



CZECH TECHNICAL UNIVERSITY IN PRAGUE  
Faculty of Nuclear Sciences and Physical Engineering



# **Real Options Valuation: A Dynamic Programming Approach**

## **Oceňování projektů metodou reálných opcí z pohledu dynamického programování**

Master's Thesis

Author: **Filip Rolenec**  
Supervisor: **Ing. Rudolf Kulhavý, DrSc.**  
Language advisor: **Ing. Rudolf Kulhavý, DrSc.**  
Academic year: 2020/2021





# ZADÁNÍ DIPLOMOVÉ PRÁCE

Student:	Bc. Filip Roleneč
Studijní program:	Aplikace přírodních věd
Obor:	Matematické inženýrství
Název práce (česky):	Oceňování projektů metodou reálných opcí z pohledu dynamického programování
Název práce (anglicky):	Real Options Valuation: A Dynamic Programming Approach

## Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstруйте jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.





Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.
2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.
3. Powell, Warren B, Approximate Dynamic Programming: Solving the Curses of Dimensionality. 2nd ed. Hoboken, NJ: Wiley, 2011.
4. Puterman, Martin L., Markov Decision Processes: Discrete Stochastic Dynamic Programming. Hoboken, NJ: Wiley, 2005.
5. Vollert, Alexander. A Stochastic Control Framework for Real Options in Strategic Valuation. Boston, MA: Birkhäuser, 2003.

Jméno a pracoviště vedoucí diplomové práce:

Ing. Rudolf Kulhavý, DrSc.

Batličkova 253/1, , 182 00 Praha 8

Jméno a pracoviště konzultanta:

Datum zadání diplomové práce: 31.10.2019

Datum odevzdání diplomové práce: 4.5.2020

Doba platnosti zadání je dva roky od data zadání.

V Praze dne 11. října 2019

.....  
garant oboru

.....  
vedoucí katedry

.....  
děkan





*Acknowledgment:*

I would like to thank my supervisor Ing. Rudolf Kulhavý, DrSc. for his professional guidance and all the advice given while creating this thesis.

*Author's declaration:*

I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, March 12, 2021

Bc. Filip Rolenec



*Název práce:*

**Oceňování projektů metodou reálných opcí z pohledu dynamického programování**

*Autor:* Bc. Filip Rolenec

*Obor:* Matematické inženýrství

*Druh práce:* Diplomová práce

*Vedoucí práce:* Ing. Rudolf Kulhavý, DrSc.

*Abstrakt:* Teorie reálných opcí (ROA) je pokročilá valuační technika pro projekty, která respektuje hodnotu možných budoucích změn (tzv. *reálných opcí*).

Domníváme se, že v současné ROA teorii neexistuje obecný algoritmus pro valuaci komplexních projektů, kterými zde myslíme projekty s více náhodnými veličinami a větším množstvím alternací projektů různého typu.

Tato diplomová práce se inspirovuje problémem valuace ve smyslu ROA a snaží se ho interpretovat jako problém dynamického rozhodování za neurčitosti z oblasti statistické teorie rozhodování (SDT). V této práci představujeme obecný valuační algoritmus, který staví na poznatcích z SDT, pokrývá problémy řešené současnou ROA teorií a navíc respektuje vliv ekonomických konceptů jako je časová hodnota peněz a averze vůči riziku investorů. Použití jmenovaného algoritmu je představeno na problému valuace plynové elektrárny, kde problém nespočetného stavového prostoru řešíme pomocí techniky přibližného dynamického programování nazývané aproximace hodnotové funkce. Hodnotové funkce modelujeme jako po částech lineární funkce, tak aby odrážely opční strukturu projektových odměn.

*Klíčová slova:* Analýza reálných opcí, Blackův-Scholesův model, Čistá současná hodnota, Dynamické programování, Energetika, Oceňování projektů, Statistická rozhodovací teorie

*Title:*

**Real Options Valuation: A Dynamic Programming Approach**

*Author:* Bc. Filip Rolenec

*Abstract:* The theory of real option analysis (ROA) is considered as an advanced project valuation technique, which respects the value of future project alternations (*real options*).

To our best knowledge, the current state of ROA does not offer a unified valuation algorithm that would be able to cover valuations of more complex projects, such as those with multiple random variables or different types and a larger number of possible actions.

This thesis takes the problem of ROA and tries to interpret it as a problem of decision making under uncertainty from the statistical decision theory (SDT). We present a general valuation algorithm that builds on the knowledge of SDT, covers the solutions proposed by ROA, and preserves the business-specific concepts as time value of money and risk-aversion of investors. This algorithm's usage is demonstrated on a problem of gas power plant valuation, where the problem of uncountable state space is solved via the approximate dynamic programming technique of value function approximation, where we use piecewise linear models to cover the option-like structure.

*Keywords:* Black-Scholes model, Dynamic programming, Net present value, Power industry, Project valuation, Real option analysis, Statistical decision theory



# Contents

<b>1</b>	<b>Introduction</b>	<b>15</b>
1.1	Historical Background - Real Options . . . . .	15
1.2	Research Motivation . . . . .	16
1.3	Outline of the Thesis . . . . .	17
<b>2</b>	<b>Preliminaries</b>	<b>19</b>
2.1	Probability Theory . . . . .	19
2.1.1	Probability Distributions . . . . .	20
2.2	Statistical Decision Theory . . . . .	21
2.2.1	Markov Decision Process (Sequential Decision Problem) . . . . .	21
2.2.2	Dynamic Programming . . . . .	22
2.2.3	Approximate Dynamic Programming . . . . .	23
2.2.4	Utility . . . . .	24
<b>3</b>	<b>Valuation - state of the art</b>	<b>27</b>
3.0.1	Financial option valuation . . . . .	28
3.0.2	Real Option Valuation . . . . .	30
<b>4</b>	<b>Project Valuation Revisited from the Statistical Decision Theory Perspective</b>	<b>33</b>
4.1	Project Valuation - Problem Definition . . . . .	33
4.2	Project Valuation in the Statistical Decision Theory Framework . . . . .	34
4.2.1	Time sets . . . . .	34
4.2.2	State sets . . . . .	35
4.2.3	Action sets . . . . .	35
4.2.4	Transition probability functions . . . . .	35
4.2.5	Reward functions . . . . .	36
4.3	Solution of the Project Valuation as SDT Problem . . . . .	37
4.3.1	Probability . . . . .	37
4.3.2	Time value of money . . . . .	39
4.3.3	Risk aversion of investors . . . . .	40
4.3.4	Approximate dynamic programming . . . . .	40
4.3.5	Summary . . . . .	41
<b>5</b>	<b>Valuation of Simple I/O Businesses</b>	<b>43</b>
5.1	General Settings of the Experiment . . . . .	43
5.1.1	Time set . . . . .	44
5.1.2	State set . . . . .	44

5.1.3	Action function . . . . .	45
5.1.4	Transition probability function . . . . .	45
5.1.5	Reward function . . . . .	47
5.1.6	Utility function . . . . .	47
5.2	Approximate Dynamic Programming . . . . .	47
5.3	Optimal Strategy Performance . . . . .	50
5.3.1	Results - initial setup . . . . .	50
5.3.2	Results - increased volatility . . . . .	51
<b>6</b>	<b>Discussion</b>	<b>53</b>
6.1	Value Function Approximation . . . . .	53
6.2	Usage of Utility . . . . .	54
6.3	Computational Complexity . . . . .	54
6.4	Experimental Results . . . . .	54
<b>7</b>	<b>Conclusion</b>	<b>57</b>

# Chapter 1

## Introduction

Investing is a problem of optimal allocation of resources (mostly money) between an investor's alternative choices. Two main groups of investments can be recognized as first: the investment in financial (or commodity) market and second: the direct investment in business ventures. The first type of investments is driven by the fairly strict rules of the specific institutions (banks, exchanges, brokers,...), whereas the second type of investments is regulated in essence only by the commercial code in the given country.

In this thesis, we will focus on the investments in business ventures, where the problem of resource allocation between various opportunities is being solved with the metric of project valuation. From the various project valuation techniques, we have chosen the theory of real options (ROA) as an inspiration for a creation of a valuation model based on the ideas of statistical decision theory (SDT).

First, we describe the historical background of the ROA valuation. Next, we focus on the reasons why we investigate the idea of SDT interpretation of ROA. In the remainder of this chapter, we present the outline of this thesis.

### 1.1 Historical Background - Real Options

The foundations of financial derivatives date back to the origins of commerce in Mesopotamia in the fourth millennium BC [46]. The derivative market consisted mainly of forward contracts<sup>1</sup> and it was introduced to the European continent through Spain in Roman times. After the expulsion of derivative trading in Spain the center of this type of commerce for Europe were the Low Lands, where at the end of the 17th century, the first ideas about options<sup>2</sup> and option trading were published by La Vega [43].

The first attempts of a mathematical option pricing come from Bachelier (1900) [5] and Bronzin (1908) [14]. Based on their work, the boom of option pricing methods in the 1970's culminated in Nobel-prize-awarded Black-Scholes model [9], which is today's standard in the option pricing theory [47].

The publicity and wide adoption of the BSM model most likely inspired an expert on capital budgeting, Stewart Myers, to introduce the term "Real Options" [31], one of two main pillars of this thesis. Myers builds on the idea that real options - the ability to alter the project's course in the future - are able to bring significantly larger value in comparison to the same project without them. Myers' approach to the real options is mostly philosophical in the sense that he stresses out the importance of thinking about the additional value options bring. At the same time he does not present any computational tool for the said value.

---

<sup>1</sup>A forward contract is a contract to purchase an asset at a fixed price on a particular date in the future [7].

<sup>2</sup>A financial option is the **ability** to buy (call option) or sell (put option) a defined volume of an asset for a specified amount of money in a given future time instant [7].

The idea of Real option analysis (ROA) as a valuation tool for projects was further developed by several influential authors in the following decades, for example Guthrie [21], Pindyck [18] or Kulatilaka [3].

The valuation of project's free cash flows with the theory of ROA is in corporate finance understood as very advanced, and its adoption in practice is slow [2]. It is argued that this slow adoption is caused mainly by misunderstanding the more difficult mathematical concept of ROA [41] and the low adoption rate of a competition: "Why should our company use a new tool that no one else is using?" [16].

## 1.2 Research Motivation

Through my studies at the FNSPE CTU, I have specialized in the theory of dynamic decision making under uncertainty. In my final years at the university I also became curious about the world of corporate finance, since I was working for a venture capital investor Presto Ventures. This combination of my professional interests is pleasantly reflected in the assignment of this thesis - the interpretation of ROA in terms of SDT.

Through the study of the ROA state of the art, we have found out that the term real options is not clearly defined. As will be illustrated in-depth in section 3.0.2 we identify three classes of ROA authors based on the level of analogy to the BSM model.

In this thesis, we focus on the class of ROA authors, which utilize only the non-arbitrage principle to determine the probabilities of used models. We build mostly on the state-of-the-art textbook from Guthrie [21] and the analysis of the potential real option representation via SDT from Vollert [45].

The goal of this thesis is to take the project valuation problem structure as is understood in ROA and look at it from the perspective of statistical decision theory (SDT). One of the challenges of this task is to implement the business-specific concepts about investors' behavior and the way they perceive value. Two main addressed concepts are the time value of money and the risk aversion of investors.

The goal of this thesis is to provide an SDT-based valuation algorithm for projects, whose value is understood as a maximal possible current cash equivalent of the uncertain future cash flows. This valuation algorithm covers the classes of problems now solved by ROA and allows for new ones.

The new SDT based valuation algorithm enables:

- seamless integration of multiple uncertainty sources;
- integration of theoretically any probability distribution as a model of uncertain variables;
- usage of a high number of possible actions (options), regardless of their nature;
- utilization of approximate dynamic programming tools for high-dimensional problems;
- preservation of the business-specific concepts as time value of money and the risk aversion of investors.

To illustrate the usage of the new SDT-based algorithm a valuation of a project from a selected class is performed. This class is denoted as *simple I/O businesses* and is defined as the type of businesses that can, for the purpose of their valuation, be described by simple input-output processes. It covers all projects whose cycle time is equal to zero, and the input-output transformation rate is constant. This class is a generalization motivated by an example of a gas power plant valuation presented by Guthrie in [21].



## 1.3 Outline of the Thesis

The thesis is structured into 5 chapters. Chapter 1 reminds the reader of the most important mathematical and economic concepts used in this thesis. Beginning with the declaration of mathematical notation, the chapter continues with key concepts of economic theory and a deeper description of the BSM model. The first chapter then follows with a summary of the last decades in ROA research, focusing on a specific level of analogy represented by Vollert [45] and Guthrie [21]. The remainder of the first chapter is reserved for key parts of SDT.

Chapter 2 represents the core of this thesis. First, we define what will be understood as a problem of ROA project valuation. We state the key features that define a project, and we limit these features accordingly (?). Then we focus on the interpretation of the problem by a general SDT framework. We illustrate the identification of ROA project features in SDT. The remainder of the second chapter is reserved for resolving the economic nuances that need to be accounted for in the SDT framework in order to make the valuation procedure consistent with the economic reality of investors' behavior.

Chapter 3 illustrates the new valuation algorithm from chapter 2 on a valuation example of a simple I/O business. A valuation of a gas power plant is chosen as the representative of this class. The first half of the chapter focuses on the value sensitivity with regard to the volume of available managerial actions in the project. The second half presents a value comparison of project alternations in different granularity of the model structure and it is aimed to demonstrate the existence of trade-of between computational complexity and precision of computed results.

Chapter 4 discusses the new findings, both theoretical and observed from the performed experiments.

Chapter 5 summarizes the thesis - reminds the motivation, underlines the main message, and lists all contributions of this thesis. Furthermore, it outlines many possible future research paths in this field as it is, to our best knowledge, the first available publication on this topic.



## Chapter 2

# Preliminaries

To properly understand a mathematical text, it is essential first to define the used symbolism and notions. The used mathematical notation comes from influential authors in their respective fields of study:

- statistical decision theory [4], [34],
- probability theory [23],
- approximate dynamic programming [33].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU demonstrated also in his previous publications [36] and [37].

### 2.1 Probability Theory

In the whole thesis, bold capital letters, such as  $\mathbf{X}$ , represent a set of all elements  $x \in \mathbf{X}$  as in [36]. The cardinality of a set  $\mathbf{X}$  is denoted with two vertical lines as  $|\mathbf{X}|$ .

Random variables, understood in the sense of the standard Kolmogorov's probability theory [26], are denoted with a tilde above the variable, e.g.,  $\tilde{x}$ . Values of random variables are denoted by the same letters as the random variable without the tilde, e.g.,  $x$ .

**Definition 2.1.** (Probability) *Let  $\tilde{x}$  be a discrete random variable. Then  $P(x)$  denotes a value of the probability mass function (p.m.f.) for the realization  $\tilde{x} = x$ . Similarly, if  $\tilde{x}$  is a continuous random variable, then  $p(x)$  denotes a probability density function (p.d.f.) at  $x$ .*

Next, we define the well-known concept of conditional probability [23].

**Definition 2.2.** (Conditional probability) *Let symbol  $P(x|y)$  represent the conditional probability of a discrete random variable. Then the  $P(x|y)$  is defined as:*

$$P(x|y) = \frac{P(x, y)}{P(y)}, \quad (2.1)$$

where  $P(x, y)$  is a joint p.m.f. of  $\tilde{x}$  and  $\tilde{y}$ .

A similar concept is also used for the continuous random variables with the name of conditional density [23].

**Definition 2.3.** (Conditional density) Let symbol  $p(x|y)$  represent the conditional density of a continuous random variable. Then the  $p(x|y)$  is defined as:

$$p(x|y) = \frac{p(x, y)}{p(y)}, \quad (2.2)$$

where  $p(x, y)$  is a joint p.d.f. of  $\tilde{x}$  and  $\tilde{y}$ .

**Definition 2.4.** (Expected value) Expected value of a discrete random variable  $\tilde{x}$  is defined as:

$$E[\tilde{x}] = \sum_{\mathbf{x}} P(x)x. \quad (2.3)$$

Similarly, for the continuous random variable  $\tilde{y}$  the expected value is defined as:

$$E[\tilde{y}] = \int_{\mathbf{Y}} p(y)y \, dy. \quad (2.4)$$

### 2.1.1 Probability Distributions

Through this thesis, the following distributions will be referenced, either as an expected model of some random variable or in a broader discussion [10].

**Bernoulli distribution** The Bernoulli distribution  $Be(p)$  models the probability of success (1) or failure (0) with  $p \in [0, 1]$  by the p.m.f.:

$$P(\tilde{x} = x, p) = \begin{cases} p & x = 1 \\ 1 - p & x = 0. \end{cases} \quad (2.5)$$

This distribution is widely used in the ROA theory. It is usually used for the modeling of the up (1) and down (0) movement of asset or commodity prices.

**Binomial distribution** Binomial distribution  $Bi(n, p)$  represents the number of successes of  $n$  independent random variables distributed by  $Be(p)$  with p.m.f.:

$$P(\tilde{x} = k, n, p) = \binom{n}{k} \cdot p^k (1 - p)^{n-k}, \quad k \in \{0, 1, 2, \dots, n\}. \quad (2.6)$$

This distribution is used in a popular ROA modeling tool - *the binomial model* [21]. The idea is that a random variable (price, quantity, ...) evolves as a realization of consequential up or down movements distributed by  $Be(p_u)$ , where  $p_u$  is the probability of the up movement. Each move then updates the variable's value by multiplying it by a given coefficient  $U$  or  $D$ . To ensure that the number of possible states rises linearly and not exponentially with the number of time epochs, the model limits the parameters as  $D = 1/U$ .

Based on the number of up movements  $n_u$  (as a realization of  $Bi(n, p_u)$ ) until the  $n$ -th time epoch, and the initial value  $v_{init}$ , the realization of the modeled variable can be expressed as:

$$v = v_{init} \cdot U^{2n_u - n}. \quad (2.7)$$

**Poisson distribution** Poisson distribution  $Po(\lambda)$  is a popular discrete distribution used in modeling of the number of successes in a given time-interval, so-called “arrivals” [10]. This distribution is used for modeling of the number of incoming emails or the number of earthquakes in a given area in one year.

Its p.m.f. given parameter  $\lambda$  is:

$$P(\tilde{x} = x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad (2.8)$$

where  $x \in \{0, 1, 2, \dots\}$ .

Also, it is worth noting that Poisson distribution  $Po(\lambda)$  with  $\lambda = np$  is the limit distribution of a  $Bi(n, p)$  for  $n \rightarrow \infty$ , whilst the product  $\lambda = np$ , which is the expected value of the number of successes from the trials, remains constant.

**Log-normal distribution** Another useful distribution for this thesis is the log-normal distribution  $LogN(\mu, \sigma)$ . It represents a continuous random variable, a logarithm of which is distributed normally.

The p.d.f. of such variable is:

$$p(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right). \quad (2.9)$$

This distribution is again used mostly for modeling of asset or commodity prices, arguably for two main reasons. The first is that its realizations are positive, and the second being that variance reflects not only the parameter  $\sigma$  but also  $\mu$ , where  $\mu$  usually represents the previous price.

## 2.2 Statistical Decision Theory

The first pillar upon which this thesis stands is the statistical decision theory (SDT). An area of applied mathematics that formalizes and studies optimal decision making of agents. As decision making under uncertainty in its broadest sense encapsulates the majority of human behavior, the class of problems it is able to solve (at least theoretically) is quite large.

The SDT’s primary focus is to determine the optimal strategy (a sequence of decisions) to act upon, generally in a dynamic and uncertain environment. The optimality of the given strategy is measured by some reward metric, in the simplest case by a pure monetary gain. This metric is, however, not sufficient for many applications, so a general metric of agent’s reward via utility (described in section 2.2.4) is used.

### 2.2.1 Markov Decision Process (Sequential Decision Problem)

In this thesis, we will be modeling the decision-making problems using the standard framework of the Markov Decision Process (MDP) [34].

**Definition 2.5.** (Markov decision process)

*Markov decision process is defined by its five building blocks:*

- The set of time epochs -  $\mathbf{T}$ ;
- The set of states in those epochs -  $\mathbf{S}_t, t \in \mathbf{T}$ ;
- The set of actions in those states -  $\mathbf{A}_{s_t}, s_t \in \mathbf{S}_t, t \in \mathbf{T}$ ;
- The reward function of transition from one state to another -  $r(s_t, a_t, s_{t-1})$ , where  $s_t, \in \mathbf{S}_t, s_{t-1}, \in \mathbf{S}_{t-1}$ , and  $a_t \in \mathbf{A}_{s_t}$ ;

- Transition probabilities governing the transition from one state to another  $p(s_t|a_t, s_{t-1})$ , where  $s_t, \in \mathbf{S}_t$ ,  $s_{t-1}, \in \mathbf{S}_{t-1}$ , and  $a_t \in \mathbf{A}_{s_t}$ .

**Remark.** The set of time epochs, states and actions is usually known, defined by the structure of the decision making problem that is being solved. Reward and mostly the transition function tend to be unknown, and they need to be estimated.

**Remark.** For further simplification of the text, we define  $\mathbf{S} = \bigcup_{t \in \mathbf{T}} \mathbf{S}_t$  and  $\mathbf{A} = \bigcup_{s \in \mathbf{S}} \mathbf{A}_s$ .

Usually, the biggest task in SDT is to correctly approach the uncertainty about transition probabilities between the different states of a decision making problem. There are two approaches to parameter estimation in statistics, a classical approach, and a Bayesian approach. Since the Bayesian approach seems to fit the decision-making format better, allowing for the notion of prior probabilities, incorporating expert knowledge, and the possibility for smooth updating on newly observed data - it is a preferable choice.

In this thesis, however, we will not go into the detail of modeling the transition probability function, and we will assume it to be known.

As outlined above, the goal of SDT is to find the optimal strategy - sequence of actions. The optimality of such strategy is defined as it having the maximal expected cumulative reward among all eligible strategies  $\Pi$ :

$$\pi^* = \arg \max_{\pi \in \Pi} E \left[ \sum_{t \in \mathbf{T}} r(s_t, a_t, s_{t-1}) \middle| \pi \right]. \quad (2.10)$$

**Remark.** This definition of the optimal strategy is used mainly for finite decision problems or problems with exponential discounting of future rewards. Alternative definitions of optimality, for example, maximal average reward per period, exist.

Due to the nature of project valuation, where projects are considered to be finite or their cash flow exponentially discounted, this thesis will focus on a metric based on the presented notion of expected cumulative reward.

### 2.2.2 Dynamic Programming

Finding the optimal policy by computing the expected reward for all policies  $\pi \in \Pi$  is due to the cardinality of  $\Pi$ :

$$|\Pi| = \prod_{t \in \mathbf{T}} \prod_{s_t \in \mathbf{S}_t} |\mathbf{A}_{s_t}| \quad (2.11)$$

a very demanding task, even for low-dimensional MDP's.

To cope with such computational complexity clever algorithms were developed, most notably the dynamic programming approach and the reinforcement learning approach. In this thesis we will focus on the first one, which is based on the idea of backward induction.

The core of dynamic programming is to define the so-called value function  $V(s)$  on each of the possible states  $s \in \mathbf{S}$ . Each of the values is computed via the Bellman equation:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A}_{s_{t-1}}} \sum_{s_t \in \mathbf{S}_t} P(s_t|a_{t-1}, s_{t-1}) [r(s_t, a_{t-1}, s_{t-1}) + V(s_t)]. \quad (2.12)$$

Value function represents the expected cumulative reward from a given state onward. The idea of computing this value through the backward induction is based on the truth that a sequence of actions is

optimal if and only if the last action is optimal. Optimal strategy comes together with the value function seemingly as a byproduct, where in each state, the optimal action is the argmax of the expression in equation 2.12.

This approach significantly reduces the computational complexity to

$$\sum_{t \in \mathbf{T}} \sum_{s_t \in \mathbf{S}_t} |\mathbf{A}_{s_t}| \quad (2.13)$$

expected value computations.

**Remark.** For a constant number of states in each time epoch  $|\mathbf{S}_t|$  and a constant number of actions in each state  $|\mathbf{A}_{s_t}|$ , this reduction is from  $|\mathbf{A}_{s_t}|^{|\mathbf{S}_t|^{|T|}}$  to  $|\mathbf{A}_{s_t}| \cdot |\mathbf{S}_t| \cdot |\mathbf{T}|$ .

The reduction of computational complexity with the DP algorithm is significant. However, for the majority of real-world applications it is not sufficient. In reality, due to the structure of decision-making problems and their formulation, the cardinality of state space can explode even for fairly simple decision problems.

The problem of remaining computational complexity of DP algorithm is in literature addressed as "three curses of dimensionality" by Powell [33], who, among others, proposes solutions under the label of approximate dynamic programming (ADP).

### 2.2.3 Approximate Dynamic Programming

The computational complexity of dynamic programming for middle and high-dimensional decision-making problems is so demanding that its results cannot be obtained in a reasonable amount of time.

To cope with this problem, a relevant topic to look at is the section of SDT called approximate dynamic programming (ADP). The ADP label can be understood as a unifying name for a number of algorithms trying to obtain quasi-optimal strategies for decision-making problems with reasonable demands for computational power<sup>1</sup>.

ADP algorithms can be divided into two main classes, policy and value iteration algorithms. In this thesis, we will be focusing on the value iteration class since we believe that due to the structure of project valuation problems (rather large  $|\mathbf{S}|$  and rather small  $|\mathbf{A}|$ ), it is a better fit.

The idea of value iteration is to have some initial heuristic value function approximation, which is being updated based on samples of possible paths.

In this section of the thesis, we present the most general form of the value function approximation model, which aims to describe the general shape of the value function by a small number of parameters<sup>2</sup>. We will model the value function in each time epoch  $v_t$  as a function of the state itself  $s$  and a parametric function  $f$  as:

$$v_t(s) = f(s, \theta_{i,t}), \quad \theta_{i,t} \in \Theta_t, \quad (2.14)$$

where the cardinality  $|\Theta_t|$  is finite and preferably small.

**Remark.** To clarify the general formula represented by the equation 2.14, we present an example, where the parametric function  $f$  is a linear combination of some basis functions  $\phi_i$ :

$$v_t(s) = \sum_i \theta_{i,t} \cdot \phi_i(s), \quad (2.15)$$

<sup>1</sup>For details, please see Powell [33].

<sup>2</sup>A value function is in classical DP similar to a look-up table. There is no simple functional relationship between states and the values of value functions.

where each of the basis functions  $\phi_i$  represents some heuristically important feature of each state  $s \in \mathbf{S}$ . A project valuation example of such basis function might be a difference between state elements representing prices of inputs and outputs or indicator function of state element representing running the production.

**Remark.** Because of the option-like structure of the investment opportunities in projects, it is worth it to define the piece-wise linear function, which is a potential candidate for a function  $f$ .

**Definition 2.6.** (Piecewise linear function) In this thesis, we consider only a two-segment piecewise linear function, which can be described by four parameters as:

$$pw(x, x_0, y_0, k_1, k_2) = \begin{cases} y_0 + k_1(x - x_0) & \text{if } x \leq x_0 \\ y_0 + k_2(x - x_0) & \text{if } x > x_0 \end{cases} \quad (2.16)$$

where  $k_1$  and  $k_2$  represent the slopes of the first and second segment and the pair  $x_0, y_0$  the single breaking point.

To determine the parameters  $\theta_{i,t}$  for each value function approximation  $v_t$  we first need to heuristically derive the parameters for the value function in the last time epoch  $\theta_{\mathbf{T},t}$ . This is done with the respect to the specific MDP and its meaning. Then, gradually, starting from the last time epochs by a backward induction we discover all the remaining  $\theta_{i,t}$  by the following steps.

- We generate a sample of states  $\mathcal{S}_t$  in time  $t \in \mathbf{T}^3$ .
- In each  $s_t \in \mathcal{S}_t$  we determine the optimal action  $a_t^*$  as the argument maximizing the Bellman equation 2.12, where the future value function  $v_{t+1}$  is understood as its approximation from the previous step.
- In each  $s_t \in \mathcal{S}_t$  we undertake the action  $a_t^*$  and simulate the transition to the following state  $s_{t+1}$ . The reward-state pair  $(s_t, r(s_{t+1}, a_t^*, s_t))$  is saved.
- Based on all state-reward pairs, we fit the parametric function  $f$ , resulting in parameters  $\theta_{i,t}$ .

With the obtained parameters  $\theta_{i,t}$  for all time epochs  $t \in \mathbf{T}$ , we are able to determine our best approximation of the expected value of each individual state and thus replicate the optimal decision making.

### 2.2.4 Utility

In many decision-making situations, rational decision-makers do not behave in a way that their decisions would maximize the expected nominal monetary value.

One of the simplest examples used to demonstrate this behavior is given by Bacci and Chiandotto [4]. Imagine an individual is given a choice, either to get 500 \$ right away or to gamble for 1000\$ in a fair coin toss. A rational decision-maker, driven only by his actions' expected value, would be indifferent to the two choices. However, most people tend to take a certain amount of 500\$, implying that the gamble's perceived value is lower than 500\$.

This effect and its implications become more understandable for substantial sums of money. There is a little difference for an average human in obtaining 10M USD and 20M USD in a fair coin toss. The change in his quality of life will be almost the same and presumably positive. However, in one case, the benefit is certain and in the other case, there is only a 50% chance of win.

<sup>3</sup>Usually based on the model of state distribution in time  $t$ .



Another interesting example of the non-linear gain perception of individuals is the famous St. Petersburg paradox first formulated by Bernoulli in 1738 [8]. This paradox is illustrated on a game with the following rules.

A fair coin is being repeatedly tossed until the result is “tails”, say in  $n$ -th toss. Then the reward for the participant is driven by the equation  $2^{n-1}$ , where for example the first appearance of “tails” in fourth toss rewards the participant with 8 units of currency.

The expected value of the proposed game is infinite. However, it is shown that people would seldom pay more than 25 USD to play it. Interestingly, that this amount corresponds with the assumption that the counterparty does not possess an infinite amount of money but rather a more reasonable amount of 16M USD [32].

By these two examples, we demonstrate that real decision-makers must, in some cases, decide based on something different than the expected value. Building on the extension of the first example, we say that decision-makers maximize their well-being measured in the utility of the given monetary rewards.

This relation between the perceived utility and monetary value is formalized by the utility function, which existence for rational decision makers is conditioned on a set of axioms (for more details, please see [4]).

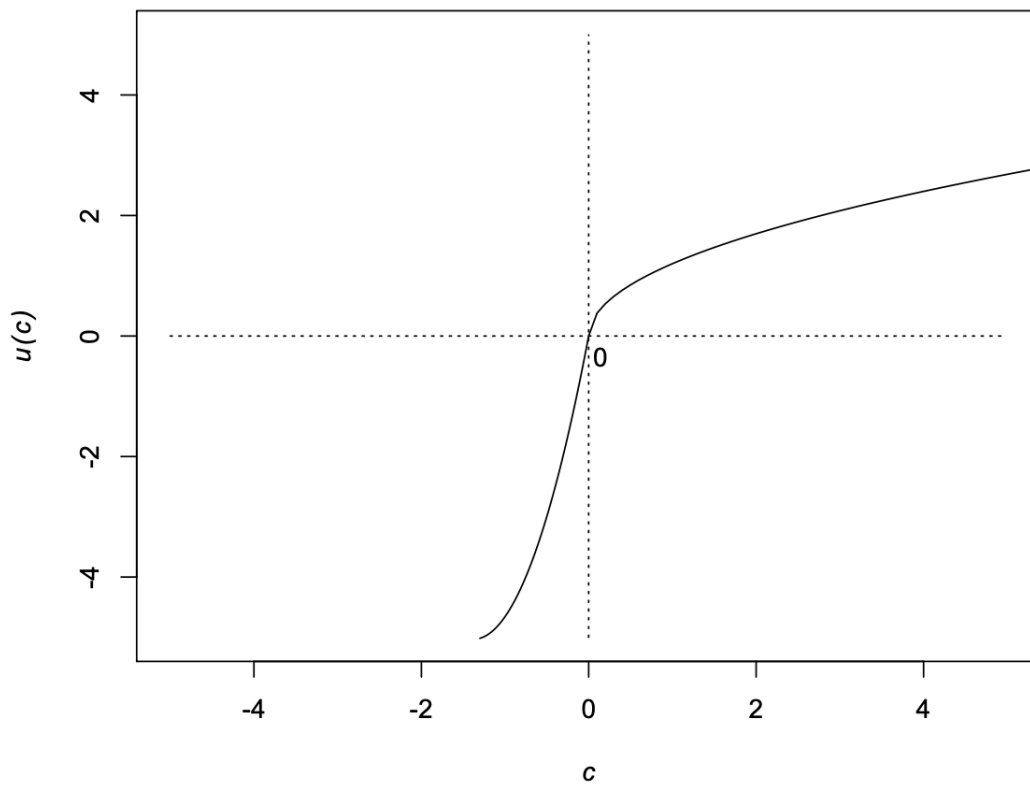
**Definition 2.7.** (*Utility function*) *The utility function is a real value function, that, in accordance with the decisional setting, represents the decision maker’s preferences. The maximum well-being of the decision maker is achieved through the action corresponding to the maximum of the utility function.*

Based on the shape of the utility function, we can define three classes of decision-makers:

- **Risk-averse** decision-makers (the majority of the population) have concave utility functions. They value uncertain monetary gain lower than its expected value.
- **Risk-neutral** decision-makers have linear utility functions. They value uncertain monetary gain exactly as its expected value.
- **Risk-seeking** decision-makers have convex utility functions. They value uncertain monetary gain more than its expected value.

The examples above illustrate that the majority of people are risk-averse. This statement is supported by many publications (for example [22]). An individual’s utility function can be obtained from a questionnaire by an algorithmic approach that ensures the individual’s consistency of responses [4].

Another relevant fact with the perception of utility is the asymmetry in human response to gains and losses. The graphical expression of this asymmetry can be seen in fig. 2.1.



**Fig. 2.1.** An example of utility function, where  $c$  represents a monetary gain and  $u(c)$  its utility. Since the function is not concave or convex on the complete interval of gains, this decision-maker cannot be labeled as any of the three classes defined above[4].

## Chapter 3

# Valuation - state of the art

This thesis is built on two main theoretical pillars. The first is the valuation principles of corporate finance [7], which culminates in real option analysis (ROA), and the second is the statistical decision theory (SDT) [4]. In this section we focus on the basic review corporate finance terminology.

**Definition 3.1. (*Process*)** A process is understood as the production of goods, purchase and trade of goods or services, driven by the supply of inputs and demand for outputs [28].

**Definition 3.2. (*Project*)** A project is defined as a sequence of actions that serve as implementation or innovation of a process, purposefully allocating existing sources to increase the economic value of a process in question.

**Definition 3.3. (*Free cash flow*)** The incremental effect of a project on the firm's available cash is the project's free cash flow [7].

**Definition 3.4. (*Economic value*)** An economic value of a project is understood to be in the future free cash flows. In this thesis, the economic value of a project is the amount of cash to which the investor is logically indifferent to having in comparison to the future cash flow vector.

**Remark.** The indifference and time value of money will be further discussed in section 4.3.2. The theory of net present value (NPV) [7] can be used as a simplification.

**Definition 3.5. (*Optimal Project*)** The goal of each project is to increase the economic value of a process. A project, if such exists, is called an optimal project if the additional economic value is maximal given the set of possible projects.

**Net present value** Net present value is an industry-standard valuation metric used in capital investment. It is simple, and it reflects the time value of money by exponential discounting of the future cash flow by a constant discount rate  $r$  (usually a risk-free interest rate):

$$NPV(c) = \sum_{t \in T} \frac{c_t}{r^t}, \quad (3.1)$$

where  $c$  is a cash flow vector with its elements  $c_t, t \in \{0, 1, \dots, |c|\}$ .

This basic form of the NPV metric is a first approximation of the real current value of cash flow vector  $c$ , meaning that its descriptive power might be limited.

In this thesis, we want to challenge the problem of constant discounting of positive and negative cash flow, which implies that the investor is able to invest in risk-free assets at the same rate at which he is able to borrow money.

We challenge this problem in the next section by presenting the *present cash equivalent* (PCE) metric.

**Present cash equivalent(??)** <Without this paragraph this section might look good> The ability to borrow money for a given interest rate  $r_b$  is individual for each investor and naturally different from the risk-free interest rate  $r_r$ . This asymmetry is not reflected in the standard NPV metric, and thus we come up with our own metric, which we call present cash equivalent, PCE.

The notion of PCE is based on an investor's logical indifference to a vector of future cash flow and some current amount of money.

For the purpose of PCE, let us also prepare the notion of the so-called responsible manager (RM). RM's task is to reinvest the positive balance (with the rate  $r_r$ ) and borrow more funds in case of a negative balance for the rate of  $r_b$ .

We represent this behavior by the RM function, where the constant rates are not explicitly denoted but rather implicitly assumed:

$$RM(b) = \begin{cases} b \cdot (1 + r_r) & \text{if } b \geq 0 \\ b \cdot (1 + r_b) & \text{otherwise} \end{cases} \quad (3.2)$$

Then, based on the assumption of completely dynamic loans<sup>1</sup>, we model the behavior of  $RM$  on a given cash flow vector  $C = \{c_0, \dots, c_t\} \ t \in \mathbb{N}$  as:

$$RM(C) = RM(\dots RM(RM(c_0) + c_1) \dots + c_t). \quad (3.3)$$

The amount of  $RM(C)$  represents the amount of money at time  $t$  obtained from the cash flow vector  $C$  and logical money management. Then, the PCE(C) can be finally derived as:

$$PCE(C) = \begin{cases} \frac{RM(C)}{(1+r_r)^t} & \text{if } RM(C) \geq 0 \\ \frac{RM(C)}{(1+r_b)^t} & \text{if } RM(C) < 0 \end{cases} \quad (3.4)$$

which represents the logic that having PCE(C) at time  $t = 0$  would result in the same outcome at time  $t$ . Thus, a manager should be logically indifferent to obtaining  $PCE(C)$  now and  $C$  by its parts at different times.

**Remark.** Based on the straightforward derivation algorithm of PCE, we argue that for each FCF vector, a unique PCE equivalent always exists.

**Risk averse investors** According to the observations made in [7], there is a positive correlation between the volatility of an investment and its average profit. This correlation is being explained by the risk aversion of investors, where investors are happy to hold onto a low-volatile investment even though the average returns are lower in the long run.

The phenomenon of risk aversion is also well documented in psychological publications, e.g., [22], where it is observed that people value the uncertain rewards below their expected value.

### 3.0.1 Financial option valuation

There exist two types of financial options, calls and puts.

**Definition 3.6.** (*Call option*) A call option gives its owner the right to buy stock (or a commodity) at a specified exercise or strike price on or before a specified maturity date. If the option can be exercised only at maturity it is known as European call, in other cases the option can be exercised at any time before maturity and it is then known as American call [12].

<sup>1</sup>Meaning that the bank will lend the investor money at any time and accept repayments at any time, all with the same interest rate.

**Example 1.** *One share of Tesla stock is traded at \$ 700 at the moment (30.5.2021). An European call option with a strike price of \$ 2500 with a maturity date of 1.1.2022 costs \$ 1.25. An investor buys this option and two things can happen on 1.1.2022. First, more probable case is that the Tesla stock price will be lower than \$ 2500 and thus, exercising the option (buying the stock at \$ 2500) would be unprofitable. Investor doesn't do anything and he lost \$ 1.25. In a second, less probable case, the stock price rises to  $x > \$2500$ . In this case, the investor will use his right to buy the stock for \$ 2500, which he is able to immediately sell on the market and pocket the difference  $x - 2500$  USD.*

**Definition 3.7.** *(Put option) Similarly to call options, which allows the investor holding them to buy the stock or the unit of a commodity on or before the maturity date, the put option allows the investor to sell for a given exercise price. American and European puts are again distinguished by the possibility to exercise before or precisely on the maturity date.*

**Example 2.** *One share of Tesla stock is traded at \$ 700 at the moment (30.5.2021). An European put option with a strike price of \$ 150 with a maturity date of 1.1.2022 costs \$ 1.25. An investor buys this option and two things can happen on 1.1.2022. First, more probable case is that the Tesla stock price will be higher than \$ 150 and thus, exercising the option (selling the stock at \$ 150) would be unprofitable. Investor doesn't do anything and he lost \$ 1.25. In a second, less probable case, the stock price drops to  $x < \$150$ . In this case, the investor is able to buy the share on the market for  $x$  and exercise the option (selling the share for \$ 150) pocketing the  $\$150 - x$  difference.*

As outlined in the first chapter, the idea of options and option trading dates back to 17th century. However, the motivation for a proper option valuation technique came with the increased adoption of derivative trading after WWII.

The famous 1973 article from Black and Scholes [9] describes today's standard tool in the financial option valuation - the BSM model.

**Remark.** *The M in BSM model stands for Merton, who expanded on the article [9] with his article [30] and received the Nobel price together with Scholes.*

In what follows, we will present the BSM model in the form of a theorem [11]. In addition, we will offer an opinion that should summarize the idea of the BSM model in few sentences.

**Theorem 1.** *(BSM model) The Black-Scholes-Merton option valuation model says that if the following list of assumption is satisfied:*

- *risk-free interest rate and volatility<sup>2</sup> of the underlying asset are constant;*
- *the underlying asset pays no dividends and its price is continuous;*
- *the asset price evolves according to a log-normal process;*
- *the markets are efficient - the no-arbitrage principle holds;*
- *the option is of the European style;*
- *there are no commission or service charges;*
- *market is perfectly liquid<sup>3</sup>;*

---

<sup>2</sup>A variance of asset price understood as a random variable.

<sup>3</sup>The investor is able to buy and sell any amount of the asset at any time and as frequently as he wishes to.

then based only on the knowledge of time to maturity  $T$ , option's strike price  $K$ , the current price of underlying asset  $S$  and its volatility  $\sigma$  the value of a call option can be computed as:

$$C = S N(d_1) - PV(K)N(d_2), \quad (3.5)$$

where  $PV(K)$  is the present value of a strike price  $K$ <sup>4</sup> and  $N(d)$  is a cumulative normal distribution, the probability that a normally distributed variable is less than  $d$ . Value of  $d_1$  and  $d_2$  are then defined as:

$$d_1 = \frac{\ln(S/PV(K))}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2}, \quad d_2 = d_1 - \sigma \sqrt{T} \quad (3.6)$$

**Remark.** The option's value is positively dependent on the volatility and the time to maturity. An increase in these parameters leads to a higher option value. On the contrary, the rise in the current stock price or strike price of the options lowers an option's value.

From the combination of remarks to the BSM model from authors like [11] [38] we formulate our own remark of what does the BSM model represent.

**Remark.** The core of the BSM model, assuming that "smooth" conditions hold, is that we are able to derive the missing parameter  $\mu$  of the asset's log-normal price model (known  $\sigma$  is an assumption). Building on this, the value of an option is then the expected value of the maximum of the difference between the strike price and the realized price of the asset and zero<sup>5</sup>, discounted by the risk-free interest rate.

Finally to illustrate the usage of BSM model we present an option valuation example from [38].

**Example 3.** Suppose that a stock's current value is 30, the risk-free interest rate 8% and the volatility of the stock is 0.2. Assume that all assumptions for the BSM model hold. Then the call option with strike price 34 that expires in three month has a value of 0.24.

### 3.0.2 Real Option Valuation

As outlined in the first chapter, the theory of real option analysis (ROA) was born after the boom in publications about financial option valuation in the 1970s. The first ideas represented by Myers [31] are of a philosophical nature - options (ability to make project changes) add value to the project.

Many publications were published on the ROA topic since. Through our studies of the state of the art, we have identified three classes of authors that differ by the level of analogy with the BSM model.

**No analogy** The first class is the class of the ROA founder, Myers. This class understands the term real option analysis as a useful lens for looking at the project valuation. Authors like Kassas [25] and Guerra [20] accentuate the value of further managerial decisions, but the valuation strategy they use is NPV with scenarios (so-called decision tree analysis DTA).

**Partial analogy** The second class of authors takes advantage of the financial option valuation's core property, which is the no-arbitrage principle. Based on this principle and further assumption of replication portfolio existence, this class of authors, represented by, e.g., Guthrie [21], Thoma [42] and Ryu

<sup>4</sup>Price of a bond paying  $K$  on the expiration day of the option.

<sup>5</sup>Since when the option is not realized no further loss occurs.

[39], derives so-called risk-neutral probabilities, which are then used for modeling of project's internal variable of the cash flow functions<sup>6</sup>.

Because we find this type of approach to real options the best and because we build on and respect the work of Guthrie [21], **this approach is the one considered as representative of the term ROA.**

Another notable author in this class is Vollert [45], who goes deep into detail building a modeling framework implementing complex conditional options. Vollert's publication is very advanced, using, for example, stochastic differential equations, which might be an obstacle for practitioners and real-world applications.

**Full analogy** The final class of the authors understands project valuation with real options as a complete analogy to the BSM model for valuation of financial options. This class of authors is predominantly represented by voluminous economic textbooks, e.g., Berk and DeMarzo [7], Brealey [12], or Crundwell [17]<sup>7</sup>.

A complete analogy means to identify all parameters of a financial option with parameters of given investment. For example in [7] the following identification table is presented:

Financial option	Real option
Stock price	Current market value of asset
Strike price	Upfront investment required
Expiration date	Final decision date
Risk-free rate	Risk-free rate
Volatility of stock	Volatility of asset value
Dividend <sup>8</sup>	FCF lost from delay

Table 3.1: Identification of financial option parameters as parameters of real option (investment opportunity). [7].

Another example published by Qu  lin [35] describes a telecommunication company valuation with the same one-to-one identification of BSM model parameters.

By focusing on the complete analogy, the authors of this class strictly limit the application scope of their approach. One of the problematic assumptions (that is in the partial-analogy class solved by the CAMP model<sup>9</sup>) is that there exists a market tradeable replicating portfolio of the asset we want to evaluate. Another limitation is that this approach considers only one possible action, which is usually to invest in the project now or later<sup>10</sup>.

In what follows, our understanding of ROA will be based on the one presented by Guthrie. This decision is based on the exceptionality of his publication [21]. A rigorous definition of ROA as we will understand it will be presented in the beginning of the following chapter.

<sup>6</sup>Where the binomial model of evolution is the most frequently used.

<sup>7</sup>Crundwell also discusses the partial analogy approach in detail.

<sup>8</sup>For the BSM model with dividends.

<sup>9</sup>Capital Asset Pricing model - for details please see [21].

<sup>10</sup>Timing option in Guthrie's terminology.





## Chapter 4

# Project Valuation Revisited from the Statistical Decision Theory Perspective

In this chapter, we develop the core idea of this thesis. The idea is to take the valuation problem as defined in ROA, reframe it as a sequential decision problem and then use the statistical decision theory to solve it while preserving the business-specific concepts of the project valuation, such as time value of money and risk aversion of investors.

We begin by clarifying the meaning of the term project valuation as is used in the publications of the ROA field, and we identify the main limitations of this approach.

Next, we focus on the identification of the project valuation in terms of the SDT framework. We define all the relevant sets and functions to be able to talk about project valuation as a structured problem of decision making under uncertainty.

The remainder of this chapter is reserved for the incorporation of the business-specific concepts to the model, namely the time value of money and the risk aversion of investors.

### 4.1 Project Valuation - Problem Definition

To be able to talk about the project valuation rigorously, we need to define what a *project* is and what we mean by its *valuation* in ROA. The inspiration for these definitions comes from examples and used rhetoric in the ROA publications, namely [21], [45] and [3].

None of the books that we have studied goes into detail to define a project as a collection of mathematical constructs, as, for example, SDT does with MDPs. Guthrie in [21] opens with three initial examples of a project and in each chapter adds new real investment opportunity. This investment opportunity is presented in such a way that it is clear what the project is, what its parameters are, and what metric is optimized by the investor.

In other books like Vollert's [45] and Kulatilaka's [3], a definition of a project is also not given. Rather, it is assumed that the used terms *project*, *capital investment*, or *investment opportunity* are clear.

It is worth noting that a definition of project *valuation* is also not deeply discussed in the ROA books. We feel like the term of valuation is assumed to be clear and is always represented by expected net present value (NPV), which, as we discussed before, is easily understandable but not a very robust metric.

As outlined above, nothing like a clear mathematical definition of a project valuation is presented in the ROA books. However, the used rhetoric is similar, and we strongly believe that the project valuation in the ROA field can be summarized as: "An amount of value that an investor is able to create with

actions that can be considered as a part of one project<sup>1</sup>, measured by the metric of expected net present value with a special non-axiomatic determination of the discount rate."

Now that the position of ROA to project valuation is clearer, we can follow with its interpretation in the SDT framework.

## 4.2 Project Valuation in the Statistical Decision Theory Framework

Trying to solve the project valuation task as a statistical decision problem means, first and foremost to identify all the necessary parts of the SDT framework in the ROA formulations. This is not a particularly hard task given the rather loose definition given above.

After this identification, the standard tool of SDT, dynamic programming (or potentially approximate dynamic programming), can be used to solve the valuation problem.

Solving a valuation problem in SDT means to define it as MDP, which consists of two parts. First, there are three sets: time set  $\mathbf{T}$ , state set  $\mathbf{S}^2$  and action set  $\mathbf{A}^3$ , which describe the structure of the decision-making problem. The second part consists of two functions: transition probability function  $p$  and reward function  $r$ , where  $p$  is responsible for describing the project's stochastic evolution and  $r$  for informing about the value gains in each time epoch.

In the following sections, we will focus on each of these five important building blocks in detail. To better illustrate each of the building blocks, and to prepare our ground for the experiment, an example concerning the valuation of a gas power plant is presented.

### 4.2.1 Time sets

Even though the SDT theory is capable of handling infinite time horizons and continuous-time modeling, these sophisticated formats are not needed for the valuation of real-life projects. The time dimension of a project can be reasonably described by a discrete set with a known finite horizon, which is valid for two reasons.

First is that observing new information and making impactful decisions by the project's manager is not done continuously at all times but discretely after some practical time intervals. No manager changes the course of a project ten times a day<sup>4</sup>.

The second reason is that managers do not think about projects as ever-lasting. Potential profits after a certain time threshold are neglected. This is given either by the finite-lifespan nature of the projects (gas power plant lifespan) or the extreme uncertainty in modeling cash flows (and their equivalent present values) in the far future. To have a good model of the project's cash flow in 100 years is wishful thinking.

Time intervals in our model reflect the frequency of influential management meetings at which the project's course can be changed significantly, e.g., week, month, or quarterly intervals. This notion is further supported by its usage in the ROA publications.

**Example 4.** *Monthly decision time intervals in a duration of gas power plant lifespan, say 25 years. The time set is then  $\mathbf{T} = \{0, 1, 2, \dots, 300\}$ .*

---

<sup>1</sup>Defined as in preliminaries.

<sup>2</sup>This global state set is a union of state set in each time  $\mathbf{S}_t$ .

<sup>3</sup>This global action set is a union of action set in each state  $\mathbf{A}_{s_t}$ .

<sup>4</sup>This thesis does not focus on the individual management of the internal processes of the project. This thesis focuses on managerial decisions that modify the project in a major way.

### 4.2.2 State sets

Defining the state set  $\mathbf{S}$  in a project valuation problem means finding a list of relevant, measurable parameters of both the project and its environment. A state  $s \in \mathbf{S}$  is then a vector of elementary states of such individual parameters.

The state set  $\mathbf{S}$  can be constant, meaning that the same parameters are measured in each time epoch. However, it might also be useful to think about dynamic state sets in time  $\mathbf{S}_t$ ,  $t \in \mathbf{T}$ , where for some particular reasons, the structure of a problem changes in time<sup>5</sup>.

It is worth noting that there are usually some elementary states that are influenceable by the managerial actions and some that are not. This classification is not reflected in our notion.

In our models, each elementary state is understood as a random variable, the probability distribution of which is conditioned on the previous state and the last action taken. This probability is described with the transition probability function  $p$ , which is discussed in detail below.

**Example 5.** *Relevant features for a gas power plant might be, for example, price of gas, price of CO2 allowances, price of power, installed capacity of the plant, or debt to be repaid. The first three elementary states would then be considered uninfluenceable by our future actions, while the last two would not.*

### 4.2.3 Action sets

In SDT structure, the action set  $\mathbf{A}$  is usually understood as an actual set, however in the format of project valuation, we find it better to represent it as an action function, whose parameter is a given state  $s_t \in \mathbf{S}_t$  and output is a set of possible actions  $\mathbf{A}_{s_t} = a(s_t)$ .

The reason for this is that possible managerial actions are most of the time strictly conditioned on the current status of the project itself. Only a small subset of all possible actions might be actually taken in a given state.

In ROA publications, the term options is used to describe possible managerial actions both current and future. Even though we believe that this terminology helps with understanding that the possibility of future managerial action has value, we do not embrace a notion of future actions since it is not consistent with the SDT terminology.

The advantage of the SDT approach in contrast to ROA is that there is no theoretical complication in adding an arbitrary amount of actions of any type (as classified in ROA by Guthrie [21], for example), possibly even conditioned on one another. The only concern that needs to be reminded is that of computational complexity, where a large number of possible managerial actions decreases the ability to compute the valuation in practice.

**Example 6.** *Possible actions for a gas power plant project might be to: build a new block of the plant, run the plant if it has some installed capacity, or wait for a better situation on the market and do nothing.*

We believe that this example clearly shows the dependency of possible actions on a given state and why it is thus better to use the function notion.

With the definition of action function, we have defined the general structure of a project, boundaries within which the project will evolve. Now, we will study the rules that guide the evolution and metrics that measure the value created.

### 4.2.4 Transition probability functions

Given the nature of projects, the evolution from one state to another is stochastic. In this thesis, we want to model the project as MDP, and thus, the probability distribution of the next state, described by

<sup>5</sup>There are more or less relevant parameters to measure.

the transition probability function, is conditioned only on the previous state and the last action taken. In mathematical notation:

$$p(s_t|a_{t-1}, s_{t-1}). \quad (4.1)$$

In this thesis we consider states being discrete or continuous random processes, described by conditional probability mass or density functions, respectively. Furthermore, as we will model each elementary state by a different distribution, we need to be able to compute the overall probability of the future state given the elementary distributions.

To make things simple, we assume individual elementary states to be represented by independent variables, and thus the probability (or probability distribution) of a next state is a product of the elementary probabilities<sup>6</sup>:

$$p(s_t|a_t, s_{t-1}) = \prod_i p(s_t^i|a_t, s_{t-1}), \quad (4.2)$$

where  $s_t^i$  is the  $i$ -th elementary state of  $s_t$ .

**Remark.** *It is possible that some elementary states are fully determined by the managerial action  $a_t$ . Such a corner case does not create a problem for the probabilistic notion above. The new elementary state  $s_t^i$  is realized with probability  $p(s_t^i|a_t, s_{t-1}) = 1$ .*

It is clear that in the majority of real-life projects, determining or estimating this transition probability function is a challenging but crucial task. Decisions will be made based on its values increasing or decreasing the value of a project.

The approach of ROA authors to the modeling of these probabilities varies a lot. Some authors like Guthrie [21] or Amram [3] use the no-arbitrage principle to determine the probabilities of their binomial models. Some authors, like Kulatilaka [27], use the principle of insufficient reasons<sup>7</sup>, where they assign 50% probability to movements in both directions of their binomial models. Some authors, like Guthrie [21] and Vollert [45], also go deeper in statistical modeling of the probabilities.

Some details of how does SDT approach the estimation of  $p(\cdot)$  will be discussed later in section 4.3.1, however, we must note that correct estimation of these probabilities is beyond the scope of this thesis.

**Example 7.** *The evolution of a gas price can be modeled as a lognormal variable, with  $\mu$  being the previous epoch's gas price and  $\sigma$  an estimate from historical data.*

#### 4.2.5 Reward functions

The final part of modeling the project valuation as MDP is the reward function. Its purpose is to assign a numerical value to the state realization  $s_t$  given the previous state  $s_{t-1}$  and last managerial action  $a_{t-1}$ , mathematically:

$$r(s_t|s_{t-1}, a_{t-1}). \quad (4.3)$$

As discussed in preliminaries, the notion of “value” is complicated. In our case of project valuation, this value is understood as FCF. Thus, the first approximation of the entity to be maximized is the expected cumulative FCF, which usually consists of expenses resulting mostly from immediate managerial actions and income, which tends to result from the environment (supply and demand for manufactured products or services) conditioned on a previous action or action sequence.

<sup>6</sup>Combination of discrete and continuous variable as elementary states results in continuously distributed global state (and thus probability distribution).

<sup>7</sup>Even though they do not call it that way.

However, as the economic theory guides us, this approximation is insufficient due to the clear preference in having money now instead of later (time value of money) and the fact that investors do not value uncertain FCF the same as their expected values (risk aversion of investors). Both of these phenomena lead us to adjust the metric that is to be optimized by the optimal strategy.

The details of this alternation, driven by the two phenomena, will be deeply discussed in the following sections 4.3.2 and 4.3.3. For now, let us simply declare that the optimal strategy will be derived from a generalization of expected cumulative FCF, respecting the individual risk preferences, borrowing, and risk-free investment opportunities of the investor.

**Example 8.** *The reward function of a gas power plant is driven by its ability to make money by transforming the gas and CO<sub>2</sub> allowances into electrical power. The initial expenses for building the individual blocks result in extreme negative rewards (driven by the action of building). At the same time, the profits are made as a multiple of installed capacity and the difference of input costs plus fixed costs and the revenue from selling the electricity (conditioned on the action of running the plant).*

This paragraph concludes the basic identification of sets required by the SDT framework. In the next section, we will focus on the solution to the project valuation problem in detail. We will discuss the sources of transition probability function, the actual incorporation of the time value of money, and the risk aversion of investors into the model.

### 4.3 Solution of the Project Valuation as SDT Problem

In the previous chapter, we have focused on the basic structure of a project valuation understood as MDP. In this chapter, we go deeper and focus on the details of the actual solution of such valuation problem.

We begin this section by looking in detail at the estimation of the transition probability function  $p$ . We outline how the SDT can not only incorporate the ideas of ROA but also help with more advanced estimation techniques.

Then, we pursue with the incorporation of the business-specific concepts - time value of money and risk aversion of investors in the form of utility maximization principle and the notion of present cash equivalent (PCE). We focus on implementation details with an accent on applicability by real-life managers.

Finally, the last part of this section addresses the computational complexity problems of classical dynamic programming, proposing an ADP class algorithm identified as the best fit for a project-valuation-style MDPs.

#### 4.3.1 Probability

In this thesis, we focus on real-life projects. Such projects are, by their nature stochastic, and except for some edge cases, the laws guiding the evolution of the relevant parameters are unknown and complex. Our search for the optimal strategy is based on the assumption that we have some model estimating the future paths of the project states, and we act as if this model was the reality. In our case, the model of this evolution is materialized in the form of transition probability function  $p$ , where for example, one of the assumptions is the Markovian property of the states.

There are many ways how to model  $p$ . Let us discuss the techniques used in ROA publications and how they can be translated into SDT terminology. Furthermore, let us also outline the more advanced techniques that SDT can offer in this section.

**Risk neutral probabilities** The idea of risk-neutral probabilities together with the binomial model is the major modeling force in the ROA publications. We observe two levels of its usage, both of which are based on the principle of arbitrage non-existence.

First, simpler approach, used by Ryu [39] or d’Amato [19], adjusts only for the time value of money represented by the risk-free rate  $r_f$ , where the probability of up move is computed as:

$$\pi_u = \frac{Xr_f - X_d}{X_u - X_d}, \quad (4.4)$$

and where  $X$ ,  $X_u$ , and  $X_d$  are the values of the asset price now, after one up move and after one down move. This equation represents the idea that the probability of up move of an asset is such that its expected appreciation is equal to the risk-free interest rate.

The more complex equation, also adjusting for the specificity of the risk of the field we invest in, called risk premium, comes from the capital asset pricing model (CAPM). This model is used by the frequently mentioned Guthrie in [21], but also by other authors, such as Lund [29].

This technique is presented exclusively in the context of binomial models, but it is easy to imagine its usage for their limit case, which is a variable with the Poisson distribution.

**Insufficient reasons** The second widely observed modeling style in ROA publications, observed again mostly with the binomial models, is to assign 50% to both up and down move. This technique can be used on different levels of the model, as the final model [27] or, for example, as a helping distribution modeling a particular supportive distribution as in [21].

It needs to be emphasized that the ROA authors do not use this terminology of insufficient reasons themselves.

**More complex models** In more mathematical publications, we observe more complex models of the future state outcomes, for example, the normal process with dynamic parameters [6] or [1], the mean-reverting process with Poisson jumps [40], or the modeling by a general Itô process [44].

It seems that these models come from authors with more mathematical than economic background and their unifying feature is the rigorous usage of random variables and continuous distributions.

**SDT interpretation** Now we would like to discuss the interpretation of the ROA probability modelling techniques in SDT. The notion of risk-neutral probabilities can be approached with the framework of “expert knowledge“, where the first expert is the one (usually the economist using the CAPM formula) who determines the individual variables like the risk-premium or expected market growth. In accordance with the philosophy of experts, the second expert is the market behaving by the non-arbitrage principle, giving us the equations for risk-neutral probabilities.

The term of insufficient reasons comes from the SDT itself, and thus, it was already interpreted above.

The class of more advanced approach that we have discussed above is easily covered with the SDT too, because of its structure. The outputs in terms of distributions, coming either from the data or again the “expert knowledge“, are easily incorporated into the SDT framework as prior (static or dynamic) probabilities.

**SDT innovation** The portfolio of SDT estimation techniques is much broader than it what was discussed so far. Because probability estimation is not the main focus of this thesis, we will only outline two of the most interesting techniques.

First is the Bayesian modeling, where each time new data are being observed, our probability model is updated by the Bayesian formula. This very influential modeling technique in SDT was not seen in the studied ROA publications, even though there is certainly a space for it.

The second modeling strategy that we want to talk about is the consistent way of information fusion from different sources. This niche part of the SDT, lead by Kárný [24], allows for the incorporation of multiple sources of prior probabilities, for example, multiple experts, data sources, and more.

**Summary** To conclude, we advice to use one of two approaches to the problem of  $p$  estimation. First, when a lot of information about the project is known, we have a strong case for the parameters behaving according to given smooth distributions. Additionally if we believe the market to compactly represent the expert knowledge, we prefer Bayesian updated risk-neutral probabilities.

On the other hand, if the project is truly innovative and there is very little data to base our model on, we prefer to use a combination of expert knowledge and principle of insufficient reasons to determine the priors.

In the end, we leave the decision of the actual modeling to the framework user, where we express the sympathy for simple models, where more “unclear” models, in spite of their better precision, might not be accepted by the investment board making the investment decision. Clearly, we do not advise using advanced SDT modeling techniques like probability distribution fusion.

#### 4.3.2 Time value of money

As outlined in preliminaries, money does not have the same value through time. This concept is one of the most important ones in project valuation and capital budgeting. The approach of the economic theory to this problem is to exponentially discount the future cash with the so-called risk-free interest rate.

In the studied ROA publications, the problem of different borrowing and risk-free investment rate is not addressed. The ROA publications assume that the investor is able to borrow at the risk-free interest rate, which is in reality not the case.

In our modeling, the first approximation of the optimized entity is the expected cumulative future FCF. Now we present the second approximation of the optimized entity, which is original for both the ROA and SDT<sup>8</sup> world.

This second approximation of the optimized entity is defined as the expected present cash equivalent (PCE). The PCE represents the amount of money the investor is logically indifferent to having instead of a vector of future cash flows. As argued in preliminaries, this value is unique and for natural borrowing and risk-free rates always defined.

The following equation represents the update of the Bellman equation 2.12, adjusting the crucial step in the optimal strategy determination:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A}_{s_{t-1}}} \sum_{s_t \in \mathbf{S}_t} P(s_t | a_t, s_{t-1}) PCE \left( r(s_t | a_t, s_{t-1}) + V(s_t) | s_{t-1}^b, r_r, r_b \right), \quad (4.5)$$

where  $s_{t-1}^b$  is the project's balance state in time  $t - 1$ ,  $r_r$  the risk-free interest rate and  $r_b$  the borrowing rate.

**Remark.** *It needs to be clarified that if we want to embrace the ROA discounting, we are able to do so. Such simple discounting is a special case of the PCE notion, with  $r_r = r_b$ .*

<sup>8</sup>SDT theory uses exponential discounting, for example, in models with infinite time sets, however, through our study, we have not encountered the idea of discounting conditioned on a state.

### 4.3.3 Risk aversion of investors

As discussed above, the nature of real-life projects is stochastic. The uncertain evolution of states results in the uncertain FCFs defined by the function  $r$ , which has implications for the manager's decision making.

As discussed in the preliminaries, section 3, the majority of investors is risk-averse, meaning that they tend to value uncertain gains lower than is their expected value.

We believe that this characteristic of investors is important to consider in the valuation of a project. Fortunately, the SDT theory already has a framework for coping with such skewness in reward perception called utility theory.

The third and final approximation of the optimized entity and the corresponding adjustment of the Bellman equation 2.12 is to transform the PCE by the utility function  $\mu(\cdot)$  of the given investor, which can be expressed as:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A}_{s_{t-1}}} \sum_{s_t \in \mathbf{S}_t} P(s_t | a_t, s_{t-1}) \cdot \mu \left( PCE_{t-1} \left( r(s_t | a_t, s_{t-1}) + V(s_t) | s_{t-1}^b, r_r, r_b \right) \right). \quad (4.6)$$

**Remark.** *Even though there are consistent methods for obtaining the utility function of the individual investors, it might be hard to get the individual investor on board with the idea of utility<sup>9</sup>. This does not present a fundamental problem for our valuation technique because we can always use the utility of the risk-neutral investor, which is unique and its usage supported by the lack of bias against uncertain outcomes.*

### 4.3.4 Approximate dynamic programming

As mentioned in the previous sections of this thesis, the classic DP algorithm can be used for the computation of MDP's optimal strategy only with rather small cardinalities of the  $\mathbf{A}$ ,  $\mathbf{S}$ , and  $\mathbf{T}$  sets. This known DP problem is in literature addressed as “three curses of dimensionality” [33].

Real-life projects, interpreted as MDPs, are usually rather complex and hardly-ever fulfill this condition. For example, when even one measured parameter is modeled to come from a continuous distribution, the DP algorithm breaks down not only from the limitation of the actual computational complexity but also theoretically.

It might be clear from the rhetoric of this thesis that our goal is to use the developed valuation technique in practice. That is why we want to address this problem with the goal to make the solvable class of project valuation problems as large as possible. We try to accomplish that with the approximate dynamic programming (ADP) approach.

From the numerous ADP techniques, we have chosen the value iteration with parameter model approximation for two main reasons, both of which originate in project valuation's fundamental characteristics.

First is that the real-life projects tend to have large state spaces (even uncountable), while on the other hand, the action set is usually limited. We cannot ask a manager to choose between 10 000 actions, for example. This argument supports the choice of value iteration over a policy iteration class of ADP.

The second reason is that we usually have a good intuition of what precisely in the given states “make money”. This allows us to build a good approximation function  $f$  with reasonable parametrization, capturing the most important parts of the model.

---

<sup>9</sup>Investor might not have time to answer the utility questionnaire as suggested by [4], or he might be discouraged by the idea of his own skewed perception of rewards.



It needs to be clarified that even though we believe that our approach is generally the best, regarding the mathematical complexity, its precision and clarity, there might be better ADP algorithms for individual projects the reader intends to value.

#### **4.3.5 Summary**

This subsection summarizes the core idea of this thesis.

Firstly we have presented the interpretation of the project valuation problem as a MDP. We have defined all its important parts in detail and offered examples for clarity.

Secondly, we have adjusted the Bellman function 2.12 to respect the business-specific concepts with the notion of PCE (respecting the time value of money) and utility function (the interpretation of the investors' risk aversion).

Lastly, we have advised the best ADP algorithm that copes with the problems of the computational complexity of the usual DP solving algorithm for non-trivial projects that can naturally be used with the adjusted Bellman equation 4.6.



## Chapter 5

# Valuation of Simple I/O Businesses

In the previous chapter, we have presented an algorithmic approach to a project valuation based on the SDT framework and its ideas. Now we want to illustrate the actual usage of this algorithmic approach on a chosen class of projects.

We have chosen the class of projects that can be labeled as an investment in simple I/O business (businesses that can, for the valuation purposes, be described by a simple input-output process). This class is characterized by a large outflow of money at the beginning, which is used for building the business (or its first functional part) and a small but long-term positive future net cash flow driven by the difference in the price of inputs and outputs and the further managerial decisions.

This class choice is supported by its appropriate level of complexity, which allows demonstrating the power of the new algorithm while at the same time not being too complicated. It is also a type of project that is substantially represented in the world of capital investment.

We could write this whole chapter using a general description of the chosen project class; however, we believe that using only one specific representative will result in a clearer picture of the situation.

Our choice of the representative - an investment into a gas power plant - is based on three grounds. First, there is certainly an influence of Guthrie's example of a similar valuation problem [21]. Second, as we will see, this valuation problem has reasonable dimensions that allow for a good presentation of the valuation algorithm. Lastly, the author of this thesis has proven domain knowledge based on his short but intensive work experience in the field of power trading.

In the first part of this chapter, we will describe the valuation problem in detail.

In the second part, we will compare the PCEs of three baseline operational strategies and the optimal one derived from the valuation algorithm. Furthermore, we will study the influence of the increased volatility of prices on the final valuation. From what we have learned about the value of options, the project's value should increase together with the volatility increase.

It needs to be emphasized that the aim of this experiment is to present the valuation algorithm, observe and describe its possible shortcoming and support its viability, not to prove any other theorems or ideas about project valuation.

### 5.1 General Settings of the Experiment

First, let us clarify what we mean by the phrase *investment in the gas power plant*. We are positioning ourselves in the role of an investment analyst of a large utility company<sup>1</sup> whose task is to evaluate the value of building and managing a new gas power plant.

---

<sup>1</sup>Companies that generate electric power usually provide also gas, water, sewage or other basic services.

For simplicity, we are considering building only one or two 200MW blocks, the lifespan of which ends 25 years from now, disregarding the time they were built [15]. We assume that each block's price is 65M EUR, which is a rough estimate based on [13].

We model the power plant to be managed in a monthly pattern. Its power is being sold by monthly contracts at the beginning of each month when the needed gas and CO2 inputs are also modeled to be purchased. This results in certain free cash flows that are assumed to be obtained immediately after taking one of the allowed actions. As was indicated earlier in this thesis, we do not want to go deep into the plant management and its internal processes.

Regarding the project's financing, we assume that the majority of the initial payment will be made by an ideally flexible loan<sup>2</sup> with an interest rate of 6%. We define the risk-free interest rate as 2%. The reality of having non-zero initial funds is reflected in the existence of the corresponding elementary state.

Now that we have clarified the project that we want to value, we can proceed with its precise definition in the MDP format.

### 5.1.1 Time set

As outlined in the introduction of this chapter, we assume that the lifespan of a gas power plant is 25 years, and it is managed on a monthly basis. Thus the time set is defined as:

$$\mathbf{T} = \{0, \dots, 300\}, \quad (5.1)$$

where the epoch 300 is understood as the final epoch, where no actions can be made, and no FCF can be obtained.

### 5.1.2 State set

The state set needs to consist of the smallest number of relevant parameters that enable us to model the process of building and running the power plant and capturing the FCF and its derived metrics for the project.

As such, we identify five elementary states as parameters describing:

- the price of gas -  $s^1$ ;
- the price of CO2 allowances -  $s^2$ ;
- the price of power -  $s^3$ ;
- number of power plant blocks built -  $s^4$ ;
- cash balance of the project (models the debt) -  $s^5$ ;

in the start of each time epoch.

The states of the state set are defined as having a constant length since there is no significant change in relevant parameters through project's lifespan.

Mathematically the state set  $\mathbf{S}$  is defined as:

$$\mathbf{S} = \{(s^1, \dots, s^5) | s^i \in \mathbf{S}^i, i \in (1, \dots, 5)\}, \quad (5.2)$$

where  $\mathbf{S}^i$  represents the limitation of the individual elementary states. These limitations will be in detail discussed now.

---

<sup>2</sup>An ideally flexible loan where we can repay any amount at any time.

For the states representing prices,  $s^i, i \in \{1, 2, 3\}$ , we define:

$$\mathbf{S}^i = \mathbb{R}_0^+. \quad (5.3)$$

The next elementary state,  $s^4$ , represents the number of blocks built, simply:

$$\mathbf{S}^4 = \{0, 1, 2\}, \quad (5.4)$$

The final elementary state, representing the financial balance of our project, is then allowed to have any real value:

$$\mathbf{S}^5 = \mathbb{R}. \quad (5.5)$$

The definition of the state set  $\mathbf{S}$  represents the problem structure, the boundaries within which the simulation of initial investment and further managerial actions will take place.

### 5.1.3 Action function

In this experiment, we consider four managerial actions. They could be understood as two-dimensional, where the first dimension represents the act of running the installed capacity of the plant (if one exists), whereas the second manages the action of building new blocks. However, we have decided to use the following one-dimensional encoding:

- 0 - do not change the current state of the project,
- 1 - run the existing installed capacity,
- 2 - run the existing capacity and build a new 200MW block,
- 3 - build a new 200MW block.

As discussed earlier in this thesis, certain actions are available only in certain states. The state that determines what actions are possible is exclusively the elementary state  $s^4$  describing the number of blocks built.

In the following list, we express the possible action set as a result of the action function in all possible elementary states  $s^4$  with a short explanation of its meaning.

- $a(s^4 = 0) = \{0, 3\}$  - when nothing is built, we can build the first block or do nothing and wait.
- $a(s^4 = 1) = \{0, 1, 2, 3\}$  - when only one block is built, all actions are possible.
- $a(s^4 = 2) = \{0, 1\}$  - when two blocks are built, the building actions are not possible.

### 5.1.4 Transition probability function

The best way to describe the model of evolution from one state to another is to assume the independence of random variables representing the individual elementary states, which allows computing the probability of state transformation as a product of transition probabilities of the individual states:

$$p(s_{t+1}|s_t, a_t) = \prod_{i=1}^5 p(s_{t+1}^i|s_t^i, a_t). \quad (5.6)$$

In our example, there are five elementary states, where the ones representing prices of gas, CO2 allowances, and power are modeled in the same way. The other two elementary states are deterministic but differ in the dependency on the previous state and action. Let us now describe the evolution of the individual elementary states in detail.

**Commodity prices** The prices of gas, CO<sub>2</sub>, and power are modeled by a Geometric Brownian motion, the model used in the BSM model and other publications like [45].

The probability of the next elementary state is thus conditioned only on the value of the previous state as:

$$p(s_{t+1}^i | s_t^i) = s_t^i \cdot \exp\left(\left(\mu - \frac{\sigma_i^2}{2}\right)dt + \sigma_i W_{dt}\right) \quad (5.7)$$

for  $i \in \{1, 2, 3\}$ , where  $dt$  represents a time fraction in years, for us  $1/12$  and  $W_t$  is a Wiener process with probability density function defined as:

$$p_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} \cdot \exp\left(\frac{-x^2}{2t}\right). \quad (5.8)$$

To be able to model the prices, we need to present the variances  $\sigma_i$  and the initial values  $s_0^i$  for all  $i \in \{1, 2, 3\}$ .

The initial prices that we use are inspired by the real prices of commodities that are being traded on the Intercontinental Exchange (ICE). In this thesis, we would like to avoid going into the details of plant efficiency and unit transformation. Thus we present the initial prices as illustrative and already transformed to EUR per MWh produced.

Similarly, the values of individual commodity variance  $\sigma_i$  are inspired by our experience with the ICE and the realization of commodity prices in recent years. However, they still remain only illustrative.

In table 5.1.4, we can see the chosen initial prices and two triplets of volatilities, which are presented due to the second goal of this experiment, determining the influence of volatility on the value of a gas power plant project. The second triplet is chosen simply as a 20% increase of the first.

	Initial price [EUR]	Volatility 1	Volatility 2
Gas	25	0.12	0.144
CO <sub>2</sub>	10	0.10	0.12
Power	37	0.15	0.18

**Installed capacity** The evolution of elementary state  $s^4$  representing the number of blocks built is deterministic and conditioned on the previous action. If the chosen action is  $a = 2$  or  $a = 3$ , representing building a new block, then the elementary state of the number of blocks increases by 1 with probability 1. If the chosen action is  $a = 0$  or  $a = 1$ , this elementary state is not changed with probability 1, mathematically:

$$P(s_{t+1}^4 = y | s_t^4 = x, a_t) = \begin{cases} 1 & \text{if } x = y \wedge a_t \in \{0, 1\} \\ 1 & \text{if } x + 1 = y \wedge a_t \in \{2, 3\} \\ 0 & \text{otherwise} \end{cases} \quad (5.9)$$

**Balance** The last model that needs to be discussed is the one representing the financial balance of the project. This state simulates the actual cash balance driven by the actions of a responsible manager (RM), the assumptions of ideal loans<sup>3</sup> and the possible risk-free investment with a constant interest rate.

Based on two types of monthly interest rates, the one for which the investor can borrow in a bank  $r_b = \sqrt[12]{0.06}$ <sup>4</sup>, and the one of risk-free interest rate  $r_r = \sqrt[12]{0.02}$ , we define the evolution of  $s^5$  as:

$$P(s_{t+1}^5 = y | s_t, a_t, s_t^5 = x) = \begin{cases} 1 & \text{if } y = RM(FCF(s_t, a_t) + x, r_b, r_r) \\ 0 & \text{otherwise,} \end{cases} \quad (5.10)$$

<sup>3</sup>With the availability to repay any amount of money at any time.

<sup>4</sup>We assume the yearly borrowing interest rate of 6%

which means that this elementary state is deterministic, and its value  $s_{t+1}^5$  is computed as a result of responsible managerial actions with the previous balance  $x$  adjusted for the FCF obtained in the previous state  $s_t$  given the action  $a_t$ .

The actual computation of  $FCF(s_t, a_t)$  will be revealed in the next section, whereas the  $RM(\cdot, r_b, r_r)$  function can be found in preliminaries. The model allows for easy interpretation of non-zero initial balance, however, in our experiment, we define  $s_0^5 = 0$ .

### 5.1.5 Reward function

The FCF model in our example, and actually in all projects in the class of simple I/O businesses, is fairly straightforward.

First, we account for the fixed price of maintenance  $C_m$ , which in our case is  $C_m = 6$  EUR/MW of installed capacity for each hour.

Then, conditioned on the action of running the business, we account for commodity input costs (here gas and CO<sub>2</sub>). Finally, we add the profit from selling our product on the market, and the computation for FCF is complete.

The mathematical expression of the sentences above can be presented as:

$$FCF(s_t, a_t) = s_t^4 \cdot \left[ -C_m + I_{\{1,2\}}(a_t) \cdot (s_t^3 - s_t^2 - s_t^1) \right] \cdot 200 \cdot h_m, \quad (5.11)$$

where  $h_m = 720$  is a constant representing approximation of hours in month, not accounting for the changes in month lengths.

**Remark.** *As discussed earlier, the optimal strategy is not being optimized for the expected cumulative FCF, but rather each action optimizes the expected utility of the present cash equivalent in the current state.*

### 5.1.6 Utility function

The chosen utility function  $v$  for this example was chosen as:

$$v(x) = \begin{cases} -(-x)^{0.9} & \text{if } x < 0, \\ x^{0.85} & \text{otherwise.} \end{cases} \quad (5.12)$$

This choice represents slight risk aversion of the investors, with an asymmetric perception of losses and gains. Its purpose is only illustrative as the details of utility function creation, its meaning, and derivation are out of the scope of this thesis.

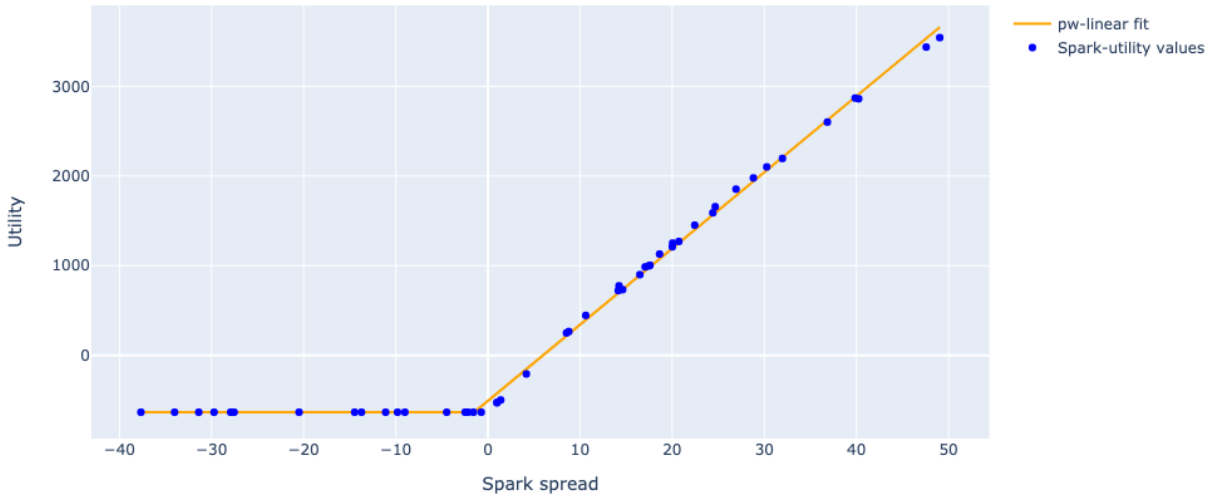
## 5.2 Approximate Dynamic Programming

In preliminaries we have discussed the algorithm for modelling the value function with some general function  $v_t(s) = f(s, \theta_{i,t})$ . Now, as we have defined our problem, we are able to outline the specific model for our case.

Based on the initial observations of state-utility pairs coming from the performed simulation (see algorithm 1 step 10 and figure 5.1) and the intuition that the FCF function, equation 5.11 gives us, we have decided to model the value function with a model of three piecewise linear models with one breaking point  $x_{i,t}, y_{i,t}$ :

$$v_t(s) = \sum_{i=0}^2 I_i(s^4) PW(s^3 - s^2 - s^1, k_{i,t}^1, k_{i,t}^2, x_{i,t}, y_{i,t}), \quad (5.13)$$

where  $I_i$  is an identifier function of installed capacity  $s^4$ , and  $k_{i,t}^1, k_{i,t}^2$  represent the slopes of the piecewise linear function model  $PW$ . For a better illustration, we present an example of such model fitted to the real data in figure 5.1.



**Fig. 5.1.** Example of a pw linear fit.

This model effectively reflects only two variables, the installed capacity of the power plant, reflected by the choice of three models and generalized "spark spread"  $s_t^3 - s_t^2 - s_t^1$ , which reflects how much money is being made by running the plant with the current costs of inputs and the market price of power.

The value function in each time epoch  $t$  can thus be represented by twelve variables  $k_{i,t}^1, k_{i,t}^2, x_{i,t}, y_{i,t}$  for  $i \in \{0, 1, 2\}$ .

To determine these parameters for our example, we will use the ADP algorithm of value iteration, which we describe in general by a pseudo-code in Algorithm 1. The important details of this algorithm are then discussed in the individual paragraphs.

**Last VF** For the consistency in notation, there is a need for the definition of the value function in the last epoch, where no action is possible anymore. Since the value function represents the metric of expected reward in the future, its value is 0. In our case, the  $v_{300}$  is represented by the same model shown in equation 5.13 with all parameters equal to 0.

**State sampling** In the third step of our algorithm, we define a state sample. This state sample serves the purpose of a reasonable coverage of the uncountable state space with a finite number of values. In this thesis, we create the sample by making samples of individual elementary states, which are then randomly put together, creating a random realization of a global state.



**Algorithm 1** ADP value iteration algorithm**Require:**  $v_{300}(s)$ 

- 1: Prepare empty list  $L$  for vf model parameters
- 2: **for**  $t \in (299, 298, \dots, 0)$  **do** ▷ Backward epoch induction
- 3:   Prepare state sample  $S_t$  ▷ See paragraph state sampling
- 4:   Declare empty list of state utility pairs  $l$
- 5:   **for**  $s \in S_t$  **do**
- 6:     **for**  $a \in a(s)$  **do** ▷ All allowed actions in given state
- 7:       Evaluate the expected utility of an action  $a$ ,  $u_a$  ▷ See expected utility paragraph  
       given  $v_{t+1}$  approximation
- 8:       Determine  $\max_{a \in a(s)} u_a = u$ .
- 9:       Save  $(s, u)$  pair in a list  $l$ .
- 10:   Fit the  $(s, u)$  pairs from  $l$  to the model described by equation 5.13
- 11:   Save the fit parameters  $k_{i,t}^1, k_{i,t}^2, x_{i,t}, y_{i,t}$  for all  $i \in \{1, 2, 3\}$  in a list  $L$
- 12: **return**  $L$

In the following table, we describe the distributions out of which the elementary realizations are taken.

Elementary state	Distribution
Gas price	Uniform (0,30)
CO2 price	Uniform (0,40)
Power price	Uniform (10,80)
Number of blocks	$p(s^4 = 0) = 0.3$ $p(s^4 = 1) = 0.35$ $p(s^4 = 2) = 0.35$
Balance	Uniform (-60M, 60M),

All the random variables are generated by the python library NumPy, and the sample size was chosen as  $|S_t| = 100$ .

**Expected utility** In step 7 of algorithm 1, we want to assign individual action  $a$  in state  $s$  the expected utility. This assignment is made with the help of the equation 4.6, respective its part:

$$\int_{s_t \in S_t} p(s_t | a_t, s_{t-1}) \mu \left( PCE_{t-1} \left( r(s_t | a_t, s_{t-1}) + V(s_t), s_{t-1}^5, r_r, r_b \right) \right) ds_t, \quad (5.14)$$

which was adjusted for our continuous case, changing the sum for an integration.

To compute this expression, we use a simple numerical integration technique. Based on the state in which we are in  $s_t$  and the transition probability function  $p(s)$  defined above, we simulate chosen number of state evolutions, here  $n = 100$ . Then the results of the numerical integration is the average of the individual utilities of PCEs of the realizations  $s_{t+1}$ .

**Model fitting** In step 10 of our ADP algorithm, the data from list  $l$  are being fitted by the model described by equation 5.13. We do this in reality by fitting three individual subsets, one for each of the plant states, by the pw linear function with one breaking point. The individual fits are done with the help of python library lmfit, which is using the least square evaluation metric.

### 5.3 Optimal Strategy Performance

Now that we have the estimation of the value function in our hands, we are able to value the project. The only thing that we need to do is to insert the chosen initial state  $s_{init} = (25, 10, 37, 0, 0)$  defined above by parts into the model as  $v_0(s_{init})$ , which gives us the result of 349M EUR.

Now, we want to verify the sensibility of this result by a simulation of the actual decision-making process and comparing it with some baseline strategies. Let us first define these strategies and then the simulation algorithm according to which we will compare them to the optimal strategy.

#### Baseline strategies

- Strategy  $B_1$  builds two blocks of the powerplant in time epoch 0 and 1 and then runs them no matter all the other factors.
- Strategy  $B_2$  builds two blocks of the power plant similarly to strategy  $B_1$ , but it runs the plant only if the prices are favorable, meaning  $s_t^3 - s_t^2 - s_t^1 - C_m > 0$ .
- Strategy  $B_3$  does not build the blocks right away but waits for more favorable market states than the  $s_{init}$  provides. It builds a new block only when the generalized spark price rises above the arbitrarily chosen amount of 40 EUR ( $s_t^3 - s_t^2 - s_t^1 - C_m > 40$ ). The rule for running the installed capacity remains the same as for the strategy  $B_2$ .

**Simulation algorithm** In this paragraph, we will describe the simulation of the decision making upon which we compare the performance of our optimal strategy and the baseline strategies  $B_i, i \in \{1, 2, 3\}$ .

---

#### Algorithm 2 Strategy performance algorithm

---

**Require:** Strategy  $B$ ,  $r_r$ ,  $r_b$

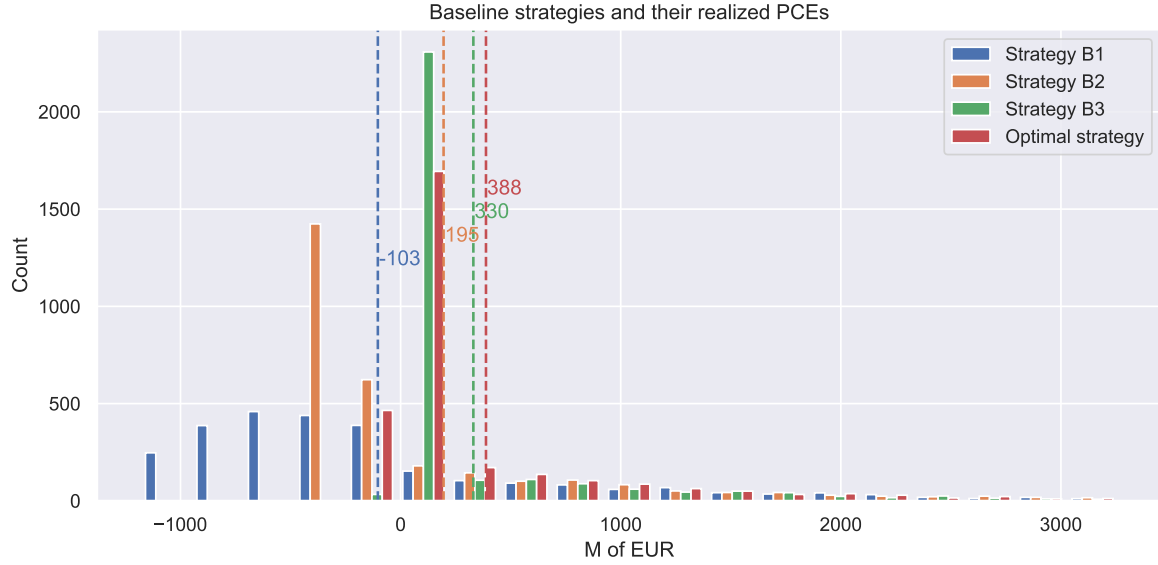
- 1: Define initial state  $s_0 = (25, 10, 37, 0, 0)$
  - 2: **for**  $t \in \{0, 1, \dots, 299\}$  **do**
  - 3:     Determine action  $a_t$  in state  $s_t$  according to strategy  $B$  ▷ See determine actions below
  - 4:     Compute  $s_{t+1}$  realization from the transition probability function  $p$  and  $s_t$
  - return**  $\text{PCE}(s_{300}^5, r_r, r_b)$  ▷ The PCE of final balance of the project
- 

**Determining actions** The general step 3 in the algorithm differs based on the type of strategy. The actions of baseline strategies are clearly defined, however, the action of the optimal strategy computed by the ADP algorithm needs to be computed again.

The action taken in the case of the optimal strategy is determined as the action with the highest expected utility, computed as a numerical approximation of the equation 5.14 with 100 numerical integration samples as before.

#### 5.3.1 Results - initial setup

Now, we are able to use algorithm 2 to compare the optimal strategy to the baseline strategies. We do this by running the simulations multiple (3000) times and comparing the average gains (in terms of PCE) of each strategy, a Monte Carlo approach. The results of the simulations can be seen in figure 5.2.



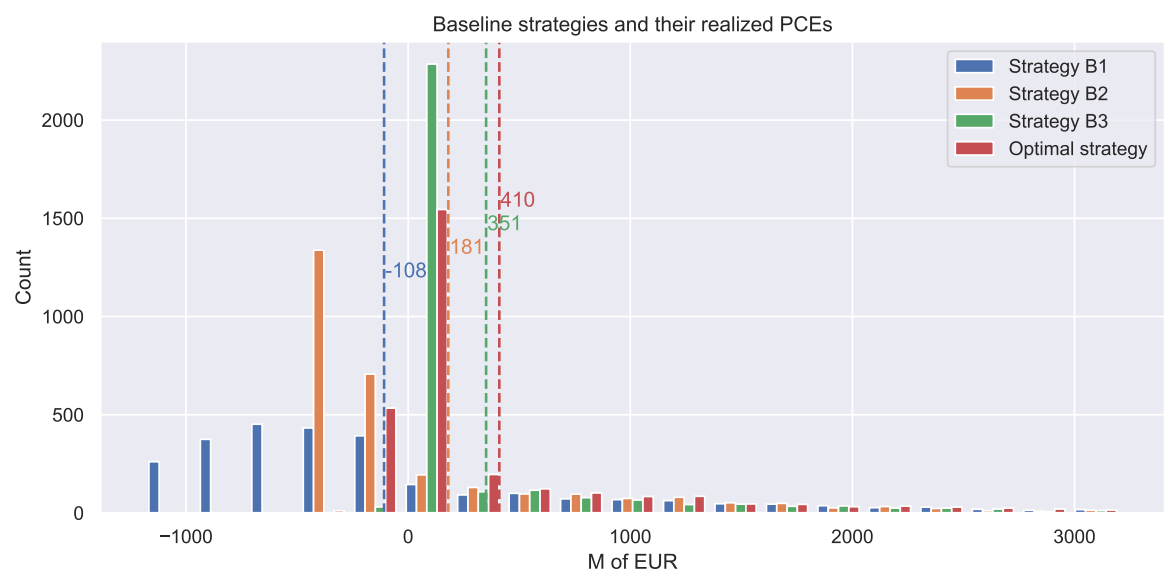
**Fig. 5.2.** Comparison of average PCE equivalents of individual strategies, realization histograms and averages with initial settings.

Setup Strategy	Baseline 1	Baseline 2	Baseline 3	Optimal strategy
Initial setup	-152M	116M	229M	317M
Increased volatility	-110M	183M	323M	406M

Table 5.1: Comparison of PCE for different strategies and settings.

### 5.3.2 Results - increased volatility

The second part of the simulation is exactly the same as the first part, the only change is in increase of all price volatilities by 20%, as can be seen in table 5.1.4. The results of this simulation can be seen in figure 5.3, while the comparison of individual results for both cases can be found in table 5.1.



**Fig. 5.3.** Comparison of average PCE equivalents of individual strategies, increased volatility.

## Chapter 6

# Discussion

We believe that in the presented experiments, we have shown the usage of the SDT-based valuation technique algorithm on an example with reasonable complexity. We have managed to cope with the uncountable state space caused by the assumption of continuous prices and also with the modeling of approximate value functions guided by the theory of ADP.

The business-specific concepts that we wanted to preserve from the field of capital investment bring after the experiment mixed feelings. The notion of PCE and its usage in the experiment seems to make logical sense. However, the notion of utility and risk-aversion of investors might make better sense in the evaluation of the project as a whole. The usage of utility will be discussed in more detail below.

### 6.1 Value Function Approximation

In the first attempts to construct the experiment for this thesis, we came up with a much more complex example than the one finally presented. There were nine elementary states concerning, for example, the government's support of renewable sources of energy, which influenced the future volatility of the power prices. We also firstly introduced actions like mothballing or selling the plant for salvage value. From the study of value function approximations, it seemed like that it will be enough to use only the linear model and choose good basis functions. Nevertheless, we were wrong.

The complexity and non-linearity of the value functions can be seen in the final simplified example as well. It is clear now that the option-like pattern appears, but the future users of this algorithm need to keep in mind is that the complexity of the project model will most likely be reflected in the complexity of the value functions.

In our case, three piecewise linear models were enough to cover the state space reasonably, but for more complex models, there might not be such clear patterns as we observed in our example.

The final choice of a simpler model was motivated by a concern that a more complex value function could take too much of the reader's attention from the main message, which is the illustration of the valuation algorithm itself.

We would like to raise a concern about the possible future adoption of the algorithm from the capital investment companies due to the complexity of the value function modeling for more real-life complex investments.

## 6.2 Usage of Utility

In this thesis, we wanted to preserve the notion of risk-averse investors. We did this by making the optimal decision strategy the one that maximizes the expected utility of the decision-maker (manager of the project) in each decision made. However, when we thought deeper about the global view of investors, the real decision-makers, the incorporation of the risk-aversion through the utility function might have been done differently.

We have used the utility function for decisions on the micro-level when the project is being further managed. However, in reality, the group of people that make the investment decision does not usually further manage the project.

On the macro level, the final valuation of a project is usually used for comparison to other investment opportunities, and we believe that this is where the utility function should come into the picture.

When looking at the results of our simulation, we, for example, see that the baseline strategy B3 is more conservative than the optimal one we derived from our algorithm. Here, we can easily imagine a very risk-averse investor who would prefer the results of strategy B3 over the optimal strategy we have derived.

Thus, we believe there might be two utility adjustments, first on the micro-level, where the plant manager optimizes the utility of each action. The second utility adjustment for the valuation is for the macro level, where the board of decision-makers will compare individual projects.

## 6.3 Computational Complexity

In section 2.2.2 we have talked about the computational complexity of dynamic programming. For our setup of the experiment (with uncountable state space), the DP approach is not even theoretically possible, but we would still like to address the computational complexity and the challenges our algorithm faces.

We needed to compute three models for each of 300 time epochs. For each model, we needed to have 100 state samples to cover the state space reasonably. For each state, we needed to numerically compute the expected utility, which was in the form of integral, to which we have also used 100 integral samples. This resulted in 90 000 state-utility pairs and 9M of integral samples created as an evolution of state with a length of 5. The computations took several hours on a 2-core device with 16GM of RAM, where it needs to be said that the code was optimized without special care.

This is to say that more complex valuation problems might also bring with them the need for either faster hardware or better techniques in numerical approximation of integrals and/or more robust models for the value function approximation.

## 6.4 Experimental Results

Even though the main focus of the experiments was to present a reasonable illustration of the valuation algorithm we can also see the individual ROA narratives in the results.

First is the notion that the value of project control in highly uncertain projects is large, since it allows to limit the "downside risk" while keeping the "upside". We can see this being confirmed by the higher expected PCE values for the second part of our experiment, where we increased the volatility of commodity prices.

By observing the shapes of histograms of different strategies, we can also observe the narrative of ROA that "options have value". The baseline strategy B1 factually represents no options in the ROA sense. We simply build the plant and run it until the end of the observation period.

The baseline strategy B2 represents the option not to run the plant when the market prices are not favorable. We can see that this option has a significant value, bringing the value of a given project from negative values to positive.

The baseline strategy B3 represents the combination of the time option (waiting for favorable commodity prices) and the option from strategy B2. We see that for our initial state, this option also brings significant value to the project.

The final remark to the experimental results is that our intuition says that the baseline strategy B3 could be a very good approximation for the optimal strategy derived by our algorithm with a better threshold than 40 EUR. The idea is that for our simple experiment, the optimal strategy simply finds the best thresholds to build the first and second block. However, in comparison to B3, the optimal strategy presumably also considers the epochs left until the 300 threshold.





## Chapter 7

# Conclusion

The core message of this thesis is to interpret the problem of project valuation in the form of statistical decision making. We can declare that the aforementioned contributions of the developed algorithm have been successfully presented by our experiment, most notably in contrast to the current valuation techniques as:

- the seamless integration of multiple uncertainty sources;
- the integration of continuous probability distribution models of variables;
- the easy representation of actions;
- improved speed of computation by the usage of ADP algorithm to compute the value function approximations;
- preservation of the domain-specific concepts.

Furthermore, through the thesis, we have discussed other relevant ideas about capital investment, where we believe we can add some value. We talk about the term of utility in a field, which is influenced by it but is not using this notion. We also present the notion of present cash equivalent (PCE), where we argue that for each FCF vector, there is an amount of present cash that the decision-maker is logically indifferent to having instead of the proposed FCF vector.

We have demonstrated the discussed ideas and the valuation algorithm on an example of gas power plant valuation. The additional value of the experiment is in the confirmation of the ROA narratives, such as that options have value and that this value rises with the inner uncertainty of the project.

We have also outlined the limitations of our approach when applied on more complex projects such as:

- complexity of value function models;
- numerical integration of more uncertain state evolutions;
- computational complexity driven by both points above.

Finally, through the time of writing this thesis, we have identified the following directions for further research:

- valuation of a more complex project in cooperation with an actual investment company;

- sensitivity analysis of price volatilities in our example;
- sensitivity analysis of the time granularity of decisions in our example;
- so-called option games, which combine elements of SDT and game-theory (studying, for example, the optimality of companies' competition vs. collaboration);
- model of investment with more complex internal processes than the simple I/O process;
- a deeper study of the macro vs. micro-optimization of the utility and its effects on consistency of decision making at individual levels of the investment.

# Bibliography

- [1] A. AlShelahi, J. Wang, M. You, F. Byon, and R. Saigal. Data-driven prediction for volatile processes based on real option theories. *International Journal of Production Economics*, 226:107605, 2020.
- [2] K. D. Ampofo. *Reasons why Real Options Analysis (ROA) is not widely adopted in the mineral industry*. PhD thesis, 2017.
- [3] M. Amram and N. Kulatilaka. *Real Options: Managing Strategic Investment in an Uncertain World*. Oxford University Press, 1998.
- [4] S. Bacci and B. Chiandotto. *Introduction to Statistical Decision Theory: Utility Theory and Causal Analysis*. CRC Press, 2019.
- [5] L. Bachelier. Théorie de la spéculation. *Annales Scientifiques de L'Ecole Normale Supérieure*, 17:21–88, 1900.
- [6] M. Benaroch and M. Whitman. Real options models for proactive uncertainty-reducing mitigations and applications in cybersecurity investment decision-making. *Information Systems Research*, forthcoming, January 2017.
- [7] J. Berk and P. DeMarzo. *Corporate Finance*. Pearson, 2014.
- [8] D. Bernoulli. Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36, 1954.
- [9] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.
- [10] J. K. Blitzstein and J. Hwang. *Introduction to probability*. Crc Press, 2019.
- [11] Z. Bodie, A. Kane, and A. J. Marcus. *Investments*. McGraw-Hill, fifth edition, 2004.
- [12] R. A. Brealey, S. C. Myers, F. Allen, and P. Mohanty. *Principles of corporate finance*. Tata McGraw-Hill Education, tenth edition, 2012.
- [13] P. Breeze. *The cost of power generation: The current and future competitiveness of renewable and traditional technologies*. Business Insights Limited, 2010.
- [14] V. Bronzin. *Theorie der Prämienengeschäfte*. Verlag Franz Deticke, 1908.
- [15] R. Cary. The future of gas power. *Critical Market and Technology Issues*, 2012.
- [16] T. Copeland and V. Antikarov. *Real Options: A Practitioner's Guide*. Texere New York, 2001.

- [17] F. Crundwell. *Finance for Engineers: Evaluation and Funding of Capital Projects*. Springer London, 2008.
- [18] R.K. Dixit and R. S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 2012.
- [19] M. d’Amato, S. Zrobek, M. R. Bilozor, M. Walacik, and G. Mercadante. Valuing the effect of the change of zoning on underdeveloped land using fuzzy real option approach. *Land Use Policy*, 86:365 – 374, 2019.
- [20] M. Guerra, P. Kort, C. Nunes, and C. Oliveira. Hysteresis due to irreversible exit: Addressing the option to mothball. *Journal of Economic Dynamics and Control*, 92, May 2017.
- [21] G. Guthrie. *Real Options in Theory and Practice*. Financial Management Association Survey and Synthesis. Oxford University Press, 2009.
- [22] C. A. Holt and S. K. Laury. Risk aversion and incentive effects: New data without order effects. *American Economic Review*, 95(3):902–912, 2005.
- [23] E. T. Jaynes. *Probability theory: The logic of science*. Cambridge University Press, 2003.
- [24] M. Kárný. On assigning probabilities to new hypotheses. *Signal processing*, 2020. submitted.
- [25] I. Kassar and P. Lasserre. Species preservation and biodiversity value: a real options approach. *Journal of Environmental Economics and Management*, 48(2):857 – 879, 2004.
- [26] A. N. Kolmogorov. *Foundations of the Theory of Probability*. Chelsea Pub Co, second edition, June 1960.
- [27] N. Kulatilaka and L. Trigeorgis. The general flexibility to switch: Real options revisited. *International Journal of Finance*, 6, January 1994.
- [28] R. Kulhavý. Personal communication, 2020.
- [29] D. Lund and R. Nymoen. Comparative statics for real options on oil: What stylized facts? *The Engineering Economist*, 63:1–12, February 2017.
- [30] R. C. Merton. Theory of rational option pricing. *The Bell Journal of Economics and Management Science*, 4(1):141–183, 1973.
- [31] S. C. Myers. Determinants of corporate borrowing. *Journal of Financial Economics*, 5(2):147 – 175, 1977.
- [32] M. Peterson. The st. petersburg paradox. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, fall 2020 edition, 2020.
- [33] W. B. Powell. *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. Wiley Series in Probability and Statistics. Wiley, 2nd edition, 2011.
- [34] M. L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. Wiley, 2005.
- [35] B. Quélin and C. Krychowski. Real options and strategic investment decisions: Can they be of use to scholars? *Academy of Management Perspectives*, 24, May 2010.

- [36] F. Rolenec. Feasible bayesian multi-armed bandit optimisation, 2018. CVUT FJFI.
- [37] F. Rolenec. Learning with forgetting preserving a guaranteed knowledge, 2019. CVUT FJFI.
- [38] S.M. Ross. *An Elementary Introduction to Mathematical Finance*. Cambridge University Press, 2011.
- [39] Y. Ryu, Y. Kim, S. B. Seo, and I. Seo. Application of real option analysis for planning under climate change uncertainty: a case study for evaluation of flood mitigation plans in korea. *Mitigation and Adaptation Strategies for Global Change*, August 2017.
- [40] R. Schneider and J. Imai. Valuing investments in digital transformation of business models. *International Journal of Real Options and Strategy*, 7:1–26, December 2019.
- [41] G. Sick and A. Gamba. Some important issues involving real options: an overview. *Multinational Finance Journal*, 14(1/2):73–123, 2010.
- [42] G. Thoma. The valuation of patent real options – an analysis using patent- and firm-level datasets. *Academy of Management Proceedings*, 2019(1):12922, 2019.
- [43] J. De La Vega and H. Kellenbenz. *Confusion de Confusiones [1688]: Portions Descriptive of the Amsterdam Stock Exchange*. Martino Publishing, 1688.
- [44] M. Visnjic. *Mineral asset valuation under price uncertainty using real options*. PhD thesis, Colorado School of Mines. Arthur Lakes Library, 2018.
- [45] A. Vollert. *A Stochastic Control Framework for Real Options in Strategic Evaluation*. Birkhäuser Boston, 2012.
- [46] E. J. Weber. *A Short History of Derivative Security Markets*, pages 431–466. Springer Berlin Heidelberg, 2009.
- [47] H. Yalincak. Criticism of the Black-Scholes model: But why is it still used?: (the answer is simpler than the formula). *SSRN Electronic Journal*, July 2012.