TRIPLE INTEGRAL

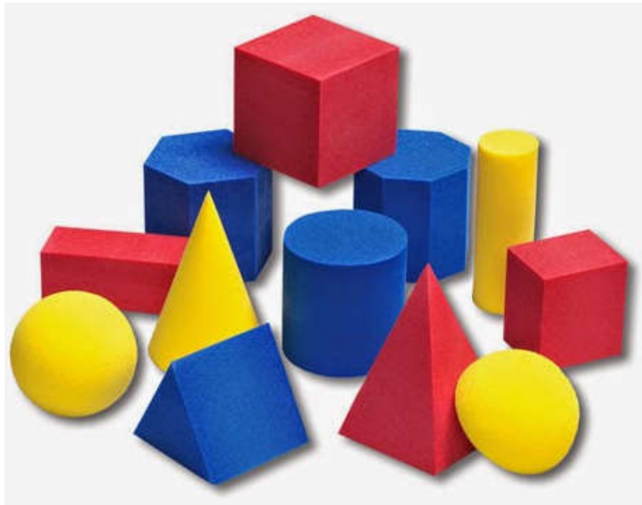
Math 55 Elementary Analysis III

From the Past

	Mass
Thin wire of liner density $\delta(x)$, where $a \leq x \leq b$	$\int_a^b \delta(x)dx$
Thin plate D of mass density $\sigma(x, y)$	$\iint_D \sigma(x, y)dA$

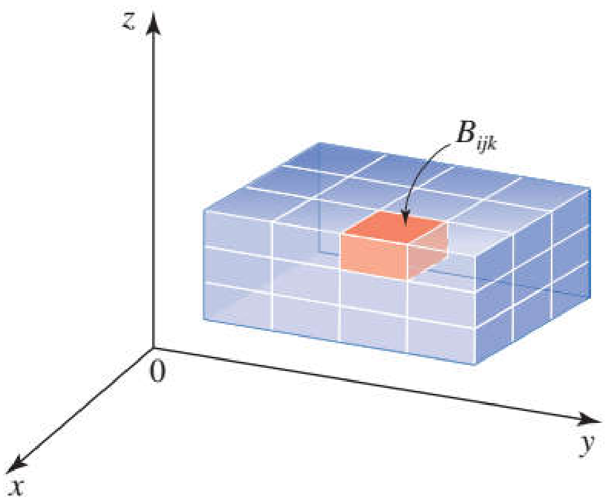
Today

Mass of a Solid T with mass density $\rho(x, y, z)$ is given by a Triple Integrals



Rectangular Box

$$B = [a, b] \times [c, d] \times [p, q] = \{(x, y, z) \mid a \le x \le b, c \le y \le d, p \le z \le q\}$$



Mass Density of the Solid: $\rho(x, yz)\ g/m^3$, $\rho > 0$ and continuous

Regular Partition:

$$a = x_0 < x_1 < \cdots < x_{i-1} < x_i < \cdots < x_l = b$$

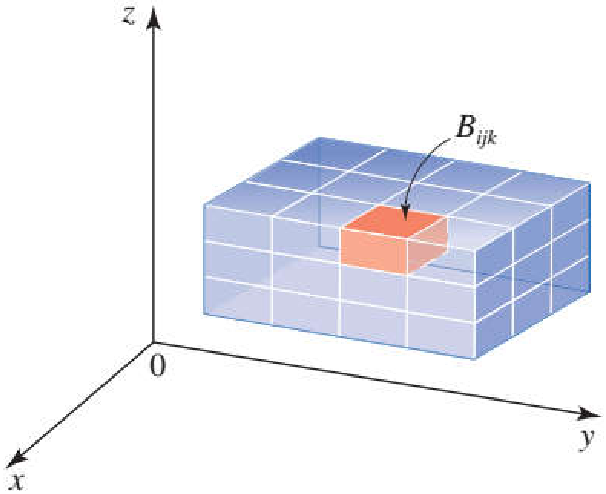
$$c = y_0 < y_1 < \cdots < y_{j-1} < y_j < \cdots < y_m = d$$

$$p = z_0 < z_1 < \cdots < z_{k-1} < z_k < \cdots < z_n = q$$

Length: $\Delta x = (b-a)/l$, $\Delta y = (d-c)/m$, $\Delta z = (q-p)/n$

Rectangular Box

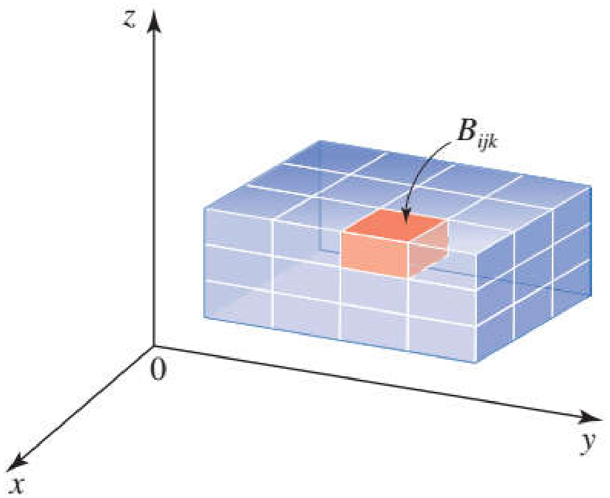
$$B = [a, b] \times [c, d] \times [p, q] = \{(x, y, z) \,|\, a \le x \le b, c \le y \le d, p \le z \le q\}$$



Observe:
The planes $x = x_i$, for $1 \le i \le l$, $y = y_j$, for $1 \le j \le m$, and $z = z_k$, for $1 \le k \le n$, parallel to the yz-, xz-, and xy- coordinate planes divide the vox B into $N = lmn$ boxes $B_{111}, B_{112}, \mathrm{B_{lmn}}$

Volume of $B_{ijk} = \Delta V = \Delta x \Delta y \Delta z$

Rectangular Box



Let $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ be an arbitrary point in B_{ijk}.
For large l, m and n, we can approximate the mas of B_{ijk} by

$$\rho(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

The mass of the box B is approximately

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\rho(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

Mass of the box B:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}\rho(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V$$

Triple Integral

DEFINITION **Triple Integral of f Over a Rectangular Box B**

Let f be a continuous function of three variables defined on a rectangular box B, and let $P = \{B_{ijk}\}$ be a partition of B.

1. A **Riemann sum of f over B** with respect to the partition P is a sum of the form

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a point in B_{ijk}.

2. The **triple integral of f over B** is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if the limit exists for all choices of $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ in B_{ijk}.

Iterated Integrals

THEOREM 1

Let f be continuous on the rectangular box

$$B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ p \le z \le q\}$$

Then

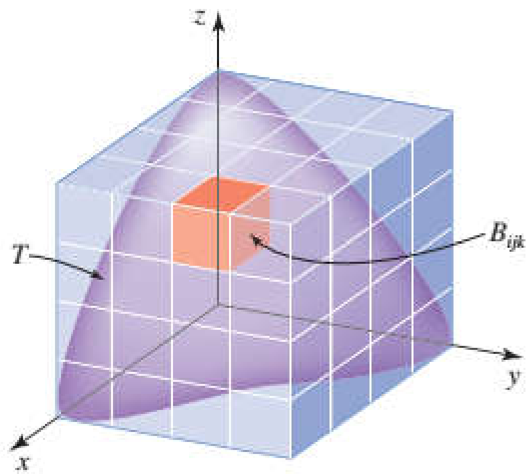
$$\iiint\limits_B f(x, y, z)\, dV = \int_p^q \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Example 1

Evaluate $\iiint_B (x^2y + yz^2)dV$, where
$B = \{(x, y, z) | -1 \leq x \leq 1, 0 \leq y \leq 3, 1 \leq z \leq 2\}$

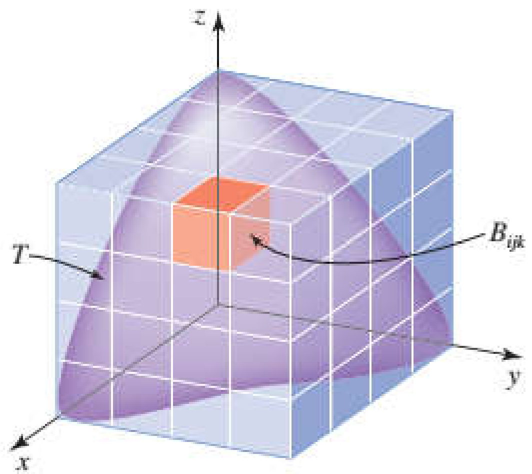
Answer: 24

Over General Bounded Solid



Let T be a bounded solid region in space. Then it can be enclosed in a rectangular box $B = [a, b] \times [c, d] \times [p, q]$. Again, let $P = \{B_{111}, B_{112}, \dots, B_{ijk}, \dots, B_{lmn}\}$ be a regular partition of B into $N = lmn$ boxes with sides Δx, Δy and Δz and volume $\Delta V = \Delta x \Delta y \Delta z$.

Over General Bounded Solid



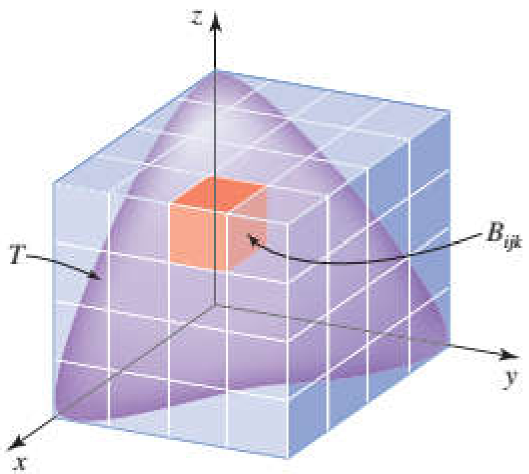
Define

$$F(x,y,z) = \begin{cases} f(x,y,z) & \text{if}(x,y,z) \text{ is in } T \\ 0 & \text{if } (x,y,z) \text{ is } B \text{ but not in } T \end{cases}$$

Riemann sum of f over T:

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} F\left(x_{ijk}^{*}, y_{ijk}^{*}, z_{ijk}^{*}\right)\Delta V$$

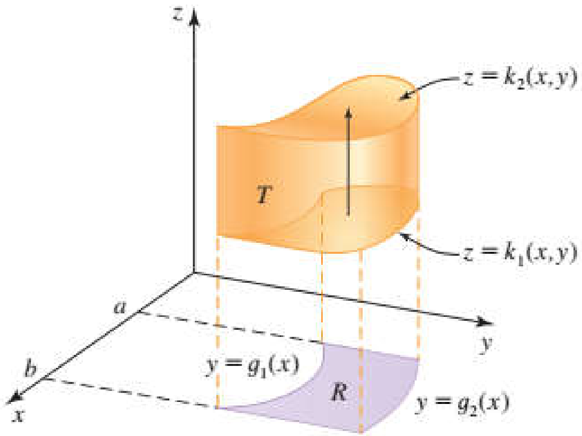
Over General Bounded Solid



Triple Integral of f over T:

$$\iiint_T f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} F\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)\Delta V$$

Evaluating Triple Integrals Over General Regions

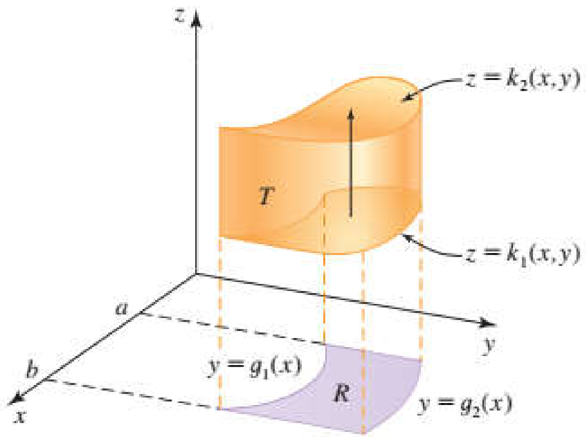


z-simple:

$$T = \{(x, y, z) |\ (x, y) \in R, k_1(x, y) \leq z \leq k_2(x, y)\}$$

Where R is the projection T onto the xy-plane

Evaluating Triple Integrals Over General Regions



If f is continuous on T, then

$$\iiint f(x,y,z)\,dV = \iint \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)dz \right] dA$$

where R is the projection T onto the xy-plane.

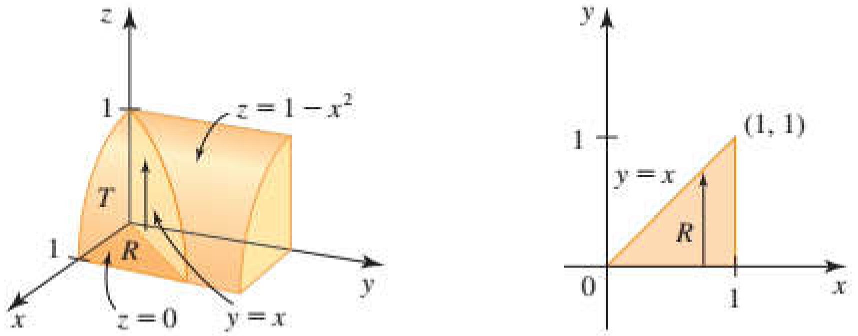
Furthermore, if R is y-simple

$R = \{(x,y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. Then

$$\iiint_T f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)\,dzdydx$$

Example 2

Evaluate $\iiint_T zdV$ where T is the solid in the first octant bounded by the graphs $z = (1 - x^2)$ and $y = x$.



(a) The solid T is z-simple.

(b) The projection of the solid T onto R in the xy-plane is y-simple.

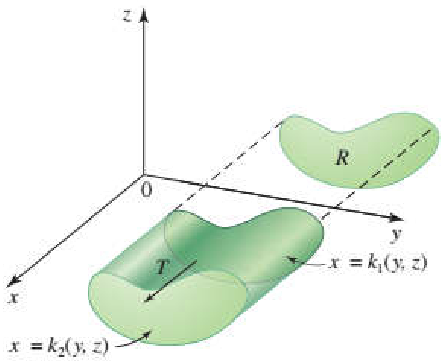
Answer:1/12

Other simple regions

- x-simple

$$T = \{(x, y, z) \mid (y, z) \in R,\ k_1(y, z) \le x \le k_2(y, z)\}$$

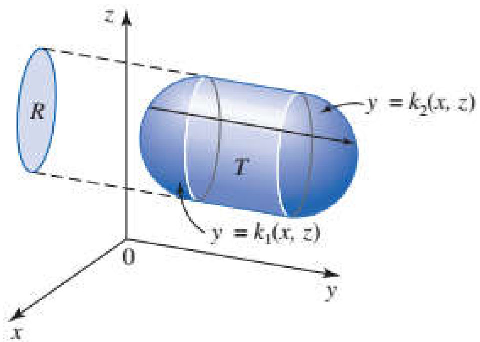
$$\iiint_T f(x, y, z)\, dV = \iint_R \left[\int_{k_1(y, z)}^{k_2(y, z)} f(x, y, z)\, dx \right] dA$$



- y-simple

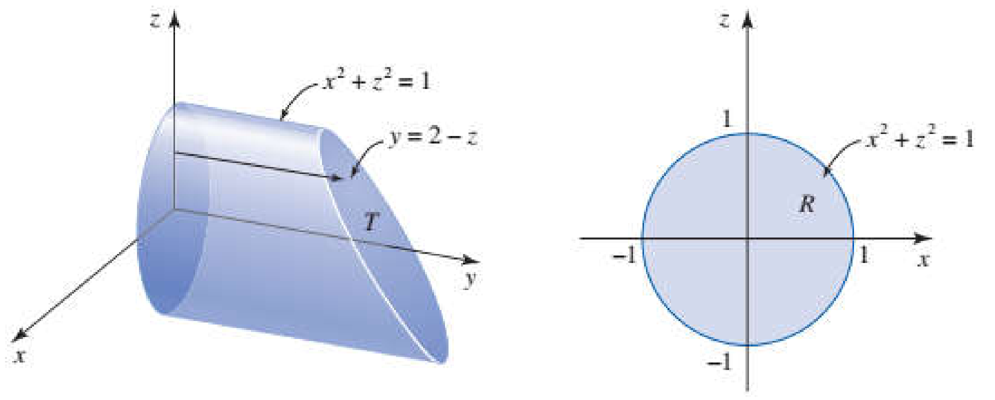
$$T = \{(x, y, z) \mid (x, z) \in R,\ k_1(x, z) \le y \le k_2(x, z)\}$$

$$\iiint_T f(x, y, z)\, dV = \iint_R \left[\int_{k_1(x, z)}^{k_2(x, z)} f(x, y, z)\, dy \right] dA$$



Example 3

Evaluate $\iiint_T \sqrt{x^2+z^2}\,dV$, where T is the region bounded by the cylinder $x^2+z^2=1$ and the planes $y+z=2$ and $y=0$.



(a) The solid T is viewed as being y-simple.

(b) The projection of the solid T onto R in the xz-plane

Volume of T

The triple integral of f over T gives the volume V of T; that is,

$$V = \iiint_T dV$$

Mass, Center of Mass and Moments of Inertia for Solids in Space

DEFINITIONS Mass, Center of Mass, and Moments of Inertia for Solids in Space

Suppose that $\rho(x, y, z)$ gives the mass density at the point (x, y, z) of a solid T. Then the **mass** m of T is

$$m = \iiint_T \rho(x, y, z)\, dV$$

Mass, Center of Mass and Moments of Inertia for Solids in Space

The **moments** of T about the three coordinate planes are

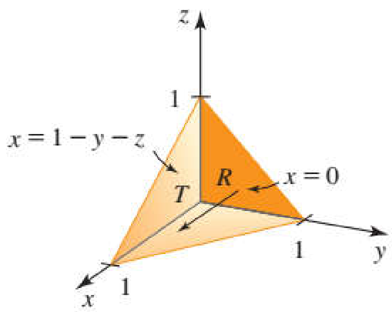
$$M_{yz} = \iiint_T x\rho(x, y, z)\, dV$$

$$M_{xz} = \iiint_T y\rho(x, y, z)\, dV$$

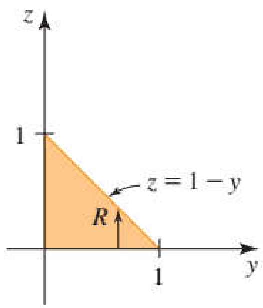
$$M_{xy} = \iiint_T z\rho(x, y, z)\, dV$$

Example 4

Let T be the solid tetrahedron bounded by the plane $x + y + z = 1$ and the three coordinate planes $x = 0, y = 0$ and $z = 0$. Find the mass of T if the mass density of T is directly proportional to the distance between a base of T and a point on T.



(a) The solid T viewed as an x-simple region

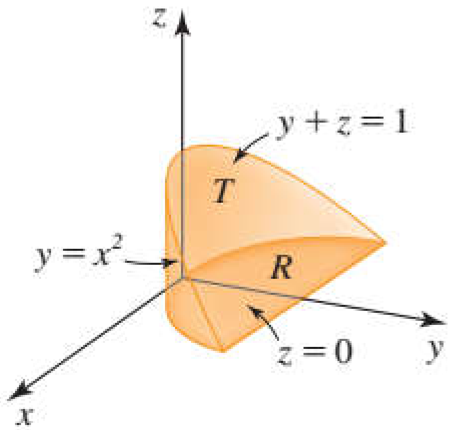


(b) The projection of the solid T onto the yz-plane viewed as a z-simple region

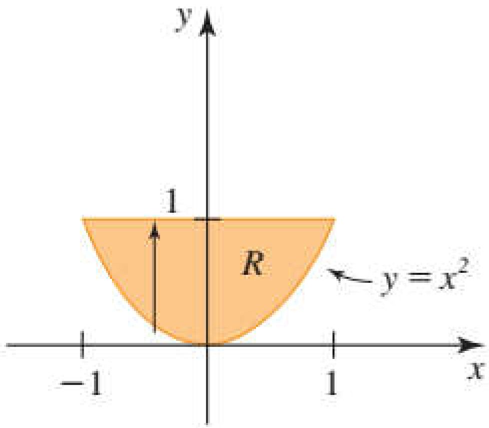
Answer: $k/24$

Example

Let T be the solid that is bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0$ and $y + z = 1$. Find the center of mass of T, given that is has uniform density $\rho(x, y, z) = 1$.



(a) The solid T is viewed as a z-simple region.

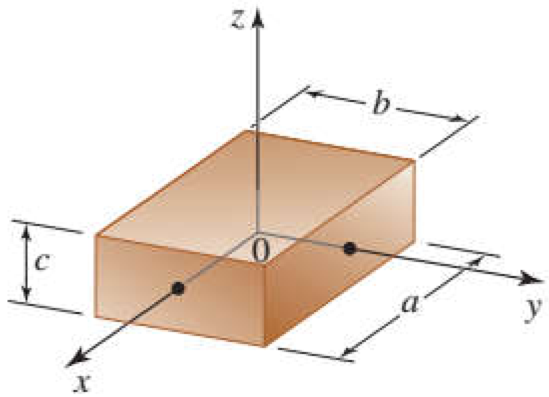


(b) The projection R of T onto the xy-plane viewed as being y-simple

Answer:
$\left(0, \frac{3}{7}, \frac{2}{7}\right)$

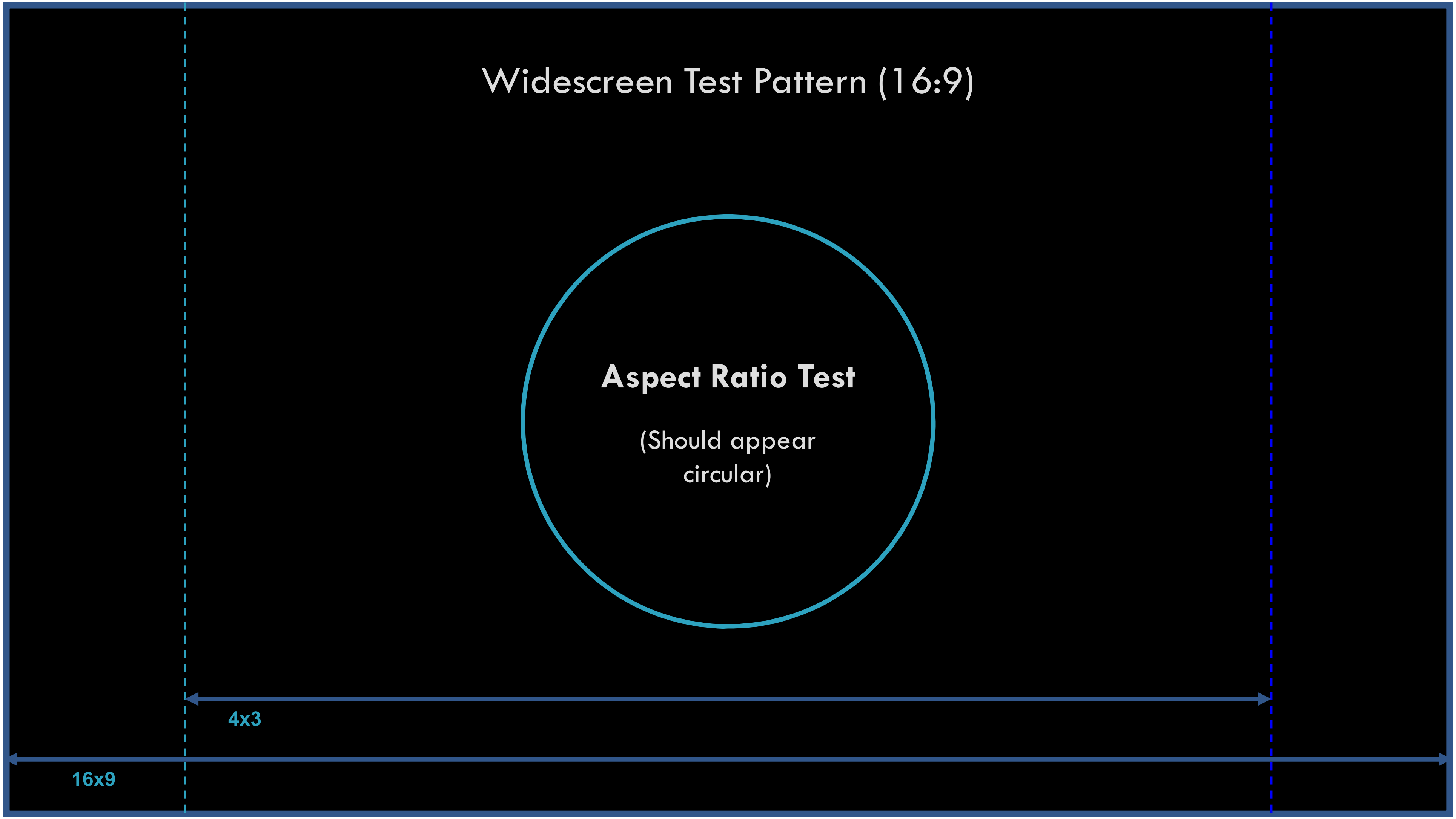
Example 6

Find the moments of inertia about the three coordinate axes for the solid rectangular parallelepiped of constant density k as show below.



Answer:

$$I_x = \frac{1}{12}m(b^2 + c^2)$$
$$I_y = \frac{1}{12}m(a^2 + c^2)$$
$$I_z = \frac{1}{12}(a^2 + b^2)$$


Widescreen Test Pattern (16:9)
Aspect Ratio Test
(Should appear circular)
4x3
16x9