Unit One Notes

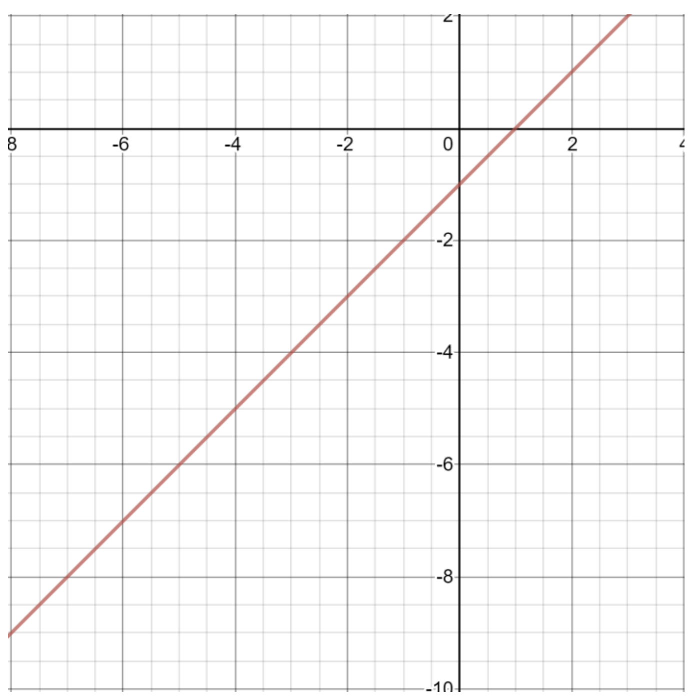
The Limit of a Function

An Introduction to Limits

To begin with limits, we will start with an introductory problem. Below is a rational function.

$$f(x) = \frac{x^2-3x+2}{x-2}$$

This function can simplify to the function $f(x) = x - 1$, however, there is a hole at $x = 1$. Graphing this function looks like this:



Keep in mind that, due to the hole, the point (1, 0) does not exist along the function. This is where a limit can tell us more about the function. A limit describes a value that is being approached, even if it is not a true output of the function.

Now that we have a graph, we can use an x value to find a limit!

We will start with this expression.

$$\lim_{x \to -7} \frac{x^2 - 3x + 2}{x - 2}$$

Notice the statement below "lim." The x → -7 gives us a value to use to find our limit. The limit asks for a y value when the x value is -7. As the x value approaches -7, the y value approaches -8.

In the case above, the point (-7, -8) exists. A special part about limits is that they do not necessarily need to use an existing point on the graph. Take this expression:

$$\lim_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$$

This limit asks for the y value as x approaches 2. As the x values approach 2, they y value approaches 1. However, due to the nature of the hole, y cannot equate to 1. The main difference between limits and variables is that the limit requests the point that is being approached as opposed to the definite answer. This means that the limit of x = 2 is (2, 1).

Directional Notation for Limits

Limits also have a special notation to denote the direction to which the x values approach a point. The chart below helps describe these. Notice that these denote the direction from which x approaches, not the direction to which it approaches.

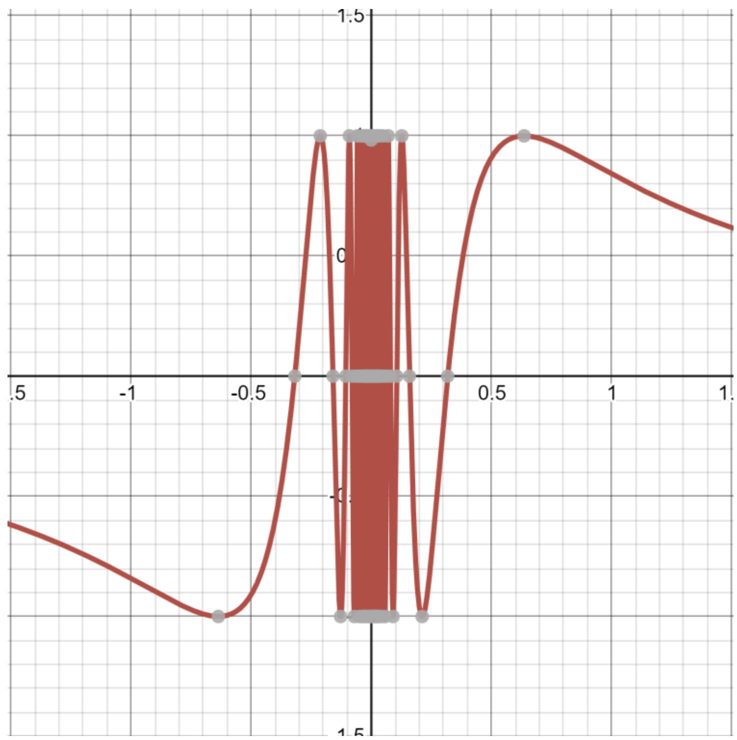
Notation	Direction x Approaches From
$x \to 0^+$	Left
$x \to 0^-$	Right
$x \to 0$	Both Directions

The direction from which x approaches will be useful later, as it will help us determine whether a limit truly exists.

Determining a Limit’s Existence

Because of the nature of limits, occasionally, a limit may not exist. The x value of a function needs to have a proper y value point to use within the limit’s coordinates. One of these cases has to do with oscillating functions; check out the expression and graph below:

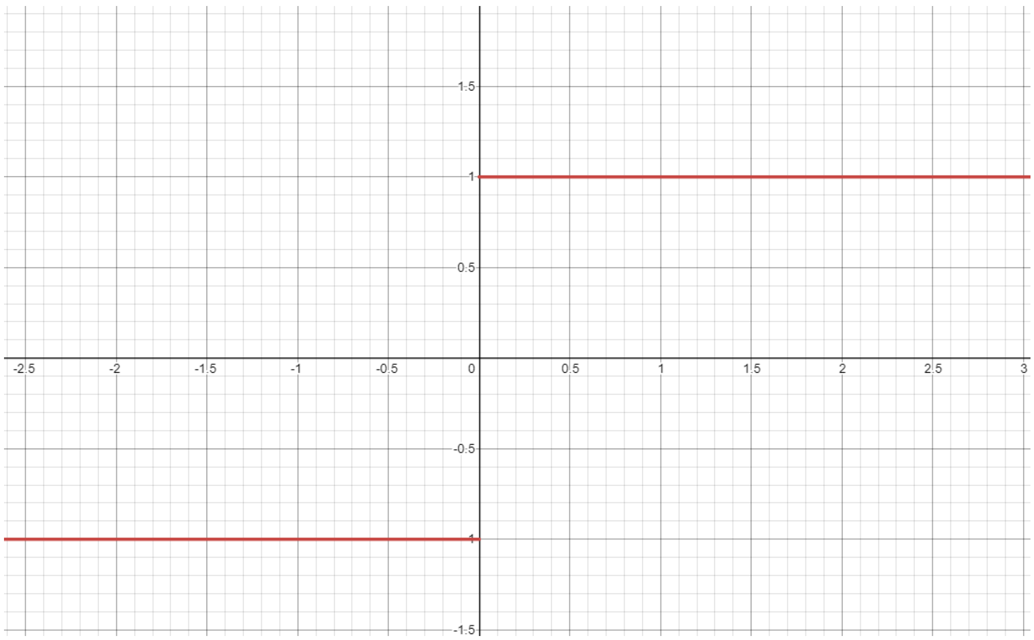
$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$



As previously stated, this is considered an oscillating function. The function continues oscillating but becomes much tighter around the y axis. This function does not seem to have a value that settles in at $x = 0$. Because of this, we can say that the limit does not exist, as we cannot find a true y value to use in our coordinate.

Another strong example of non-existent limits is with piecewise functions. To keep things simple, using a rational absolute function will do.

$$\lim_{x \to 0} \left(\frac{|x|}{x}\right)$$



Notice that you would acquire a different value when approaching from either direction. When approaching from the left, y approaches -1. When approaching from the right, y approaches 1. Because these two directional limits are different, we cannot find a defined limit. Therefore, the overall limit does not exist.

Defining a Limit

Using the examples from above, we can now define a limit. For a limit to be valid, three things need to be true:

1. The limit exists when x approaches from the left.
2. The limit exists when x approaches from the right.
3. The left-approaching limit and the right-approaching limit are the same.

Another way to right this is:

1. $\lim_{x \to a^+} f(x)$ exists.
2. $\lim_{x \to a^+} f(x)$ exists.
3. $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$

If all three of these statements hold true, the limit exists!

In Summary

A limit is a set of coordinate points that represents the x and y values that are found when approaching a certain x value. For a limit to exist, there needs to be an existent limit when approaching from both the left and right, and those limits need to be the same.