1 Structure of the observables

Let us start from Eq. (2.6) of Ref. [1], that is the fully differential cross section for lepton-pair production in the region in which the TMD factorisation applies, *i.e.* $q_T \ll Q$. It reads:

$$\frac{d\sigma}{dQ^2 dy dq_T^2} = \frac{4\pi\alpha^2(Q)\mathcal{P}}{3N_c Q^4} H(Q,\mu) \sum_q C_q(Q) \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} x_1 F_q(x_1,\mathbf{b};\mu,\zeta) x_2 F_{\bar{q}}(x_2,\mathbf{b};\mu,\zeta)\,. \tag{1}$$

Setting aside the unimportant factors, the interesting part has the form of the integral:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} x_1 F_i(x_1,\mathbf{b};\mu,\zeta) x_2 F_j(x_2,\mathbf{b};\mu,\zeta)\,. \tag{2}$$

where $F_{i(j)}$ are the evolved TMD PDFs and:

$$x_{1,2} = \frac{Q}{\sqrt{s}} e^{\pm y}\,, \tag{3}$$

being $\sqrt{s}$ the centre-of-mass energy of the collision. In actual fact, $F_{i(j)}$ only depend of the absolute value of $\mathbf{b}$, therefore eq. (2) can be written as:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\, b J_0(bq_T) x_1 F_i(x_1,b;\mu,\zeta) x_2 F_j(x_2,b;\mu,\zeta)\,. \tag{4}$$

where J_0 is the zero-th order Bessel function of the first kind whose integral representation is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{ix\cos(\theta)}\,. \tag{5}$$

The single evolved TMD PDF at the final scales μ and ζ F_i, with $i = q, \bar{q}$([1]), is obtained by multiplying the same TMD PDF at the initial scales μ_0 and ζ_0 by a single evolution factor $R_q(\mu_0,\zeta_0 \to \mu,\zeta;b)$ (no convolutions and no mixing), that is:

$$xF_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0 \to \mu,\zeta;b) xF_i(x,b;\mu_0,\zeta_0) \quad \text{with } i = g, q(=\bar{q})\,. \tag{6}$$

so that eq. (4) becomes:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\, b J_0(bq_T) \left[R_q(\mu_0,\zeta_0 \to \mu,\zeta;b)\right]^2 x_1 F_i(x_1,b;\mu_0,\zeta_0) x_2 F_j(x_2,b;\mu_0,\zeta_0)\,. \tag{7}$$

The initial scale TMD PDFs at LO in the OPE expansion, that is for $b \ll B$ where B is an unknown non-perturbative parameter that represents the intrinsic hadron scale (see eq. (2.27) of Ref. [1]), can be written as:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 \frac{dy}{y} C_{ij}(y;\mu_0,\zeta_0) f_j\left(\frac{x}{y},\mu_0\right)\,, \tag{8}$$

where f_j are the collinear PDFs (including the gluon) and C_{ij} are the so-called matching functions that are perturbatively computable and are currently known to NNLO, *i.e.* $\mathcal{O}(\alpha_s^2)$. In order account for higher orders in the OPE and non-perturbative effects where the OPE is not valid, one usually introduces a phenomenological non-perturbative function $f_{\rm NP}(x,b)$ that modifies the convolution in eq. (8) (for now we assume that this function does not depend on the flavour of the parton, but in principle it could). The way how $f_{\rm NP}(x,b)$ is introduced in not unique. Indeed, one can either multiply eq. (8) by this function, that is:

$$xF_i(x,b;\mu_0,\zeta_0) \to \sum_{j=g,q(\bar{q})} xf_{\rm NP}(x,b) \int_x^1 \frac{dy}{y} C_{ij}(y;\mu_0,\zeta_0) f_j\left(\frac{x}{y},\mu_0\right)\,. \tag{9}$$

[1]Note that in eq. (1) the gluon TMD PDF F_g is not involved.

or this function can be inserted in the convolution integral. This can be done in two ways: in the first way one "associates" $f_{\rm NP}$ to the collinear PDFs so that eq. (8) becomes:

$$xF_i(x,b;\mu_0,\zeta_0) \to \sum_{j=g,q(\bar{q})} x\int_x^1 \frac{dy}{y} C_{ij}(y;\mu_0,\zeta_0)\left[f_{\rm NP}\left(\frac{x}{y},b\right) f_j\left(\frac{x}{y},\mu_0\right)\right]\,, \tag{10}$$

while in the second way $f_{\rm NP}$ is associated to the matching operators, that is:

$$xF_i(x,b;\mu_0,\zeta_0) \to \sum_{j=g,q(\bar{q})} x\int_x^1 \frac{dy}{y} \left[f_{\rm NP}\left(y,b\right) C_{ij}(y;\mu_0,\zeta_0)\right] f_j\left(\frac{x}{y},\mu_0\right)\,. \tag{11}$$

While eqs. (9) and (10) appear to be valid alternatives, eq. (11) looks unnatural because it modifies the structure of the matching coefficients that are computable quantities (as a matter of fact this is also problematic to implement due to the presence of plus-prescripted and δ-function terms in C_{ij}). Nevertheless, this seems to be the approach taken (perhaps inadvertently) in Ref. [1].

In the following we will consider the case of eq. (10). So we define:

$$\widetilde{f}_i\left(x,\mu_0,b\right) = xf_{\rm NP}\left(x,b\right) f_i\left(x,\mu_0\right)\,, \tag{12}$$

so that:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\widetilde{f}_i\left(\frac{x}{y},\mu_0,b\right)\,. \tag{13}$$

At this point, it is opportune to mention that, despite there seem to be many variables, the variables μ_0 and ζ_0 are usually taken to be functions of the impact parameter b. Therefore, eq. (13) is a function of two variables only that we rewrite in a simpler way as:

$$xF_i(x,b) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y,b)\widetilde{f}_i\left(\frac{x}{y},b\right)\,. \tag{14}$$

This kind of convolutions can be computed using standard interpolation techniques by which one approximates the function $\widetilde{f}_i$ as:

$$\widetilde{f}_i(x,b) = \sum_{\alpha} w_\alpha(x)\widetilde{f}_i(x_\alpha,b) \tag{15}$$

where x_α is the α-th node of an interpolation grid and w_α is the interpolating function associated to that node. Assuming for now that x coincides with the β-th node of the grid and introducing another grid in the b dimension whose nodes are indexed by τ, eq. (14) can be written as:

$$\hat{F}_{i,\beta}^{\tau} = \hat{C}_{ij,\beta\alpha}^{\tau}\hat{f}_{j,\alpha}^{\tau}\,. \tag{16}$$

where now all sums over repeated indices are understood (except τ) and where we have used the following definitions:

$$\hat{F}_{i,\beta}^{\tau} \equiv x_\beta F_i(x_\beta,b_\tau)\,,\quad \hat{C}_{ij,\beta\alpha}^{\tau} \equiv \int_{x_\beta}^1 dy\, C_{ij}(y,b_\tau)w_\alpha\left(\frac{x_\beta}{y}\right)\,,\quad \hat{f}_{j,\alpha}^{\tau} \equiv \widetilde{f}_i(x_\alpha,b_\tau)\,. \tag{17}$$

Since we have to integrate over the impact parameter b (see eq. (4)), we need to be able to reconstruct the dependence of the function F_i on b. This can be done using the same interpolation technique. In particular, we write:

$$x_\alpha F_i(x_\alpha,b)x_\beta F_j(x_\beta,b) = \sum_\tau \widetilde{w}_\tau(b)\hat{F}_{i,\alpha}^{\tau}\hat{F}_{j,\beta}^{\tau} = \sum_\tau \widetilde{w}_\tau(b)\hat{C}_{ik,\alpha\gamma}^{\tau}\hat{C}_{jl,\beta\delta}^{\tau}\hat{f}_{k,\gamma}^{\tau}\hat{f}_{l,\delta}^{\tau}\,. \tag{18}$$

Keeping in mind that μ_0 and ζ_0 are functions of the impact parameter b, eq. (7) takes the form:

$$I_{ij}(x_\alpha,x_\beta,q_T;\mu,\zeta) = \sum_\tau K_\tau(\mu,\zeta;q_T)\hat{C}_{ik,\alpha\gamma}^{\tau}\hat{C}_{jl,\beta\delta}^{\tau}\hat{f}_{k,\gamma}^{\tau}\hat{f}_{l,\delta}^{\tau}\,, \tag{19}$$

where we have defined:

$$K_\tau(\mu,\zeta;q_T) \equiv \frac{1}{2}\int_0^\infty db\, b J_0(bq_T)\left[R_q(\mu,\zeta;b)\right]^2 \widetilde{w}_\tau(b)\,. \tag{20}$$

Since the final scales μ and ζ are usually functions of the kinematics (the typical choice is $\mu = \sqrt{\zeta} = Q$), the factors K_τ can be precomputed for any given experiment. It should also be noted that the function $\widetilde{w}_\tau(b)$ is typically different from zero only over a finite interval in b, say $\widetilde{w}_\tau(b) \neq 0 \quad \forall\, b \in [a_\tau, b_\tau]$, therefore the integral in eq. (20) reduces to:

$$K_\tau(\mu,\zeta;q_T) \equiv \frac{1}{2}\int_{a_\tau}^{b_\tau} db\, b J_0(bq_T)\left[R_q(\mu,\zeta;b)\right]^2 \widetilde{w}_\tau(b)\,, \tag{21}$$

and the sum over τ in eq. (19), that is supposed to run over an infinite number of nodes, has to be truncated.

References

[1] I. Scimemi and A. Vladimirov, arXiv:1706.01473 [hep-ph].