CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Real Options Valuation: A Dynamic Programming Approach

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Master's Thesis

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Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstrujte jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.

Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.
2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.
3. Powell, Warren B, Approximate Dynamic Programming: Solving the Curses of Dimensionality. 2nd ed. Hoboken, NJ: Wiley, 2011.
4. Puterman, Martin L., Markov Decision Processes: Discrete Stochastic Dynamic Programming. Hoboken, NJ: Wiley, 2005.
5. Vollert, Alexander. A Stochastic Control Framework for Real Options in Strategic Valuation. Boston, MA: Birkhäuser, 2003.

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I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

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Chapter 1

Historical background and research motivation

The foundations of financial derivatives date back to the origins of commerce in Mesopotamia in the fourth millennium BC [33]. The derivative market consisted mainly of forward contracts [1] and it was introduced to the European continent through Spain in Roman times. After the expulsion of derivative trading in Spain the center of this type of commerce for Europe were the Low Lands. There, in the end of the 17th century, the first ideas about options [2] and option trading are published by La Vega [15].

The first attempts of a mathematical option pricing come from Bachelier (1900) [6] and Bronzin (1908) [13]. Based on their work the boom of option pricing methods in the 1970's culminated by Nobel-price-winning Black-Scholes model [10], which is today's standard in the option pricing theory[34].

The publicity and wide adoption of the BSM model most likely inspired an expert on capital budgeting, Stewart Myers, to introduce the term "Real Options" [22], which is one of two main pillars of this thesis. Myers builds on the idea that part of the project's value is hidden in the form of real options - ability to change the course of the project in the future. Myers' approach to the real options is mostly philosophical in a sense that he stresses the importance of thinking about the additional value options bring, while he does not present any computational tool for the said value.

The idea of Real Option Analysis (ROA) as a valuation tool for projects was further developed by several influential authors in the following decades, for example Guthrie [18], Vollert [32] Pindyck [16] and Kulatilaka [4].

The valuation of project's free cash flows with the theory of ROA is in the economical world understood as very advanced and its adoption in practice is slow [3]. It is argued that this slow adoption is caused mainly by misunderstanding the more difficult mathematical concept of ROA [30] and the low adoption rate of a competition: "Why should our company use a new tool that no one else is using?" [1].

Through the study of the ROA state of the art, we have identified that there is a discrepancy in understanding what ROA actually is. As will be illustrated in depth in section ... we identify three classes of ROA authors based on the level of analogy to the BSM model.

In this thesis we focus on the second class of ROA authors represented by brilliant publications of Graeme Guthrie [18] and Alexander Vollert [32].[3]

[1] A forward contract is a contract to purchase an asset at a fixed price on a particular date in the future. [8]

[2] A financial option is the **ability** to buy (call option) or sell (put option) a defined volume of an asset for a defined amount of money in a given future time instant. [8]

[3] Second class utilizes only the no-arbitrage principle to determine probabilities of the models.

Through my studies at the FNSPE CTU, I have specialized in the theory of dynamic decision making under uncertainty [27], [28]. When studying ROA it came only natural to understand the project valuation as an optimal decision making problem.

The goal of this thesis is to take the project valuation problem structure as is understood in ROA and look at it from the perspective of stochastic decision theory (SDT). One of the challenges of this task is to implement domain specific economical truths about the behavior of investors and the way they perceive value. Two main addressed concepts are: the time value of money and the risk-aversion of investors.

The goal of this thesis is to provide an SDT-based valuation algorithm for projects, whose value is understood as a maximal possible current cash equivalent of the uncertain future cash flows. This valuation algorithm will cover the classes of problems now solved by ROA and allow for new classes.

The new SDT based valuation algorithm will enable:

- seamless integration of multiple uncertainty sources;
- integration of theoretically any probability distribution as a model of uncertain variables;
- usage of high number of possible actions, regardless of their nature;
- utilization of approximate dynamic programming tools for high-dimensional problems;
- preservation the economical truths as time value of money and the risk-aversion of investors.

To illustrate the usage of the new SDT-based algorithm a valuation of a project from a selected class is performed. This class is denoted as *facilities with simple input-output process models*. It covers all projects whose cycle time is equal zero and the input-output transformation rate is constant. This class is a generalization motivated by an example of gas power plant valuation presented by Guthrie in [18].

The thesis is structured in 5 chapters. Chapter 1 reminds the reader of the most important mathematical and economical concepts used in this thesis. Beginning with the declaration of mathematical notation the chapter continues with key concepts of economical theory and deeper description of the BSM model. The first chapter continues with a summary of the last decades in ROA research with the focus on a specific level of analogy represented by Vollert [32] and Guthrie [18]. The reminder of the first chapter is reserved for key parts of stochastic decision theory.

Chapter 2 represents the core of this thesis. First, we define what will be understood as a problem of ROA project valuation. We state the key features that define a project and we limit this features accordingly (?). Then we focus on the interpretation of the problem by a general SDT framework. We illustrate the identification of ROA project features in SDT. The reminder of the second chapter is reserved for resolving the economical nuances that need to be accounted for in the SDT framework in order to make the valuation procedure consistent with the economical reality of investors' behavior.

Chapter 3 illustrates the new valuation algorithm from chapter 2 on an example of valuation of a facility with a simple input-output process model. A valuation of a gas power plant is chosen as the representative of this class. The first half of the chapter focuses on the value sensitivity with regard to the volume of available managerial actions in the project. The second half presents a value comparison of project alternations in different granularity of the model structure and it is aimed to demonstrate the existence of trade-of between computational complexity and precision of computed results.

Chapter 4 discusses the new findings both theoretical and observed from the performed experiments.

Chapter 5 summarizes the thesis - reminds the motivation, underlines the main message and lists all contributions of this thesis. Furthermore, it outlines many possible future research paths in this field as it is, to our best knowledge, the first available publication on this topic.

Chapter 2

Preliminaries

To properly understand a mathematical text it is important to first define the used notions and symbolism. Since this thesis is based on many different authors, from both financial and mathematical world, a short unifying overview of the used theory is important.

The notation used in this thesis comes predominantly from the most influential authors in the respective fields of study:

- general economy [8];
- real options [18];
- stochastic decision theory [5].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU and its proven applicability in his previous publications [27] and [28].

2.1 Used mathematical symbolism

In the whole thesis, bold capital letters, such as $\mathbf{X}$, represent a set of all elements $x \in \mathbf{X}$ as in [27]. The cardinality of a set $\mathbf{X}$ is denoted with two vertical lines as $|\mathbf{X}|$. Random variables, understood in a sense of the standard Kolmogorov's probability theory [21][1], are represented with a tilde above the variable, i.e. $\tilde{x}$. Realizations of random variables are denoted by the same letters as the random variable without the tilde, i.e. x.

Definition 2.1. *(Probability) Let $\tilde{x}$ be a random discrete variable. Then $P(x)$ denotes a probability that the realization of $\tilde{x} = x$. Similarly if $\tilde{x}$ is a continuous random variable, then $p(x)$ denotes a probability density of the realization $\tilde{x} = x$.*

Remark. *To rigorously unify the notation and simplify the formulas a Radon-Nikodým (RN) density [26] is introduced with the notation $p(x)$ and the name "probability density". The dominating measure of this RN density is either the counting measure (in discrete case) or the Lebesgue measure (in continuous case). The notation $P(X)$ is reserved only for the cases when the discreteness of the argument needs to be emphasized.*

The last general definition is the definition of well known concept of conditional probability [19].

[1] Does this citation make sense?

Definition 2.2. *(Conditional probability) Let, depending on the context, symbol $p(x|y)$ represent the conditional probability density of a random variable variables. Then the $p(x|y)$ is defined as:*

$$p(x|y) = \frac{p(x,y)}{p(y)}, \tag{2.1}$$

where $p(x,y)$ is a joint probability density of $\tilde{x}$ and $\tilde{y}$.

Remark. *The definition of conditional probability expressed by the equation (2.1) corresponds with the classic definitions of the conditional probability and conditional probability density in both the discrete and continuous case.*

Definition 2.3. *(Expected value) Expected value of a random variable $\tilde{x}$ is defined as:*

$$\int_{\mathbf{X}} p(x)xd\mu, \tag{2.2}$$

where μ is the dominating measure of the RN density and $\mathbf{X}$ is the set of all possible realizations of $\tilde{x}$.

<Probably some other definitions that will be needed in the following chapters >

2.2 General economics

This thesis is built on two main theoretical pillars, the theory of corporate finance [8] and stochastic decision theory (SDT) [5]. A basic review of corporate finance terminology is presented in this section with a focus on project valuation techniques.

Definition 2.4. *(**Process**) A process is understood as the production of goods, purchase and trade of goods or services, driven by the supply of inputs and demand for outputs.* [2]

Definition 2.5. *(**Project**) A project is defined as an sequence of actions that serve as an implementation or a innovation of a process, purposefully allocating existing sources to increase the economical value of given project.*

Definition 2.6. *(**Free cash flow**) The incremental effect of a project on the firm's available cash is the project's free cash flow [8].*

Definition 2.7. *(**Economical value**) An economical value of a project is understood to be in the future free cash flows. In this thesis the economical value of a project is the amount of cash to which is the investor logically indifferent to having in comparison to the future cash flow vector.*

Remark. *The indifference and time value of money will be further discussed in section ... The theory of net present value (NPV) [8] can be used as a simplification.*

Definition 2.8. *(**Optimal Project**) The goal of each project is to increase the economical value of a process. A project, if such exists, is called optimal project if the additional economical value is maximal given the set of possible projects.*

[2] How to cite Mr. Kulhavy?

2.2.1 Net present value and other valuation metrics (?)

<Say that this section is for illustration of current state of thinking about investments>
<Say that NPV is the standard in this field>
<Define NPV and say it is for illustration purposes>
<A short paragraph about other metrics used in the decision making about a project, DTA, IRR, WACC>

2.2.2 Risk averse investors

According to the observations made in [8] there is a positive correlation between the volatility of an investment and its average profit. This correlation is being explained by the risk aversion of investors, where investors are happy to hold onto a low-volatile investment even though the average returns are lower in the long run.

The phenomenon of risk aversion is well documented also in psychological publications, i.e. [2], where we observe that people do not like uncertainty and they do value uncertain rewards lower than their expected value.

2.2.3 Option valuation - Black-Scholes-Merton model

As outlined in the first chapter, the motivation for option valuation technique came with the increased adoption of derivative trading after the WWII. The famous 1973 article from Black and Scholes [10] presented today's standard in the option valuation - the BSM model.

Remark. *The M in BSM model stands for Merton, as the publication of Black and Scholes "relied on earlier work by Robert Merton."* [3]

In what follows we will present the BSM model in a form of a theorem [11]. In addition we will present an opinion that should summarize the idea of BSM model in few sentences.

Theorem 1. *(BSM model) The Black-Scholes-Merton option valuation model says that if the following list of assumption is satisfied:*

- *risk-free interest rate and volatility of the underlying asset are constant;*
- *the underlying asset pays no dividends and its price is continuous;*
- *the asset price evolves according to a log-normal process;*
- *the markets are efficient - the no-arbitrage principle holds;*
- *the option has European style;*
- *there are no commission or transaction costs;*
- *market is perfectly liquid;*

[3] I want to talk about it because I want to explain why BSM and not BS model, where I do not want to use BS model for obvious reasons.

then based only on the knowledge of time to maturity T, option's strike price K, the current price of underlying asset S and its volatility σ the value of a call option can be computed as:

$$C = SN(d_1) - PV(K)N(d_2), \tag{2.3}$$

where $PV(K)$ is the present value of a strike price K [4] and $N(d)$ is a cumulative normal distribution, probability that a normally distributed variable is less than d. Value of d_1 and d_2 is then defined as:

$$d_1 = \frac{ln(S/PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{2.4}$$

Remark. *The dependency of the price of an option is positive in case of volatility and time to maturity Increasing these parameters leads to a higher option price. On the contrary the rise in current stock price or strike price of the options lowers the value of an option.*

Now, we would like to present our opinion about what the BSM model actually says.

Remark. *The core of BSM model, assuming that "smooth" conditions hold, represents a way to derive a parameter in a log-normal model of the underlying asset. Based on the no-arbitrage principle and assumption of known volatility σ, only the parameter μ of log-normal process is missing. Building on this, the value of an option is the expected value of the maximum of difference between strike price and the realized price of the asset and zero [5], discounted by the risk-free interest rate.*

2.3 Real option analysis

As outlined in the first chapter, the theory of real option analysis (ROA) was born after the boom in publications about valuation of financial options in the 1970's. The first ideas represented by Myers [22] are of a philosophical nature - options (ability to make project changes) adds value to the project.

Many publications were published on the ROA topic since. Through our studies of the state of the art, we have identified three classes of authors which differ by the level of analogy with the BSM model.

No analogy The first class is the class of the ROA founder, Myers. This class understands the term real option analysis as a useful lens for looking at the project valuation. Authors like [20] and [17] accentuate the value of further managerial decisions, but the valuation strategy they use is NPV with scenarios (so called decision tree analysis DTA).

Partial analogy The second class of authors takes advantage of the core property of the financial option valuation and that is the no-arbitrage principle. Based on this principle and further assumption of replication portfolio existence, this class of authors, e.g. [18], [31] and [29], derives so called risk-neutral probabilities, which are then used for modeling of some project's internal variable of the cash flow functions.

Because we find this type of approach to real options as the most appropriate one and because we build on and respect the work of Guthrie [18], **this approach is the one considered as representative of the term ROA.**

Another author in this class whose work is notable is publication [32] from Vollert, who goes deep into detail with modeling framework implementing complex conditional options. Vollert's publication is very advanced, using i.e. stochastic differential equations, which might be an obstacle for practitioners and real world applications.

[4] Price of a bond paying K on the expiration day of the option

[5] Since the option does not have to be realized, no further loss will occur.

Full analogy The final class of the authors understands project valuation with real options as a complete analogy to thee BSM model for valuation of financial options. This class of authors is predominantly represented by voluminous economical textbooks, e.g. [8], [12] or [14][6].

A complete analogy means to identify all parameters of financial option with a parameter of investment. For example in [8] the following identification table is presented:

Financial option	Real option
Stock price	Current market value of asset
Strike price	Upfront investment required
Expiration date	Final decision date
Risk-free rate	Risk-free rate
Volatility of stock	Volatility of asset value
Dividend[7]	FCF lost from delay

Table 2.1: Identification of parameters for real options with respect to the financial option [8].

Another example can be found in [25], where a telecommunication company is being valued by the same one-to-one identification of BSM model parameters.

By focusing on the complete analogy, the authors of this class strictly limit the application scope of the ROA valuation technique (as they understand it). One of the problematic assumptions (that is in the partial-analogy class solved by the CAMP model [8]) is that there exists a market tradeable replicating portfolio of the asset we want to valuate. Another limitation is that this approach considers only one decision, which is usually to invest in the project now, or later [9].

In what follows, our understanding of ROA will be based on the one presented by Guthrie. This decision is based on the exceptionality of his publication [18]. A rigorous definition of ROA as we will understand it will be presented in the beginning of the following chapter.

2.4 Statistical decision theory

The second pillar upon which this thesis stands is the statistical decision theory (SDT). An area of applied mathematics that formalizes and studies optimal decision making of agents. As decision making under uncertainty in its broadest sense encapsulates the majority of human behavior, the class of problems it is able to solve (at least theoretically) is quite large.

The SDT's main focus is to determine the optimal strategy (a sequence of decisions) to act upon, generally in dynamic and uncertain environment. In this thesis we will be modeling the decision making problems by the standard framework of Markov Decision Process (MDP).

Definition 2.9. *(Markov decision process)*

Markov decision process is defined by its five building blocks:

- *Set of time epochs -* $\mathbf{T}$*;*
- *Set of states in those epochs -* $\mathbf{S_t}$, $t \in \mathbf{T}$*;*

[6]Crundwell also discusses the partial analogy approach in detail.

[7]For the BSM model with dividends

[8]Capital Asset Pricing model - for details please see [18]

[9]Timing option in Guthrie's terminology

- *Set of actions in those states -* $\mathbf{A_{s_t}}$*,* $s_t \in \mathbf{S_t}$*,* $t \in \mathbf{T}$*;*
- *Reward function of transition from one state to another -* $r(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*
- *Transition probabilities governing the transitions from one state to another* $p(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*

Remark. *The set of time epochs, states, actions is usually known, defined by the structure of the decision problem that is being solved. Reward and transition functions tend to be unknown in solving these problems and they need to be often somehow estimated.*

Remark. *For further simplification of the text, we define* $\mathbf{S} = \bigcup_{t \in \mathbf{T}} \mathbf{S_t}$ *and* $\mathbf{A} = \bigcup_{s \in \mathbf{S}} \mathbf{A_s}$*.*

Usually, the biggest task in SDT is to correctly approach the uncertainty about transition probabilities between the different states of a decision making problem. There are two approaches to parameter estimation in statistics, classical approach and a Bayesian approach. Since the Bayesian approach seems to fit the format of decision making better - allowing for notion of prior probabilities, incorporating experts knowledge and possibility for smooth updating on newly observed data - it is used in this thesis.

As outlined above, the goal of SDT is to find the optimal strategy - sequence of actions. The optimality of such strategy is defined as it having the maximal expected cumulative reward among all eligible strategies $\mathbf{\Pi}$:

$$\pi^* = \arg\max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} r(s_t|a_t, s_{t-1})|\pi\right]. \tag{2.5}$$

Remark. *This definition of optimal strategy is used mainly in finite decision problems or problems with exponential discounting of future rewards. Alternative definitions of optimality, for example maximal average reward per period, exist.*

Due to the nature of project valuation, where projects are considered to be finite or their cash flow exponentially discounted, this thesis will focus on the total cumulative reward.

2.4.1 Dynamic programming

Finding the optimal policy by computing the expected reward for all policies $\pi \in \mathbf{\Pi}$ is due to the cardinality of $\mathbf{\Pi}$:

$$|\mathbf{\Pi}| = \prod_{t \in \mathbf{T}} \prod_{s_t \in \mathbf{S_t}} |\mathbf{A_{s_t}}| \tag{2.6}$$

very demanding task even for low-dimensional problems.

To cope with such computational complexity a clever idea of backward induction, called dynamic programming, is used. The core of dynamic programming is to define so called value function $V(s)$ on each of the possible states $s \in \mathbf{S}$. Each of the values is computed via the Bellman equation:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A_{s_{t-1}}}} \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1})[r(s_t|a_t, s_{t-1}) + V(s_t)]. \tag{2.7}$$

Value function represents the expected cumulative reward from given state onward. The idea of computing this value through the backward induction is based on the truth that a sequence of actions is optimal if and only if the last action is optimal. Optimal strategy comes together with the value function seemingly as a byproduct, where in each state, the optimal action is the argmax of the expression in equation 2.7.

This clever approach significantly reduces the computational complexity to

$$\sum_{t \in \mathbf{T}} \sum_{s_t \in \mathbf{S_t}} |\mathbf{A_{s_t}}| \tag{2.8}$$

expected values.

Remark. *For constant number of states and actions in each time epoch, this reduction is from* $|\mathbf{T}|^{|\mathbf{S_t}|^{|\mathbf{A_{s_t}}|}}$ *to* $|\mathbf{T}| \cdot |\mathbf{S_t}| \cdot |\mathbf{A_{s_t}}|$.

The reduction of computational complexity with the DP algorithm is significant. However, for the majority of real world applications this reduction is not enough. In reality, due to the structure of decision making problems and their formulation, the cardinality of state space can explode even for fairly simple decision problems.

The problem of remaining computational complexity of DP algorithm is in literature addressed as "three curses of dimensionality" [24] and various solutions under the label of approximate dynamic programming (ADP) were proposed.

2.4.2 Approximate dynamic programming

The computational complexity of dynamic programming for middle and high-dimensional decision making problems is so demanding that its results cannot be obtained in a reasonable amount of time.

To cope with this problem a relevant topic to look at is the section of SDT called approximate dynamic programming (ADP). The ADP label can be understood as a unifying name for a number of algorithms[10] that are trying to obtain quasi-optimal strategies for decision making problems with reasonable demands for computational power.

ADP algorithms can be divided into two main classes, policy and value iteration algorithms. In this thesis we will be focusing on the value iteration class, since we believe that due to the structure of project valuation problems (rather large $|\mathbf{S}|$ and small $|\mathbf{A}|$) it is a better fit.

The idea of value iteration is to have some initial heuristic value function approximation, which is being updated based on the sample of possible paths.

In this thesis, we will focus on the most simple approximation model of the value function, that enables to parameterize the value function with only a small number of parameters[11]. We will model the value function in each time epoch v_t as a linear function of basis functions ϕ_i and parameters $\theta_{i,t}$:

$$v_t(s) = \sum_i \theta_{i,t} \cdot \phi_i(s), \tag{2.9}$$

where each of the basis functions ϕ_i represents some heuristically important feature of each state $s \in \mathbf{S}$. A project valuation example of such basis function might be a difference between state elements representing prices of inputs and outputs or indicator function of state element representing running production.

The update of v_t then unfolds as follows:

- Sample of states $\mathcal{S}$ in time $t \in \mathbf{T}$ is generated[12]

[10] for details see Powell [24].

[11] A value function is in classical DP similar to a look-up table, there is no simple relationship between states and the values of value functions.

[12] Usually based on the model of state distribution in time t.

- In each $s_t \in \mathcal{S}$ the optimal action a_t^* is determined as argument maxima of the Bellman equation, where the value function is understood as its last approximation.
- In each $s_t \in \mathcal{S}$ undertaking the action a_t^* and transition to the following state s_{t+1} is simulated. The reward-state pair $(s_t, r(s_t, a_t^*, s_{t+1}))$ is saved.
- Based on all state-reward pairs a new linear model is constructed, resulting in new parameters $\theta_{i,t}$[13].

By updating the parameters $\theta_{i,t}$ in all $t \in \mathbf{T}$ (in this thesis we will be updating v_t from the horizon) the value function approximation is getting more and more precise[14]. The algorithm ends when a predefined stopping rule is met.

An example of such rule can be that the sum of parameter changes is lower than some predefined threshold.

After the stopping rule is met, we are in possession all parameters $\theta_{i,t}$ with which we are able to determine our best approximation of the expected value of each individual state. One particularly relevant state is the one that describes the current state of the world and thus its value is our estimate of the project's valuation.

2.4.3 Bayesian statistics (?)

The field of mathematical statistics can be divided into two branches, classical (also called frequentist) and Bayesian. The philosophies of each one are fundamentally different and they can be used with a various level of success in different applications.

In this thesis, we are not trying to broadly discuss the internal philosophies of the Bayesian and classical approach as this is not a short and clear discussion. Instead, our approach is to simply use the Bayesian statistics, because it is a better fit for the narrative of learning and decision making under uncertainty.

In general, statistical theory is used to determine a distribution from which the observed data come from. In majority of cases, it is assumed that the data are realizations of a random variable with a distribution from some parameterized class - normal, log-normal, Poisson, etc. The goal is then to determine, with some level of confidence, the parameters that fit the observed data in some sense the best. [15]

The main difference between the Bayesian and classical statistics is how the parameters of a distribution are perceived by the statistician. In the classical theory, it is assumed that observed data come from some distribution with some firm but unknown parameters Θ. In contrast, the Bayesian view on the parameters is such that they are perceived as random variables $\tilde{\Theta}$.

This terminology twist can be a source of initial confusion for frequentist statisticians, but it allows a simple and elegant update of parameter estimates with the Bayesian formula:

$$p(\Theta|d) = \frac{p(d|\Theta)p(\Theta)}{p(d)}, \tag{2.10}$$

where Θ is generally a multivariate parameter and d are observed data. [16]

[13] We could also introduce learning here, meaning that new parameters are a weighted average of the last ones and the newly determined.

[14] Really? I tried to look in Powell, but did not find any convergence theorems to real vf, or at least the optimal parameters of the approximation

[15] Large simplification, statistics can be used in many different ways.

[16] The p(d) in denominator needs to be rewritten as integral if this formula is really to be used.

The interpretation of Bayes formula, is that the distribution of parameter $p(\Theta)$ called the prior distribution, is updated for the newly observed data d, providing new, posterior, distribution $p(\Theta|d)$.

This update can be understood as learning about the "true value" of a parameter, which is very useful structure for dynamic decision problems.

Since the Bayesian theory tells us only how to update an already existing distribution, a prior distribution needs to be given, even though no data were measured yet.

This problem is in Bayesian statistics understood as an advantage, since one can use his knowledge about the problem that is being solved and incorporate it to the prior distribution, which is then updated on the measured data.

The task of consistent creation of prior distribution is a complicated topic and can be found in more detail in [7]. Furthermore, the prior information always exists, as Peterka [23] puts it: “No prior information is a fallacy: an ignorant has no problems to solve“.

2.4.4 Utility

In many decision making situations, rational decision makers do not behave in a way that their decisions would maximize the expected nominal monetary value.

One of the simplest examples used to demonstrate this behavior is given by Bacci and Chiandotto [5]. Imagine an individual is given a choice, either to get 500 $ right away or to gamble for 1000$ in a fair coin toss. A rational decision maker, driven only by the expected value of his actions would be indifferent to the two choices. However, the majority of people ten to take the certain amount of 500$, suggesting that the perceived value of the gamble is lower than 500$.

This effect and its implications becomes more understandable for very large sums of money. There is a little difference for an average human in obtaining 10M USD and 20M USD in a fair coin toss. The change in his quality of life will be almost the same and presumably positive. However one result is certain and the other one has only a 50% probability.

Another interesting example of the non-linear gain perception of individuals is the famous St. Petersburg paradox first formulated by Bernoulli in 1738, [9]. An expected value of the proposed game is infinite, however it is shown that people would seldom pay more than 25 USD to play it. It is interesting that this amount corresponds with the assumption that the counterparty does not posses infinite amount of money, but rather a more reasonable amount of 16.5M USD []. [17]

By these two examples we demonstrate that real decision makers must, in some cases, decide based on something different than the expected value. Building on the extension of the first example, we say that decision makers maximize their well-being measured in utility of the given monetary rewards.

This relation between the perceived utility and monetary value if formalized by the **utility function**.

Based on the shape of utility function we are able to define three classes of decision makers:

- **Risk-averse** decision makers (majority of the population) have concave utility functions. They value uncertain monetary gain lower than its expected value.

- **Risk-neutral** linear utility function. They value uncertain monetary gain exactly as its expected value.

- **Risk-seeking** convex utility function. They value uncertain monetary gain more than its expected value.

[17]This is from wikipedia, find more cool sources. Interesting, but does not have to be in the thesis

The examples above illustrate that the majority of people are risk-averse. This statement is supported by many publications (for example [2]). An individuals utility function can be obtained from a questionnaire by an algorithmic approach that ensures the consistency of responses given by the individual [5].

Another relevant fact with the perception of utility is the asymmetry in human response to gains and losses. The graphical expression of this asymmetry can be seen in fig. 2.1

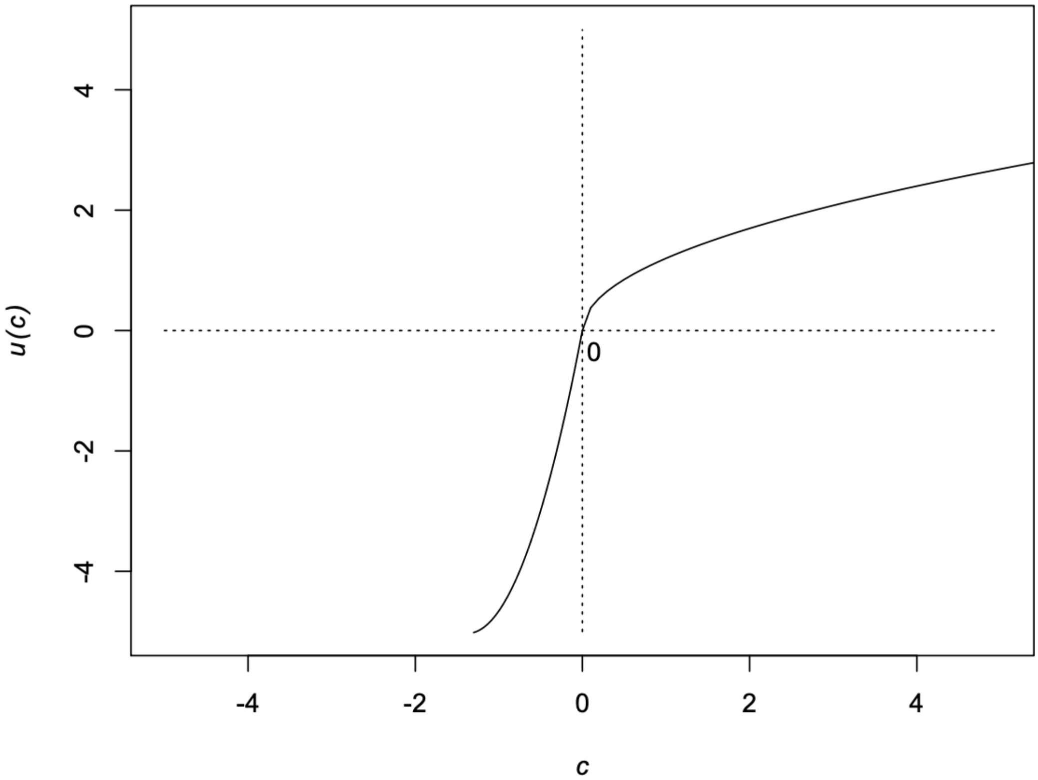

Fig. 2.1. An example of utility function [5].

Chapter 3

Project valuation as stochastic decision problem

In this chapter the core idea of this thesis is developed. This idea is to take the valuation problem as defined in ROA and solve it with SDT, while preserving the economical truths about project valuation, such as time value of money and risk-aversion of investors.

We begin by a clear definition of a project inspired by publications in the field of ROA. We define project valuation problem as a collection of mathematical constructs and identify the main limitations of this approach.

Next, we focus on the identification of the project valuation in terms of SDT framework. We define all the relevant sets and functions to be able to talk about project valuation as a structured decision making under uncertainty.

The reminder of this chapter is reserved for the incorporation of the economical truths to the model, namely the time value of money and the risk aversion of investors.

3.1 Project valuation - problem definition

To be able to rigorously talk about the project valuation we need to define what is a *project* and what do we mean by its *valuation* in ROA. The inspiration for these definitions comes from examples and used rhetoric in the ROA publications, namely [18], [32] and [4].

None of the books that we have studied goes in detail to define a project as a collection of mathematical constructs, as for example SDT does with MDPs. Guthrie in [18] illustrates what a project is with three initial examples and in each chapter presents a real investment opportunity in a way that it is "clear" what is meant by this specific project and what are its parameters.

In other books like Vollert's [32] and Kulatilaka's [4] (???) a definition of a project is also not given, rather, it is assumed that the terms*project* or *capital investment* are clear.

It is worth noting that a definition of project *valuation* is also not deeply discussed in the ROA books. We feel like the term of valuation is assumed to be clear and a is always being understood as expected net present value (NPV).

NPV as a concept is easily understandable, however as we will discuss later it has some features that are not ideal, for example the unclear value of a discount rate or not reflecting the individual perception of risk.

As explained above, nothing like a clear mathematical definition of a project valuation is not presented in the ROA books. However, their rhetoric is similar and we strongly believe that the project valuation can be understood as: "An amount of value that I am able to create with actions that can be considered as a part of one project, measured in a metric of expected net present value with a special non-axiomatic determination of the discount rate." [1]

Now, that the position of ROA to project valuation is clearer, we can follow with its interpretation in the SDT framework.

3.2 Project valuation in the stochastic decision theory framework

Trying to solve the project valuation task as a stochastic decision problem means first and foremost to identify all the necessary structure of the SDT framework in the ROA formulations. This in not a particularly hard task given the rather loose definition given above.

After this identification the standard tool of SDT, dynamic programming (or potentially approximate dynamic programming), can be used to solve the valuation problem.

Solving a valuation problem in SDT means to define it as MDP, which is defined by two parts. First there are three sets: Time set $\mathbf{T}$, State set $\mathbf{S}$ [2], Action set $\mathbf{A}$ [3], which express the structure of the decision making problem. The second part consists of two functions: transition probability function and reward function, where the first one is responsible for describing the probability evolution of the project and the second one for informing about the value gains in each time epoch.

In the following paragraphs we will focus on each of these five important building blocks of the project valuation decision problem in detail. In each paragraph we will present an example from valuation of gas power-plant, helping the reader to understand and preparing the ground for our experiments that will follow.

Time set Even though the SDT theory is capable of handling infinite time horizons and continuous time modeling, these sophisticated formats are not needed when valuing a real life projects. The time element of a project can be reasonably described by a discrete set with known finite horizon for two reasons.

First is that observing new information and making impactful decisions by project's manager is not done at any time but after some reasonable time chunks. No manager changes the course of a project 10 times a day.

Second reason is that managers do not think about project valuations after a certain time interval. This is given by the finite time-span nature of projects (Gas powerplant life-span) or the notion that in today's uncertain world modeling cash flow of a project in 100 years and further a wishful thinking.

Time intervals usually correspond with the intervals of large management meetings at which the course of the project can be changed significantly, e.g. week, month or quarterly intervals. This truth is usually reflected in the ROA publications.

Example 1. *Monthly time intervals in a scope of gas power plant lifespan which is 25 years. The time set is then* $\mathbf{T} = \{0, 1, 2, ..., 300\}$

State set Defining the state set $\mathbf{S}$ in a project valuation problem means to find a list of relevant measurable parameters of both the project itself and its environment. A state $s \in \mathbf{S}$ is then a vector of elementary states of individual parameters.

[1] I am really not sure here, it does not seem that authors actually defined what a project valuation is...

[2] This global state set is a union of state set in each time $\mathbf{S_t}$

[3] This global action set is a union of Action set in each state $\mathbf{A_{s_t}}$

The state set **S** can be constant, meaning that in each time epoch we are measuring the same parameters and the state vector has a constant length. However, it might be useful to think about variable state sets in each time epoch $t \in \mathbf{T}$ as $\mathbf{S_t}$ where for some reason the states in different time epoch have variable length [4].

It is worth noting that there are usually some elementary states that are influenceable by the managerial actions and some states that are not. In our notation we do not distinguish between them.

In our models, each elementary state is understand as a random variable, which probability distributions are conditioned on the previous state and the last action taken. This probability is described with the transition probability function that follows.

Example 2. *Relevant features for a gas power plant might be for example: price of gas, price of CO2 allowances, price of power, installed capacity of the plant, debt to be returned or if the plant is currently running or not. The first four elementary states would then be considered as uninfluenceable by our future actions, while the last three would not.*

Remark. *remark*

Action set In SDT structure the action set **A** is usually understood as an actual set, however in the format of project valuation, we find it better to represent it as an action function, whose parameter is given state $s_t \in \mathbf{S_t}$ and output is a set of possible actions $\mathbf{A(s_t)}$.

The reason for this is that possible managerial actions are most of the time conditioned on the current status of the project itself. Only a small portion of all possible actions might be actually performed in given state.

In ROA publications the term of options is used to describe possible managerial actions now or in the future. Even though we believe that this terminology helps with understanding that possibility of future managerial action has value, we do not embrace it in this thesis and we will use the standard SDT terminology.

The advantage of the SDT approach in contrast to ROA is that there is no theoretical complication in adding as many actions as possible of any type (as classified in ROA by Guthrie [18] for example) possibly even conditioned on one another. The only concern that needs to be reminded is that of computational complexity.

Example 3. *Actions in gas power plant valuation might be to: build some stage of the plant if there is some stage left to built, run the plant if it has some installed capacity, mothball or sell the power plant for salvage value. We believe that this example clearly shows the dependency of possible actions on given state and why is it thus better to use the function notion from now on.*

Transition probability function <Is a function that, conditioned on actions of the manager, defines the probability distribution of all paths the project can take>

<The probabilities in this thesis are discrete>

<We assume that individual transformations of the state vector elements are independent. Thus the probabilities can be computed as a product of individual transition probabilities of each state vector element>

<Product equation>

<The details of obtaining the probabilities will be discussed in one of the next sections>

[4] More or less features are relevant for the project in different stages

Reward function <Is a function of states and actions>

<Represents FCF. Expenses are usually a result of immediate actions and profits tend to be result of the environment (supply, demand) conditioned on a previous action or a set of actions>

<FCF can be modeled in various level of complexity. For us the FCF is a reward for taking given actions with respect to the probabilistic evolution in the project environment>

This paragraph concludes the basic identification of sets required by the SDT framework. In the next section, we will focus on a solution to the project valuation problem in detail. We will discuss the sources of transition probability function, the actual implementation of dynamic programming and the risk-aversion of investors to the model.

3.3 Solution of the project valuation as SDT problem

<We have the identification now, we can solve the valuation problem as SDT now>

<If we would do that we are ignoring the economical truths about how money in time works and how investors think>

<Next we will discuss the details of probability estimate origin>

<Finally, since the complexity of real problems is large and we will need ADP we will discuss what is the best from the ones that I have presented in preliminaries>

3.3.1 Time value of money

As outlined in the preliminaries, cash does not have the same value through time. This economical truth is one of the most important ones in project valuation and capital budgeting. The effective rate of discounting depends on an information if we own money or we hold money.

<Since the reward function represents the free cash flow and thus money, we need to adjust for that>

<To incorporate the time value of money to the dynamic programming we need to introduce the discounting factor>

<The usage of such factor is nothing new for DP. In the problems with infinite horizon, it helps to converge the sum of rewards and it generally represents the widely accepted psychological truth (cite) that reward now is better than reward later.>

<Present the updated bellman equation>

[Problem with the discounting factor as a function of debt>]

3.3.2 Risk-aversion of investors

<There is inherent uncertainty in each project>

<The basic reasons and observations about risk-aversion were already stated in the preliminaries>

<Investors are risk-averse and we want to model that>

<A natural framework for that is the notion of utility from SDT. We will maximize utility instead of actual monetary value. >

<New bellman function, where we maximize utility over actions, not the monetary reward>

<It might be a very complex task to get the utility function from the investor in reality, sad truth.>

3.3.3 Probability

<The actual transition probability function is unknown or it's notion might not make sense at all in reality (maybe the assumption of prices coming from a distribution is not fulfilled). We still need to quantify our beliefs about the project future, usually backed by data and simple models in economics.>

<There are many ways to obtain the estimate of probabilities.>

<Models with data used in both SDT and Economy. How do they do it?>

<Risk neutral probabilities in ROA, where does it come from? >

<Experts and Bayesian updating and insufficient reasons are standard in SDT. There are procedures that enable us to mix different sources of information>

<It is up to the statistician using this new valuation technique to choose>

<We cannot say what is optimal and the details of the choice are out of scope of this thesis>

<We prefer Bayesian updated risk-neutral probabilities (Which can be understood as data-based expert knowledge) for the variables where there is a lot of data and there is a strong case of the model working>

<We prefer expert knowledge with theory of insufficient reasons for the cases when there is little to no relevant data to base our model on>

This section finalizes the core idea of this thesis. With the presented techniques and detailed description of each part of the framework we are now theoretically able to run the algorithm and get the valuation of any project that can be defined as **??**.

A relevant aspect that needs to be taken into account is the actual plausibility of running the DP algorithm due to its possibly extreme computational complexity. This issue will be illustrated [or not] in the next chapter.

Chapter 4

Valuation of facilities with simple cycle time model

In this chapter we aim to illustrate the new perspective on the problem of project valuation on <this example> because <a reason>.

First we will focus on the dependency between project value and the amount of available managerial actions, allowed by its structure.

In the second part of the experiment, we will take one of the first three setups regarding available project actions and investigate at the sensitivity of project value on the granularity of its structure. First we experiment with changing the time granularity corresponding to the ability to act in smaller time intervals. Next we increase the granularity of the price modeling of inputs and outputs. Finally we fuse these two project structure changes into a final experiment.

In the first half of this chapter we expect to gain more valuable projects with the increasing managerial ability to act. On the contrary, there is no clear expectation of results in the second part of the experiments. We have no strong opinion about what the increased granularity in price modeling, time intervals or their combination will do with the valuation of the alternative projects.

4.1 General settings of the experiment

<First identify the initial common parameters for the experiment>

Time set <Time set mathematical expression, with the reasoning behind this decision, i.e. Gas power plant, decisions per month, lifespan of the plant 20 years.>

State set <State set, defined in "rectangular way", some states are logically unreachable. Vector with encoded state of the environment. For example [built/not-built plant, index of gas price increase/decrease, index of power price increase/decrease>

<Mathematical expression for the state set>

<Detail elaboration what the encoding means for each encoded state>

<Discussion about the model of states. Up movements volatility could be taken from the actual volatility on the market, we are just putting some values.>

Action function <In the most simple case presented for now, we expect only timing option to be available>

<Mathematical expression of the action function>

<We will increase the size of this set by adding new possible managerial actions in two levels in the following paragraphs>

Transition probability function <We have to cope with four sources of uncertainty: price of gas, price of CO2, price of power>

<We are modeling all of them by the historical prices and the log-normal model, since that is standard>

Reward function : <The FCF model for the gas power plant and other facilities in its class is rather simple>

<We do not assume the price of labor which tends not to be very volatile>

<Mathematical model of the FCF, easy and clear>

4.2 Sensitivity towards potential action set

<In this section we will focus on the effects of adding managerial actions on the valuation, in both complexity of computation and results>

<We will add two levels of possible managerial actions and we will investigate the changes>

Action set 1 : <We are adding these types of actions to the model>

<The reward function changes in this way>

<The probability function changes in this way>

<Results>

Action set 2 <The reward function changes in this way>

<The probability function changes in this way>

<Results>

Action set 3 <The reward function changes in this way>

<The probability function changes in this way>

<Results>

4.3 Sensitivity toward time epoch granularity

<Now we will study the sensitivity of the valuation for the length of the time epoch>

<The structure for this model is the same as in example ... above, because <reason»

<We will change the time set to <Mathematical expression»

<Results>

With this result we finish the chapter of experiments. Its results and implications will be discussed in the next chapter.

Chapter 5

Discussion

In this chapter we will discuss both the results of the 6 variations of a single project from chapter 3 and its theoretical background from the Chapter 2.

Comparing the results of different granularity of … we can state that …

Looking back at the formulation … there is a potential for improvement in …

With the newly obtained knowledge we can state that the new valuation technique helps with … and is more general than the techniques used nowadays. All this while preserving … and hopefully not exceeding the mathematical capabilities of the potential users.

Chapter 6

Conclusions

The core message of this thesis is to interpret the problem of project valuation in the form of stochastic decision making. The contributions of the newly presented valuation algorithm in contrast to already existing techniques are:

- Usage of general distributions
- Theoretically any number and type of actions
- ...

Furthermore, the thesis copes with the problem of computational complexity, arising as a result of high-dimensional problems, with identifying a <ADP theory> as the best fitting algorithm from the class of ADP for the problem of project valuation.

The new approach to project valuation is demonstrated on six variations of one project type, which show its real applicability in real world. First three examples confirm the expected sensitivity of the project's value on the level of possible managerial actions, endorsing the idea that projects with higher degree of managerial action space have more value. The second trio of experiments shows how is the valuation sensitive on the choice of SDT framework. We conclude that . . . <probably not much>

The limitations of this approach are:

Finally, through the time of writing this thesis I have identified the following directions for further research as:

- ...

Bibliography

[1]

[2]

[3] Kwasi Darkwa Ampofo. Reasons why real options analysis (roa) is not widely adopted in the mineral industry. 2017.

[4] Martha Amram and Nalin Kulatilaka. *Real Options:: Managing Strategic Investment in an Uncertain World*. Oxford University Press, 1998.

[5] S. Bacci and B. Chiandotto. *Introduction to Statistical Decision Theory: Utility Theory and Causal Analysis*. CRC Press, 2019.

[6] Louis Bachelier. Théorie de la spéculation. *Annales Scientifiques de L'Ecole Normale Supérieure*, 17:21–88, 1900. Reprinted in P. H. Cootner (ed), 1964, The Random Character of Stock Market Prices, Cambridge, Mass. MIT Press.

[7] J.O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer, NY, 1985.

[8] J. Berk and P. DeMarzo. *Corporate Finance*. Pearson, 2014.

[9] Dino Bernoulli. Exposition of a new theory on the measurement of risk. 22(1):23–36.

[10] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[11] Z. Bodie, A. Kane, and A.J. Marcus. *Investments*. McGraw-Hill/Irwin series in finance, insurance, and real estate. McGraw-Hill Irwin, 2004.

[12] Richard A Brealey, Stewart C Myers, Franklin Allen, and Pitabas Mohanty. *Principles of corporate finance*. Tata McGraw-Hill Education, 2012.

[13] V. Bronzin. *Theorie der Prämiengeschäfte*. Verlag Franz Deticke, 1908.

[14] F. Crundwell. *Finance for Engineers: Evaluation and Funding of Capital Projects*. Springer London, 2008.

[15] J. De La Vega and H. Kellenbenz. *Confusion de Confusiones [1688]: Portions Descriptive of the Amsterdam Stock Exchange*. Martino Publishing, 1688.

[16] R.K. Dixit and R.S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, 2012.

[17] Manuel Guerra, Peter Kort, Cláudia Nunes, and Carlos Oliveira. Hysteresis due to irreversible exit: Addressing the option to mothball. *Journal of Economic Dynamics and Control*, 92, 05 2017.

[18] G. Guthrie. *Real Options in Theory and Practice*. Financial Management Association Survey and Synthesis. Oxford University Press, 2009.

[19] E. T. Jaynes. *Probability theory: The logic of science*. Cambridge University Press, Cambridge, 2003.

[20] Ilhem Kassar and Pierre Lasserre. Species preservation and biodiversity value: a real options approach. *Journal of Environmental Economics and Management*, 48(2):857 – 879, 2004.

[21] Andrey N. Kolmogorov. *Foundations of the Theory of Probability*. Chelsea Pub Co, 2 edition, June 1960.

[22] Stewart C. Myers. Determinants of corporate borrowing. *Journal of Financial Economics*, 5(2):147 – 175, 1977.

[23] V. Peterka. Bayesian approach to system identification. In *in Trends and Progress in System Identification, P. Eykhoff, Ed*, pages 239–304. Peron Press, 1981.

[24] Warren B. Powell. *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. Wiley Series in Probability and Statistics. Wiley, Hoboken, NJ, USA, 2nd edition, 2011.

[25] Bertrand Quélin and Charlotte Krychowski. Real options and strategic investment decisions: Can they be of use to scholars? *Academy of Management Perspectives*, 24, 05 2010.

[26] M.M. Rao. *Measure Theory and Integration*. John Wiley, NY, 1987.

[27] F. Rolenec. Feasible bayesian multi-armed bandit optimisation, 2018. CVUT FJFI.

[28] F. Rolenec. Learning with forgetting preserving a guaranteed knowledge, 2019. CVUT FJFI.

[29] Young Ryu, Young-Oh Kim, Seung Beom Seo, and Il Seo. Application of real option analysis for planning under climate change uncertainty: a case study for evaluation of flood mitigation plans in korea. *Mitigation and Adaptation Strategies for Global Change*, 08 2017.

[30] Gordon Sick and Andrea Gamba. Some important issues involving real options: an overview. *Multinational Finance Journal*, 14(1/2):73–123, 2010.

[31] Grid Thoma. The valuation of patent real options – an analysis using patent- and firm-level datasets. *Academy of Management Proceedings*, 2019(1):12922, 2019.

[32] A. Vollert. *A Stochastic Control Framework for Real Options in Strategic Evaluation*. Birkhäuser Boston, 2012.

[33] Ernst Juerg Weber. *A Short History of Derivative Security Markets*, pages 431–466. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[34] Hakan Yalincak. Criticism of the black-scholes model: But why is it still used?: (the answer is simpler than the formula). *SSRN Electronic Journal*, 07 2012.