0)}_{jl}(z) \\
\hat{B}^{(2,1)}_{ij,kl} &= (H^{(1)}S^{(1,1)} + \widetilde{S}^{(2,1)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ (H^{(1)} + S^{(1,0)})\left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
&+ S^{(1,1)}\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,1)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) + \mathcal{C}^{(1,0)}_{ik}(x)\mathbb{C}^{(1,1)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,1)}_{jl}(z) \\
\hat{B}^{(2,2)}_{ij,kl} &= (H^{(1)}S^{(1,2)} + \widetilde{S}^{(2,2)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ \widetilde{S}^{(1,2)}\left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ S^{(1,1)}\left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,2)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,1)}_{ik}(x)\mathbb{C}^{(1,1)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,2)}_{jl}(z) \\
\hat{B}^{(2,3)}_{ij,kl} &= \widetilde{S}^{(2,3)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + S^{(1,2)}\left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
\hat{B}^{(2,4)}_{ij,kl} &= \widetilde{S}^{(2,4)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)
\end{aligned} \tag{31}$$

In order to obtain a differential cross section in q_T, we need to take the Fourier transform of $\hat{B}_{ij,kl}$, that is:

$$\hat{B}_{ij,kl}(x,z;q_T) \equiv \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T}\hat{B}_{ij,kl}(x,z;b) = \sum_{n=0}^{2} a_s^n(Q)\sum_{p=0}^{2n}\hat{B}^{(n,p)}_{ij,kl}(x,z)I_p(q_T)\,, \tag{32}$$

where I have defined:

$$I_p(q_T) = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T}\mathcal{L}^p = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T}\ln^p\left(\frac{b^2Q^2}{C_0^2}\right) = \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)\ln^p\left(\frac{b^2Q^2}{C_0^2}\right)\,. \tag{33}$$

Results for I_p have been computed up to $p = 4$ in Eq. (136) of Appendix B of Ref. [4]. Specifically, and including the trivial tranform with $p = 0$, they read:

$$
\begin{aligned}
I_0(q_T) &= \delta(q_T)\,,\\
I_1(q_T) &= -\frac{1}{q_T^2}\,,\\
I_2(q_T) &= -\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\,,\\
I_3(q_T) &= -\frac{3}{q_T^2}\ln^2\left(\frac{Q^2}{q_T^2}\right)\,,\\
I_4(q_T) &= -\frac{4}{q_T^2}\left[\ln^3\left(\frac{Q^2}{q_T^2}\right)-4\zeta_3\right]\,.
\end{aligned}
\tag{34}
$$

As clear from the transforms above, all terms with $p = 0$ will be proportional to $\delta(q_T)$. We do not need to consider these terms because analogous terms are not included in the fixed-order calculation and thus does not need to be subtracted. Therefore, we write:

$$
\hat{B}_{ij,kl}(x,z;q_T) = \sum_{n=1}^{2} a_s^n(Q)\sum_{p=1}^{2n}\hat{B}_{ij,kl}^{(n,p)}(x,z)I_p(q_T) + \left(\sum_{n=0}^{2} a_s^n(Q)\hat{B}_{ij,kl}^{(n,0)}(x,z)\right)\delta(q_T) + \mathcal{O}(a_s^3)\,, \tag{35}
$$

and we are not going to consider the term proportional to $\delta(q_T)$, even though all terms have been derived above.

As clear from Eq. (35), removing all terms proportional to $\delta(q_T)$ also means removing the full $\mathcal{O}(1)$ terms such that leading-order term is now $\mathcal{O}(a_s)$.

In order to validate the results above, it is opportune to compare the $\mathcal{O}(a_s)$ expressions to those present in the literature. To this end, we write explicitly the expression for $\hat{B}_{ij,kl}^{(1)}$ in q_T space without the $\delta(q_T)$ term:

$$
\begin{aligned}
\hat{B}_{ij,kl}^{(1)}(x,z;q_T) &= -\hat{B}_{ij,kl}^{(1,1)}(x,z)\frac{1}{q_T^2} - \hat{B}_{ij,kl}^{(1,2)}(x,z)\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\\
&= \frac{1}{q_T^2}\left[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\right.\\
&+ \left.\mathcal{P}_{ik}^{(1)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(1)}(z)\right]\,,
\end{aligned}
\tag{36}
$$

so that:

$$
\begin{aligned}
B_{ij}(x,z;q_T) &= a_s(Q)\sum_{kl}\hat{B}_{ij,kl}^{(1)}\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q) + \mathcal{O}(a_s^2)\\
&= a_s(Q)\frac{1}{q_T^2}\left[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x,Q)d_j(z,Q)\right.\\
&+ \left.\left(\sum_k \mathcal{P}_{ik}^{(1)}(x)\underset{x}{\otimes} f_k(x,Q)\right)d_j(z,Q) + f_i(x,Q)\left(\sum_l \mathbb{P}_{jl}^{(1)}(z)\underset{z}{\otimes} d_l(z,Q)\right)\right] + \mathcal{O}(a_s^2)\,.
\end{aligned}
\tag{37}
$$

This result, up to som pre-factor and to selecting the right flavour combinations,[3] nicely agrees with that of, *e.g.*, Refs. [5, 7, 6].

[3] We have purposely left the flavour indices i and j unspecified because they do not necessarily run over the quark flavours u, d, etc. In fact, it turns out to be useful to adopt a different basis (usual referred to as evolution basis) that simplifies the structure of the products between, for example, PDFs/FFs and splitting functions.

In order to check that the matching is actually removing the double counting terms, it is instructive to derive Eq. (37) extracting the asymptote from the fixed-order computation at $\mathcal{O}(a_s)$. We take the expressions for the coefficient functions from Eqs. (106)-(109) of Appendix B of Ref. [7] or from Eqs. (4.6)-(4.20) of Ref. [8].Referring to the second reference, some simplifications apply. First, we consider cross sections with unpolarised projectiles ($\lambda_e = 0$) over unpolarised targets ($S_\perp^\mu = 0$) and integrated over the azimuthal angles ϕ_H and ϕ_S. By doing so and after a simple manipulation, the cross section simplifies greatly and can be written in terms of structure functions as:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{2\pi\alpha^2}{xyQ^2}\left[Y_+ F_{UU,T} + 2(1-y)F_{UU,L}\right] = \frac{2\pi\alpha^2}{xyQ^2}Y_+\left[F_{UU,2} - \frac{y^2}{Y_+}F_{UU,L}\right], \tag{38}$$

with:

$$Y_+ \equiv 1 + (1-y)^2\,, \tag{39}$$

and where we have defined the structure function:

$$F_{UU,2} \equiv F_{UU,T} + F_{UU,L}\,. \tag{40}$$

Notice that, as compared to Ref. [8], we have factored out from the structure functions a further factor $1/(\pi z^2)^4$ so that they factorize as:

$$\begin{aligned} F_{UU,S} \;=\;& a_s \frac{x}{Q^2}\sum_i e_i^2 \int_x^1 \frac{d\bar{x}}{\bar{x}}\int_z^1 \frac{d\bar{z}}{\bar{z}}\delta\left(\frac{q_T^2}{Q^2} - \frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)\left[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_i\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right)\right. \\ &+ \left.\hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_g\left(\frac{x}{\bar{x}}\right) d_i\left(\frac{z}{\bar{z}}\right) + \hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x},\bar{z},q_T) f_i\left(\frac{x}{\bar{x}}\right) d_g\left(\frac{z}{\bar{z}}\right)\right] + \mathcal{O}(a_s^2)\,. \end{aligned} \tag{41}$$

with $S = 2, L$ and where the sum over i runs over the active quark and antiquark flavours and e_i is the electric charge of the i-th flavour. The expression for the coefficient functions are:

$$\begin{aligned} &\hat{B}_{qq}^{2,\mathrm{FO}}(x,z,q_T) = 2C_F\left[(1-x)(1-z) + 4xz + \frac{1+x^2z^2}{xz}\frac{Q^2}{q_T^2}\right], \\ &\hat{B}_{qq}^{L,\mathrm{FO}}(x,z,q_T) = 8C_F xz\,, \\ &\hat{B}_{qg}^{2,\mathrm{FO}}(x,z,q_T) = 2T_R\left[[x^2+(1-x)^2][z^2+(1-z)^2]\frac{1-x}{xz^2}\frac{Q^2}{q_T^2} + 8x(1-x)\right], \\ &\hat{B}_{qg}^{L,\mathrm{FO}}(x,z,q_T) = 16T_R x(1-x)\,. \\ &\hat{B}_{gq}^{2,\mathrm{FO}}(x,z,q_T) = 2C_F\left[(1-x)z + 4x(1-z) + \frac{1+x^2(1-z)^2}{xz}\frac{1-z}{z}\frac{Q^2}{q_T^2}\right], \\ &\hat{B}_{gq}^{L,\mathrm{FO}}(x,z,q_T) = 8C_F x(1-z)\,, \end{aligned} \tag{42}$$

These expressions are enough to compute the SIDIS cross section at $\mathcal{O}(a_s)$ in the region $q_T \lesssim Q$. In order to match Eq. (37) one has to take the limit $q_T \ll Q$ and retain in the coefficient functions only the terms enhanced as Q^2/q_T^2. This automatically means that $F_{UU,L}$ does not contribute in the $q_T \leftarrow 0$ limit:

$$F_{UU,L} \underset{q_T \ll Q}{\longrightarrow} 0\,. \tag{43}$$

[4]Factor z^2 is the consequence of the fact that we are writing the cross section differential in q_T^2 that is the transverse momentum of the exchanged photon while in Ref. [8] the cross section is differentila in p_T^2 that is the transverse momentum of the of the outgoing hadrons. Since $p_T = zq_T$, the factor z^2 cancels.

Another crucial observation is that the δ-function in Eq. (41) can be expanded as follows[5]:

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-x)(1-z)}{xz}\right)\underset{q_T^2/Q^2\to 0}{\longrightarrow}\ln\left(\frac{Q^2}{q_T^2}\right)\delta(1-x)\delta(1-z)+\frac{x\delta(1-z)}{(1-x)_+}+\frac{z\delta(1-x)}{(1-z)_+}\,, \tag{44}$$

so that:

$$\begin{aligned}F_{UU,2}\quad&\underset{q_T\ll Q}{\longrightarrow}\quad a_s\frac{x}{q_T^2}\sum_i e_i^2\int_x^1\frac{d\bar{x}}{\bar{x}}\int_z^1\frac{d\bar{z}}{\bar{z}}\Big[\hat{B}_{qq}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)\\&+\quad\hat{B}_{qg}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_g\left(\frac{x}{\bar{x}}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}_{gq}^{2,\mathrm{asy}}(\bar{x},\bar{z},q_T)f_i\left(\frac{x}{\bar{x}}\right)d_g\left(\frac{z}{\bar{z}}\right)\Big]+\mathcal{O}(a_s^2)\,.\end{aligned} \tag{45}$$

with:

$$\begin{aligned}\hat{B}_{qq}^{2,\mathrm{asy}}(x,z,q_T)&=2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)+\frac{1+x^2}{(1-x)_+}\delta(1-z)+\delta(1-x)\frac{1+z^2}{(1-z)_+}\right]\\&=2C_F\left[2\ln\left(\frac{Q^2}{q_T^2}\right)-3\right]\delta(1-x)\delta(1-z)+\mathcal{P}_{qq}^{(1)}(x)\delta(1-z)+\delta(1-x)\mathbb{P}_{qq}^{(1)}(z)\,,\\\hat{B}_{qg}^{2,\mathrm{asy}}(x,z,q_T)&=2T_R\left[x^2+(1-x)^2\right]\delta(1-z)=\mathcal{P}_{qg}^{(1)}(x)\delta(1-z)\,,\\\hat{B}_{gq}^{2,\mathrm{asy}}(x,z,q_T)&=\delta(1-x)2C_F\left[\frac{1+(1-z)^2}{z}\right]=\delta(1-x)\mathbb{P}_{qg}^{(1)}(z)\,.\end{aligned} \tag{46}$$

It is thus easy to see that we can rewrite Eq. (45) as:

$$\begin{aligned}F_{UU,2}\quad&\underset{q_T\ll Q}{\longrightarrow}\quad a_s\frac{x}{q_T^2}\sum_i e_i^2\Bigg[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i\left(x\right)d_i\left(z\right)\\&+\quad\left(\sum_{k=q,g}\mathcal{P}_{qk}^{(1)}(x)\otimes f_k\left(x\right)\right)d_i\left(z\right)+f_i\left(x\right)\left(\sum_{k=q,g}\mathbb{P}_{qk}^{(1)}(z)\otimes d_k\left(z\right)\right)\Bigg]+\mathcal{O}(a_s^2)\,.\end{aligned} \tag{47}$$

Eq. (47) agrees with Eq. (37). This confirms that this term removes the double-counting terms when doing the matching.

In order to provide a version of Eq. (41) that can be readily implemented, we need to perform one of the integrals making use of the δ-function. We integrate over $\bar{x}$ so that we write:

$$\delta\left(\frac{q_T^2}{Q^2}-\frac{(1-\bar{x})(1-\bar{z})}{\bar{x}\bar{z}}\right)=\frac{\bar{z}\bar{x}_0^2}{1-\bar{z}}\delta(\bar{x}-\bar{x}_0)\,, \tag{48}$$

with:

$$\bar{x}_0=\frac{1-\bar{z}}{1-\bar{z}\left(1-\frac{q_T^2}{Q^2}\right)}\,. \tag{49}$$

This allows us to write:

$$\begin{aligned}F_{UU,S}\quad&=\quad a_s\frac{x}{Q^2}\sum_i e_i^2\int_z^{z_{\max}}\frac{d\bar{z}}{1-\bar{z}}\bar{x}_0\Big[\hat{B}_{qq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)\\&+\quad\hat{B}_{qg}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_g\left(\frac{x}{\bar{x}_0}\right)d_i\left(\frac{z}{\bar{z}}\right)+\hat{B}_{gq}^{S,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)f_i\left(\frac{x}{\bar{x}_0}\right)d_g\left(\frac{z}{\bar{z}}\right)\Big]+\mathcal{O}(a_s^2)\,,\end{aligned} \tag{50}$$

[5]The proof of the relation in is given in Appendix A.

with:

$$z_{\max} = \frac{1-x}{1-x\left(1-\frac{q_T^2}{Q^2}\right)} \,. \tag{51}$$

Now I would like to rewrite the cross section above in such a way that it matches that at $\mathcal{O}(a_s)$ of Ref. [9]. That would allow us to confidently use the $\mathcal{O}(a_s^2)$ calculation presented right in that reference for the matching to the resummed calculation. This is made a little tricky by the different notation used in Ref. [9] and from the fact that in that paper the cross section is differential in a different set of variables. Specifically, we would like it to be differential in x, y, z, and q_T^2 while in Ref. [9] it is differential in x, Q^2, η, and p_T^2, where the last two are the rapidity and the transverse momentum of the outgoing hadron, respectively. Eq. (13) of Ref. [9], can be translated into our notation by noticing that:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{x}{zQ^2}\sum_{i,j}\int_z^{z_{\max}} \frac{d\bar{z}}{1-\bar{z}} f_i\left(\frac{x}{\bar{x}_0}\right) d_j\left(\frac{z}{\bar{z}}\right) \frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta} + \mathcal{O}(a_s^2) \tag{52}$$

where we have exploited the δ-functions in Eqs. (18)-(20) to get rid of the integral over z.[6]

The $\mathcal{O}(a_s)$ partonic cross sections in Eqs. (18)-(20) of Ref. [9], setting $\varepsilon = 0$, can be written as:

$$\frac{d\sigma_{ij}^{(1)}}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2 a_s e_q^2 \bar{x}_0}{xQ^4} Y_+ \left[\underbrace{\left(F_{UU,M}^{ij}(\bar{x}_0,\bar{z}) + \frac{3}{2}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right)}_{F_{UU,2}^{ij}} - \frac{y^2}{Y_+}F_{UU,L}^{ij}(\bar{x}_0,\bar{z})\right] . \tag{53}$$

One can verify that $F_{UU,2}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qq}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,L}^{qg}(\bar{x}_0,\bar{z})$, $F_{UU,M}^{gq}(\bar{x}_0,\bar{z})$, and $F_{UU,L}^{gq}(\bar{x}_0,\bar{z})$ exactly correspond to $\hat{B}_{qq}^{2,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{qg}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, $\hat{B}_{gq}^{M,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$, and $\hat{B}_{gq}^{L,\mathrm{FO}}(\bar{x}_0,\bar{z},q_T)$ of Eq. (42). It is crucial to notice that the correspondence holds only if $\bar{x}_0$ defined in Eq. (49) as a function of $\bar{z}$ is used. As an example, taking into account the different factors (a factor 2 for $F_{UU,M}$ and a factor 4 for $F_{UU,L}$) and using our notation for the integration variables ($y \to \bar{z}$, $\rho(z=0) \to \bar{x}_0$), we read off from Eq. (18) of Ref. [9]:

$$\begin{aligned} F_{UU,M}^{qq}(\bar{x}_0,\bar{z}) &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2 + 2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})}\right] , \\ F_{UU,L}^{qq}(\bar{x}_0,\bar{z}) &= 8C_F\bar{x}_0\bar{z} , \end{aligned} \tag{54}$$

so that:

$$\begin{aligned} F_{UU,2}^{qq}(\bar{x}_0,\bar{z}) &= F_{UU,M}^{qq}(\bar{x}_0,\bar{z}) + \tfrac{3}{2}F_{UU,L}^{qq}(\bar{x}_0,\bar{z}) \\ &= 2C_F\left[\frac{(\bar{x}_0+\bar{z})^2 + 2(1-\bar{x}_0-\bar{z})}{(1-\bar{x}_0)(1-\bar{z})} + 3\bar{x}_0\bar{z}\right] \\ &= 2C_F\left[(1-\bar{x}_0)(1-\bar{z}) + 4\bar{x}_0\bar{z} + \frac{1+\bar{x}_0^2\bar{z}^2}{\bar{x}_0\bar{z}}\left(\frac{\bar{x}_0\bar{z}}{(1-\bar{x}_0)(1-\bar{z})}\right)\right] . \end{aligned} \tag{55}$$

Using Eq. (49), it is easy to see that the factor in round brackets in the last line of the equation above is equal to Q^2/q_T^2. Therefore, it reduces exactly to the first relation in Eq. (42). The same holds also for the remaining two partonic channels.

Putting all pieces together, Eq. (52) can be recast as:

$$\frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{zxQ^4}Y_+\left[F_{UU,2} - \frac{y^2}{Y_+}F_{UU,L}\right] , \tag{56}$$

[6]Notice that the z variable of Ref. [9] does not coincide with our definition.

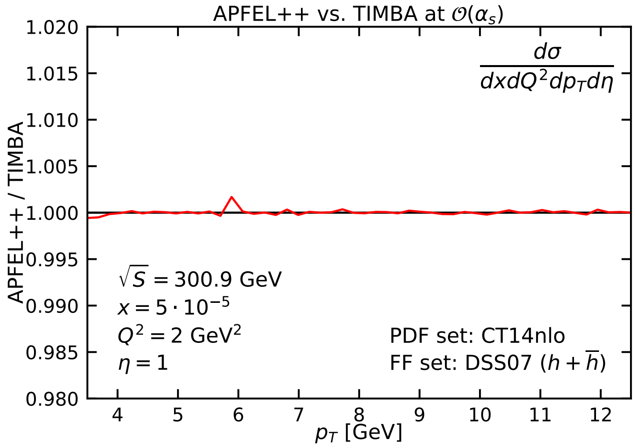

Figure 1: Comparison at $\mathcal{O}(\alpha_s)$ between the `TIMBA` code, implementation of the results of Ref. [9], and the implementation of the expressions in Eq. (50) in the `APFEL++` code [10].

with:

$$\begin{aligned} F_{UU,S} &= a_s \frac{x}{Q^2} \sum_i e_i^2 \int_z^{z_{\max}} \frac{d\bar{z}}{1-\bar{z}} \bar{x}_0 \left[F_{UU,L}^{qq}(\bar{x}_0, \bar{z}) f_i\left(\frac{x}{\bar{x}_0}\right) d_i\left(\frac{z}{\bar{z}}\right) \right. \\ &+ \left. F_{UU,L}^{qg}(\bar{x}_0, \bar{z}) f_g\left(\frac{x}{\bar{x}_0}\right) d_i\left(\frac{z}{\bar{z}}\right) + F_{UU,L}^{gq}(\bar{x}_0, \bar{z}) f_i\left(\frac{x}{\bar{x}_0}\right) d_g\left(\frac{z}{\bar{z}}\right) \right] + \mathcal{O}(a_s^2) . \end{aligned} \tag{57}$$

Therefore, the structure of the observables is exactly the same. What is left to work out is the Jacobian to express the cross section as differential in the same variable as in Eq. (50). What we need is to know how the variables Q^2, p_T, and η are related to y, z, and q_T. The relevant relations are:

$$\left\{ \begin{array}{rcl} Q^2 &=& xyS_H \\ p_T^2 &=& z^2 q_T^2 \\ \eta &=& \dfrac{1}{2} \ln\left(\dfrac{y(1-x)S_H}{q_T^2}\right) \end{array} \right. \quad \Longrightarrow \quad dQ^2 dp_T^2 d\eta = \frac{zQ^2}{y} dydzdq_T^2 . \tag{58}$$

where S_H is the squared center-of-mass energy of the collision, so that:

$$\frac{d\sigma}{dxdydzdq_T^2} = \frac{zQ^2}{y} \frac{d\sigma}{dxdQ^2dp_T^2d\eta} = \frac{2\pi\alpha^2}{xyQ^2} Y_+ \left[F_{UU,2} - \frac{y^2}{Y_+} F_{UU,L} \right] , \tag{59}$$

exactly like in Eq. (38). This confirms that the computation of Ref. [9] can be matched to the resummed calculation provided that the correct Jacobian is taken into account.

In Fig. 1, a comparison between the $\mathcal{O}(a_s)$ computation of Ref. [9] for the differential cross section in Eq. (52), implemented in the computer code `TIMBA`, and the implementation in the `APFEL++` framework [10] of the expressions in Eq. (50) is presented. The cross section as a function of the hadron transverse momentum p_T obtained with `APFEL++` is displayed as ratios to `TIMBA` for a representative values of the kinematic variables. It is clear that the agreement between the two codes is excellent. Since we will be using `TIMBA` for the NLO (*i.e.* $\mathcal{O}(a_s^2)$) cross section calculation to be matched to the NNLL resummed computation, this provides a solid ground to start from.

The next step is to compare the asymptotic behaviour worked out in Eq. (47) to the fixed-order computation. This is done Fig. 2 where the differential cross section in both cases is plotted as a function of the photon tranverve momentum q_T for a representative set of values of the kinematic

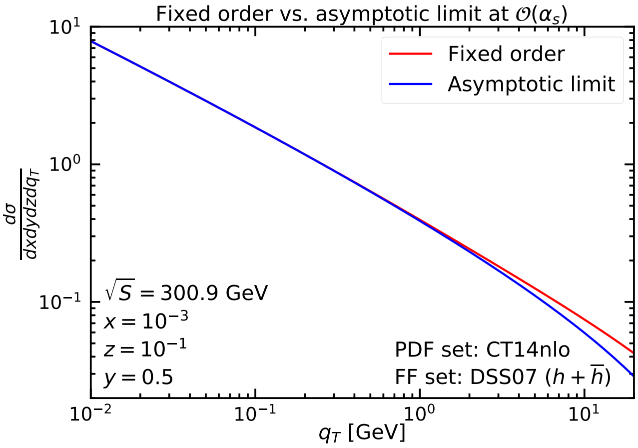

Figure 2: Comparison at $\mathcal{O}(\alpha_s)$ between the asymptotic limit in Eq. (47) to the fixed-order computation expressions in Eq. (50).

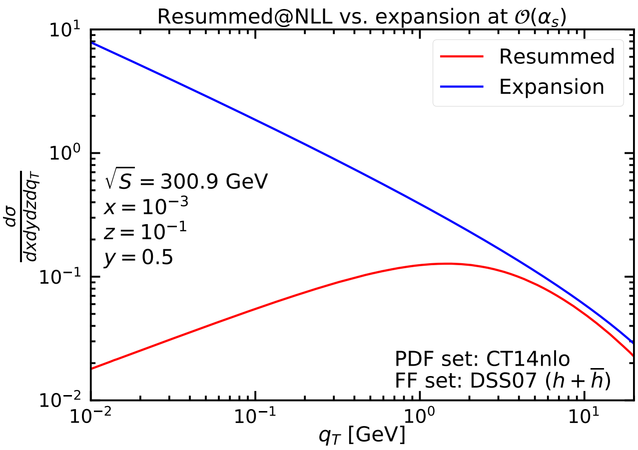

Figure 3: Comparison between the NLL resummed calculation and the expansion to $\mathcal{O}(\alpha_s)$ in Eq. (47).

variables. As expected, the two curves are very close to each other at small values of q_T where the logarithmically enhanced terms dominate. At larger values of q_T instead the two curves tend to depart indicating that power corrections are important in that region.

The conclusive step is the comparison between the resummed computation and its fixed-order order expansion that, as we have shown above, coincides with the asymptote of the full fixed-oder. This is done in Fig. (3) where the resummed calculation a NLL accuracy is compared to its fixed-order expansion to $\mathcal{O}(\alpha_s)$, that coincides with Eq. (47). As expected, while the two curves are far apart at low q_T they tend to converge towards larger values of q_T. However, the convergence is not so "accurate" as in the case of the fixed-order calculation versus its asymptotic limit. The reason is that in this case the two computations converge up to subleading terms that in this case are $\mathcal{O}(\alpha_s^2)$ that are effectively NLO and thus numerically large. We will see that the situation improves when going one order up.

We can now formulate the matching procedure to consistently combine the resummed calculation with the fixed-order one subtracting the double counting terms. The particular way in which the matching is implemented is defined up to subleading and power-suppressed terms. One can then exploit this arbitrariness to adjust the matching in such a way that the possibly large subleading terms are mitigated by some *ad-hoc* prescription. The most natural, despite not unique, prescription

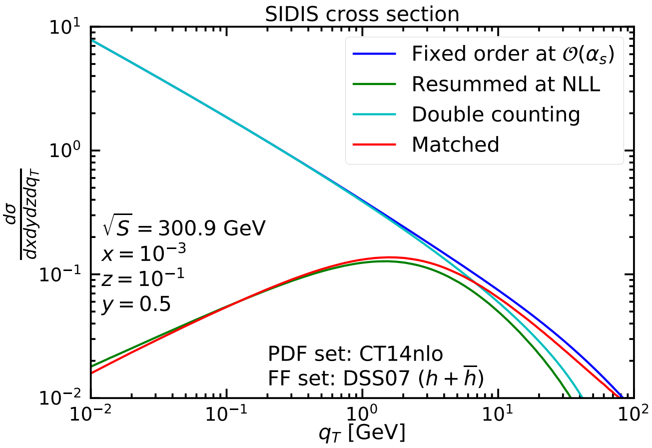

Figure 4: Summary of the computations: fixed order (blue curve), resummed (green curve), double counting (cyan curve), and matched according to Eq. (60) (red curve).

is the *additive* matching that reads:

$$\sigma^{\rm Match}(q_T, Q) = \sigma^{\rm FO}(q_T, Q) + \sigma^{\rm Res}(q_T, Q) - \sigma^{\rm Asy}(q_T, Q)\,. \tag{60}$$

where $\sigma^{\rm Match}$ is obtained adding up the fixed-order computation $\sigma^{\rm FO}$ to some perturbative accuracy (say $\mathcal{O}(\alpha_s^n)$), valid for $q_T \simeq Q$, to the resummed calculation $\sigma^{\rm Res}$ at some logarithmic accuracy, valid for $q_T \ll Q$, and subtracting the double-counting terms $\sigma^{\rm Asy}$. By the arguments discussed above, it is easy to see that:

$$\begin{aligned} \sigma^{\rm Match}(q_T, Q) \quad &\underset{q_T \simeq Q}{\longrightarrow} \quad \sigma^{\rm FO}(q_T, Q) + \mathcal{O}\left[\alpha_s^{n+1}(Q)\right]\,, \\ \sigma^{\rm Match}(q_T, Q) \quad &\underset{q_T \ll Q}{\longrightarrow} \quad \sigma^{\rm Res}(q_T, Q) + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)\,. \end{aligned} \tag{61}$$

However, as we have seen above, for low-order computations the $\mathcal{O}(\alpha_s^{n+1})$ residual for $q_T \simeq Q$ may be numerically relevant. On the other hand, one would expect that when increasing the perturbative accuracy these differences tend to shrink. Therefore, for the moment I refrain from form introducing any prescription for the suppression of subleading terms. I will come back on this point in the next section where I will study the matching of the NLO fixed-order computation to the NNLL resummed one. Fig. 4 shows a summary of the computations consideres so far, including the matched one according to Eq. (60): this provides the lowest order matching LO+NLL. The behaviour of the red curve is the expected one. In particular, in the low-q_T region it is close to the resummed (green) curve while in the high-q_T region it tends to the fixed-order (blue) curve. In the next section we will raise the perturbative accuracy to NLO+NNLL.

A Expansion of the kinematic δ-function

In order to prove the equality in Eq. (44), we consider the following integral:

$$I(\varepsilon) = \int_0^1 dx \int_0^1 dy\, \delta(xy - \varepsilon) f(x, y)\,, \tag{62}$$

where $f(x, y)$ is a test function "well-behaved" over the integration region. Now I split the integral as follows:

$$I(\varepsilon) = \left(\int_0^1 dx \int_x^1 dy + \int_0^1 dy \int_y^1 dx\right)\, \delta(xy - \varepsilon) f(x, y)\,, \tag{63}$$

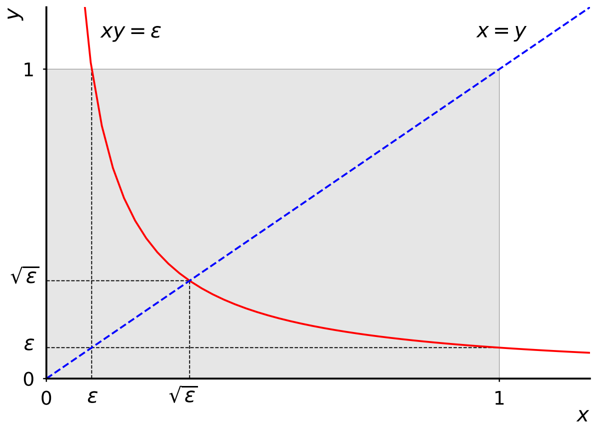

Figure 5: Integration region of the integral in Eq. (62). The integral is along the red curve defined by the δ-function.

where the first term in the r.h.s. corresponds to the integral over the grey region *above* the blue line while the second over the grey region *below* the blue line in Fig. 5.

Now I use the following equalities:

$$\delta(xy-\varepsilon)=\begin{cases}\dfrac{1}{x}\delta\left(y-\dfrac{\varepsilon}{x}\right)\theta(y-\sqrt{\varepsilon}) & \text{integral over } y,\\[2ex] \dfrac{1}{y}\delta\left(x-\dfrac{\varepsilon}{y}\right)\theta(x-\sqrt{\varepsilon}) & \text{integral over } x.\end{cases} \tag{64}$$

The θ-functions arise from the fact that the first integral has to be done along the upper branch on the red curve while the second along the lower branch. The two branches are joint at the point $x=y=\sqrt{\varepsilon}$ and thus the integration ranges are bounded from below by this point. Therefore, I find:

$$I(\varepsilon)=\int_{\sqrt{\varepsilon}}^{1}\frac{dx}{x}f\left(x,\frac{\varepsilon}{x}\right)+\int_{\sqrt{\varepsilon}}^{1}\frac{dy}{y}f\left(\frac{\varepsilon}{y},y\right). \tag{65}$$

It is crucial to realise that in the first and the second integral the following conditions hold: $\varepsilon<\sqrt{\varepsilon}\le x$ and $\varepsilon<\sqrt{\varepsilon}\le y$, respectively. Therefore, in the limit $\varepsilon\to 0$, the arguments ε/x and ε/y of the function f will tend to zero. I now add and subtract a term proportional to $f(0,0)$ to both integrals, so that:

$$I(\varepsilon)=\int_{\sqrt{\varepsilon}}^{1}\frac{dx}{x}\left[f\left(x,\frac{\varepsilon}{x}\right)-f(0,0)\right]+\int_{\sqrt{\varepsilon}}^{1}\frac{dy}{y}\left[f\left(\frac{\varepsilon}{y},y\right)-f(0,0)\right]+2f(0,0)\underbrace{\int_{\sqrt{\varepsilon}}^{1}\frac{d\xi}{\xi}}_{-\ln\sqrt{\varepsilon}}. \tag{66}$$

Finally, I take the limit for $\varepsilon\to 0$:

$$\lim_{\varepsilon\to 0}I(\varepsilon)=\int_0^1\frac{dx}{x}\left[f\left(x,0\right)-f(0,0)\right]+\int_0^1\frac{dy}{y}\left[f\left(0,y\right)-f(0,0)\right]-\ln\left(\varepsilon\right)f(0,0)\,, \tag{67}$$

that I rewrite as:

$$\lim_{\varepsilon\to 0}I(\varepsilon)=\int_0^1 dx\int_0^1 dy\left\{\frac{\delta(y)}{[x]_+}+\frac{\delta(x)}{[y]_+}-\ln\left(\varepsilon\right)\delta(x)\delta(y)\right\}f(x,y)\,. \tag{68}$$

Comparing the equation above with Eq. (62), one deduces that:

$$\delta(xy-\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow}\frac{\delta(y)}{[x]_+}+\frac{\delta(x)}{[y]_+}-\ln\left(\varepsilon\right)\delta(x)\delta(y)\,. \tag{69}$$

Finally, substituting:

$$x \to \frac{1-x}{x}\,, \quad y \to \frac{1-z}{z}\,, \quad \text{and} \quad \varepsilon \to \frac{q_T^2}{Q^2}\,, \tag{70}$$

it is easy to recover Eq. (44). In particular, one needs to use the fact that:

$$\delta\left(\frac{1-x}{x}\right) = \delta(1-x)\,. \tag{71}$$

References

[1] I. Scimemi and A. Vladimirov, arXiv:1706.01473 [hep-ph].

[2] M. G. Echevarria, I. Scimemi and A. Vladimirov, JHEP **1609** (2016) 004 doi:10.1007/JHEP09(2016)004 [arXiv:1604.07869 [hep-ph]].

[3] W. L. van Neerven and A. Vogt, Nucl. Phys. B **588** (2000) 345 doi:10.1016/S0550-3213(00)00480-6 [hep-ph/0006154].

[4] G. Bozzi, S. Catani, D. de Florian and M. Grazzini, Nucl. Phys. B **737** (2006) 73 doi:10.1016/j.nuclphysb.2005.12.022 [hep-ph/0508068].

[5] R. Meng, F. I. Olness and D. E. Soper, Phys. Rev. D **54** (1996) 1919 doi:10.1103/PhysRevD.54.1919 [hep-ph/9511311].

[6] J. Collins, L. Gamberg, A. Prokudin, T. C. Rogers, N. Sato and B. Wang, Phys. Rev. D **94** (2016) no.3, 034014 doi:10.1103/PhysRevD.94.034014 [arXiv:1605.00671 [hep-ph]].

[7] P. M. Nadolsky, D. R. Stump and C. P. Yuan, Phys. Rev. D **61** (2000) 014003 Erratum: [Phys. Rev. D **64** (2001) 059903] doi:10.1103/PhysRevD.64.059903, 10.1103/PhysRevD.61.014003 [hep-ph/9906280].

[8] A. Bacchetta, D. Boer, M. Diehl and P. J. Mulders, JHEP **0808** (2008) 023 doi:10.1088/1126-6708/2008/08/023 [arXiv:0803.0227 [hep-ph]].

[9] A. Daleo, D. de Florian and R. Sassot, Phys. Rev. D **71** (2005) 034013 doi:10.1103/PhysRevD.71.034013 [hep-ph/0411212].

[10] V. Bertone, arXiv:1708.00911 [hep-ph].