1 Structure of the observables

Let us start from Eq. (2.6) of Ref. [1], that is the fully differential cross section for lepton-pair production in the region in which the TMD factorisation applies, *i.e.* $q_T \ll Q$. After some minor manipulations, it reads:

$$\frac{d\sigma}{dQdydq_T} = \frac{16\pi\alpha^2(Q)q_T\mathcal{P}(q_T,Q)}{3N_cQ^3}H(Q,\mu)\sum_q C_q(Q)\int\frac{d^2\mathbf{b}}{4\pi}e^{i\mathbf{b}\cdot\mathbf{q}_T}x_1F_q(x_1,\mathbf{b};\mu,\zeta)x_2F_{\bar{q}}(x_2,\mathbf{b};\mu,\zeta)\,, \tag{1}$$

where Q, y, and q_T are the invariant mass, the rapidity, and the transverse momentum of the lepton pair, respectively, while $N_c = 3$ is the number of colours, α is the electromagnetic coupling, H is the appropriate QCD form factor that can be perturbatively computed, and C_q are the effective electroweak charges. In addition, the variables x_1 and x_2 are functions of Q and y and are given by:

$$x_{1,2} = \frac{Q}{\sqrt{s}}e^{\pm y}\,, \tag{2}$$

being $\sqrt{s}$ the centre-of-mass energy of the collision. The kinematic factor $\mathcal{P}$ takes into account the reduction of the integration leptonic phase space due to possible cuts on the leptons and thus it depends on q_T, y, and Q as well as on the numerical values of the cut parameters. Finally, the scales μ and ζ are introduced through TMD factorisation to factorise collinear and rapidity divergences. As usual, despite they are arbitrary scales, they are typically chosen $\mu = \sqrt{\zeta} = Q$. Therefore, for all practical purposes their presence is fictitious.

The computation-intensive part of eq.(1) has the form of the integral:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \int\frac{d^2\mathbf{b}}{4\pi}e^{i\mathbf{b}\cdot\mathbf{q}_T}x_1F_i(x_1,\mathbf{b};\mu,\zeta)x_2F_j(x_2,\mathbf{b};\mu,\zeta)\,. \tag{3}$$

where $F_{i(j)}$ are combinations of evolved TMD PDFs. At this stage, for convenience, i and j do not coincide with q and $\bar{q}$ but they are linked through a simple linear transformation. The integral over the bidimensional impact parameter $\mathbf{b}$ has to be taken. However, $F_{i(j)}$ only depend on the absolute value of $\mathbf{b}$, therefore eq. (3) can be written as:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\,bJ_0(bq_T)x_1F_i(x_1,b;\mu,\zeta)x_2F_j(x_2,b;\mu,\zeta)\,. \tag{4}$$

where J_0 is the zero-th order Bessel function of the first kind whose integral representation is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}d\theta e^{ix\cos(\theta)}\,. \tag{5}$$

The single evolved TMD PDF F_i at the final scales μ and ζ is obtained by multiplying the same TMD PDF at the initial scales μ_0 and ζ_0 by a single evolution factor R_q([1]), that is:

$$xF_i(x,b;\mu,\zeta) = R_q(\mu_0,\zeta_0\to\mu,\zeta;b)xF_i(x,b;\mu_0,\zeta_0)\,. \tag{6}$$

so that eq. (4) becomes:

$$I_{ij}(x_1,x_2,q_T;\mu,\zeta) = \frac{1}{2}\int_0^\infty db\,bJ_0(bq_T)\left[R_q(\mu_0,\zeta_0\to\mu,\zeta;b)\right]^2x_1F_i(x_1,b;\mu_0,\zeta_0)x_2F_j(x_2,b;\mu_0,\zeta_0)\,. \tag{7}$$

The initial scale TMD PDFs at LO in the OPE region, that is for $b \ll B$ where B is an unknown non-perturbative parameter that represents the intrinsic hadron scale (see eq. (2.27) of Ref. [1]), can be written as:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})}x\int_x^1\frac{dy}{y}C_{ij}(y;\mu_0,\zeta_0)f_j\left(\frac{x}{y},\mu_0\right)\,, \tag{8}$$

[1] Note that in eq. (1) the gluon TMD PDF F_g is not involved. If also the gluon TMD PDF was involved, it would evolve by means of a different evolution factor R_g.

where f_j are the collinear PDFs (including the gluon) and C_{ij} are the so-called matching functions that are perturbatively computable and are currently known to NNLO, *i.e.* $\mathcal{O}(\alpha_s^2)$. If we define:

$$\widetilde{f}_i\left(x,\mu_0\right) = x f_i\left(x,\mu_0\right)\,, \tag{9}$$

eq. (8) can be written as:

$$xF_i(x,b;\mu_0,\zeta_0) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y;\mu_0,\zeta_0)\widetilde{f}_i\left(\frac{x}{y},\mu_0\right)\,. \tag{10}$$

At this point, it is opportune to mention that the variables μ_0 and ζ_0 are usually taken to be functions of the impact parameter b. Therefore, eq. (10) is a function of two variables only that we rewrite way as:

$$xF_i(x,b) = \sum_{j=g,q(\bar{q})} \int_x^1 dy\, C_{ij}(y,b)\widetilde{f}_i\left(\frac{x}{y},b\right)\,. \tag{11}$$

This kind of convolutions can be computed using standard interpolation techniques by which one approximates the function $\widetilde{f}_i$ as:

$$\widetilde{f}_i(x,b) = \sum_\alpha w_\alpha(x)\widetilde{f}_i(x_\alpha,b) \tag{12}$$

where x_α is the α-th node of an interpolation grid and w_α is the interpolating function associated to that node. Assuming for now that x coincides with the β-th node of the grid and introducing another grid in the b dimension whose nodes are indexed by τ, eq. (11) can be written as:

$$\hat{F}^\tau_{i,\beta} = \sum_j \sum_\alpha \hat{C}^\tau_{ij,\beta\alpha}\hat{f}^\tau_{j,\alpha}\,. \tag{13}$$

where we have used the following definitions:

$$\hat{F}^\tau_{i,\beta} \equiv x_\beta F_i(x_\beta,b_\tau)\,,\quad \hat{C}^\tau_{ij,\beta\alpha} \equiv \int_{x_\beta}^1 dy\, C_{ij}(y,b_\tau) w_\alpha\left(\frac{x_\beta}{y}\right)\,,\quad \hat{f}^\tau_{j,\alpha} \equiv \widetilde{f}_i(x_\alpha,b_\tau)\,. \tag{14}$$

Since we have to integrate over the impact parameter b (see eq. (4)), we need to be able to reconstruct the dependence of the function F_i on b. This can be done using the same interpolation technique. In particular, we write:

$$x_\alpha F_i(x_\alpha,b) x_\beta F_j(x_\beta,b) = \sum_\tau \widetilde{w}_\tau(b)\hat{F}^\tau_{i,\alpha}\hat{F}^\tau_{j,\beta} = \sum_\tau \widetilde{w}_\tau(b) \sum_{kl}\sum_{\gamma\delta} \hat{C}^\tau_{ik,\alpha\gamma}\hat{C}^\tau_{jl,\beta\delta}\hat{f}^\tau_{k,\gamma}\hat{f}^\tau_{l,\delta}\,. \tag{15}$$

Keeping in mind that μ_0 and ζ_0 are functions of the impact parameter b and that $\mu = \sqrt{\zeta} = Q$, eq. (7) takes the form:

$$I_{ij}(x_\alpha,x_\beta,q_T;Q) = \sum_\tau K_\tau(Q;q_T) \sum_{kl}\sum_{\gamma\delta} \hat{C}^\tau_{ik,\alpha\gamma}\hat{C}^\tau_{jl,\beta\delta}\hat{f}^\tau_{k,\gamma}\hat{f}^\tau_{l,\delta}\,, \tag{16}$$

where we have defined:

$$K_\tau(Q;q_T) \equiv \frac{1}{2}\int_0^\infty db\, bJ_0(bq_T)\left[R_q(Q;b)\right]^2 \widetilde{w}_\tau(b)\,, \tag{17}$$

being $R_q(Q;b) \equiv R_q(\mu_0,\zeta_0 \to \mu,\zeta;b)$. It should be noticed that $\widetilde{w}_\tau$ is a piecewise function different from zero only over a finite interval in b, say $[c_\tau,d_\tau]$. In practice, $\widetilde{w}_\tau$ extends over $k+1$ intervals on the grid in b, being k the interpolation degree, around the node b_τ so that, typically $c_\tau = b_{\tau-k}$ and $d_\tau = b_{\tau+1}$. Therefore the integral in eq. (17) reduces to:

$$K_\tau(Q;q_T) \equiv \frac{1}{2}\int_{c_\tau}^{d_\tau} db\, bJ_0(bq_T)\left[R_q(Q;b)\right]^2 \widetilde{w}_\tau(b)\,, \tag{18}$$

and the sum over τ in eq. (16), that is supposed to run over an infinite number of nodes, has to be truncated.

As customary in QCD, the most convenient flavour basis, that is the one that minimises the mixing between operators, is the so-called "evolution" basis (*i.e.* Σ, V, T_3, V_3, etc.). In fact, in this basis the operators matrix C_{ij} is almost diagonal with the only exception of crossing terms that couple the gluon and the singlet Σ distributions. This greatly simplifies the sums over k and l in eq. (16). On the other hand, given that the TMDs that appear in eq. (1) are in the so-called "physical" basis (*i.e.* d, $\bar{d}$, u, $\bar{u}$, etc.), we need to rotate the quantity in eq. (16) from the evolution basis, over which the indices i and j run, to the physical basis. This is done by means of an appropriate constant matrix T, so that:

$$I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \sum_\tau \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} \,. \tag{19}$$

In order account for higher orders in the OPE and non-perturbative effects where the OPE is not valid, one usually introduces a phenomenological non-perturbative function $f_{\rm NP}$ that modifies the convolution in eq. (8) The way how $f_{\rm NP}(x,b)$ is introduced in not unique. Here we choose to follow the most traditional approach in which TMDs get corrected by a multiplicative function, that is to say:

$$xF_i(x,b) \to f_{\rm NP}(x,b) xF_i(x,b) \,. \tag{20}$$

This can be easily introduced in eq. (23) by defining:

$$f^\tau_{{\rm NP},\alpha} \equiv f_{\rm NP}(x_\alpha, b_\tau) \,, \tag{21}$$

so that:

$$I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \sum_\tau \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,. \tag{22}$$

The computation of $I_{q\bar{q}}$ for a generic x_1 and x_2 is achieved by interpolation as:

$$\begin{aligned} I_{q\bar{q}}(x_1, x_2, q_T; Q) &= \sum_{\alpha\beta} w_\alpha(x_1) w_\beta(x_2) I_{q\bar{q}}(x_\alpha, x_\beta, q_T; Q) = \\ &\sum_\tau \sum_{\alpha\beta} \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) w_\alpha(x_1) w_\beta(x_2) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,. \end{aligned} \tag{23}$$

Keeping in mind eq. (2), one realises that the variables x_1 and x_2 are functions of Q and y and thus one can simply write:

$$I_{q\bar{q}}(Q, y, q_T) = \sum_\tau \sum_{\alpha\beta} W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) f^\tau_{{\rm NP},\alpha} f^\tau_{{\rm NP},\beta} \,, \tag{24}$$

where we have defined:

$$W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) \equiv \sum_{kl} \sum_{\gamma\delta} \sum_{ij} K_\tau(Q; q_T) w_\alpha\left(\frac{Q}{\sqrt{s}} e^y\right) w_\beta\left(\frac{Q}{\sqrt{s}} e^{-y}\right) T_{qi} T_{\bar{q}j} \hat{C}^\tau_{ik,\alpha\gamma} \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{k,\gamma} \hat{f}^\tau_{l,\delta} \,. \tag{25}$$

Unsurprisingly, the W factors can be factorised as:

$$W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) \equiv K_\tau(Q; q_T) w_\alpha\left(\frac{Q}{\sqrt{s}} e^y\right) \left(\sum_i T_{qi} \sum_k \sum_\gamma \hat{C}^\tau_{ik,\alpha\gamma} \hat{f}^\tau_{k,\gamma}\right) w_\beta\left(\frac{Q}{\sqrt{s}} e^{-y}\right) \left(\sum_j T_{\bar{q}j} \sum_l \sum_\delta \hat{C}^\tau_{jl,\beta\delta} \hat{f}^\tau_{l,\delta}\right) \,. \tag{26}$$

This equation emphasises that $I_{q\bar{q}}$ is a function of three independent kinematics variables Q, y, and q_T. This is relevant when integrating the cross section over the experimental bins as we will discuss in the next section. With eq. (24) at hand, eq. (1) can be written as:

$$\frac{d\sigma}{dQdydq_T} = \sum_\tau \sum_{\alpha\beta} \left[\frac{16\pi\alpha^2(Q) q_T \mathcal{P}(q_T, Q)}{3N_c Q^3} H(Q) \sum_q C_q(Q) W^\tau_{q\bar{q},\alpha\beta}(Q, y, q_T) \right] f^\tau_{\mathrm{NP},\alpha} f^\tau_{\mathrm{NP},\beta} \,. \tag{27}$$

Crucially, the quantity inside the squared brackets is fully determined by the kinematics and the leading-twist component of the process, while the non-perturbative part is fully factorised. Clearly, this is extremely useful if one wants to fit the non-perturbative component to data.

2 Integrating over the final-state kinematic variables

Despite eq. (19) provides a powerful tool for a fast computation of cross sections, it is often not sufficient to allow for a direct comparison to experimental data. The reason is that experimental measurements of differential distributions are usually delivered integrated over finite regions of the final-state kinematic phase space. In other words, experiments measure quantities like:

$$\widetilde{\sigma} = \int_{Q_{\min}}^{Q_{\max}} dQ \int_{y_{\min}}^{y_{\max}} dy \int_{q_{T,\min}}^{q_{T,\max}} dq_T \left[\frac{d\sigma}{dQdydq_T} \right] . \tag{28}$$

As a consequence, in order to guarantee performance, we need to include the integrations above in the precomputed factors.

2.1 Integrating over q_T

The integration over q_T is relatively simple to implement because the full dependence on q_T in eq. (18) is given by the factors q_T, $\mathcal{P}$, and K_τ. Therefore, integrating over q_T simply amounts of computing the integrals:

$$\widetilde{K}_\tau(Q) \equiv \int_{q_{T,\min}}^{q_{T,\max}} dq_T \, q_T \mathcal{P}(q_T, Q) K_\tau(Q; q_T) \tag{29}$$

2.2 Integrating over y

The dependence on y of the cross section in eq. (1) exclusively happens through the variables x_1 and x_2 defined in eq. (2). Since this dependence is reconstructed through interpolation in eq. (23), what we need to do is computing the following integrals:

$$u_{\alpha\beta}(Q) \equiv \int_{y_{\min}}^{y_{\max}} dy \, w_\alpha \left(\frac{Q}{\sqrt{s}} e^y \right) w_\beta \left(\frac{Q}{\sqrt{s}} e^{-y} \right) \tag{30}$$

and replace $w_\alpha(x_1) w_\beta(x_2)$ in eq. (23) with $u_{\alpha\beta}(Q)$.

2.3 Integrating over Q

The integration over Q has finally to be done by brute force due to the fact that the dependence on Q of the expression we are considering is not localised and involves essentially all ingredients (we remind that we are assuming $\mu = \sqrt{\zeta} = Q$). One alternative solution is to use the so-called narrow-width approximation (NWA) in which one assumes that the width of the Z boson Γ_Z is much smaller that its mass M_Z. This way one can approximate the peaked behaviour of the couplings $C_q(Q)$ around $Q = M_Z$ with a δ-function, *i.e.* $C_q(Q) \sim \delta(Q^2 - M_Z^2)$, so that the integration over Q can be done analytically essentially setting $Q = M_Z$ everywhere in the expression. This approximation, though, is usable only for data around the Z peak and, of course, it is only an approximation and thus might lead to substantial inaccuracies. Therefore, it is useful to be able to carry out the integration over Q explicitly.

To this end, we start by writing explicitly the cross section integrated over q_T and y making use of the definitions given in the previous subsections:

$$\frac{d\sigma}{dQ} = \sum_{\tau}\sum_{\alpha\beta}\left[\frac{16\pi}{3N_c}\sum_{kl}\sum_{\gamma\delta}\sum_{ij}\sum_{q}\left(\frac{\alpha^2(Q)}{Q^3}H(Q)u_{\alpha\beta}(Q)\widetilde{K}_{\tau}(Q)C_q(Q)\right)T_{qi}T_{\bar{q}j}\hat{C}^{\tau}_{ik,\alpha\gamma}\hat{C}^{\tau}_{jl,\beta\delta}\hat{f}^{\tau}_{k,\gamma}\hat{f}^{\tau}_{l,\delta}\right] f^{\tau}_{\mathrm{NP},\alpha}f^{\tau}_{\mathrm{NP},\beta}\,, \tag{31}$$

where we have purposely enclosed between round brakets the Q-dependant factors. In fact, if we define:

$$S^{\tau}_{q,\alpha\beta} \equiv \frac{16\pi}{3N_c}\int_{Q_{\min}}^{Q_{\max}} dQ\,\frac{\alpha^2(Q)}{Q^3}H(Q)\widetilde{K}_{\tau}(Q)u_{\alpha\beta}(Q)C_q(Q)\,, \tag{32}$$

we have that:

$$\widetilde{\sigma} = \sum_{\tau}\sum_{\alpha\beta}\left[\sum_{q}S^{\tau}_{q,\alpha\beta}\left(\sum_{i}T_{qi}\sum_{k}\sum_{\gamma}\hat{C}^{\tau}_{ik,\alpha\gamma}\hat{f}^{\tau}_{k,\gamma}\right)\left(\sum_{j}T_{\bar{q}j}\sum_{l}\sum_{\delta}\hat{C}^{\tau}_{jl,\beta\delta}\hat{f}^{\tau}_{l,\delta}\right)\right] f^{\tau}_{\mathrm{NP},\alpha}f^{\tau}_{\mathrm{NP},\beta}\,. \tag{33}$$

so that, defining:

$$\overline{F}^{\tau}_{q(\bar{q}),\alpha} \equiv \sum_{i}T_{q(\bar{q})i}\sum_{k}\sum_{\gamma}\hat{C}^{\tau}_{ik,\alpha\gamma}\hat{f}^{\tau}_{k,\gamma}\,, \tag{34}$$

eq. (33) can be readily written as:

$$\widetilde{\sigma} = \sum_{\tau}\sum_{\alpha\beta}\left[\sum_{q}S^{\tau}_{q,\alpha\beta}\overline{F}^{\tau}_{q,\alpha}\overline{F}^{\tau}_{\bar{q},\beta}\right] f^{\tau}_{\mathrm{NP},\alpha}f^{\tau}_{\mathrm{NP},\beta} = \sum_{\tau}\sum_{\alpha\beta}M^{\tau}_{\alpha\beta}f^{\tau}_{\mathrm{NP},\alpha}f^{\tau}_{\mathrm{NP},\beta}\,, \tag{35}$$

with:

$$M^{\tau}_{\alpha\beta} \equiv \sum_{q}S^{\tau}_{q,\alpha\beta}\overline{F}^{\tau}_{q,\alpha}\overline{F}^{\tau}_{\bar{q},\beta}\,. \tag{36}$$

References

[1] I. Scimemi and A. Vladimirov, arXiv:1706.01473 [hep-ph].