1 Expansion of the Sudakov form factor

In this section I work out the expansion of the quark Sudakov form factor to $\mathcal{O}(\alpha_s^2)$. The Sudakov form factor is independent of the process and thus the same expansion can be used for example for Drell-Yan and SIDIS.[1] One crucial point is that the expansion will be done in the so-called ζ-prescription introduced in Ref. [1].

The Sudakov form factor we are considering here is essentially the square of the quark evolution factor deriving from the solution of the Collins-Soper (CS) equations. As is well known, the CS equations govern the evolution of TMD distributions in two independent factorisation scales μ and ζ. Usually, μ and ζ are related to each other by assuming that $\mu = \sqrt{\zeta}$. This particular choice reduces the degrees of freedom of the problem. On the other hand, as observed in Ref. [1], this choice does not help cure large logarithms that appear in the matching functions. The mutual independence of μ and ζ can be exploited to guarantee that such large logarithms are reabsorbed in the definition of the scale ζ as a function of μ (or viceversa). This is the basic observation at the base of the ζ-prescription that achieves this goal by assuming $\zeta \equiv \zeta(\mu)$ and requiring that the total derivative of TMD distribution F with respect to μ is zero:

$$\mu^2 \frac{dF\left(x, b; \mu, \zeta(\mu)\right)}{d\mu^2} = 0. \tag{1}$$

This naturally leads to a differential equation in $\zeta(\mu)$ involving the CS anomalous dimensions and $\Gamma_{\rm cusp}$ that can be solved order by order in α_s. The exact form of the function $\zeta(\mu)$ has to be taken into account when expanding the Sudakov form factor.

The exact form of the Sudakov form factor, that we remind to be the squared evolution factor deriving from the solution of the CS equations and the evolves a pair of quark TMD distributions in b space from the scales (μ_i, ζ_i) to (μ_f, ζ_f), is:

$$\left[R(\mu_i, \zeta_i \to \mu_f, \zeta_f; b)\right]^2 = \exp\left[\int_{\mu_i^2}^{\mu_f^2} \frac{d\mu^2}{\mu^2}\left(-\gamma_V + \Gamma_{\rm cusp} \ln\left(\frac{\mu^2}{\zeta_f}\right)\right) - 2\mathcal{D}\ln\left(\frac{\zeta_f}{\zeta_i}\right)\right]. \tag{2}$$

where the following perturbative expansion hold:

$$\gamma_V = \sum_{n=1} a_s^n \gamma_V^{(n)} \qquad \Gamma_{\rm cusp} = \sum_{n=1} a_s^n \Gamma_{\rm cusp}^{(n)}\,, \tag{3}$$

and:

$$\mathcal{D} = \sum_{n=1} a_s^n \sum_{k=0}^{n} d^{(n,k)} L^k\,, \tag{4}$$

where I have defined:

$$a_s(\mu) = \frac{\alpha_s(\mu)}{4\pi} \quad \text{and} \quad L \equiv \ln\left(\frac{b^2\mu_i^2}{4e^{-2\gamma_E}}\right)\,. \tag{5}$$

The perturbative coefficients, $\gamma_V^{(n)}$, $\Gamma_{\rm cusp}^{(n)}$, and $d^{(n,k)}$ are know up to the order necessary to implement NNLL evolution.[2] The final scales μ_f and ζ_f are usually taken to be equal to $\mu_f = \sqrt{\zeta} = \kappa Q$, where Q is, for example the virtuality of the exchanged photon in DIS and the invariant mass of the lepton pair in Drell-Yan. In addition, using the ζ-prescription, we set:

$$\zeta_i \equiv \zeta(\mu_i) = \mu_i^2 \exp\left(\sum_{n=0} a_s^n \sum_{k=0}^{n+1} \ell^{(n,k)} L^k\right) \tag{6}$$

where the coefficients $\ell^{(n,k)}$ are constants.

Before expanding the exponential we need to write its argument as a polynomial in α_s (or better a_s) computed in Q. These terms will in turn multiply powers of $\ln(Q)$. In view of the integral over

[1] Note that for gluon-initiated processes the expression will be different.

[2] In fact, with the only exception of $\Gamma_{\rm cusp}^{(4)}$, it would be possible to implement evolution up to N^{3}LL.

the impact parameter b needed to obtain cross sections differential in the transverse momentum q_T, we need to assign μ_i a dependence on b. Despite this dependence is arbitrary and needs only be dimensionally correct , the most natural choice is:

$$\mu_i = \frac{C_0}{b}\,, \quad \text{with} \quad C_0 = 2e^{-\gamma_E}\,. \tag{7}$$

This choice is such that L defined in Eq. (5) vanishes so that:

$$\mathcal{D} = \sum_{n=1} a_s^n d^{(n,0)} \tag{8}$$

and:

$$\zeta(\mu_i) = \frac{C_0^2}{b^2}\exp\left(\sum_{n=0} a_s^n \ell^{(n,0)}\right)\,. \tag{9}$$

This way the Sudakov form factor in Eq. (2) reads:

$$\begin{aligned}[R(Q,b)]^2 &= \exp\Bigg\{-\sum_{n=1}^{2}\int_{C_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}a_s^n(\mu)\left[\gamma_V^{(n)}+\Gamma_{\text{cusp}}^{(n)}\ln\left(\frac{Q^2}{\mu^2}\right)\right]\\ &+ 2\sum_{n=1}^{2}a_s^n\left(\frac{C_0}{b}\right)d^{(n,0)}\left[-\ln\left(\frac{b^2Q^2}{C_0^2}\right)+\ell^{(0,0)}+a_s\left(\frac{C_0}{b}\right)\ell^{(1,0)}\right]\Bigg\}\,,\end{aligned} \tag{10}$$

where we limited the contributions in the exponential to $\mathcal{O}(a_s^2)$ that is the order we need. Now, using the RGE:

$$\mu^2\frac{da_s}{d\mu^2} = -\beta_0 a_s^2(\mu)\,, \tag{11}$$

whose solution is:

$$a_s(\mu) = \frac{a_s(Q)}{1+a_s(Q)\beta_0\ln(\mu^2/Q^2)} \simeq a_s(Q)\left[1+a_s(Q)\beta_0\ln(Q^2/\mu^2)+\mathcal{O}(a_s^2)\right]\,, \tag{12}$$

we write every instance of a_s appearing in Eq. (10) in terms of $a_s(Q)$ and finally retain only terms up to $a_s^2(Q)$:

$$\begin{aligned}[R(Q,b)]^2 &= \exp\Bigg\{-a_s(Q)\int_{C_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[\gamma_V^{(1)}+\Gamma_{\text{cusp}}^{(1)}\ln\left(\frac{Q^2}{\mu^2}\right)\right]\\ &- a_s^2(Q)\int_{C_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\left[\gamma_V^{(2)}+\left(\Gamma_{\text{cusp}}^{(2)}+\gamma_V^{(1)}\beta_0\right)\ln\left(\frac{Q^2}{\mu^2}\right)+\Gamma_{\text{cusp}}^{(1)}\beta_0\ln^2\left(\frac{Q^2}{\mu^2}\right)\right]\\ &+ 2a_s(Q)\left[d^{(1,0)}\ell^{(0,0)}-d^{(1,0)}\ln\left(\frac{b^2Q^2}{C_0^2}\right)\right]\\ &+ 2a_s^2(Q)\left[d^{(2,0)}\ell^{(0,0)}+d^{(1,0)}\ell^{(1,0)}+\left(-d^{(2,0)}+d^{(1,0)}\beta_0\ell^{(0,0)}\right)\ln\left(\frac{b^2Q^2}{C_0^2}\right)-d^{(1,0)}\beta_0\ln^2\left(\frac{b^2Q^2}{C_0^2}\right)\right]\Bigg\}\,.\end{aligned} \tag{13}$$

The final step before carrying out the expansion is that of resolving the integrals:

$$\int_{C_0^2/b^2}^{Q^2}\frac{d\mu^2}{\mu^2}\ln^k\left(\frac{Q^2}{\mu^2}\right) = \int_{\ln(C_0^2/b^2)}^{\ln Q^2} d\ln\mu^2\ln^k\left(\frac{Q^2}{\mu^2}\right) = \int_0^{\ln(b^2Q^2/C_0^2)} dx\, x^k = \frac{1}{k+1}\ln^{k+1}\left(\frac{b^2Q^2}{C_0^2}\right)\,. \tag{14}$$

If I define:

$$\mathcal{L} \equiv \ln\left(\frac{b^2Q^2}{C_0^2}\right)\,, \tag{15}$$

the Sudakov form fact in Eq. 13 reads:

$$
\begin{aligned}
[R(Q,\mathcal{L})]^2 &= \exp\left\{a_s(Q)\left[2d^{(1,0)}\ell^{(0,0)} - \left(\gamma_V^{(1)} + 2d^{(1,0)}\right)\mathcal{L} - \frac{1}{2}\Gamma^{(1)}_{\rm cusp}\mathcal{L}^2\right]\right. \\
&+ a_s^2(Q)\left[2d^{(2,0)}\ell^{(0,0)} + 2d^{(1,0)}\ell^{(1,0)} + \left(-2d^{(2,0)} + 2d^{(1,0)}\beta_0\ell^{(0,0)} - \gamma_V^{(2)}\right)\mathcal{L}\right. \\
&- \left.\left.\frac{1}{2}\left(4d^{(1,0)}\beta_0 + \Gamma^{(2)}_{\rm cusp} + \gamma_V^{(1)}\beta_0\right)\mathcal{L}^2 - \frac{1}{3}\Gamma^{(1)}_{\rm cusp}\beta_0\mathcal{L}^3\right]\right\},
\end{aligned}
\tag{16}
$$

that can be conveniently written as:

$$
[R(Q,\mathcal{L})]^2 = \exp\left\{\sum_{n=1}^{2} a_s^n(Q)\sum_{k=0}^{n+1} S^{(n,k)}\mathcal{L}^k\right\},
\tag{17}
$$

with:

$$
\begin{aligned}
&S^{(1,0)} = 2d^{(1,0)}\ell^{(0,0)} = 0\,, \quad S^{(1,1)} = -\left(\gamma_V^{(1)} + 2d^{(1,0)}\right) = 6C_F\,, \quad S^{(1,2)} = -\frac{1}{2}\Gamma^{(1)}_{\rm cusp} = -2C_F\,, \\
&S^{(2,0)} = 2d^{(2,0)}\ell^{(0,0)} + 2d^{(1,0)}\ell^{(1,0)}\,, \quad S^{(2,1)} = \left(-2d^{(2,0)} + 2d^{(1,0)}\beta_0\ell^{(0,0)} - \gamma_V^{(2)}\right), \\
&S^{(2,2)} = -\frac{1}{2}\left(4d^{(1,0)}\beta_0 + \Gamma^{(2)}_{\rm cusp} + \gamma_V^{(1)}\beta_0\right), \quad S^{(2,3)} = -\frac{1}{3}\Gamma^{(1)}_{\rm cusp}\beta_0\,.
\end{aligned}
\tag{18}
$$

The values of the coefficients of the anomalous dimensions and beta function can be read from Appendix D of Ref. [2]. The coefficients of the expansion of $\zeta(\mu)$ are instead reported in Eq. (2.29) of Ref. [1]. For the $\mathcal{O}(a_s)$ coefficients I reported the explicit values. This will help check the result against those in the literature.

Eq. (17) can be easily expanded as up to order a_s^2 as:

$$
\begin{aligned}
[R(Q,\mathcal{L})]^2 &= 1 + a_s(Q)\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k + a_s^2(Q)\left[\sum_{k=0}^{3} S^{(2,k)}\mathcal{L}^k + \frac{1}{2}\left(\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k\right)^2\right] + \mathcal{O}(a_s^3) \\
&= 1 + a_s(Q)\sum_{k=0}^{2} S^{(1,k)}\mathcal{L}^k + a_s^2(Q)\sum_{k=0}^{4}\widetilde{S}^{(2,k)}\mathcal{L}^k + \mathcal{O}(a_s^3) \\
&\equiv 1 + a_s(Q)R^{(1)} + a_s^2(Q)R^{(2)} + \mathcal{O}(a_s^3)\,,
\end{aligned}
\tag{19}
$$

with:

$$
\begin{aligned}
&\widetilde{S}^{(2,0)} = S^{(2,0)} + \frac{1}{2}\left[S^{(1,0)}\right]^2, \\
&\widetilde{S}^{(2,1)} = S^{(2,1)} + S^{(1,0)}S^{(1,1)} \\
&\widetilde{S}^{(2,2)} = S^{(2,2)} + \frac{1}{2}\left[S^{(1,1)}\right]^2 + S^{(1,2)}S^{(1,0)}\,, \\
&\widetilde{S}^{(2,3)} = S^{(2,3)} + S^{(1,1)}S^{(1,2)} \\
&\widetilde{S}^{(2,4)} = \frac{1}{2}\left[S^{(1,2)}\right]^2.
\end{aligned}
\tag{20}
$$

Now that the expansion of the Sudakov form factor is done to $\mathcal{O}(a_s^2)$, it can be combined to the rest of the perturbative quantities entering the computation of the cross section. In the following, we will concentrate on SIDIS that involves both TMD PDFs F and TMD FFs D. Specifically, in b space the SIDIS cross section is a combination of terms having the following structure:

$$
\begin{aligned}
B_{ij} &= H(Q)\,F_i(x,b;Q)\,zD_j(z,b;Q) = H(Q)\,\left[R(Q,\mathcal{L})\right]^2\,F_i\left(x,b;\frac{C_0}{b}\right)\,zD_j\left(z,b;\frac{C_0}{b}\right)\\
&= H(Q)\,\left[R(Q,\mathcal{L})\right]^2\left[\sum_k \mathcal{C}_{ik}(x,\mathcal{L})\underset{x}{\otimes} f_k\,(x;Q)\right]\left[\sum_l \mathbb{C}_{jl}(z,\mathcal{L})\underset{z}{\otimes} d_l\,(z;Q)\right]\\
&\equiv \sum_{kl}\hat{B}_{ij,kl}(x,z;b)\underset{x}{\otimes} f_k \underset{z}{\otimes} d_l\,,
\end{aligned}
\tag{21}
$$

where f_k and d_l are the (non-perturbative) collinear PDFs and FFs, respectively. The other terms, including R^2, are perturbatively computable as:

$$
\begin{aligned}
&H(Q) = 1 + a_s(Q)H^{(1)} + a_s^2(Q)H^{(2)} + \mathcal{O}(a_s^3)\,,\\
&\mathcal{C}_{ik}(x,\mathcal{L}) = \delta_{ik}\delta(1-x) + a_s(Q)\mathcal{C}_{ik}^{(1)}(x,\mathcal{L}) + a_s^2(Q)\mathcal{C}_{ik}^{(2)}(x,\mathcal{L}) + \mathcal{O}(a_s^3)\,,\\
&\mathbb{C}_{jl}(z,\mathcal{L}) = \delta_{jl}\delta(1-z) + a_s(Q)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L}) + a_s^2(Q)\mathbb{C}_{jl}^{(2)}(z,\mathcal{L}) + \mathcal{O}(a_s^3)\,.
\end{aligned}
\tag{22}
$$

I now need to put everything together and truncate to order a_s^2 in such a way that:

$$
\hat{B}_{ij,kl}(x,z;b) = \hat{B}_{ij,kl}^{(0)}(x,z;b) + a_s(Q)\hat{B}_{ij,kl}^{(1)}(x,z;b) + a_s^2(Q)\hat{B}_{ij,kl}^{(2)}(x,z;b) + \mathcal{O}(a_s^3)\,,
\tag{23}
$$

with:

$$
\begin{aligned}
\hat{B}_{ij,kl}^{(0)} &= \delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\,,\\
\hat{B}_{ij,kl}^{(1)} &= (H^{(1)}+R^{(1)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\\
&+ \mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L})\\
\hat{B}_{ij,kl}^{(2)} &= (H^{(2)}+H^{(1)}R^{(1)}+R^{(2)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\\
&+ (H^{(1)}+R^{(1)})\left[\mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1)}(z,\mathcal{L})\right]\\
&+ \mathcal{C}_{ik}^{(2)}(x,\mathcal{L})\delta_{jl}\delta(1-z) + \mathcal{C}_{ik}^{(1)}(x,\mathcal{L})\mathbb{C}_{jl}^{(1)}(z,\mathcal{L}) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(2)}(z,\mathcal{L})
\end{aligned}
\tag{24}
$$

Despite we derived the expansion of the resummed cross section up to $\mathcal{O}(a_s^2)$, these are not yet the final formulas. The reason is that, in order some of the terms of the expansion above do not depend on the impact parameter b. This means that these terms, upon Fourier transform needed to obtain the expression in the transverse moment q_T, will give rise to terms proportional to $\delta(q_T)$. In view of the matching procedure, since the fixed-order expression we are going to match to does not include any $\delta(q_T)$ terms, we need to make sure that they are not subtracted.Therefore, we need to identify and remove any term that does not depend on b. In fact, by construction, this is equivalent to leave only terms proportional to a power of $\mathcal{L}$. In order to do so, we start observing that the hard factor H does not contain any $\mathcal{L}$. In addition, we have to pay attention to the coefficients $R^{(n)}$ because they are polynomials in $\mathcal{L}$ but also include a constant term ($\mathcal{L}^0$). Finally, the ζ-prescription, Eq. (1), provides us with a simple recipe to compute the logarithmic terms of the matching functions $\mathcal{C}_{ik}^{(n)}$ and $\mathbb{C}_{jl}^{(n)}$. Specifically, the condition of independence of the TMD PDFs and FFs from the factorisation scale μ is such that TMDs behave like physical observable (*e.g.* deep-inelastic-scattering or single-inclusive-annihilation structure functions) and thus obey the standard scale variation rules derived, for example, in Eq. (2.17) of Ref. [3]. More in particular, one finds that the matching function coefficients s have the usual logarithmic structure:

$$\mathcal{C}_{ij}^{(n)}(x,\mathcal{L}) = \sum_{k=0}^{n} \mathcal{C}_{ij}^{(n,k)}(x)\mathcal{L}^k \quad \text{and} \quad \mathbb{C}_{ij}^{(n)}(x,\mathcal{L}) = \sum_{k=0}^{n} \mathbb{C}_{ij}^{(n,k)}(x)\mathcal{L}^k\,, \tag{25}$$

where the non-logarithmic terms $\mathcal{C}_{ij}^{(n,0)}$ and $\mathbb{C}_{ij}^{(n,0)}$ have to be computed explicitly while the other terms proportional to a positive power of $\mathcal{L}$ can be expressed in terms of the non-logarithmic term of the previous orders and of the coefficients of the DGLAP splitting functions and of the QCD β function:

$$\begin{aligned}
\mathcal{C}_{ij}^{(1,1)}(x) &= -\mathcal{P}_{ij}^{(1)}(x)\,,\\
\mathcal{C}_{ij}^{(2,1)}(x) &= -\left(\mathcal{P}_{ij}^{(2)}(x) + \mathcal{C}_{ik}^{(1,0)}(x)\otimes\mathcal{P}_{kj}^{(1)}(x) - \beta_0\mathcal{C}_{ij}^{(1,0)}(x)\right)\,,\\
\mathcal{C}_{ij}^{(2,2)}(x) &= \frac{1}{2}\left(\mathcal{P}_{ik}^{(1)}(x)\otimes\mathcal{P}_{kj}^{(1)}(x) - \beta_0\mathcal{P}_{ij}^{(1)}(x)\right)\,,
\end{aligned} \tag{26}$$

and:

$$\begin{aligned}
\mathbb{C}_{ij}^{(1,1)}(x) &= -\mathbb{P}_{ij}^{(1)}(x)\,,\\
\mathbb{C}_{ij}^{(2,1)}(x) &= -\left(\mathbb{P}_{ij}^{(2)}(x) + \mathbb{C}_{ik}^{(1,0)}(x)\otimes\mathbb{P}_{kj}^{(1)}(x) - \beta_0\mathbb{C}_{ij}^{(1,0)}(x)\right)\,,\\
\mathbb{C}_{ij}^{(2,2)}(x) &= \frac{1}{2}\left(\mathbb{P}_{ik}^{(1)}(x)\otimes\mathbb{P}_{kj}^{(1)}(x) - \beta_0\mathbb{P}_{ij}^{(1)}(x)\right)\,,
\end{aligned} \tag{27}$$

where $\mathcal{P}_{ij}^{(n)}$ and $\mathbb{P}_{ij}^{(n)}$ are the coefficients of the a_s^n terms of the space- and time-like splitting functions, respectively. With this information at hand, knowing the logarithmic expansion of the coefficients $R^{(n)}$, and keeping in mind that the hard coefficients $H^{(n)}$ do not contain any logarithms, we can organize the coefficients inEq. (24) in terms of powers of $\mathcal{L}$. Specifically, we find that:

$$\hat{B}_{ij,kl}^{(n)} = \sum_{p=0}^{2n} \hat{B}_{ij,kl}^{(n,p)}\mathcal{L}^p\,, \tag{28}$$

with the $\mathcal{O}(1)$ coefficient being:

$$\hat{B}_{ij,kl}^{(0,0)} = \delta_{ik}\delta_{kl}\delta(1-x)\delta(1-z)\,, \tag{29}$$

the $\mathcal{O}(a_s)$ coefficients being:

$$\begin{aligned}
\hat{B}_{ij,kl}^{(1,0)} &= (H^{(1)} + S^{(1,0)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + \mathcal{C}_{ik}^{(1,0)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,0)}(z)\,,\\
\hat{B}_{ij,kl}^{(1,1)} &= S^{(1,1)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + \mathcal{C}_{ik}^{(1,1)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}_{jl}^{(1,1)}(z)\,,\\
\hat{B}_{ij,kl}^{(1,2)} &= S^{(1,2)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)
\end{aligned} \tag{30}$$

and the $\mathcal{O}(a_s^2)$ coefficients being:

$$
\begin{aligned}
\hat{B}^{(2,0)}_{ij,kl} &= (H^{(2)} + H^{(1)} S^{(1,0)} + \widetilde{S}^{(2,0)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\,, \\
&+ (H^{(1)} + S^{(1,0)}) \left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,0)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,0)}_{jl}(z) \\
\hat{B}^{(2,1)}_{ij,kl} &= (H^{(1)} S^{(1,1)} + \widetilde{S}^{(2,1)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ (H^{(1)} + S^{(1,0)}) \left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
&+ S^{(1,1)} \left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,1)}_{ik}(x)\mathbb{C}^{(1,0)}_{jl}(z) + \mathcal{C}^{(1,0)}_{ik}(x)\mathbb{C}^{(1,1)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,1)}_{jl}(z) \\
\hat{B}^{(2,2)}_{ij,kl} &= (H^{(1)} S^{(1,2)} + \widetilde{S}^{(2,2)})\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) \\
&+ \widetilde{S}^{(1,2)} \left[\mathcal{C}^{(1,0)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,0)}_{jl}(z)\right] \\
&+ S^{(1,1)} \left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
&+ \mathcal{C}^{(2,2)}_{ik}(x)\delta_{jl}\delta(1-z) + \mathcal{C}^{(1,1)}_{ik}(x)\mathbb{C}^{(1,1)}_{jl}(z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(2,2)}_{jl}(z) \\
\hat{B}^{(2,3)}_{ij,kl} &= \widetilde{S}^{(2,3)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z) + S^{(1,2)} \left[\mathcal{C}^{(1,1)}_{ik}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{C}^{(1,1)}_{jl}(z)\right] \\
\hat{B}^{(2,4)}_{ij,kl} &= \widetilde{S}^{(2,4)}\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)
\end{aligned}
\tag{31}
$$

In order to obtain a differential cross section in q_T, we need to take the Fourier transform of $\hat{B}_{ij,kl}$, that is:

$$
\hat{B}_{ij,kl}(x,z;q_T) \equiv \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \hat{B}_{ij,kl}(x,z;b) = \sum_{n=0}^{2} a_s^n(Q) \sum_{p=0}^{2n} \hat{B}^{(n,p)}_{ij,kl}(x,z) I_p(q_T)\,, \tag{32}
$$

where I have defined:

$$
I_p(q_T) = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \mathcal{L}^p = \int \frac{d^2\mathbf{b}}{4\pi} e^{i\mathbf{b}\cdot\mathbf{q}_T} \ln^p\left(\frac{b^2Q^2}{C_0^2}\right) = \frac{1}{2}\int_0^\infty db\, b J_0(bq_T) \ln^p\left(\frac{b^2Q^2}{C_0^2}\right)\,. \tag{33}
$$

Results for I_p have been computed up to $p=4$ in Eq. (136) of Appendix B of Ref. [4]. Specifically, and including the trivial tranform with $p = 0$, they read:

$$
\begin{aligned}
I_0(q_T) &= \delta(q_T)\,,\\
I_1(q_T) &= -\frac{1}{q_T^2}\,,\\
I_2(q_T) &= -\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\,,\\
I_3(q_T) &= -\frac{3}{q_T^2}\ln^2\left(\frac{Q^2}{q_T^2}\right)\,,\\
I_4(q_T) &= -\frac{4}{q_T^2}\left[\ln^3\left(\frac{Q^2}{q_T^2}\right)-4\zeta_3\right]\,.
\end{aligned}
\tag{34}
$$

As clear from the transforms above, all terms with $p = 0$ will be proportional to $\delta(q_T)$. We do not need to consider these terms because analogous terms are not included in the fixed-order calculation and thus does not need to be subtracted. Therefore, we write:

$$
\hat{B}_{ij,kl}(x,z;q_T) = \sum_{n=1}^{2} a_s^n(Q)\sum_{p=1}^{2n}\hat{B}_{ij,kl}^{(n,p)}(x,z)I_p(q_T) + \left(\sum_{n=0}^{2} a_s^n(Q)\hat{B}_{ij,kl}^{(n,0)}(x,z)\right)\delta(q_T) + \mathcal{O}(a_s^3)\,, \tag{35}
$$

and we are not going to consider the term proportional to $\delta(q_T)$, even though all terms have been derived above.

As clear from Eq. (35), removing all terms proportional to $\delta(q_T)$ also means removing the full $\mathcal{O}(1)$ terms such that leading-order term is now $\mathcal{O}(a_s)$.

In order to validate the results above, it is opportune to compare the $\mathcal{O}(a_s)$ expressions to those present in the literature. To this end, we write explicitly the expression for $\hat{B}_{ij,kl}^{(1)}$ in q_T space without the $\delta(q_T)$ term:

$$
\begin{aligned}
\hat{B}_{ij,kl}^{(1)}(x,z;q_T) &= -\hat{B}_{ij,kl}^{(1,1)}(x,z)\mathcal{L}\frac{1}{q_T^2} - \hat{B}_{ij,kl}^{(1,2)}(x,z)\frac{2}{q_T^2}\ln\left(\frac{Q^2}{q_T^2}\right)\\
&= \frac{1}{q_T^2}\left[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)\delta_{ik}\delta_{jl}\delta(1-x)\delta(1-z)\right.\\
&+ \left.\mathcal{P}_{ik}^{(1)}(x)\delta_{jl}\delta(1-z) + \delta_{ik}\delta(1-x)\mathbb{P}_{jl}^{(1)}(z)\right]\,,
\end{aligned}
\tag{36}
$$

so that:

$$
\begin{aligned}
B_{ij}(x,z;q_T) &= a_s(Q)\sum_{kl}\hat{B}_{ij,kl}^{(1)}\underset{x}{\otimes} f_k(x,Q)\underset{z}{\otimes} d_l(z,Q) + \mathcal{O}(a_s^2)\\
&= a_s(Q)\frac{1}{q_T^2}\left[4C_F\left(\ln\left(\frac{Q^2}{q_T^2}\right)-\frac{3}{2}\right)f_i(x,Q)d_j(z,Q)\right.\\
&+ \left.\left(\sum_k \mathcal{P}_{ik}^{(1)}(x)\underset{x}{\otimes} f_k(x,Q)\right)d_j(z,Q) + f_i(x,Q)\left(\sum_l \mathbb{P}_{jl}^{(1)}(z)\underset{z}{\otimes} d_l(z,Q)\right)\right] + \mathcal{O}(a_s^2)\,.
\end{aligned}
\tag{37}
$$

This result nicely agrees with that of, *e.g.*, Refs. [5, 6].

Let us start from Eq. (2.6) of Ref. [1], that is the fully differential cross section for lepton-pair production in the region in which the TMD factorisation applies, *i.e.* $q_T \ll Q$. After some minor manipulations, it reads:

$$\frac{d\sigma}{dQdydq_T} = \frac{16\pi\alpha^2(Q)q_T\mathcal{P}(q_T,Q)}{3N_cQ^3}H(Q,\mu)\sum_q C_q(Q)\int\frac{d^2\mathbf{b}}{4\pi}e^{i\mathbf{b}\cdot\mathbf{q}_T}x_1F_q(x_1,\mathbf{b};\mu,\zeta)x_2F_{\bar{q}}(x_2,\mathbf{b};\mu,\zeta)\,, \tag{38}$$

where Q, y, and q_T are the invariant mass, the rapidity, and the transverse momentum of the lepton pair, respectively, while $N_c = 3$ is the number of colours, α is the electromagnetic coupling, H is the appropriate QCD form factor that can be perturbatively computed, and C_q are the effective electroweak charges. In addition, the variables x_1 and x_2 are functions of Q and y and are given by:

$$x_{1,2} = \frac{Q}{\sqrt{s}}e^{\pm y}\,, \tag{39}$$

being $\sqrt{s}$ the centre-of-mass energy of the collision. The kinematic factor $\mathcal{P}$ takes into account the reduction of the integration leptonic phase space due to possible cuts on the leptons and thus it depends on q_T, y, and Q as well as on the numerical values of the cut parameters. Finally, the scales μ and ζ are introduced through TMD factorisation to factorise collinear and rapidity divergences. As usual, despite they are arbitrary scales, they are typically chosen $\mu = \sqrt{\zeta} = Q$. Therefore, for all practical purposes their presence is fictitious.

The computation-intensive part of eq.(38) has the form of the integral:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \int\frac{d^2\mathbf{b}}{4\pi}e^{i\mathbf{b}\cdot\mathbf{q}_T}x_1F_i(x_1,\mathbf{b};\mu,\zeta)x_2F_j(x_2,\mathbf{b};\mu,\zeta)\,. \tag{40}$$

where $F_{i(j)}$ are combinations of evolved TMD PDFs. At this stage, for convenience, i and j do not coincide with q and $\bar{q}$ but they are linked through a simple linear transformation. The integral over the bidimensional impact parameter $\mathbf{b}$ has to be taken. However, $F_{i(j)}$ only depend on the absolute value of $\mathbf{b}$, therefore eq. (40) can be written as:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)x_1F_i(x_1,b;\mu,\zeta)x_2F_j(x_2,b;\mu,\zeta)\,. \tag{41}$$

where J_0 is the zero-th order Bessel function of the first kind whose integral representation is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}d\theta e^{ix\cos(\theta)}\,. \tag{42}$$

The single evolved TMD PDF F_i at the final scales μ and ζ is obtained by multiplying the same TMD PDF at the initial scales μ_0 and ζ_0 by a single evolution factor R_q([3]), that is:

$$xF_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0\to\mu,\zeta;b)xF_i(x,b;\mu_0,\zeta_0)\,. \tag{43}$$

so that eq. (41) becomes:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)\left[R_q(\mu_0,\zeta_0\to\mu,\zeta;b)\right]^2 x_1F_i(x_1,b;\mu_0,\zeta_0)x_2F_j(x_2,b;\mu_0,\zeta_0)\,. \tag{44}$$

The initial scale TMD PDFs at LO in the OPE region, that is for $b \ll B$ where B is an unknown non-perturbative parameter that represents the intrinsic hadron scale (see eq. (2.27) of Ref. [1]), can be written as:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} x\int_x^1\frac{dy}{y}C_{ij}(y;\mu_0,\zeta_0)f_j\left(\frac{x}{y},\mu_0\right)\,, \tag{45}$$

[3] Note that in eq. (38) the gluon TMD PDF F_g is not involved. If also the gluon TMD PDF was involved, it would evolve by means of a different evolution factor R_g.

where f_j are the collinear PDFs (including the gluon) and C_{ij} are the so-called matching functions that are perturbatively computable and are currently known to NNLO, *i.e.* $\mathcal{O}(\alpha_s^2)$. If we define:

$$\widetilde{f}_i\left(x,\mu_0\right) = x f_i\left(x,\mu_0\right)\,, \tag{46}$$

eq. (45) can be written as:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\widetilde{f}_i\left(\frac{x}{y},\mu_0\right)\,. \tag{47}$$

At this point, it is opportune to mention that the variables μ_0 and ζ_0 are usually taken to be functions of the impact parameter b. Therefore, eq. (47) is a function of two variables only that we rewrite way as:

$$xF_i(x,b) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y,b)\widetilde{f}_i\left(\frac{x}{y},b\right)\,. \tag{48}$$

This kind of convolutions can be computed using standard interpolation techniques by which one approximates the function $\widetilde{f}_i$ as:

$$\widetilde{f}_i(x,b) = \sum_\alpha w_\alpha(x)\widetilde{f}_i(x_\alpha,b) \tag{49}$$

where x_α is the α-th node of an interpolation grid and w_α is the interpolating function associated to that node. Assuming for now that x coincides with the β-th node of the grid and introducing another grid in the b dimension whose nodes are indexed by τ, eq. (48) can be written as:

$$\hat{F}_{i,\beta}^\tau = \sum_j\sum_\alpha \hat{C}_{ij,\beta\alpha}^\tau \hat{f}_{j,\alpha}^\tau\,. \tag{50}$$

where we have used the following definitions:

$$\hat{F}_{i,\beta}^\tau \equiv x_\beta F_i(x_\beta,b_\tau)\,,\quad \hat{C}_{ij,\beta\alpha}^\tau \equiv \int_{x_\beta}^1 dy\, C_{ij}(y,b_\tau)w_\alpha\left(\frac{x_\beta}{y}\right)\,,\quad \hat{f}_{j,\alpha}^\tau \equiv \widetilde{f}_i(x_\alpha,b_\tau)\,. \tag{51}$$

Since we have to integrate over the impact parameter b (see eq. (41)), we need to be able to reconstruct the dependence of the function F_i on b. This can be done using the same interpolation technique. In particular, we write:

$$x_\alpha F_i(x_\alpha,b)x_\beta F_j(x_\beta,b) = \sum_\tau \widetilde{w}_\tau(b)\hat{F}_{i,\alpha}^\tau\hat{F}_{j,\beta}^\tau = \sum_\tau \widetilde{w}_\tau(b)\sum_{kl}\sum_{\gamma\delta}\hat{C}_{ik,\alpha\gamma}^\tau\hat{C}_{jl,\beta\delta}^\tau\hat{f}_{k,\gamma}^\tau\hat{f}_{l,\delta}^\tau\,. \tag{52}$$

Keeping in mind that μ_0 and ζ_0 are functions of the impact parameter b and that $\mu = \sqrt{\zeta} = Q$, eq. (44) takes the form:

$$I_{ij}(x_\alpha,x_\beta,q_T;Q) = \sum_\tau K_\tau(Q;q_T)\sum_{kl}\sum_{\gamma\delta}\hat{C}_{ik,\alpha\gamma}^\tau\hat{C}_{jl,\beta\delta}^\tau\hat{f}_{k,\gamma}^\tau\hat{f}_{l,\delta}^\tau\,, \tag{53}$$

where we have defined:

$$K_\tau(Q;q_T) \equiv \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)\left[R_q(Q;b)\right]^2\widetilde{w}_\tau(b)\,, \tag{54}$$

being $R_q(Q;b) \equiv R_q(\mu_0,\zeta_0 \to \mu,\zeta;b)$. It should be noticed that $\widetilde{w}_\tau$ is a piecewise function different from zero only over a finite interval in b, say $[c_\tau,d_\tau]$. In practice, $\widetilde{w}_\tau$ extends over $k+1$ intervals on the grid in b, being k the interpolation degree, around the node b_τ so that, typically $c_\tau = b_{\tau-k}$ and $d_\tau = b_{\tau+1}$. Therefore the integral in eq. (54) reduces to:

$$K_\tau(Q;q_T) \equiv \frac{1}{2}\int_{c_\tau}^{d_\tau} db\, bJ_0(bq_T)\left[R_q(Q;b)\right]^2\widetilde{w}_\tau(b)\,, \tag{55}$$

and the sum over τ in eq. (53), that is supposed to run over an infinite number of nodes, has to be truncated.

As customary in QCD, the most convenient flavour basis, that is the one that minimises the mixing between operators, is the so-called "evolution" basis (*i.e.* Σ, V, T_3, V_3, etc.). In fact, in this basis the operators matrix C_{ij} is almost diagonal with the only exception of crossing terms that couple the gluon and the singlet Σ distributions. This greatly simplifies the sums over k and l in eq. (53). On the other hand, given that the TMDs that appear in eq. (38) are in the so-called "physical" basis (*i.e.* d, $\bar{d}$, u, $\bar{u}$, etc.), we need to rotate the quantity in eq. (53) from the evolution basis, over which the indices i and j run, to the physical basis. This is done by means of an appropriate constant matrix T, so that:

$$I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \sum_\tau \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} \,. \tag{56}$$

In order account for higher orders in the OPE and non-perturbative effects where the OPE is not valid, one usually introduces a phenomenological non-perturbative function $f_{\rm NP}$ that modifies the convolution in eq. (45) The way how $f_{\rm NP}(x, b)$ is introduced in not unique. Here we choose to follow the most traditional approach in which TMDs get corrected by a multiplicative function, that is to say:

$$xF_i(x, b) \rightarrow f_{\rm NP}(x, b) xF_i(x, b) \,. \tag{57}$$

This can be easily introduced in eq. (60) by defining:

$$f^\tau_{{\rm NP},\alpha} \equiv f_{\rm NP}(x_\alpha, b_\tau) \,, \tag{58}$$

so that:

$$I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \sum_\tau \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,. \tag{59}$$

The computation of $I_{q\bar{q}}$ for a generic x_1 and x_2 is achieved by interpolation as:

$$\begin{gathered} I_{q\bar{q}}(x_1, x_2, q_T; Q) = \sum_{\alpha\beta} w_\alpha(x_1) w_\beta(x_2) I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \\ \sum_\tau \sum_{\alpha\beta} \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) w_\alpha(x_1) w_\beta(x_2) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,. \end{gathered} \tag{60}$$

Keeping in mind eq. (39), one realises that the variables x_1 and x_2 are functions of Q and y and thus one can simply write:

$$I_{q\bar{q}}(Q, y, q_T) = \sum_\tau \sum_{\alpha\beta} W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,, \tag{61}$$

where we have defined:

$$W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) \equiv \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) w_\alpha\left(\frac{Q}{\sqrt{s}} e^y\right) w_\beta\left(\frac{Q}{\sqrt{s}} e^{-y}\right) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} \,. \tag{62}$$

Unsurprisingly, the W factors can be factorised as:

$$W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) \equiv K_\tau(Q; q_T) w_\alpha\left(\frac{Q}{\sqrt{s}} e^y\right) \left(\sum_i T_{qi} \sum_k \sum_\gamma \hat{C}^\tau_{ik,\alpha\gamma} \hat{f}^\tau_{k,\gamma}\right) w_\beta\left(\frac{Q}{\sqrt{s}} e^{-y}\right) \left(\sum_j T_{\bar{q}j} \sum_l \sum_\delta \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{l,\delta}\right) \,. \tag{63}$$

This equation emphasises that $I_{q\bar{q}}$ is a function of three independent kinematics variables Q, y, and q_T. This is relevant when integrating the cross section over the experimental bins as we will discuss in the next section. With eq. (61) at hand, eq. (38) can be written as:

$$\frac{d\sigma}{dQdydq_T} = \sum_{\tau}\sum_{\alpha\beta}\left[\frac{16\pi\alpha^2(Q)q_T\mathcal{P}(q_T,Q)}{3N_cQ^3}H(Q)\sum_q C_q(Q)W^{\tau}_{q\bar{q},\alpha\beta}(Q,y,q_T)\right]f^{\tau}_{\mathrm{NP},\alpha}f^{\tau}_{\mathrm{NP},\beta}\,. \tag{64}$$

Crucially, the quantity inside the squared brackets is fully determined by the kinematics and the leading-twist component of the process, while the non-perturbative part is fully factorised. Clearly, this is extremely useful if one wants to fit the non-perturbative component to data.

2 Integrating over the final-state kinematic variables

Despite eq. (56) provides a powerful tool for a fast computation of cross sections, it is often not sufficient to allow for a direct comparison to experimental data. The reason is that experimental measurements of differential distributions are usually delivered integrated over finite regions of the final-state kinematic phase space. In other words, experiments measure quantities like:

$$\widetilde{\sigma} = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{y_{\min}}^{y_{\max}} dy \int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T}\right]\,. \tag{65}$$

As a consequence, in order to guarantee performance, we need to include the integrations above in the precomputed factors.

2.1 Integrating over q_T

The integration over q_T is relatively simple to implement because the full dependence on q_T in eq. (55) is given by the factors q_T, $\mathcal{P}$, and K_τ. Therefore, integrating over q_T simply amounts of computing the integrals:

$$\widetilde{K}_\tau(Q) \equiv \int_{q_{T,\min}}^{q_{T,\max}} dq_T\, q_T\mathcal{P}(q_T,Q)K_\tau(Q;q_T) \tag{66}$$

2.2 Integrating over y

The dependence on y of the cross section in eq. (38) exclusively happens through the variables x_1 and x_2 defined in eq. (39). Since this dependence is reconstructed through interpolation in eq. (60), what we need to do is computing the following integrals:

$$u_{\alpha\beta}(Q) \equiv \int_{y_{\min}}^{y_{\max}} dy\, w_\alpha\left(\frac{Q}{\sqrt{s}}e^{y}\right) w_\beta\left(\frac{Q}{\sqrt{s}}e^{-y}\right) \tag{67}$$

and replace $w_\alpha(x_1)w_\beta(x_2)$ in eq. (60) with $u_{\alpha\beta}(Q)$.

2.3 Integrating over Q

The integration over Q has finally to be done by brute force due to the fact that the dependence on Q of the expression we are considering is not localised and involves essentially all ingredients (we remind that we are assuming $\mu = \sqrt{\zeta} = Q$). One alternative solution is to use the so-called narrow-width approximation (NWA) in which one assumes that the width of the Z boson Γ_Z is much smaller that its mass M_Z. This way one can approximate the peaked behaviour of the couplings $C_q(Q)$ around $Q = M_Z$ with a δ-function, *i.e.* $C_q(Q) \sim \delta(Q^2 - M_Z^2)$, so that the integration over Q can be done analytically essentially setting $Q = M_Z$ everywhere in the expression. This approximation, though, is usable only for data around the Z peak and, of course, it is only an approximation and thus might lead to substantial inaccuracies. Therefore, it is useful to be able to carry out the integration over Q explicitly.

To this end, we start by writing explicitly the cross section integrated over q_T and y making use of the definitions given in the previous subsections:

$$\frac{d\sigma}{dQ} = \sum_\tau \sum_{\alpha\beta} \left[\frac{16\pi}{3N_c} \sum_{kl} \sum_{\gamma\delta} \sum_{ij} \sum_q \left(\frac{\alpha^2(Q)}{Q^3} H(Q) u_{\alpha\beta}(Q) \widetilde{K}_\tau(Q) C_q(Q) \right) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} \right] f^\tau_{\mathrm{NP},\alpha} f^\tau_{\mathrm{NP},\beta}\,, \tag{68}$$

where we have purposely enclosed between round brakets the Q-dependant factors. In fact, if we define:

$$S^\tau_{q,\alpha\beta} \equiv \frac{16\pi}{3N_c} \int_{Q_{\min}}^{Q_{\max}} dQ\, \frac{\alpha^2(Q)}{Q^3} H(Q) \widetilde{K}_\tau(Q) u_{\alpha\beta}(Q) C_q(Q)\,, \tag{69}$$

we have that:

$$\widetilde{\sigma} = \sum_\tau \sum_{\alpha\beta} \left[\sum_q S^\tau_{q,\alpha\beta} \left(\sum_i T_{qi} \sum_k \sum_\gamma \hat{C}^\tau_{ik,\alpha\gamma} \hat{f}^\tau_{k,\gamma} \right) \left(\sum_j T_{\bar{q}j} \sum_l \sum_\delta \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{l,\delta} \right) \right] f^\tau_{\mathrm{NP},\alpha} f^\tau_{\mathrm{NP},\beta}\,. \tag{70}$$

so that, defining:

$$\overline{F}^\tau_{q(\bar{q}),\alpha} \equiv \sum_i T_{q(\bar{q})i} \sum_k \sum_\gamma \hat{C}^\tau_{ik,\alpha\gamma} \hat{f}^\tau_{k,\gamma}\,, \tag{71}$$

eq. (70) can be readily written as:

$$\widetilde{\sigma} = \sum_\tau \sum_{\alpha\beta} \left[\sum_q S^\tau_{q,\alpha\beta} \overline{F}^\tau_{q,\alpha} \overline{F}^\tau_{\bar{q},\beta} \right] f^\tau_{\mathrm{NP},\alpha} f^\tau_{\mathrm{NP},\beta} = \sum_\tau \sum_{\alpha\beta} M^\tau_{\alpha\beta} f^\tau_{\mathrm{NP},\alpha} f^\tau_{\mathrm{NP},\beta}\,, \tag{72}$$

with:

$$M^\tau_{\alpha\beta} \equiv \sum_q S^\tau_{q,\alpha\beta} \overline{F}^\tau_{q,\alpha} \overline{F}^\tau_{\bar{q},\beta}\,. \tag{73}$$

References

[1] I. Scimemi and A. Vladimirov, arXiv:1706.01473 [hep-ph].

[2] M. G. Echevarria, I. Scimemi and A. Vladimirov, JHEP **1609** (2016) 004 doi:10.1007/JHEP09(2016)004 [arXiv:1604.07869 [hep-ph]].

[3] W. L. van Neerven and A. Vogt, Nucl. Phys. B **588** (2000) 345 doi:10.1016/S0550-3213(00)00480-6 [hep-ph/0006154].

[4] G. Bozzi, S. Catani, D. de Florian and M. Grazzini, Nucl. Phys. B **737** (2006) 73 doi:10.1016/j.nuclphysb.2005.12.022 [hep-ph/0508068].

[5] R. Meng, F. I. Olness and D. E. Soper, Phys. Rev. D **54** (1996) 1919 doi:10.1103/PhysRevD.54.1919 [hep-ph/9511311].

[6] J. Collins, L. Gamberg, A. Prokudin, T. C. Rogers, N. Sato and B. Wang, Phys. Rev. D **94** (2016) no.3, 034014 doi:10.1103/PhysRevD.94.034014 [arXiv:1605.00671 [hep-ph]].