PhD Report 2015

F. Gatti[1,2]

[1]Laboratoire de Mécanique des Sols, Structures et Matériaux
CNRS UMR 8579 CentraleSupélec - Université Paris-Saclay , Chatenay-Malabry, France
[2]Dipartimento di Ingegneria Civile e Ambientale
Politecnico di Milano, Italy

February 27, 2016

Contents

List of Figures

List of Tables

1 Cyclic Non-Linear constitutive model for SEM codes

1.1 Introduction

A non-linear kinematic hardening rule is presented here within the framework of thermodynamic principles. The derived kinematic hardening evolution equation has two distinct terms: a strain hardening term and a dynamic recovery term that operates at all times. The proposed hardening rule refers to the Fredrick and Armstrong kinematic hardening rule. This kinematic hardening rule is incorporated in a material constitutive model based on the von Mises plasticity type (or J_2 plasticity). Numerical integration of the incremental elasto-plastic constitutive relationship is based on the *sub-stepping forward Euler* method proposed first by Sloan [1] and further modified by [2].

1.2 Non-Linear strain hardening constitutive model

PRELIMINARIES

$\underline{\underline{\boldsymbol{I}}} = \sum_{n=1}^{3} e_n \otimes e_n$	2^{nd}order Identity Tensor
$\underline{\underline{\boldsymbol{A}}} = \underline{\underline{\boldsymbol{A}}}_D + \frac{1}{3}\mathcal{I}_1^A \underline{\underline{\boldsymbol{I}}}_2$	Tensor decomposition: deviatoric and spherical part
$\underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{A}}} = Tr\left(\underline{\underline{\boldsymbol{A}}}\,\underline{\underline{\boldsymbol{A}}}^T\right)$	2^{nd}order Tensor Double Contraction
$\mathcal{I}_1^A = Tr\left(\underline{\underline{\boldsymbol{A}}}\right) = \underline{\underline{\boldsymbol{I}}} : \underline{\underline{\boldsymbol{A}}}$	Tensor 1^{st}Invariant
$\mathcal{J}_1^A = Tr\left(\underline{\underline{\boldsymbol{A}}}_D\right) = 0$	Deviatoric Tensor 1^{st}Invariant
$\mathcal{J}_2^A\left(\underline{\underline{\boldsymbol{A}}}\right) = \frac{1}{2}\underline{\underline{\boldsymbol{A}}}_D : \underline{\underline{\boldsymbol{A}}}_D$	Deviatoric Tensor 2^{nd}Invariant

$f\left(\underline{\underline{\boldsymbol{A}}}\right)$	Tensor Valued Scalar Function
$\underline{\underline{\boldsymbol{\nabla}_A \boldsymbol{f}}} = \sum_{m,n=1}^{3} \frac{\partial f}{\partial A_{mn}} e_m \otimes e_n$	Gradient of Tensor Valued Scalar Function
$f\left(\underline{\underline{\boldsymbol{A}}}; \underline{\underline{\boldsymbol{B}}}\right)$	Tensor Bi-Valued Scalar Function
$\mathcal{K}_1^{A-B} = \underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}}$	Tensor 1^{st}Joint Invariant
$\mathcal{K}_2^{A-B} = \underline{\underline{\boldsymbol{A}}} : \underline{\underline{\boldsymbol{B}}}^2$	Tensor 2^{nd}Joint Invariant
$\mathcal{K}_3^{A-B} = \underline{\underline{\boldsymbol{A}}}^2 : \underline{\underline{\boldsymbol{B}}}$	Tensor 3^{rd}Joint Invariant
$\mathcal{K}_4^{A-B} = \underline{\underline{\boldsymbol{A}}}^2 : \underline{\underline{\boldsymbol{B}}}^2$	Tensor 4^{th}Joint Invariant

$$f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)=f\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right) \qquad \text{Tensor Bi-Valued Scalar Function}$$
$$\mathcal{L}_1^{A-B}=\underline{\underline{\boldsymbol{I}}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right) \qquad \text{Tensor } 1^{st}\text{Invariant}$$
$$\mathcal{L}_2^{A-B}=\underline{\underline{\boldsymbol{I}}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)^2 \qquad \text{Tensor } 2^{nd}\text{Invariant}$$
$$\mathcal{L}_3^{A-B}=\underline{\underline{\boldsymbol{I}}}:\left(\underline{\underline{\boldsymbol{A}}}-\underline{\underline{\boldsymbol{B}}}\right)^3 \qquad \text{Tensor } 3^{st}\text{Invariant}$$

$$f\left(\underline{\underline{\boldsymbol{A}}};\underset{\sim}{\boldsymbol{\xi}}\right) \qquad \underset{\sim}{\boldsymbol{\xi}} \text{ Structural Hidden Variables}$$
$$\underset{\sim}{\boldsymbol{\xi}}=\left[\underline{\underline{\boldsymbol{A}}};\underline{\mathbf{a}};a\right] \qquad \text{Hidden Variables: Tensors,Vectors and Scalars}$$

Objectivity:

- $f\left(\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}\underset{\sim}{\boldsymbol{\xi}}\right)=f\left(\underline{\underline{\boldsymbol{A}}};\underset{\sim}{\boldsymbol{\xi}}\right)\forall\underline{\underline{\boldsymbol{Q}}}\in \mathit{SO(3)}$
  $\rightarrow \underline{\underline{\boldsymbol{Q}}}\underset{\sim}{\boldsymbol{\xi}}=\left[\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}\underline{\mathbf{a}};a\right]$

- $f\left(\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{B}}}\underline{\underline{\boldsymbol{Q}}}\right)=f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)\forall\underline{\underline{\boldsymbol{Q}}}\in \mathit{SO(3)}$
  $\rightarrow f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)=f\left(\mathcal{I}_m^A;\mathcal{I}_m^B;\mathcal{K}_m^{A-B}\right)$

- $f\left(\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{A}}}\underline{\underline{\boldsymbol{Q}}};\underline{\underline{\boldsymbol{Q}}}^T\underline{\underline{\boldsymbol{B}}}\underline{\underline{\boldsymbol{Q}}}\right)=f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)\forall\underline{\underline{\boldsymbol{Q}}}\in \mathit{SO(3)}$
  $\rightarrow f\left(\underline{\underline{\boldsymbol{A}}};\underline{\underline{\boldsymbol{B}}}\right)=f\left(\mathcal{L}_m^{A-B}\right)$

SMALL STRAIN REGIME

$$\underline{\underline{\boldsymbol{\varepsilon}}}=\underline{\underline{\boldsymbol{e}}}+\frac{1}{3}\mathcal{I}_1^{\varepsilon}\underline{\underline{\boldsymbol{I}}} \qquad \text{Deviatoric and Spherical decomposition}$$
$$\mathcal{I}_1^{\varepsilon}=Tr\left(\underline{\underline{\boldsymbol{\varepsilon}}}\right)=\varepsilon_{vol} \qquad 1^{st} \text{ Strain Tensor Invariant = Volumetric Strain}$$
$$\mathcal{J}_2^{\varepsilon}\left(\underline{\underline{\boldsymbol{\varepsilon}}}\right)=\frac{1}{2}\underline{\underline{\boldsymbol{e}}}:\underline{\underline{\boldsymbol{e}}} \qquad 2^{nd} \text{ Strain Tensor Invariant}$$
$$\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}=\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}^{el}+\dot{\underline{\underline{\boldsymbol{\varepsilon}}}}^{pl} \qquad \text{Small Strain Additivity}$$

HARDENING VARIABLES

$$\underset{\sim}{\boldsymbol{\eta}}=\left[\underline{\underline{\boldsymbol{\alpha}}};r\right] \qquad \text{Kinematic Internal Variables}$$
$$\underset{\sim}{\boldsymbol{\chi}}\left(\underline{\underline{\boldsymbol{\eta}}}\right)=\left[\underline{\underline{\boldsymbol{X}}}\left(\underline{\underline{\boldsymbol{\alpha}}}\right);R\left(r\right)\right] \qquad \text{Static Internal Variables}$$

$$\underline{\underline{\boldsymbol{X}}}\left(\underline{\underline{\boldsymbol{\alpha}}}\right) \qquad \text{Back Stress (Kinematic Hardening)}$$

$$R\left(r\right) \qquad \text{Yield Limit (Isotropic Hardening)}$$

Assumptions:

- $\mathcal{I}_1^{\varepsilon^{pl}} = 0 \rightarrow \underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} = \underline{\underline{\dot{\boldsymbol{e}}}}^{pl}$
- $\underset{\sim}{\chi} = -\dot{\lambda}\boldsymbol{\nabla}_{\underset{\sim}{\eta}}\boldsymbol{g}$

J2 PLASTICITY

- Von Mises yield locus:

$$f\left(\underline{\underline{\boldsymbol{\sigma}}};\underset{\sim}{\chi}\right) \leq 0$$

$$f = \sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)} - R\left(r\right)$$

$$\underline{\underline{\boldsymbol{D_\sigma f}}} = \frac{3}{2}\frac{\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}}{\sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)}}$$

$$\dot{\lambda} = \sqrt{\frac{2}{3}}\|\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl}\| = \sqrt{\frac{2}{3}}\|\underline{\underline{\dot{\boldsymbol{e}}}}^{pl}\|$$

- Non-associative flow-rule

$$g\left(\underline{\underline{\boldsymbol{\sigma}}};\underline{\underline{\boldsymbol{\chi}}}\right) = f\left(\underline{\underline{\boldsymbol{\sigma}}};\underline{\underline{\boldsymbol{\chi}}}\right) + \underbrace{\frac{3}{4}\frac{\kappa}{C}Tr\left(\underline{\underline{\boldsymbol{X}}}.\underline{\underline{\boldsymbol{X}}}^T\right)}_{\textbf{fading memory}}$$

$$\underline{\underline{\dot{\boldsymbol{\varepsilon}}}}^{pl} = \underline{\underline{\dot{\boldsymbol{e}}}}^{pl} = \dot{\lambda}\underline{\underline{\boldsymbol{D_\sigma g}}}$$

HARDENING LAW   Deviatoric stress plane:

- $\underline{\underline{\boldsymbol{X}}}$: back-stress **centre** of moving yield locus
- R: evolving **radius** of yield locus
- σ_{yld}: first yielding limit
- Prager kinematic hardening:

$$\underline{\underline{\boldsymbol{X}}} = \frac{2}{3}C\underline{\underline{\boldsymbol{\alpha}}} \rightarrow \underline{\underline{\dot{\boldsymbol{X}}}} = \frac{2}{3}C\underline{\underline{\dot{\boldsymbol{\alpha}}}}$$

$$\underline{\underline{\dot{\boldsymbol{\alpha}}}} = -\dot{\lambda}\underline{\underline{\boldsymbol{D_X g}}} = \dot{\lambda}\left(\underline{\underline{\boldsymbol{D_\sigma g}}} - \underbrace{\frac{3}{2}\frac{\kappa}{C}\underline{\underline{\boldsymbol{X}}}}_{\textbf{recall term}}\right)$$

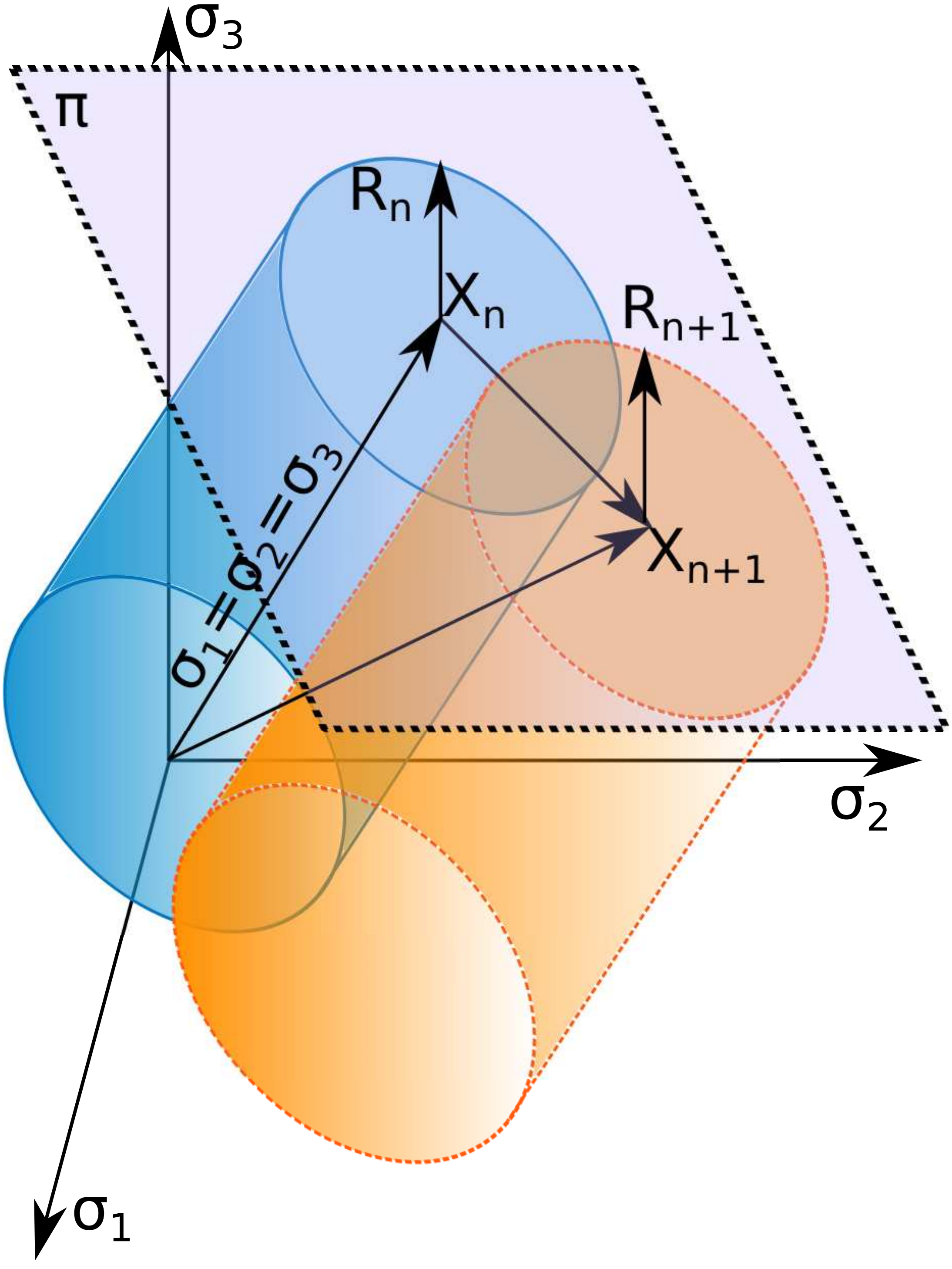
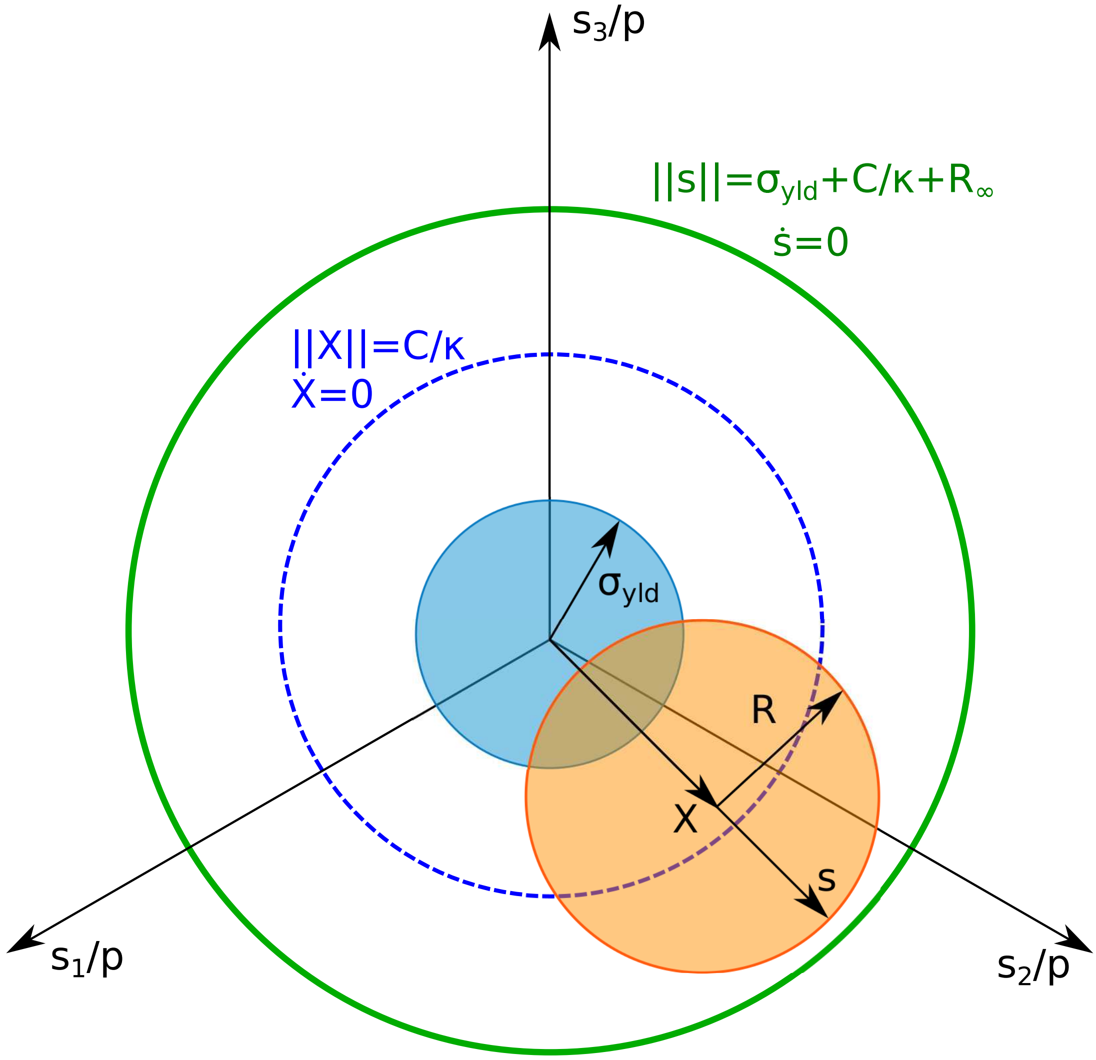


Figure 1: Von Mises yield locus in principal stress 3D space

- Isotropic hardening:



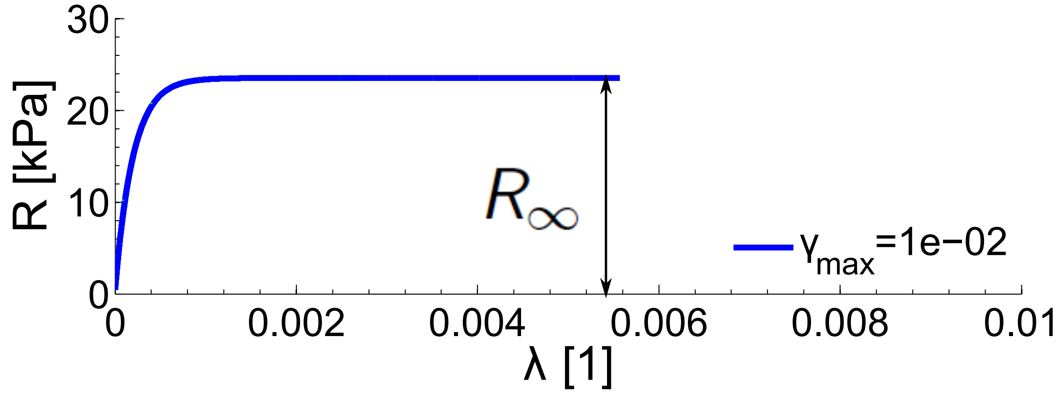
$$R = R(r) = R\left(\int_0^t \|\dot{\underline{\underline{\varepsilon}}}^{pl}\| dt\right) = \sigma_{yld} + R_\infty\left(1 - e^{-br}\right)$$

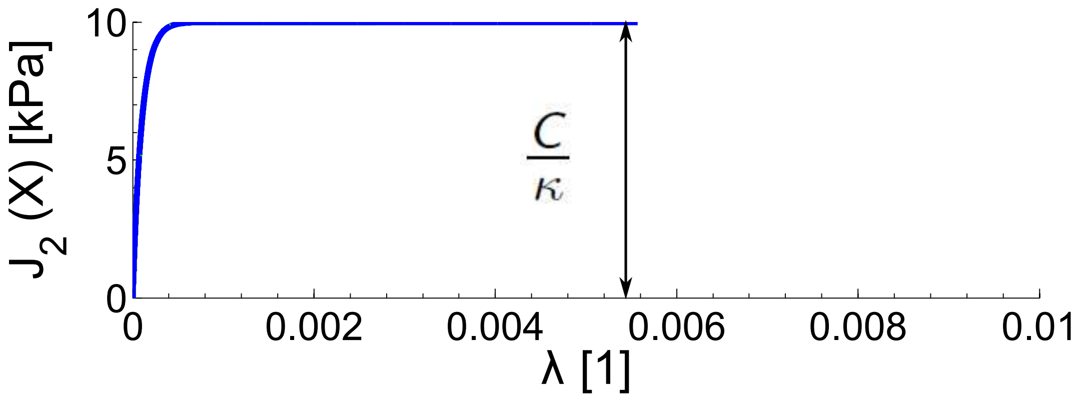
$$\dot{R} = b\left(R_\infty + \sigma_{yld} - R\right)\dot{r}$$

$$\dot{r} = -\dot{\lambda}\frac{\partial g}{\partial R} = \dot{\lambda} = \sqrt{\frac{2}{3}}\|\dot{\underline{\underline{\varepsilon}}}^{pl}\|$$

Hardening saturation:

- Isotropic hardening
  $\dot{R} = 0 \rightarrow R = R_\infty + \sigma_{yld}$
- Kinematic hardening
  $\|\dot{\underline{\underline{\boldsymbol{X}}}}\| = 0 = \|\dot{\underline{\underline{\boldsymbol{\alpha}}}}\|$
  $\rightarrow \|\underline{\underline{\boldsymbol{X}}}\| = \frac{C}{\kappa}$





ELASTIC-PLASTIC SOLUTION

- **Hardening modulus**

$$h = C - \frac{3}{2} \frac{\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right) : \underline{\underline{\boldsymbol{X}}}}{\sqrt{3\mathcal{J}_2\left(\underline{\underline{\boldsymbol{\sigma}}}^D - \underline{\underline{\boldsymbol{X}}}\right)}} + b\left(R_\infty + \sigma_{yld} - R\right)$$

- **Plastic multiplier** $\dot{\lambda}$

$$\dot{\lambda} = \frac{\langle \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{f}}} : \mathbb{D}^{el} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \rangle}{h + \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{f}}} : \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{g}}}}$$

- **Elastic-Plastic stiffness matrix** $\mathbb{D}^{ep}$

$$\underline{\underline{\dot{\boldsymbol{\sigma}}}} = \mathbb{D}^{ep} : \underline{\underline{\dot{\boldsymbol{\varepsilon}}}} \Longrightarrow \mathbb{D}^{ep} = \mathbb{D}^{el} - \frac{\mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{g}}} \otimes \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{f}}}}{h + \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{f}}} : \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\nabla}_\sigma \boldsymbol{g}}}}$$

1.2.1 Numerical Integration

KEY POINTS

1. Updating current state: t_n

2. Strain increment: $\underline{\underline{\boldsymbol{\Delta\varepsilon}}}_n = \int_{t_n}^{t_{n+1}} \underline{\underline{\dot{\varepsilon}}}(s)\, ds = \frac{1}{\Delta t_n} \int_0^1 \left[\underline{\underline{\dot{\varepsilon}}}^{el}(s) + \underline{\underline{\dot{\varepsilon}}}^{pl}(s)\right] ds$

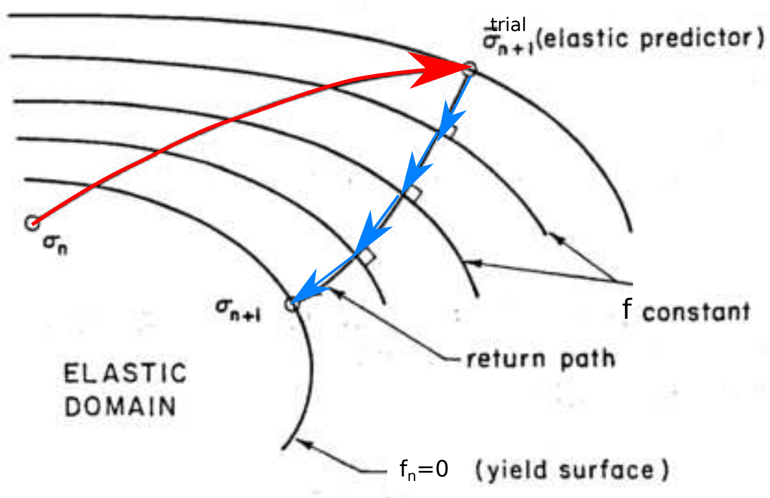
3. Elastic trial stress state prediction: $\underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial} = \mathbb{D}^{el} : \underline{\underline{\boldsymbol{\Delta\varepsilon}}}_n$

4. Plasticity-Check: $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial}; \underset{\sim}{\boldsymbol{\chi}}_n\right)$

   - $f < 0$: elastic step $\rightarrow$ next strain increment
   - $f \geq 0$: elastic-plastic step $\rightarrow$ **PLASTIC CORRECTION!**

Plastic Correction:
Sub-stepping explicit algorithm ($\Delta t_n \ll$) [2]

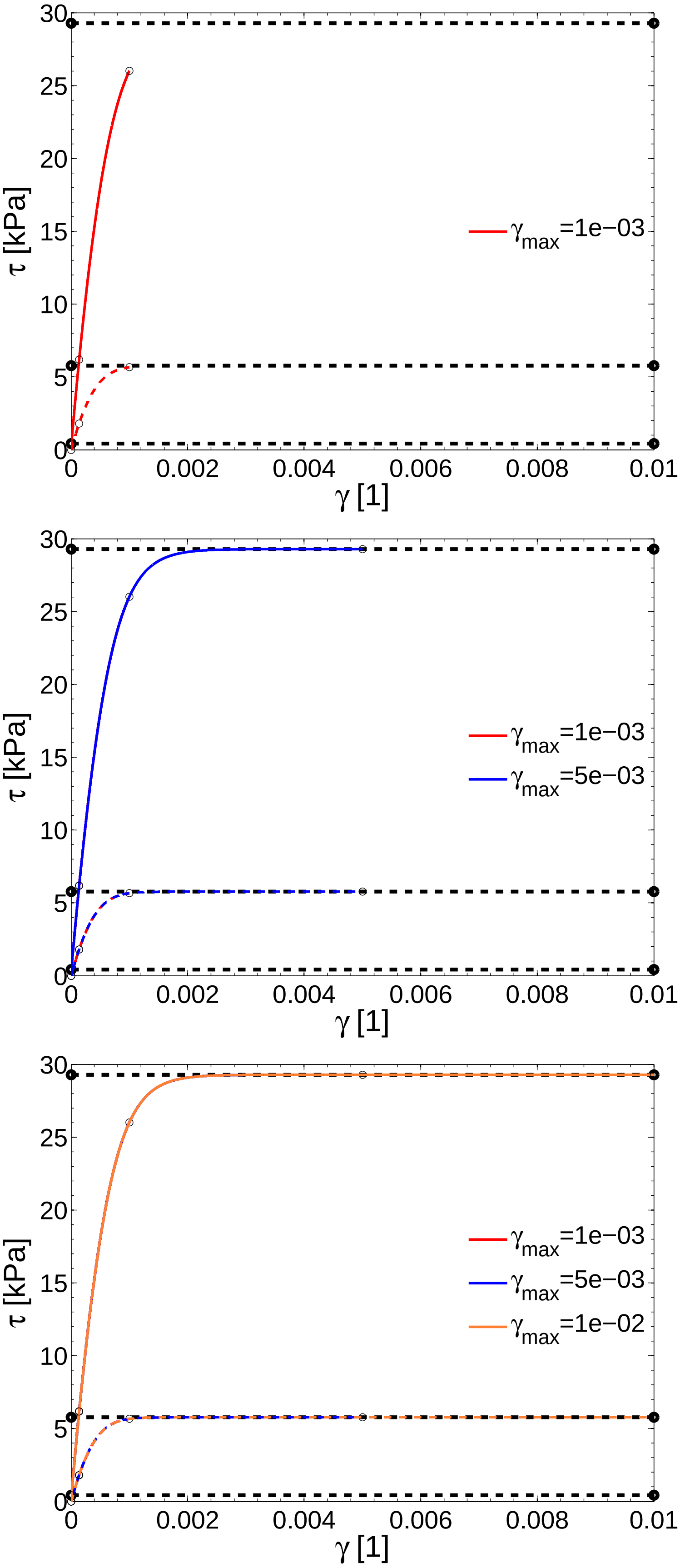
- sub-stepping method (radial return) **Sloan 1987**: $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial-k}; \underline{\underline{\boldsymbol{\chi}}}_n\right) = 0$
- hardening update: $\underset{\sim}{\boldsymbol{\chi}}_n \rightarrow \underset{\sim}{\boldsymbol{\chi}}_{n+1}$
- drift correction: $f\left(\underline{\underline{\boldsymbol{\sigma}}}_n + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{trial-k} + \underline{\underline{\boldsymbol{\Delta\sigma}}}_n^{drift}; \underset{\sim}{\boldsymbol{\chi}}_{n+1}\right) = 0$

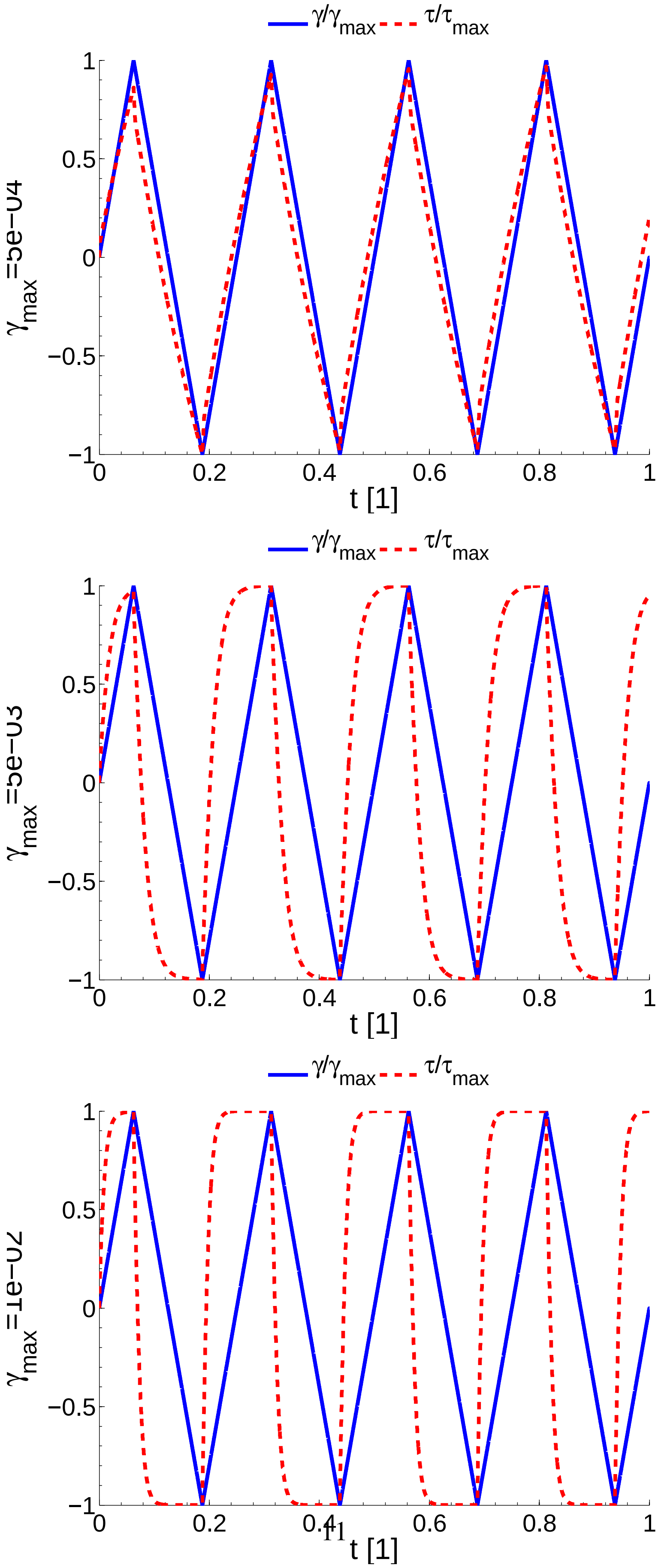


1.3 Examples

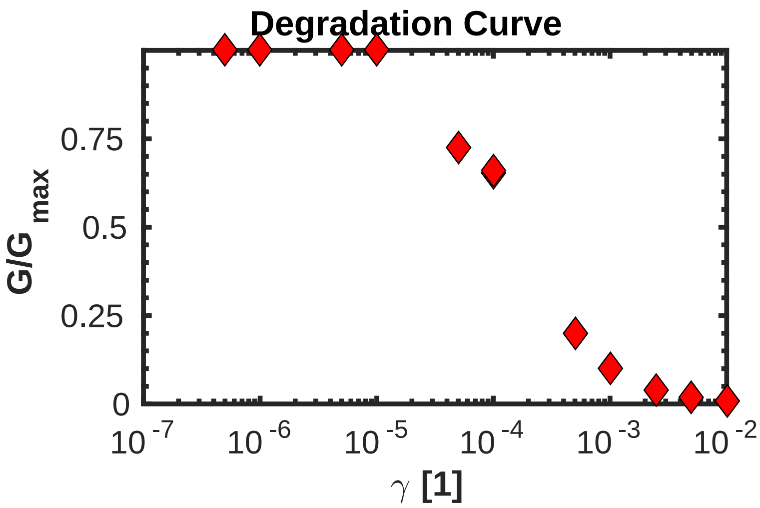
MONOTONIC LOADING

SAW-TOOTH LOADING


30
25
20
15
10
5
0
τ [kPa]
0
0.002
0.004
0.006
0.008
0.01
γ [1]
γmax=1e−03
γmax=1e−03
γmax=5e−03
γmax=1e−03
γmax=5e−03
γmax=1e−02



γ/γ_{max} τ/τ_{max}
γ_{max}=5e−04
t [1]
γ/γ_{max} τ/τ_{max}
γ_{max}=5e−03
t [1]
γ/γ_{max} τ/τ_{max}
γ_{max}=1e−02
t [1]



Degradation Curve
G/G_{max}
0.75
0.5
0.25
0
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
γ [1]

References

[1] S.W. Sloan, "Substepping schemes for the numerical integration of elasto-plastic stress–strain relations", *International Journal for Numerical Methods in Engineering*, 24(5): 893–911, 1987, ISBN 1097-0207, ISSN 1097-0207, URL `http://onlinelibrary.wiley.com/doi/10.1002/nme.1620240505/abstract$\backslash$nhttp://onlinelibrary.wiley.com/store/10.1002/nme.1620240505/asset/1620240505{\_}ftp.pdf?v=1{\&}t=hgq46e6y{\&}s=46743ad1a9e1b7731bba8a835a8714f6594341db`.

[2] S. Sloan, A. Abbo, D. Sheng, "Refined explicit integration of elastoplastic models with automatic error control", *Engineering Computations*, 18(1/2): 121–154, 2001.