Mechanics

Brazilian test

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- Algebra Recap
- Recap Lecture 2
    - Stress
- Hands on session
    - 2.3 Brazilian test

Algebra Recap

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Tensor/Vector product

- Scalar product :

$$\underline{a}, \underline{b} \in \mathbb{R}^a \qquad \left\langle \underline{a}, \underline{b} \right\rangle = \sum_{m=1}^{a} a_m b_m$$

- Tensors and tensor product :

$$\underline{\underline{A}} \in \mathbb{R}^a \to \mathbb{R}^b \qquad \underline{\underline{A}} = \underline{a} \otimes \underline{b}, \underline{a} \in \mathbb{R}^a, \underline{b} \in \mathbb{R}^b$$

- Tensor application to vector :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \to \mathbb{R}^b, \underline{c} \in \mathbb{R}^b \qquad \underline{\underline{A}}.\underline{c} = \underline{a} \left\langle \underline{b}, \underline{c} \right\rangle$$

- Tensor/Tensor product :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \to \mathbb{R}^b, \underline{\underline{B}} = \underline{c} \otimes \underline{d} \in \mathbb{R}^b \otimes \mathbb{R}^c \qquad \underline{\underline{A}}.\underline{\underline{B}} = \underline{a} \otimes \underline{d} \left\langle \underline{b}, \underline{c} \right\rangle$$

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Tensor/Vector product

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}} = \sum_{m=1}^{a} \frac{\partial \underline{\boldsymbol{a}}}{\partial x_m} \otimes \underline{\boldsymbol{e}}_m$$

- Vector divergence :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}}\right) = \langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \rangle = \sum_{m=1}^{a} \frac{\partial a_m}{\partial x_m}$$

- Tensor divergence :

$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^a \to \mathbb{R}^b \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \sum_{m=1}^{b} \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial x_m} . \underline{\boldsymbol{e}}_m$$

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Tensor/Vector product in cylindrical coordinates

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}} = \frac{\partial \underline{\boldsymbol{a}}}{\partial r} \otimes \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta} \otimes \frac{\underline{\boldsymbol{i}}_\theta}{r}(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial z} \otimes \underline{\boldsymbol{i}}_z$$

- Vector divergence :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}}\right) = \langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \rangle = \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial r}, \underline{\boldsymbol{i}}_r \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta}, \frac{\underline{\boldsymbol{i}}_\theta}{r} \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial z}, \underline{\boldsymbol{i}}_z \right\rangle$$

- Tensor divergence :

$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial r} . \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial \theta} . \frac{\underline{\boldsymbol{i}}_\theta}{r} + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial z} . \underline{\boldsymbol{i}}_z$$

Stress

Stress

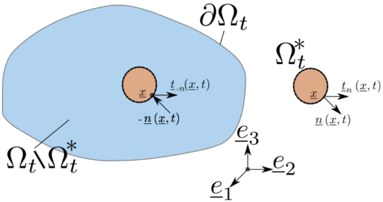
Modelling contact forces exerted by $\Omega_t \setminus \Omega_t^*$ on Ω_t^* :

- local contact force density exerted by proximal subdomains
- virtual cutting surface $\partial\Omega_t^*$
- influence of local surface orientation : normal vector $\underline{\boldsymbol{n}}$ defining the tangent plane at $\underline{\boldsymbol{x}}$
- Traction vector definition

$$\underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \partial\Omega_t^*\right) = \underline{\boldsymbol{f}}_{\partial\Omega_t^*}\left(\underline{\boldsymbol{x}}; t\right) \quad \forall \underline{\boldsymbol{x}} \in \partial\Omega_t^*, \forall \partial\Omega_t^* \tag{1}$$

Boundary conditions :

$$\underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \partial\Omega_t\right) = \underline{\boldsymbol{f}}_s\left(\underline{\boldsymbol{x}}; t\right)$$



- Cauchy's postulate :

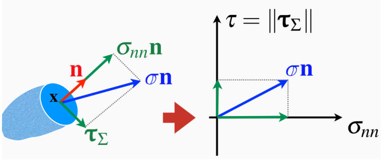
$$\underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \partial\Omega_t^*\right) = \underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \underline{\boldsymbol{n}}\right), \forall \underline{\boldsymbol{x}} \in \partial\Omega_t^* \tag{1}$$

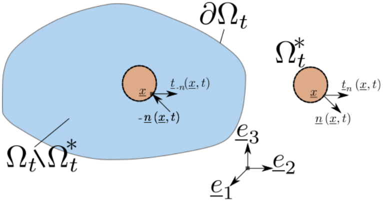
Surface curvature has no influence on the traction vector !

- Cauchy's theorem :

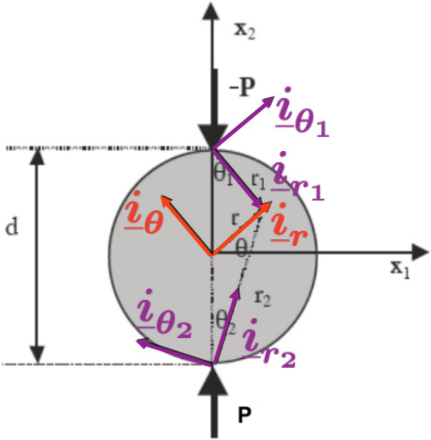
$$\underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; -\underline{\boldsymbol{n}}\right) + \underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \underline{\boldsymbol{n}}\right) = \underline{\boldsymbol{0}}, \forall \underline{\boldsymbol{x}} \in \partial\Omega_t^* \tag{2}$$

$$\underline{\boldsymbol{t}}\left(\underline{\boldsymbol{x}}, t; \underline{\boldsymbol{n}}\right) = \underline{\boldsymbol{t}}_n\left(\underline{\boldsymbol{x}}; t\right) = \underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}}; t\right).\underline{\boldsymbol{n}}, \forall \underline{\boldsymbol{x}} \in \partial\Omega_t^* \tag{3}$$





2.3 Brazilian test



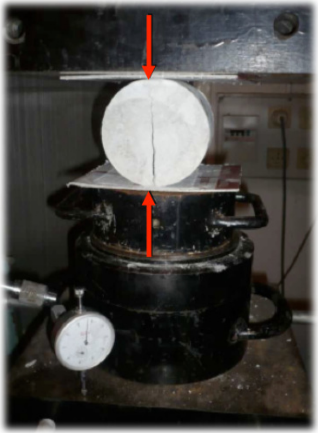
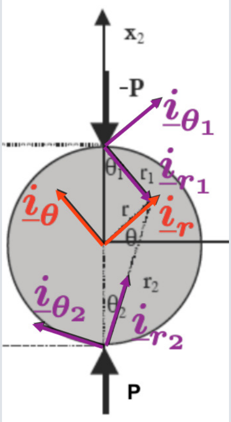


Figure – Set up [Credits : G. Puel]

Stress description

- $\underline{x} = r\underline{i}_r(\theta) + z\underline{i}_z, \qquad (r,\theta,z) \in \Omega_t = \left]0, \frac{D}{2}\right[\times \left]0, 2\pi\right[\times \left]0, H\right[$
- $\underline{\underline{\sigma}}(\underline{x}) = k\frac{\cos\theta_1}{r_1}\underline{i}_{r_1}(\theta_1)\otimes\underline{i}_{r_1}(\theta_1) + k\frac{\cos\theta_2}{r_2}\underline{i}_{r_2}(\theta_2)\otimes\underline{i}_{r_2}(\theta_2) - \frac{k}{D}\left(\underline{\underline{I}} - \underline{i}_z\otimes\underline{i}_z\right)$



Check equilibrium

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Question 1 : $div_{x}\left(\underline{\underline{\sigma}}\left(\underline{x}\right)\right) = 0$?

- $div_{x}\left(k\frac{\cos\theta}{r}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)\right) = 0$?
  - $\square$ $\frac{\partial}{\partial r}\left(k\frac{\cos\theta}{r}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)\right) = -k\frac{\cos\theta}{r^2}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)$
  - $\boxtimes$ $\frac{\partial}{\partial\theta}\left(k\frac{\cos\theta}{r}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)\right) = -k\frac{\sin\theta}{r^2}\underline{i}_{r}\otimes\underline{i}_{r} + 2k\frac{\cos\theta}{r}\underline{i}_{r}\otimes_S\underline{i}_{\theta}$
  - $\boxplus$ $\frac{\partial}{\partial z}\left(k\frac{\cos\theta}{r}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)\right) = 0$

$\longrightarrow div_{x}\left(k\frac{\cos\theta}{r}\underline{i}_{r}(\theta)\otimes\underline{i}_{r}(\theta)\right) = \square.\underline{i}_{r} + \boxtimes.\frac{\underline{i}_{\theta}}{r} + \boxplus.\underline{i}_{z} =$
$-k\frac{\cos\theta}{r^2}\underline{i}_{r} + k\frac{\cos\theta}{r^2}\underline{i}_{r} + 0 = 0$

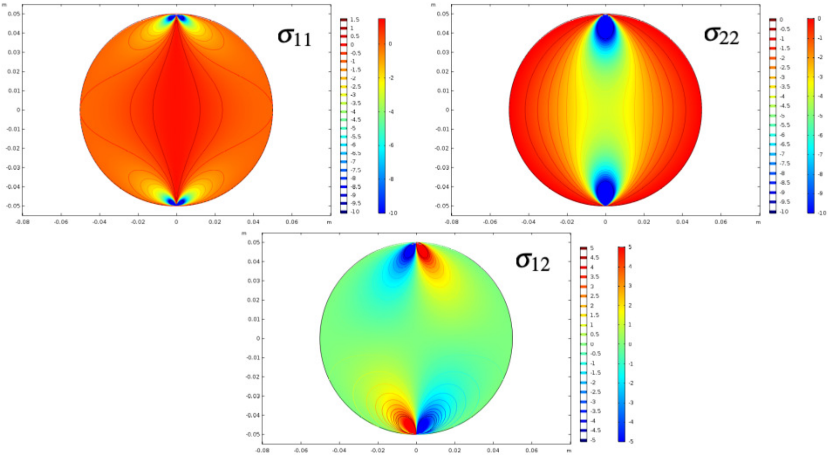
$$div_{x}\left(\underline{\underline{\sigma}}\left(\underline{x}\right)\right) = div_{x}\left(k\frac{\cos\theta_1}{r_1}\underline{i}_{r_1}(\theta_1)\otimes\underline{i}_{r_1}(\theta_1)\right) + div_{x}\left(k\frac{\cos\theta_2}{r_2}\underline{i}_{r_2}(\theta_2)\otimes\underline{i}_{r_2}(\theta_2)\right) +$$

$$+div_{x}\left(\frac{k}{D}\left(\underline{\underline{I}} - \underline{i}_{z}\otimes\underline{i}_{z}\right)\right) = 0$$

Question 2 : $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\partial\Omega_t} = 0$?

1. $\underline{\underline{\sigma}}|_{\partial\Omega_t} = \underline{\underline{\sigma}}\left(\frac{D}{2}, \theta, z\right)$
2. $\underline{\boldsymbol{n}} = \underline{\boldsymbol{i}}_r$
3. on $\partial\Omega_t$: $\theta_2 = \frac{\pi}{2} - \theta_1, \quad r_i = \frac{D}{2}\cos\theta_i (i = 1, 2), \quad \left\langle \underline{\boldsymbol{i}}_{r_1}, \underline{\boldsymbol{i}}_{r_2} \right\rangle$
4. $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}} = \underline{\underline{\sigma}}.\underline{\boldsymbol{i}}_r = k\sum_{i=1}^{2}\left(\frac{\cos\theta_i}{r_i}\left\langle \underline{\boldsymbol{i}}_{r_i}, \underline{\boldsymbol{i}}_r \right\rangle \underline{\boldsymbol{i}}_{r_i}\right) - \frac{k}{D}\underline{\boldsymbol{i}}_r$

   $\underline{\underline{\sigma}}.\underline{\boldsymbol{i}}_r|_{\partial\Omega_t} = \frac{2k}{D}\sum_{i=1}^{2}\left(\frac{\cos\theta_i}{\cos\theta_i}\left\langle \underline{\boldsymbol{i}}_{r_i}, \underline{\boldsymbol{i}}_r \right\rangle \underline{\boldsymbol{i}}_{r_i}\right) - \frac{k}{D}\underline{\boldsymbol{i}}_r = 0$

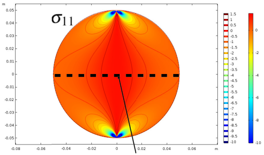

σ_{11}
σ_{22}
σ_{12}

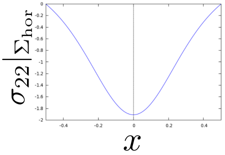
Question 3 : Compute force $\underline{\boldsymbol{F}}^{\mathrm{UP}\rightarrow\mathrm{DOWN}}$

- $\underline{\boldsymbol{F}}^{\mathrm{UP}\rightarrow\mathrm{DOWN}} = \int_{\Sigma_{\mathrm{hor}}} \underline{\underline{\sigma}}\,\underline{\boldsymbol{n}}\, ds = \int_{\Sigma_{\mathrm{hor}}} \underline{\underline{\sigma}}\,\underline{\boldsymbol{i}}_2 ds$
- on Σ_{hor} :
  $r_1 \cos\theta_1 = r_2 \cos\theta_2, \quad x_1 \in \left]-\frac{D}{2}, \frac{D}{2}\right[, x_1 = r_1 \sin\theta_1 = r_2 \sin\theta_2 (\theta_1 = \theta_2 = \alpha, r_1 = r_2 = \rho)$
- on Σ_{hor} : $\left\langle \underline{\boldsymbol{i}}_{r_1}, \underline{\boldsymbol{n}} \right\rangle = \left\langle \underline{\boldsymbol{i}}_{r_1}, \underline{\boldsymbol{i}}_2 \right\rangle = \cos\alpha; \quad \left\langle \underline{\boldsymbol{i}}_{r_2}, \underline{\boldsymbol{n}} \right\rangle = \left\langle \underline{\boldsymbol{i}}_{r_2}, \underline{\boldsymbol{i}}_2 \right\rangle = -\cos\alpha$
- $\underline{\underline{\sigma}}|_{\Sigma_{\mathrm{hor}}} = \frac{kD}{2\rho^2}\cos\alpha\left(\underline{\boldsymbol{i}}_{r_1} - \underline{\boldsymbol{i}}_{r_2}\right) - \frac{k}{D}\underline{\boldsymbol{i}}_2 = \frac{kD}{\rho^2}\cos^2\alpha\,\underline{\boldsymbol{i}}_2 - \frac{k}{D}\underline{\boldsymbol{i}}_2 = \frac{kD^3}{4\left(\frac{D^2}{4}+x_1^2\right)^2}\underline{\boldsymbol{i}}_2 - \frac{k}{D}\underline{\boldsymbol{i}}_2 =$

$$\frac{k}{D}\left(\frac{1}{4\left(1+x^2\right)^2} - 1\right)\underline{\boldsymbol{i}}_2 \Rightarrow \underline{\boldsymbol{F}}^{\mathrm{UP}\rightarrow\mathrm{DOWN}} = \frac{k}{D}\left(\frac{1}{4}\left(\frac{x}{2\left(1+x^2\right)} + \frac{a\tan x}{2}\right)\Bigg|_{-\frac{D}{2}}^{+\frac{D}{2}} - D\right)\underline{\boldsymbol{i}}_2 = -P\underline{\boldsymbol{i}}_2$$

$$k = -\frac{2P}{\pi}$$





Traction forces on vertical cut

Question 3 : Compute force $\underline{\boldsymbol{F}}^{\text{RIGHT}\rightarrow\text{LEFT}}$

- $\underline{\boldsymbol{F}}^{\text{RIGHT}\rightarrow\text{LEFT}} = \int_{\Sigma_{\text{ver}}} \underline{\underline{\sigma}}\,\underline{\boldsymbol{n}}\,ds = \int_{\Sigma_{\text{ver}}} \underline{\underline{\sigma}}\,\underline{\boldsymbol{i}}_1\,ds$
- on Σ_{ver} : $\theta_1 = \theta_2 = 0, \quad x_2 \in \left]-\frac{D}{2}, \frac{D}{2}\right[, x_2 = D - r_1 = r_2 - \frac{D}{2} \quad \underline{\boldsymbol{i}}_{r_1} = \underline{\boldsymbol{i}}_2 = -\underline{\boldsymbol{i}}_{r_2}$
- on Σ_{ver} : $\frac{\cos\theta_1}{r_1}\underline{\boldsymbol{i}}_{r_1} \otimes \underline{\boldsymbol{i}}_{r_1} = \frac{1}{r_1}\underline{\boldsymbol{i}}_{r_1} \otimes \underline{\boldsymbol{i}}_{r_1}; \frac{\cos\theta_2}{r_2}\underline{\boldsymbol{i}}_{r_2} \otimes \underline{\boldsymbol{i}}_{r_2} = \frac{1}{r_2}\underline{\boldsymbol{i}}_{r_1} \otimes \underline{\boldsymbol{i}}_{r_1}$
- $\underline{\underline{\sigma}}.\underline{\boldsymbol{i}}_1|_{\Sigma_{\text{ver}}} = k\frac{r_1+r_2}{r_1 r_2}\underline{\boldsymbol{i}}_2 \otimes \underline{\boldsymbol{i}}_2.\underline{\boldsymbol{i}}_1 - \frac{k}{D}\underline{\boldsymbol{i}}_1 = -\frac{k}{D}\underline{\boldsymbol{i}}_1 > 0$ TRACTION !