Continuum Mechanics

Design of a Gravity Dam

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Outline

- Algebra Recap
- Recap Lecture 3
    - Failure Criteria
- Hands on session
    - 3.1 Gravity Dam

Algebra Recap

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Tensor/Vector product

- Scalar product :

$$\underline{a}, \underline{b} \in \mathbb{R}^a \qquad \langle \underline{a}, \underline{b} \rangle = \sum_{m=1}^{a} a_m b_m$$

- Tensors and tensor product :

$$\underline{\underline{A}} \in \mathbb{R}^a \to \mathbb{R}^b \qquad \underline{\underline{A}} = \underline{a} \otimes \underline{b}, \underline{a} \in \mathbb{R}^a, \underline{b} \in \mathbb{R}^b$$

- Tensor application to vector :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \to \mathbb{R}^b, \underline{c} \in \mathbb{R}^b \qquad \underline{\underline{A}}.\underline{c} = \underline{a} \langle \underline{b}, \underline{c} \rangle$$

- Tensor/Tensor product :

$$\underline{\underline{A}} = \underline{a} \otimes \underline{b} \in \mathbb{R}^a \to \mathbb{R}^b, \underline{\underline{B}} = \underline{c} \otimes \underline{d} \in \mathbb{R}^b \otimes \mathbb{R}^c \qquad \underline{\underline{A}}.\underline{\underline{B}} = \underline{a} \otimes \underline{d} \langle \underline{b}, \underline{c} \rangle$$

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Tensor/Vector product

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}} = \sum_{m=1}^{a} \frac{\partial \underline{\boldsymbol{a}}}{\partial x_m} \otimes \underline{\boldsymbol{e}}_m$$

- Vector divergence :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^a \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}}\right) = \langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \rangle = \sum_{m=1}^{a} \frac{\partial a_m}{\partial x_m}$$

- Tensor divergence :

$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^a \rightarrow \mathbb{R}^b \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \sum_{m=1}^{b} \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial x_m} . \underline{\boldsymbol{e}}_m$$

Tensor/Vector product in cylindrical coordinates

- Vector gradient :

$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad \underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}} = \frac{\partial \underline{\boldsymbol{a}}}{\partial r} \otimes \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta} \otimes \frac{\underline{\boldsymbol{i}}_\theta}{r}(\theta) + \frac{\partial \underline{\boldsymbol{a}}}{\partial z} \otimes \underline{\boldsymbol{i}}_z$$

- Vector divergence :

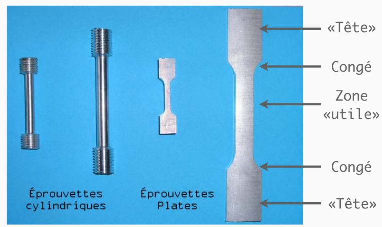
$$\underline{\boldsymbol{a}} \in \mathbb{R}^3 \quad div_x\left(\underline{\boldsymbol{a}}\right) = Tr\left(\underline{\underline{\mathbb{D}_{\boldsymbol{x}}\boldsymbol{a}}}\right) = \left\langle \underline{\boldsymbol{\nabla}}_x, \underline{\boldsymbol{a}} \right\rangle = \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial r}, \underline{\boldsymbol{i}}_r \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial \theta}, \frac{\underline{\boldsymbol{i}}_\theta}{r} \right\rangle + \left\langle \frac{\partial \underline{\boldsymbol{a}}}{\partial z}, \underline{\boldsymbol{i}}_z \right\rangle$$

- Tensor divergence :

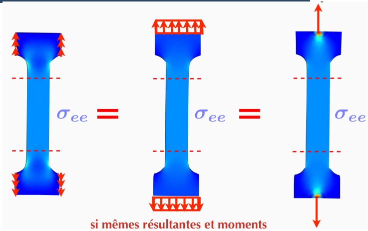
$$\underline{\underline{\boldsymbol{A}}} \in \mathbb{R}^3 \to \mathbb{R}^3 \quad div_x\left(\underline{\underline{\boldsymbol{A}}}\right) = \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial r} . \underline{\boldsymbol{i}}_r(\theta) + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial \theta} . \frac{\underline{\boldsymbol{i}}_\theta}{r} + \frac{\partial \underline{\underline{\boldsymbol{A}}}}{\partial z} . \underline{\boldsymbol{i}}_z$$

Failure Citeria

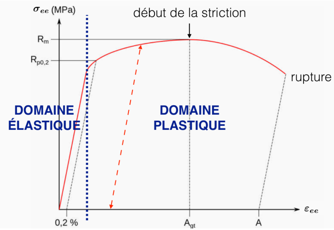
Rupture



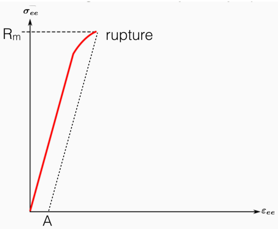
a



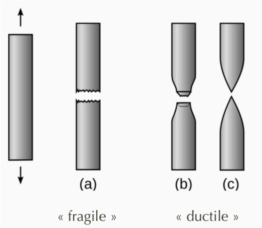
b



c ductile



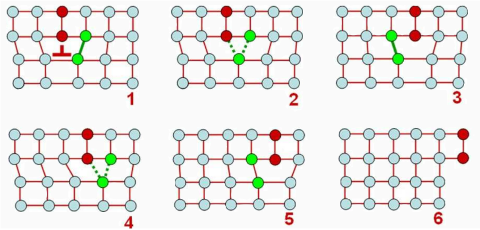
d fragile



e

Figure – Credits G. Puel

a



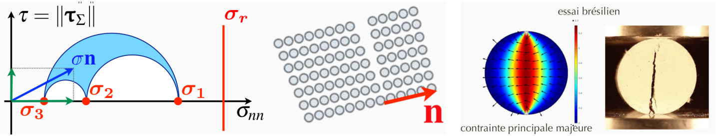
b

Figure – Credits G. Puel

- Separation of two atomic planes
- Driven by normal stress component σ_{nn}
- Random atomic planes
  $\longrightarrow$ orientation that maximizes σ_{nn}

$$f\left(\underline{\underline{\sigma}}\right) = \max_{\|\underline{\boldsymbol{n}}\|=1} \sigma_{nn}\left(\underline{\boldsymbol{x}}\,;t\right) - \sigma_r \leq 0 \tag{1}$$

- for fragile materials (concrete)



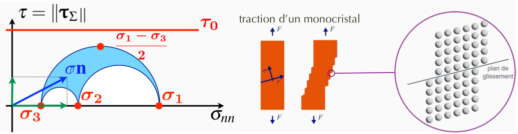
a b

Figure – Credits G. Puel

- plastic shearing along a certain plane
- induced by shearing stress overruns the threshold (Schmidt's law) σ_{nn}

1. maximum tangent traction component $\longrightarrow$ Tresca's criterion

$$
\begin{aligned}
f\left(\underline{\underline{\sigma}}\right) &= \max_{\|\underline{\boldsymbol{n}}\|=1} \|\underline{\boldsymbol{\tau}}_{\Sigma}\|\left(\underline{\boldsymbol{x}};t\right) - \tau_0 \leq 0 \\
\underline{\boldsymbol{n}} &= \frac{1}{\sqrt{2}}\left(\underline{\boldsymbol{n}}_I + \underline{\boldsymbol{n}}_{III}\right)
\end{aligned} \tag{2}
$$



a

Figure – Credits G. Puel

- plastic shearing along a certain plane
- induced by shearing stress overruns the threshold (Schmidt's law) σ_{nn}

1. Potential energy of elastic distortion $\longrightarrow$ Von Mises criterion

$$
\begin{aligned}
& f\left(\underline{\underline{\sigma}}\right) = \sigma_{eq} - \tau_0 \leq 0, \quad \sigma_{eq} = \sqrt{3J_2\left(\underline{\underline{s}}^{\sigma}\right)} \\
& \underline{\underline{\sigma}} = \sigma_{ee}\underline{e} \otimes \underline{e} \longrightarrow \sigma_{eq} = \sigma_{ee} \\
& \underline{\underline{\sigma}} = \tau_{em}\underline{e} \otimes_s \underline{m} \longrightarrow \sigma_{eq} = \sqrt{3}\tau_{em}
\end{aligned} \tag{2}
$$

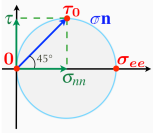
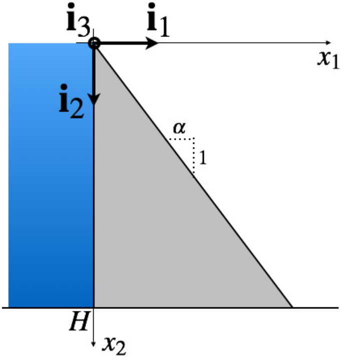


Figure – Credits G. Puel

3.1 Gravity Dam

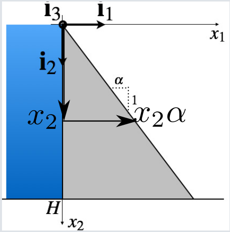


Barrage de Willow Creek (Oregon, USA)

Figure – Set up [Credits : G. Puel]

Stress description

- $\underline{\boldsymbol{x}} = x_1\underline{\boldsymbol{i}}_1 + x_2\underline{\boldsymbol{i}}_2 + x_3\underline{\boldsymbol{i}}_3, \qquad (x_1, x_2, x_3) \in \Omega_t =]0, x_2\alpha[\times]0, H[\times]0, L[$
- $\underline{\underline{\sigma}}\left(\underline{\boldsymbol{x}}\right) = b_1 x_2 \underline{\boldsymbol{i}}_1 \otimes \underline{\boldsymbol{i}}_1 + (a_2 x_1 + b_2 x_2) \underline{\boldsymbol{i}}_2 \otimes \underline{\boldsymbol{i}}_2 - 2\left(\rho_b g + b_2\right) x_1 \underline{\boldsymbol{i}}_1 \otimes_S \underline{\boldsymbol{i}}_2, \forall \underline{\boldsymbol{x}} \in \Omega_t$
- ρ_e=1000 kg/m^3, ρ_b=2200 kg/m^3, g=9.81 m/s^2
- $b_1 = -\rho_e g, \quad a_2 = \frac{\rho_b g}{\alpha} - \frac{2\rho_e g}{\alpha^3}, \quad b_2 = -\rho_b g + \frac{\rho_e g}{\alpha^2} = -\alpha a_2 - \frac{\rho_e g}{\alpha^2}$



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Question 1 : $div_x\left(\underline{\underline{\sigma}}\left(\underline{x}\right)\right) + \underline{f}_v = 0$ (static)?

- $\underline{f}_v\left(\underline{x};t\right) = \rho_b g \underline{i}_2$
    - $\square$ $\dfrac{\partial \underline{\underline{\sigma}}}{\partial x_1} = a_2 \underline{i}_2 \otimes \underline{i}_2 - 2\left(\rho_b g + b_2\right)\underline{i}_1 \otimes_S \underline{i}_2$
    - $\boxtimes$ $\dfrac{\partial \underline{\underline{\sigma}}}{\partial x_2} = b_1 \underline{i}_1 \otimes \underline{i}_1 + b_2 \underline{i}_2 \otimes \underline{i}_2$
    - $\boxplus$ $\dfrac{\partial \underline{\underline{\sigma}}}{\partial x_3} = 0$

$\rightarrow div_x\left(\underline{\underline{\sigma}}\right) + \underline{f}_v = \square.\underline{i}_1 + \boxtimes.\underline{i}_2 + \boxplus.\underline{i}_3 + \rho_b g \underline{i}_2 =$
$-\left(\rho_b g + b_2\right)\underline{i}_2 + b_2 \underline{i}_2 + 0 + \rho_b g \underline{i}_2 = 0$

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Question 2 : $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\Gamma_{\text{downstream}}} = 0$?

- $\Gamma_{\text{downstream}} : x_1 - \alpha x_2 = 0$
  1. $\underline{\boldsymbol{n}}(\alpha) = \frac{1}{\sqrt{1+\alpha^2}}\left(\underline{\boldsymbol{i}}_1 - \alpha \underline{\boldsymbol{i}}_2\right), \quad \underline{\boldsymbol{n}}(\alpha \to 0) = \underline{\boldsymbol{i}}_1, \underline{\boldsymbol{n}}(\alpha \to \infty) = -\frac{\alpha}{|\alpha|}\underline{\boldsymbol{i}}_2$
  2. $\underline{\underline{\sigma}}|_{\Gamma_{\text{downstream}}} = -\rho_e g x_2 \underline{\boldsymbol{i}}_1 \otimes \underline{\boldsymbol{i}}_1 - \frac{\rho_e g x_2}{\alpha^2}\underline{\boldsymbol{i}}_2 \otimes \underline{\boldsymbol{i}}_2 - 2\frac{\rho_e g x_2}{\alpha}\underline{\boldsymbol{i}}_1 \otimes_S \underline{\boldsymbol{i}}_2$
  3. $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\Gamma_{\text{downstream}}} = -\frac{\rho_e g x_2}{\sqrt{1+\alpha^2}}\left(\underline{\boldsymbol{i}}_1 - \frac{1}{\alpha}\underline{\boldsymbol{i}}_2 + \frac{1}{\alpha}\underline{\boldsymbol{i}}_2 - \underline{\boldsymbol{i}}_1\right) = 0$

Question 3 : $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\Gamma_{\text{upstream}}} = -pi_1$?

- $\Gamma_{\text{upstream}} : x_1 = 0$
  1. $\underline{\boldsymbol{n}} = -\underline{\boldsymbol{i}}_1, \quad \underline{\boldsymbol{f}}_s|_{\Gamma_{\text{upstream}}} = -p\underline{\boldsymbol{i}}_1 (p < 0)$
  2. $\underline{\underline{\sigma}}|_{\Gamma_{\text{upstream}}} = b_1 x_2 \underline{\boldsymbol{i}}_1 \otimes \underline{\boldsymbol{i}}_1 + b_2 x_2 \underline{\boldsymbol{i}}_2 \otimes \underline{\boldsymbol{i}}_2 \to$ NO SHEAR STRESS !
  3. $\underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\Gamma_{\text{upstream}}} = -b_1 x_2 \underline{\boldsymbol{i}}_1 = \rho_e g x_2 \underline{\boldsymbol{i}}_1 = -p\underline{\boldsymbol{i}}_1 \Rightarrow p = -\rho_e g x_2 < 0$
  4. $\max_{x_2 \in [0,H]} \underline{\underline{\sigma}}.\underline{\boldsymbol{n}}|_{\Gamma_{\text{upstream}}} = \rho_e g H$

Question 4 : $\max_{\|\underline{\boldsymbol{n}}\|=1} \sigma_{nn} \leq \sigma_r$?

- $\Gamma_{\text{upstream}} : x_1 = 0$
  1. $\underline{\underline{\sigma}}|_{\Gamma_{\text{upstream}}} = b_1 x_2 \underline{\boldsymbol{i}}_1 \otimes \underline{\boldsymbol{i}}_1 + b_2 x_2 \underline{\boldsymbol{i}}_2 \otimes \underline{\boldsymbol{i}}_2 \rightarrow$ NO SHEAR STRESS !
  2. $\sigma_I = \max \sigma_{11}, \sigma_{22} = x_2 \max b_1(\alpha), b_2(\alpha)$
  3. $\sigma_{nn}|_{\Gamma_{\text{upstream}}} = \left\langle \underline{\underline{\sigma}}|_{\Gamma_{\text{upstream}}} \cdot \underline{\boldsymbol{n}}, \underline{\boldsymbol{n}} \right\rangle = b_1 x_2 n_1^2 + b_2 x_2 n_2^2 =$
     $x_2 \left(-\rho_e g + \rho_b g - \frac{\rho_e g}{\alpha^2}\right) n_1^2 + \left(\frac{\rho_e g}{\alpha^2} - \rho_b g\right) x_2$