Continuum Mechanics

Tunnel Strength

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Outline

- Algebra Recap
- Recap Lecture 4
  - Linear Elasticity
- Hands on session
  - 4.2 Tunnel strength

Algebra Recap

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Tensor/Vector product

- Scalar product :

$$\underline{a}, \underline{b} \in \mathbb{R}^a \qquad \langle \underline{a}, \underline{b} \rangle = \sum_{m=1}^{a} a_m b_m$$

- Tensors and tensor product :

$$\underline{\underline{A}} \in \mathbb{R}^a \rightarrow \mathbb{R}^b \qquad \underline{\underline{A}} = \underline{a} \otimes \underline{b}, \underline{a} \in \mathbb{R}^a, \underline{b} \in \mathbb{R}^b$$

- Tensor application to vector :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \rightarrow \mathbb{R}^b, \underline{c} \in \mathbb{R}^b \qquad \underline{\underline{A}}.\underline{c} = \underline{a} \langle \underline{b}, \underline{c} \rangle$$

- Tensor/Tensor product :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \rightarrow \mathbb{R}^b, \underline{\underline{B}} = \underline{c} \otimes \underline{d} \in \mathbb{R}^b \otimes \mathbb{R}^c \qquad \underline{\underline{A}}.\underline{\underline{B}} = \underline{a} \otimes \underline{d} \langle \underline{b}, \underline{c} \rangle$$

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Tensor/Vector product

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}} = \sum_{m=1}^{a} \frac{\partial \underline{\boldsymbol{a}}}{\partial x_m} \otimes \underline{\boldsymbol{e}}_m$$

- Vector divergence :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}}\right) = \left\langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \right\rangle = \sum_{m=1}^{a} \frac{\partial a_m}{\partial x_m}$$

- Tensor divergence :

$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^a \to \mathbb{R}^b \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \sum_{m=1}^{b} \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial x_m} . \underline{\boldsymbol{e}}_m$$

Tensor/Vector product in cylindrical coordinates

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}} \boldsymbol{a}}} = \frac{\partial \underline{\boldsymbol{a}}}{\partial r} \otimes \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta} \otimes \frac{\underline{\boldsymbol{i}}_\theta}{r}(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial z} \otimes \underline{\boldsymbol{i}}_z$$

- Vector divergence :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}} \boldsymbol{a}}}\right) = \left\langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \right\rangle = \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial r}, \underline{\boldsymbol{i}}_r \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta}, \frac{\underline{\boldsymbol{i}}_\theta}{r} \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial z}, \underline{\boldsymbol{i}}_z \right\rangle$$

- Tensor divergence :

$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial r} . \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial \theta} . \frac{\underline{\boldsymbol{i}}_\theta}{r} + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial z} . \underline{\boldsymbol{i}}_z$$

Linear Elasticity

$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$$

$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$$

- Linearity $\longrightarrow g\left(\alpha\underline{\underline{\varepsilon}}_1^{el} + \beta\underline{\underline{\varepsilon}}_2^{el}\right) = \alpha\mathcal{F}\left(\underline{\underline{\varepsilon}}_1^{el}\right) + \beta\mathcal{F}\left(\underline{\underline{\varepsilon}}_2^{el}\right)$:
  - Stiffness :

$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathbf{D}^{el}\left(\underline{\boldsymbol{x}}\right) : \underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right) \tag{1}$$

  - Compliance :

$$\underline{\underline{\varepsilon}}^{el} = \mathbf{C}^{el}\left(\underline{\boldsymbol{x}}\right) : \underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) \tag{2}$$

$\mathbf{A}$: 4^{th}-order tensor (real symmetric)

$\mathbf{A} = \underline{\underline{\boldsymbol{A}}}_l \otimes \underline{\underline{\boldsymbol{A}}}_r$, $\quad \mathbf{A} : \underline{\underline{\boldsymbol{B}}} = \underline{\underline{\boldsymbol{A}}}_l Tr\left(\underline{\underline{\boldsymbol{A}}}_r^T \underline{\underline{\boldsymbol{B}}}\right)$

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$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right)=\mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$$

- Isotropic $\left\{\mathbf{I},\underline{\underline{\boldsymbol{I}}}\otimes\underline{\underline{\boldsymbol{I}}}\right\}$:
  - Stiffness :

$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right)=\lambda\left(\underline{\boldsymbol{x}}\right)Tr\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)\underline{\underline{\boldsymbol{I}}}+2\mu\left(\underline{\boldsymbol{x}}\right)\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right) \tag{3}$$

$$\mathbf{D}^{el}\left(\underline{\boldsymbol{x}}\right)=\lambda\left(\underline{\boldsymbol{x}}\right)\underline{\underline{\boldsymbol{I}}}\otimes\underline{\underline{\boldsymbol{I}}}+2\mu\left(\underline{\boldsymbol{x}}\right)\mathbf{I} \tag{4}$$

  - Compliance :

$$\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)=\frac{1+\nu\left(\underline{\boldsymbol{x}}\right)}{E\left(\underline{\boldsymbol{x}}\right)}Tr\left(\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right)\right)\underline{\underline{\boldsymbol{I}}}-\frac{\nu\left(\underline{\boldsymbol{x}}\right)}{E\left(\underline{\boldsymbol{x}}\right)}\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) \tag{5}$$

$$\mathbf{C}^{el}\left(\underline{\boldsymbol{x}}\right)=\frac{1+\nu\left(\underline{\boldsymbol{x}}\right)}{E\left(\underline{\boldsymbol{x}}\right)}\underline{\underline{\boldsymbol{I}}}\otimes\underline{\underline{\boldsymbol{I}}}-\frac{\nu\left(\underline{\boldsymbol{x}}\right)}{E\left(\underline{\boldsymbol{x}}\right)}\mathbf{I} \tag{6}$$

$\mathbf{I}=\sum_{i=1}^{3}\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_i\otimes\underline{\boldsymbol{e}}_i$

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$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$

- Spherical and deviatoric decomposition :
  - Spherical component (pression) σ_m :

$$\sigma_m\left(\underline{\boldsymbol{x}};t\right) = \frac{Tr\left(\underline{\underline{\sigma}}\right)}{3} = \frac{\left(3\lambda\left(\underline{\boldsymbol{x}}\right)+2\mu\left(\underline{\boldsymbol{x}}\right)\right)}{3}Tr\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right) = K\left(\underline{\boldsymbol{x}}\right)\varepsilon_{vol}^{el}\left(\underline{\boldsymbol{x}};t\right) \tag{7}$$

  - Deviatoric component $\underline{\underline{\boldsymbol{s}}}^{\sigma}$:

$$\underline{\underline{\boldsymbol{s}}}^{\sigma}\left(\underline{\boldsymbol{x}};t\right) = \underline{\underline{\sigma}} - \sigma_m\underline{\underline{\boldsymbol{I}}} = 2\mu\left(\underline{\boldsymbol{x}}\right)\underline{\underline{\varepsilon}}^{el} - \frac{2}{3}\mu\left(\underline{\boldsymbol{x}}\right)\varepsilon_{vol}^{el}\left(\underline{\boldsymbol{x}};t\right) = 2\mu\left(\underline{\boldsymbol{x}}\right)\underline{\underline{\boldsymbol{e}}}^{el}\left(\underline{\boldsymbol{x}};t\right) \tag{8}$$

Direct relationship spherical/deviatoric
ε_{vol}^{el} : volumetric strain
$\underline{\underline{\boldsymbol{e}}}^{el}$: deviatoric strains

$\bar{e} = \sqrt{\frac{4}{3}J_2\left(\underline{\underline{\varepsilon}}\right)}$: equivalent deviatoric strain

$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$

- Homogeneity :

$$\mathbf{D}^{el}\left(\underline{\boldsymbol{x}}\right) = \mathbf{D}^{el}, \quad \mathbf{C}^{el}\left(\underline{\boldsymbol{x}}\right) = \mathbf{C}^{el} \tag{9}$$

- (λ, μ) : Lamé parameters (stiffness)
- (E, ν) : Young's modulus and Poisson's coefficient (compliance)
- $E > 0, \mu > 0, \quad 3\lambda + 2\mu > 0, \quad -1 < \nu < 0.5$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{10}$$

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$$\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}};t\right) = \mathcal{F}\left(\underline{\underline{\varepsilon}}^{el}\left(\underline{\boldsymbol{x}};t\right)\right)$$

$$\mathbf{D}^{el} = \frac{E}{(1+\nu)}\begin{bmatrix} \frac{1-\nu}{(1-2\nu)} & \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{(1-2\nu)} & \frac{\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & \frac{1-\nu}{(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{C}^{el} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix}$$

Figure – Elastic tensors with (E, ν).

4.2 Tunnel Strength

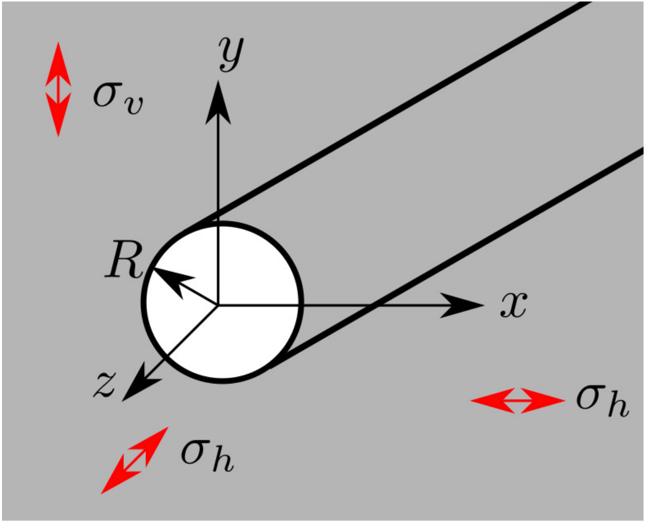

Figure – Set up [Credits : G. Puel]

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Stress description

- $\underline{\boldsymbol{x}} = x\underline{\boldsymbol{i}}_x + y\underline{\boldsymbol{i}}_y + z\underline{\boldsymbol{i}}_z = r\underline{\boldsymbol{i}}_r + z\underline{\boldsymbol{i}}_z, \qquad (r, \theta, z) \in \Omega_t =]0, R[\times]0, 2\pi[\times]0, L[$
- $x = r\cos\theta =, y = r\sin\theta, \quad r = \sqrt{x^2 + y^2}$
- Initial stress

$$\underline{\underline{\sigma}}_0 = \sigma_h \underline{\underline{\boldsymbol{I}}} + (\sigma_v - \sigma_h)\, \underline{\boldsymbol{i}}_y \otimes \underline{\boldsymbol{i}}_y, \quad \sigma_v \leq \sigma_h < 0$$

- $\underline{\boldsymbol{f}}_v = 0, \underline{\boldsymbol{a}} = 0$

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Question 1 : principal stresses $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$?

$$\sigma_I = \sigma_{II} = \sigma_{xx} = \sigma_{zz} = \sigma_h, \sigma_{III} = \sigma_v$$

Question 2 : $\underline{\boldsymbol{n}}$ that maximizes $\|\underline{\boldsymbol{\tau}}_\Sigma\|$?

- $\underline{\boldsymbol{\tau}}_\Sigma = \underline{\underline{\sigma}}.\underline{\boldsymbol{n}} - \sigma_{nn}\underline{\boldsymbol{n}} = \sigma_h\underline{\boldsymbol{n}} + (\sigma_v - \sigma_h)n_y\underline{\boldsymbol{i}}_y - \left(\sigma_h + (\sigma_v - \sigma_h)n_y^2\right)\underline{\boldsymbol{n}} = (\sigma_v - \sigma_h)n_y\left(\underline{\boldsymbol{i}}_y - n_y\underline{\boldsymbol{n}}\right)$
- $\|\underline{\boldsymbol{\tau}}_\Sigma\| = |\sigma_v - \sigma_h| \cdot |n_y| \cdot \sqrt{1 - n_y^2} \quad (n_y \in [0,1])$
- $\tau_{\max} = \|\underline{\boldsymbol{\tau}}_\Sigma\| \big|_{n_y = \pm\frac{\sqrt{2}}{2}} = \frac{|\sigma_v - \sigma_h|}{2}$ Mohr's circle with $\alpha = \arccos n_y = \pm\frac{\pi}{4}$

Question 3 : failure criteria?

- Splitting : $\max_{\|\underline{\boldsymbol{n}}\|=1} \sigma_{nn} = \sigma_I = \sigma_h \leq 0 \leq \sigma_R, \quad$ with $\sigma_R \geq 0$
- Maximum shear (Tresca) : $\max_{\|\underline{\boldsymbol{n}}\|=1} \|\underline{\boldsymbol{\tau}}_\Sigma\| = \frac{|\sigma_v - \sigma_h|}{2} \leq \tau_R$

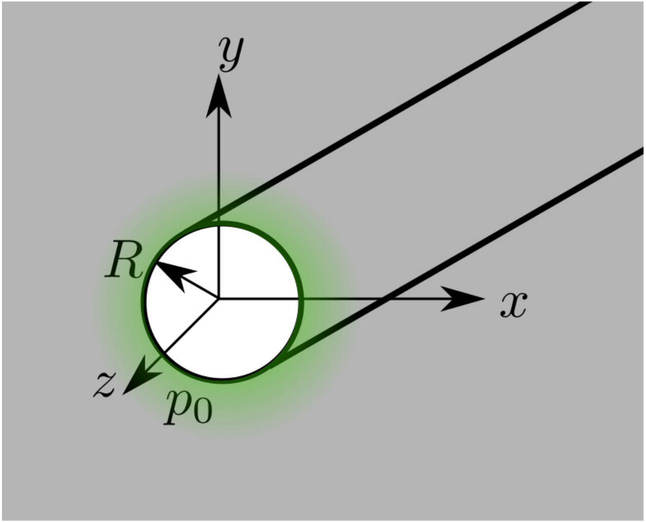

Figure – Set up [Credits : G. Puel]

Auxiliary problem $r > R$

- $\underline{u}(r,\theta) = \alpha\frac{R^2}{r}\underline{i}_r(\theta), r > R$
- $\underline{f}_S = -p_0\underline{n}, r = R$
- $\underline{f}_v = 0, \underline{a} = 0$

Question 4/5/6/7 : verify auxiliary problem

4 Strain tensor : $\underline{\underline{\varepsilon}} = \underline{\underline{\mathbb{D}_{\boldsymbol{x}}^{S}\boldsymbol{u}}} = \frac{\partial\underline{u}}{\partial r}\otimes\underline{i}_r + \frac{\partial\underline{u}}{\partial\theta}\otimes\frac{\underline{i}_\theta}{r} = \alpha\left(\frac{R}{r}\right)^2\left(\underline{i}_\theta\otimes\underline{i}_\theta - \underline{i}_r\otimes\underline{i}_r\right)$

5 Stress tensor (linear elastic homogenenous isotropic) :
$\underline{\underline{\sigma}} = \lambda Tr\left(\underline{\underline{\varepsilon}}\right)\underline{\underline{I}} + 2\mu\underline{\underline{\varepsilon}} = 0 + 2\mu\alpha\left(\frac{R}{r}\right)^2\left(\underline{i}_\theta\otimes\underline{i}_\theta - \underline{i}_r\otimes\underline{i}_r\right)$

6 $div_x\left(\underline{\underline{\sigma}}\right) = \frac{\partial\underline{\underline{\sigma}}}{\partial r}.\underline{i}_r + \frac{\partial\underline{\underline{\sigma}}}{\partial\theta}.\frac{\underline{i}_\theta}{r} + \frac{\partial\underline{\underline{\sigma}}}{\partial z}.\underline{i}_z = 4\mu\alpha\frac{R^2}{r^3}\underline{i}_r - 4\mu\alpha\frac{R^2}{r^3}\underline{i}_r + 0 = 0$

7 $\underline{\underline{\sigma}}.\underline{n}|_{r=R} = -\underline{\underline{\sigma}}|_{r=R}.\underline{i}_r = 2\mu\alpha\underline{i}_r = \underline{f}_s = -p_0\underline{n} = p_0\underline{i}_r \Rightarrow \alpha = \frac{p_0}{2\mu}$

Isotropic stress state

- $\sigma_v = \sigma_h = -\rho g H < 0 \quad \underline{\underline{\sigma}}_0 = \sigma_v \underline{\underline{I}}, \underline{\underline{\sigma}}_t = \underline{\underline{\sigma}}_0 + \underline{\underline{\sigma}}$

Question 8/9/10/11 : failure criteria?

8 $div_x\left(\underline{\underline{\sigma}}_t\right) = div_x\left(\underline{\underline{\sigma}}_0\right) + div_x\left(\underline{\underline{\sigma}}\right) = 0$

$\underline{\underline{\sigma}}_t . \underline{n}|_{r=R} = -\underline{\underline{\sigma}}_t . \underline{i}_r = -\sigma_v \underline{i}_r + p_0 \underline{i}_r = 0 \rightarrow p_0 = \sigma_v < 0$

9 $\underline{\underline{\sigma}}_t = \sigma_v \underline{\underline{I}} + \sigma_v \left(\frac{R}{r}\right)^2 \left(\underline{i}_\theta \otimes \underline{i}_\theta - \underline{i}_r \otimes \underline{i}_r\right)$ (diagonal) $\rightarrow \sigma_I = \sigma_v \left(1 - \left(\frac{R}{r}\right)^2\right); \sigma_{III} = \sigma_v \left(1 + \left(\frac{R}{r}\right)^2\right), \tau_{\max} = \frac{\sigma_I - \sigma_{III}}{2} = -\sigma_v \leq \tau_c$

10 In plane $(\underline{i}_r, \underline{i}_\theta)$, tangent to tunnel's cross section, fractures at $\alpha = \pm\frac{\pi}{4}$)

11 $\sigma_I < 0 = \sigma_R \quad \tau_{\max} = -\sigma_v \leq \tau_c \rightarrow H \leq \frac{\tau_c}{\rho g}$

$\underline{u}_t = \underline{u}_0 + \underline{u} = 0 + \frac{\rho g H}{2\mu}\frac{R^2}{r}\underline{i}_r \rightarrow u_{\max} = \left\langle \underline{u}_t, \underline{i}_y \right\rangle|_{r=R} = \frac{\rho g H R}{2\mu} \leq \frac{\tau_c R}{2\mu}$