CZECH TECHNICAL UNIVERSITY IN PRAGUE
Faculty of Nuclear Sciences and Physical Engineering

Real Options Valuation: A Dynamic Programming Approach

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Master's Thesis

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Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstrujte jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.

Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.
2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.
3. Powell, Warren B, Approximate Dynamic Programming: Solving the Curses of Dimensionality. 2nd ed. Hoboken, NJ: Wiley, 2011.
4. Puterman, Martin L., Markov Decision Processes: Discrete Stochastic Dynamic Programming. Hoboken, NJ: Wiley, 2005.
5. Vollert, Alexander. A Stochastic Control Framework for Real Options in Strategic Valuation. Boston, MA: Birkhäuser, 2003.

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Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, October 24, 2020 Bc. Filip Rolenec

Název práce:

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Autor: Bc. Filip Rolenec

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Vedoucí práce: Ing. Rudolf Kulhavý, DrSc.

Abstrakt: **BUDE DOPLNĚNO**

Klíčová slova: Analýza reálných opcí, Blackův-Scholesův model, Čistá současná hodnota, Dynamické programování, Energetika, Oceňování projektů, Stochastická rozhodovací teorie

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Author: Bc. Filip Rolenec

Abstract: **WILL BE ADDED**

Keywords: Black-Scholes model, Dynamic programming, Net present value, Power industry, Project valuation, Real option analysis, Stochastic decision control

Contents

Chapter 1

Historical background and research motivation

The foundations of financial derivatives date back to the origins of commerce in Mesopotamia in the fourth millennium BC [32]. The derivative market consisted mainly of forward contracts [1] and it was introduced to the European continent through Spain in Roman times. After the expulsion of derivative trading in Spain the center of this type of commerce for Europe were the Low Lands. There, in the end of the 17th century, the first ideas about options [2] and option trading are published by La Vega [15].

The first attempts of a mathematical option pricing come from Bachelier (1900) [6] and Bronzin (1908) [13]. Based on their work the boom of option pricing methods in the 1970's culminated by Nobel-price-winning Black-Scholes model [10], which is today's standard in the option pricing theory[33].

The publicity and wide adoption of the BSM model most likely inspired an expert on capital budgeting, Stewart

s, to introduce the term "Real Options" [22], which is one of two main pillars of this thesis. Myers builds on the idea that part of the project's value is hidden in the form of real options - ability to change the course of the project in the future. Myers' approach to the real options is mostly philosophical in a sense that he stresses the importance of thinking about the additional value options bring, while he does not present any computational tool for the said value.

The idea of Real Option Analysis (ROA) as a valuation tool for projects was further developed by several influential authors in the following decades, for example Guthrie [18], Vollert [31] Pindyck [16] and Kulatilaka [4].

The valuation of project's free cash flows with the theory of ROA is in the economical world understood as very advanced and its adoption in practice is slow [3]. It is argued that this slow adoption is caused mainly by misunderstanding the more difficult mathematical concept of ROA [**?**] and the low adoption rate of a competition: "Why should our company use a new tool that no one else is using?" [1].

Through the study of the ROA state of the art, we have identified that there is a discrepancy in understanding what ROA actually is. As will be illustrated in depth in section ... we identify three classes of ROA authors based on the level of analogy to the BSM model.

In this thesis we focus on the second class of ROA authors represented by brilliant publications of Graeme Guthrie [18] and Alexander Vollert [31].[3]

[1]A forward contract is a contract to purchase an asset at a fixed price on a particular date in the future. [8]

[2]A financial option is the **ability** to buy (call option) or sell (put option) a defined volume of an asset for a defined amount of money in a given future time instant. [8]

[3]Second class utilizes only the no-arbitrage principle to determine probabilities of the models.

Through my studies at the FNSPE CTU, I have specialized in the theory of dynamic decision making under uncertainty [27], [28]. When studying ROA it came only natural to understand the project valuation as an optimal decision making problem.

The goal of this thesis is to take the project valuation problem structure as is understood in ROA and look at it from the perspective of stochastic decision theory (SDT). One of the challenges of this task is to implement domain specific economical truths about the behavior of investors and the way they perceive value. Two main addressed concepts are: the time value of money and the risk-aversion of investors.

The goal of this thesis is to provide an SDT-based valuation algorithm for projects, whose value is understood as a maximal possible current cash equivalent of the uncertain future cash flows. This valuation algorithm will cover the classes of problems now solved by ROA and allow for new classes.

The new SDT based valuation algorithm will enable:

- seamless integration of multiple uncertainty sources;
- integration of theoretically any probability distribution as a model of uncertain variables;
- usage of high number of possible actions, regardless of their nature;
- utilization of approximate dynamic programming tools for high-dimensional problems;
- preservation the economical truths as time value of money and the risk-aversion of investors.

To illustrate the usage of the new SDT-based algorithm a valuation of a project from a selected class is performed. This class is denoted as *facilities with simple input-output process models*. It covers all projects whose cycle time is equal zero and the input-output transformation rate is constant. This class is a generalization motivated by an example of gas power plant valuation presented by Guthrie in [18].

The thesis is structured in 5 chapters. Chapter 1 reminds the reader of the most important mathematical and economical concepts used in this thesis. Beginning with the declaration of mathematical notation the chapter continues with key concepts of economical theory and deeper description of the BSM model. The first chapter continues with a summary of the last decades in ROA research with the focus on a specific level of analogy represented by Vollert [31] and Guthrie [18]. The reminder of the first chapter is reserved for key parts of stochastic decision theory.

Chapter 2 represents the core of this thesis. First, we define what will be understood as a problem of ROA project valuation. We state the key features that define a project and we limit this features accordingly (?). Then we focus on the interpretation of the problem by a general SDT framework. We illustrate the identification of ROA project features in SDT. The reminder of the second chapter is reserved for resolving the economical nuances that need to be accounted for in the SDT framework in order to make the valuation procedure consistent with the economical reality of investors' behavior.

Chapter 3 illustrates the new valuation algorithm from chapter 2 on an example of valuation of a facility with a simple input-output process model. A valuation of a gas power plant is chosen as the representative of this class. The first half of the chapter focuses on the value sensitivity with regard to the volume of available managerial actions in the project. The second half presents a value comparison of project alternations in different granularity of the model structure and it is aimed to demonstrate the existence of trade-of between computational complexity and precision of computed results.

Chapter 4 discusses the new findings both theoretical and observed from the performed experiments.

Chapter 5 summarizes the thesis - reminds the motivation, underlines the main message and lists all contributions of this thesis. Furthermore, it outlines many possible future research paths in this field as it is, to our best knowledge, the first available publication on this topic.

Chapter 2

Preliminaries

To properly understand a mathematical text it is important to first define the used notions and symbolism. Since this thesis is based on many different authors, from both financial and mathematical world, a short unifying overview of the used theory is important.

The notation used in this thesis comes predominantly from the most influential authors in the respective fields of study:

- general economy [8];
- real options [18];
- stochastic decision theory [5].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU and its proven applicability in his previous publications [27] and [28].

2.1 Used mathematical symbolism

In the whole thesis, bold capital letters, such as $\mathbf{X}$, represent a set of all elements $x \in \mathbf{X}$ as in [27]. The cardinality of a set $\mathbf{X}$ is denoted with two vertical lines as $|\mathbf{X}|$. Random variables, understood in a sense of the standard Kolmogorov's probability theory [21][1], are represented with a tilde above the variable, i.e. $\tilde{x}$. Realizations of random variables are denoted by the same letters as the random variable without the tilde, i.e. x.

Definition 2.1. *(Probability) Let $\tilde{x}$ be a random discrete variable. Then $P(x)$ denotes a probability that the realization of $\tilde{x} = x$. Similarly if $\tilde{x}$ is a continuous random variable, then $p(x)$ denotes a probability density of the realization $\tilde{x} = x$.*

Remark. *To rigorously unify the notation and simplify the formulas a Radon-Nikodým (RN) density [26] is introduced with the notation $p(x)$ and the name "probability density". The dominating measure of this RN density is either the counting measure (in discrete case) or the Lebesgue measure (in continuous case). The notation $P(X)$ is reserved only for the cases when the discreteness of the argument needs to be emphasized.*

The last general definition is the definition of well known concept of conditional probability [19].

[1] Does this citation make sense?

Definition 2.2. *(Conditional probability) Let, depending on the context, symbol $p(x|y)$ represent the conditional probability density of a random variable variables. Then the $p(x|y)$ is defined as:*

$$p(x|y) = \frac{p(x, y)}{p(y)}, \tag{2.1}$$

where $p(x, y)$ is a joint probability density of $\tilde{x}$ and $\tilde{y}$.

Remark. *The definition of conditional probability expressed by the equation (2.1) corresponds with the classic definitions of the conditional probability and conditional probability density in both the discrete and continuous case.*

Definition 2.3. *(Expected value) Expected value of a random variable $\tilde{x}$ is defined as:*

$$\int_{\mathbf{X}} p(x) x d\mu, \tag{2.2}$$

where μ is the dominating measure of the RN density and $\mathbf{X}$ is the set of all possible realizations of $\tilde{x}$.

<Probably some other definitions that will be needed in the following chapters >

2.2 General economics

This thesis is built on two main theoretical pillars, the theory of corporate finance [8] and stochastic decision theory (SDT) [5]. A basic review of corporate finance terminology is presented in this section with a focus on project valuation techniques.

Definition 2.4. *(**Process**) A process is understood as the production of goods, purchase and trade of goods or services, driven by the supply of inputs and demand for outputs.* [2]

Definition 2.5. *(**Project**) A project is defined as an sequence of actions that serve as an implementation or a innovation of a process, purposefully allocating existing sources to increase the economical value of given project.*

Definition 2.6. *(**Free cash flow**) The incremental effect of a project on the firm's available cash is the project's free cash flow [8].*

Definition 2.7. *(**Economical value**) An economical value of a project is understood to be in the future free cash flows. In this thesis the economical value of a project is the amount of cash to which is the investor logically indifferent to having in comparison to the future cash flow vector.*

Remark. *The indifference and time value of money will be further discussed in section ... The theory of net present value (NPV) [8] can be used as a simplification.*

Definition 2.8. *(**Optimal Project**) The goal of each project is to increase the economical value of a process. A project, if such exists, is called optimal project if the additional economical value is maximal given the set of possible projects.*

[2] How to cite Mr. Kulhavy?

2.2.1 Net present value and other valuation metrics (?)

<Say that this section is for illustration of current state of thinking about investments>

<Say that NPV is the standard in this field>

<Define NPV and say it is for illustration purposes>

<A short paragraph about other metrics used in the decision making about a project, DTA, IRR, WACC>

2.2.2 Risk averse investors

According to the observations made in [8] there is a positive correlation between the volatility of an investment and its average profit. This correlation is being explained by the risk aversion of investors, where investors are happy to hold onto a low-volatile investment even though the average returns are lower in the long run.

The phenomenon of risk aversion is well documented also in psychological publications, i.e. [2], where we observe that people do not like uncertainty and they do value uncertain rewards lower than their expected value.

2.2.3 Option valuation - Black-Scholes-Merton model

As outlined in the first chapter, the motivation for option valuation technique came with the increased adoption of derivative trading after the WWII. The famous 1973 article from Black and Scholes [10] presented today's standard in the option valuation - the BSM model.

Remark. *The M in BSM model stands for Merton, as the publication of Black and Scholes "relied on earlier work by Robert Merton."* [3]

In what follows we will present the BSM model in a form of a theorem [11]. In addition we will present an opinion that should summarize the idea of BSM model in few sentences.

Theorem 1. *(BSM model) The Black-Scholes-Merton option valuation model says that if the following list of assumption is satisfied:*

- *risk-free interest rate and volatility of the underlying asset are constant;*
- *the underlying asset pays no dividends and its price is continuous;*
- *the asset price evolves according to a log-normal process;*
- *the markets are efficient - the no-arbitrage principle holds;*
- *the option has European style;*
- *there are no commission or transaction costs;*
- *market is perfectly liquid;*

[3] I want to talk about it because I want to explain why BSM and not BS model, where I do not want to use BS model for obvious reasons.

then based only on the knowledge of time to maturity T, option's strike price K, the current price of underlying asset S and its volatility σ the value of a call option can be computed as:

$$C = SN(d_1) - PV(K)N(d_2), \tag{2.3}$$

where $PV(K)$ is the present value of a strike price K [4] *and $N(d)$ is a cumulative normal distribution, probability that a normally distributed variable is less than d. Value of d_1 and d_2 is then defined as:*

$$d_1 = \frac{ln(S/PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{2.4}$$

Remark. *The dependency of the price of an option is positive in case of volatility and time to maturity Increasing these parameters leads to a higher option price. On the contrary the rise in current stock price or strike price of the options lowers the value of an option.*

Now, we would like to present our opinion about what the BSM model actually says.

Remark. *The core of BSM model, assuming that "smooth" conditions hold, represents a way to derive a parameter in a log-normal model of the underlying asset. Based on the no-arbitrage principle and assumption of known volatility σ, only the parameter μ of log-normal process is missing. Building on this, the value of an option is the expected value of the maximum of difference between strike price and the realized price of the asset and zero* [5]*, discounted by the risk-free interest rate.*

2.3 Real option analysis

As outlined in the first chapter, the theory of real option analysis (ROA) was born after the boom in publications about valuation of financial options in the 1970's. The first ideas represented by Myers [22] are of a philosophical nature - options (ability to make project changes) adds value to the project.

Many publications were published on the ROA topic since. Through our studies of the state of the art, we have identified three classes of authors which differ by the level of analogy with the BSM model.

No analogy The first class is the class of the ROA founder, Myers. This class understands the term real option analysis as a useful lens for looking at the project valuation. Authors like [20] and [17] accentuate the value of further managerial decisions, but the valuation strategy they use is NPV with scenarios (so called decision tree analysis DTA).

Partial analogy The second class of authors takes advantage of the core property of the financial option valuation and that is the no-arbitrage principle. Based on this principle and further assumption of replication portfolio existence, this class of authors, e.g. [18], [30] and [29], derives so called risk-neutral probabilities, which are then used for modeling of some project's internal variable of the cash flow functions.

Because we find this type of approach to real options as the most appropriate one and because we build on and respect the work of Guthrie [18], **this approach is the one considered as representative of the term ROA.**

Another author in this class whose work is notable is publication [31] from Vollert, who goes deep into detail with modeling framework implementing complex conditional options. Vollert's publication is very advanced, using i.e. stochastic differential equations, which might be an obstacle for practitioners and real world applications.

[4] Price of a bond paying K on the expiration day of the option

[5] Since the option does not have to be realized, no further loss will occur.

Full analogy The final class of the authors understands project valuation with real options as a complete analogy to thee BSM model for valuation of financial options. This class of authors is predominantly represented by voluminous economical textbooks, e.g. [8], [12] or [14][6].

A complete analogy means to identify all parameters of financial option with a parameter of investment. For example in [8] the following identification table is presented:

Financial option	Real option
Stock price	Current market value of asset
Strike price	Upfront investment required
Expiration date	Final decision date
Risk-free rate	Risk-free rate
Volatility of stock	Volatility of asset value
Dividend[7]	FCF lost from delay

Table 2.1: Identification of parameters for real options with respect to the financial option [8].

Another example can be found in [25], where a telecommunication company is being valued by the same one-to-one identification of BSM model parameters.

By focusing on the complete analogy, the authors of this class strictly limit the application scope of the ROA valuation technique (as they understand it). One of the problematic assumptions (that is in the partial-analogy class solved by the CAMP model [8]) is that there exists a market tradeable replicating portfolio of the asset we want to valuate. Another limitation is that this approach considers only one decision, which is usually to invest in the project now, or later [9].

In what follows, our understanding of ROA will be based on the one presented by Guthrie. This decision is based on the exceptionality of his publication [18]. A rigorous definition of ROA as we will understand it will be presented in the beginning of the following chapter.

2.4 Statistical decision theory

The second pillar upon which this thesis stands is the statistical decision theory (SDT). An area of applied mathematics that formalizes and studies optimal decision making of agents. As decision making under uncertainty in its broadest sense encapsulates the majority of human behavior, the class of problems it is able to solve (at least theoretically) is quite large.

The SDT's main focus is to determine the optimal strategy (a sequence of decisions) to act upon, generally in dynamic and uncertain environment. In this thesis we will be modeling the decision making problems by the standard framework of Markov Decision Process (MDP).

Definition 2.9. *(Markov decision process)*

Markov decision process is defined by its five building blocks:

- *Set of time epochs* - $\mathbf{T}$*;*
- *Set of environment states in those epochs* - $\mathbf{S_t}$, $t \in \mathbf{T}$*;*

[6] Crundwell also discusses the partial analogy approach in detail.
[7] For the BSM model with dividends
[8] Capital Asset Pricing model - for details please see [18]
[9] Timing option in Guthrie's terminology

- *Set of actions in those states -* $\mathbf{A_{s_t}}$*,* $s_t \in \mathbf{S_t}$*;*
- *Reward function of transition from one state to another -* $r(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*
- *Transition probabilities governing the transitions from one state to another* $p(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*

Remark. *The set of time epochs, states, actions is usually known, defined by the structure of the decision problem that is being solved. Reward and transition functions tend to be unknown in solving these problems and they need to be often somehow estimated.*

Usually, the biggest task in SDT is to correctly approach the uncertainty about transition probabilities between the different states of a decision making problem. There are two approaches to parameter estimation in statistics, classical approach and a Bayesian approach. Since the Bayesian approach seems to fit the format of decision making better - allowing for notion of prior probabilities, incorporating experts knowledge and possibility for smooth updating on newly observed data - it is used in this thesis.

As outlined above, the goal of SDT is to find the optimal strategy - sequence of actions. The optimality of such strategy is defined as it having the maximal expected cumulative reward among all eligible strategies $\mathbf{\Pi}$:

$$\pi^* = \arg\max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} r(s_t|a_t, s_{t-1})|\pi\right]. \tag{2.5}$$

Remark. *This definition of optimal strategy is used mainly in finite decision problems or problems with exponential discounting of future rewards. Alternative definitions of optimality, for example maximal average reward per period, exist.*

Due to the nature of project valuation, where projects are considered to be finite or their cash flow exponentially discounted, this thesis will focus on the total cumulative reward.

2.4.1 Dynamic programming

Finding the optimal policy by computing the expected reward for all policies $\pi \in \mathbf{\Pi}$ is due to the cardinality of $\mathbf{\Pi}$:

$$|\mathbf{\Pi}| = \prod_{t \in \mathbf{T}} \prod_{s_t \in \mathbf{S_t}} |\mathbf{A_{s_t}}| \tag{2.6}$$

very demanding task even for low-dimensional problems.

ENDED HERE

Thus, a clever idea of backward induction called dynamic programming is used. A function, called the value function is defined on the set of all possible states $\mathbf{S}$. This function represents the expected cumulative reward to be obtained from the given state onwards. The idea of backward induction is based on the truth that a sequence of actions is optimal if and only if the last action is optimal.

This clever computation of value functions from the problem horizon backwards through all the possible states of the problem decreases the complexity from exponential to polynomial. Instead of maximizing over $|\mathbf{A}|^{|\mathbf{S}|^{|\mathbf{T}|}}$ possible strategies at once, one needs to compute significantly less demanding complexity of $|\mathbf{A}| \cdot |\mathbf{T}| \cdot |\mathbf{S}|$. [10]

The formula representing the backward induction is called the Bellman equation:

[10] Check this

$$V(s_{t-1}) = \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1})[r(s_t|a_t, s_{t-1}) + V(s_t)]. \tag{2.7}$$

By defining the value function on the horizon, we can compute value functions of states with lower and lower time indexes, until we get to the time 0, which represents the present. Not only that we have the expected value of the optimal decision making, but we have also derived the optimal strategy for every possible path through the state space.

The backward induction reduces the computation complexity significantly. However, for even a moderate-dimensional decision problems, the number of computations is still extremely large.

The problem of computational complexity of dynamic programming is called "three curses of dimensionality" [24] and various solutions have been proposed. These solutions are as a group referenced as approximate dynamic programming.

2.4.2 Approximate dynamic programming

The computational complexity of dynamic programming for moderate and high-dimensional decision making problems is so demanding that results cannot be obtained in a reasonable amount of time.

The response to this problem comes in a form of approximate dynamic programming, a section of decision making under uncertainty, that is represented by a number of algorithms that are trying to obtain quasi-optimal strategies with more reasonable demand for computation power.

There are many different algorithms, that try to obtain approximate results of the precise dynamic programming represented by the bellman equation. In this thesis the ADP algorithm called <Q-learning, SARSA...> is used because of its high performance in ..., while being still relatively easy to implement. A longer discussion of its choice is left to its corresponding chapter **??**.

<Q-Learning, SARSA,..> <Detailed description of the chosen ADP algorithm>

2.4.3 Bayesian statistics

The field of mathematical statistics can be divided into two branches, classical (also called frequentist) and Bayesian. The philosophies of each one are fundamentally different, however in principle, they can serve for revealing new truths of the measured data in a similar fashion.

Mathematical statistics is a very broad topic, not possible to summarize it in one paragraph. The use of Bayesian statistics in this thesis is only as a tool, no broader discussions about the internal philosophy of different approaches are presented.

In general, statistical theory is used to determine a distribution from which the observed data come from. In majority of cases, it is assumed that the data are realizations of a random variable with a distribution from some parameterized class - normal, log-normal, poisson, etc. The goal is then to determine, with some level of confidence, the parameters that fit the observed data in some sense the best. [11]

The main difference between the Bayesian and classical statistics is how the parameters of a distribution are perceived by the statistician. In the classical theory, it is assumed that observed data come from some distribution with some firm but unknown parameters Θ. In contrast, the Bayesian view on the parameters is such that they are perceived as random variables $\tilde{\Theta}$.

This terminology twist can be a source of initial confusion for frequentist statisticians, but it allows a simple and elegant update of parameter estimates with the Bayes formula.

[11] Large simplification, statistics can be used in many different ways.

$$p(\Theta|d) = \frac{p(d|\Theta)p(\Theta)}{p(d)}, \tag{2.8}$$

where Θ is generally a multivariate parameter and d are observed data. [12]

The interpretation of Bayes formula, is that the distribution of parameter $p(\Theta)$ called the prior distribution, is updated for the newly observed data d, providing new, posterior, distribution $p(\Theta|d)$.

This update can be understood as learning about the "true value" of a parameter, which is very useful structure for dynamic decision problems.

Since the Bayesian theory tells us only how to update an already existing distribution, a prior distribution needs to be given, even though no data were measured yet.

This problem is in Bayesian statistics understood as an advantage, since one can use his knowledge about the problem that is being solved and incorporate it to the prior distribution, which is then updated on the measured data.

The task of consistent creation of prior distribution is a complicated topic and can be found in more detail in [7]. Furthermore the prior information always exists, as Peterka [23] puts it: "No prior information is a fallacy: an ignorant has no problems to solve".

2.4.4 Utility

The concept of utility instead of monetary or other globally measurable gain comes in when the gains are valued non-linearly.

Multiple studies show [13], that the majority of people are risk-averse, meaning that the value of uncertain monetary gain is not equal to its expected value.

One of the simplest example to demonstrate the usage of utility is given by [5]. Imagine an individual is given a choice, either to get 500$ right away or to gamble for 1000$ in a fair coin toss. A rational decision maker driven only by the expected value of his actions would be indifferent to the two choices. However, the majority of people tend to take the certain amount instead of gambling.

This example can be reformulated as follows: How much money would the decision maker need to obtain for certain so that he would be indifferent to gamble for a 1000$. In other words, how much the risk-averse person values that gamble.

The non-linearity of utility obtained from large amounts of money is only more understandable for very large sums of money. There is a little difference for an average human in obtaining 10M USD and 20M USD. The change in the person's life will be almost the same and presumably positive. However one result is certain and the other one has only a probability of 1/2.

Another interesting example of the risk-aversion of people is the famous St. Petersburg paradox first formulated by Bernoulli in 1738, [9]. A risk-neutral [14] decision maker would be willing to pay any amount of money to be able to play a game defined by the paradox. However it is shown that people seldom value the game more than 25 USD, which corresponds to a case that the initiator of the bet does not have an infinite amount of money, rather only 16,5M USD []. [15]

Regarding to utility there is also an interesting asymmetry in human psychology about obtaining gains and incurring losses. The graphical expression of this asymmetry can be found in [5]. [16]

[12] The p(d) in denominator needs to be rewritten as integral if this formula is really to be used.

[13] Find citations (?) or omit this formulation

[14] Define risk-neutral (?)

[15] This is from wikipedia, find more cool sources. Interesting, but does not have to be in the thesis

[16] Put the picture here, or cite the exact page?

The utility function of each decision maker is different and an approximation of its shape can be obtained by an algorithm based on a questionnaire, which also ensures the consistency of responses of a given individual.

Chapter 3

Project valuation as stochastic decision problem

In this chapter we develop the core idea of this thesis. This idea is to take the valuation problem as stated in ROA and solve it with the tools of SDT, while preserving the economical truths about project valuation, such as time value of money and risk-aversion of investors.

We begin by a clear definition of a project inspired by the publications in field od ROA. We define project valuation problem as a collection of mathematical constructs, and we also present the limits of this model. Next we focus the actual identification of the problem in the SDT framework. We define all the relevant sets and functions needed for a structured decision making problem. The reminder of this chapter is reserved for the incorporation of the economical truth to the model, namely the time value of money and the risk-aversion of investors.

3.1 Project valuation - problem definition

To be able to rigorously talk about the project valuation we need to set up boundaries of what is considered a project and what do we mean by its valuation. The inspiration for our definitions comes from the examples that are being solved in the economical publications. The format and framework that is used in many different publications namely [] and [] and [] can be presented as follows:

<Define the problem that is being solved. With sets, relationships between them, transition function etc. purposely not making the connection from the beginning>

Now that the problem is identified, we can follow with its interpretation in the SDT framework.

3.2 Project valuation in the stochastic decision theory framework

Trying to solve the project valuation task as a stochastic decision problem means first and foremost to identify all the necessary structure of the SDT framework with the project valuation problem as defined above. After this identification the standard tools of SDT for solving the problem, dynamic programming (possibly approximate dynamic programming) can be used for obtaining the results.

The SDT framework consists of two parts. First there are three sets: Time set, State set and Action set, which express the structure of the decision making problem. The second part consists of two functions: transition probability function and reward function, which express the relationships between nodes of the structure and most importantly the rewards that are being optimized.

The following paragraphs will focus on each of these five important building blocks od a decision problem in detail.

Time set <Discrete set with a known horizon>

<Horizon matches with the perception of projects in economy>

<Time intervals coincide with the decision making and adaptability of the project, moths, quarters,... >

State set <Dependent on time set or static structure>

<State is a vector of elementary states>

<Elementary state describes with one value one truth about project environment or the status of the project itself>

<There are states that are possible to be influenced by managerial actions and those who are not, we do not distinguish between them in the notation>

<State is understood as a random variable where probabilities of realizations are conditioned by the previous states and actions. This probability is described by the transition probability function>

Action set <Say that usually the action set is understood as really a set, but in the format of project valuation, it makes more sense to make it as action function>

<Action function takes state as an argument and returns the possible actions>

<This is usually because actions in a project are conditioned by the already achieved project status (i.e. action to produce if the plant has been already built)>

<ROA terminology of having options, reflecting the influence of its origins in valuation of derivative markets, emerges as simply having an action in the action set of a given state>

<Because of the structure of SDT framework, there is no problem in adding theoretically unlimited number of actions, even conditioned on one another. Increasing the number of possible actions makes from theoretical point of view no complications, which is in contrast of the ROA framework. It it worth noting that additional actions influence the computational complexity of the dynamic programming algorithm>

Transition probability function <Is a function that, conditioned on actions of the manager, defines the probability distribution of all paths the project can take>

<The probabilities in this thesis are discrete>

<We assume that individual transformations of the state vector elements are independent. Thus the probabilities can be computed as a product of individual transition probabilities of each state vector element>

<Product equation>

<The details of obtaining the probabilities will be discussed in one of the next sections>

Reward function <Is a function of states and actions>

<Represents FCF. Expenses are usually a result of immediate actions and profits tend to be result of the environment (supply, demand) conditioned on a previous action or a set of actions>

<FCF can be modeled in various level of complexity. For us the FCF is a reward for taking given actions with respect to the probabilistic evolution in the project environment>

This paragraph concludes the basic identification of sets required by the SDT framework. In the next section, we will focus on a solution to the project valuation problem in detail. We will discuss the sources of transition probability function, the actual implementation of dynamic programming and the risk-aversion of investors to the model.

3.3 Solution of the project valuation as SDT problem

<We have the identification now, we can solve the valuation problem as SDT now>

<If we would do that we are ignoring the economical truths about how money in time works and how investors think>

<Next we will discuss the details of probability estimate origin>

<Finally, since the complexity of real problems is large and we will need ADP we will discuss what is the best from the ones that I have presented in preliminaries>

3.3.1 Time value of money

As outlined in the preliminaries, cash does not have the same value through time. This economical truth is one of the most important ones in project valuation and capital budgeting. The effective rate of discounting depends on an information if we own money or we hold money.

<Since the reward function represents the free cash flow and thus money, we need to adjust for that>

<To incorporate the time value of money to the dynamic programming we need to introduce the discounting factor>

<The usage of such factor is nothing new for DP. In the problems with infinite horizon, it helps to converge the sum of rewards and it generally represents the widely accepted psychological truth (cite) that reward now is better than reward later.>

<Present the updated bellman equation>

[Problem with the discounting factor as a function of debt>]

3.3.2 Risk-aversion of investors

<There is inherent uncertainty in each project>

<The basic reasons and observations about risk-aversion were already stated in the preliminaries>

<Investors are risk-averse and we want to model that>

<A natural framework for that is the notion of utility from SDT. We will maximize utility instead of actual monetary value. >

<New bellman function, where we maximize utility over actions, not the monetary reward>

<It might be a very complex task to get the utility function from the investor in reality, sad truth.>

3.3.3 Probability

<The actual transition probability function is unknown or it's notion might not make sense at all in reality (maybe the assumption of prices coming from a distribution is not fulfilled). We still need to quantify our beliefs about the project future, usually backed by data and simple models in economics.>

<There are many ways to obtain the estimate of probabilities.>

<Models with data used in both SDT and Economy. How do they do it?>

<Risk neutral probabilities in ROA, where does it come from? >

<Experts and Bayesian updating and insufficient reasons are standard in SDT. There are procedures that enable us to mix different sources of information>

<It is up to the statistician using this new valuation technique to choose>

<We cannot say what is optimal and the details of the choice are out of scope of this thesis>

<We prefer Bayesian updated risk-neutral probabilities (Which can be understood as data-based expert knowledge) for the variables where there is a lot of data and there is a strong case of the model working>

<We prefer expert knowledge with theory of insufficient reasons for the cases when there is little to no relevant data to base our model on>

This section finalizes the core idea of this thesis. With the presented techniques and detailed description of each part of the framework we are now theoretically able to run the algorithm and get the valuation of any project that can be defined as **??**.

A relevant aspect that needs to be taken into account is the actual plausibility of running the DP algorithm due to its possibly extreme computational complexity. This issue will be illustrated [or not] in the next chapter.

Chapter 4

Valuation of facilities with simple cycle time model

In this chapter we aim to illustrate the new perspective on the problem of project valuation on <this example> because <a reason>.

First we will focus on the dependency between project value and the amount of available managerial actions, allowed by its structure.

In the second part of the experiment, we will take one of the first three setups regarding available project actions and investigate at the sensitivity of project value on the granularity of its structure. First we experiment with changing the time granularity corresponding to the ability to act in smaller time intervals. Next we increase the granularity of the price modeling of inputs and outputs. Finally we fuse these two project structure changes into a final experiment.

In the first half of this chapter we expect to gain more valuable projects with the increasing managerial ability to act. On the contrary, there is no clear expectation of results in the second part of the experiments. We have no strong opinion about what the increased granularity in price modeling, time intervals or their combination will do with the valuation of the alternative projects.

4.1 General settings of the experiment

<First identify the initial common parameters for the experiment>

Time set <Time set mathematical expression, with the reasoning behind this decision, i.e. Gas power plant, decisions per month, lifespan of the plant 20 years.>

State set <State set, defined in "rectangular way", some states are logically unreachable. Vector with encoded state of the environment. For example [built/not-built plant, index of gas price increase/decrease, index of power price increase/decrease>

<Mathematical expression for the state set>

<Detail elaboration what the encoding means for each encoded state>

<Discussion about the model of states. Up movements volatility could be taken from the actual volatility on the market, we are just putting some values.>

Action function <In the most simple case presented for now, we expect only timing option to be available>

<Mathematical expression of the action function>

<We will increase the size of this set by adding new possible managerial actions in two levels in the following paragraphs>

Transition probability function <We have to cope with four sources of uncertainty: price of gas, price of CO2, price of power>

<We are modeling all of them by the historical prices and the log-normal model, since that is standard>

Reward function : <The FCF model for the gas power plant and other facilities in its class is rather simple>

<We do not assume the price of labor which tends not to be very volatile>

<Mathematical model of the FCF, easy and clear>

4.2 Sensitivity towards potential action set

<In this section we will focus on the effects of adding managerial actions on the valuation, in both complexity of computation and results>

<We will add two levels of possible managerial actions and we will investigate the changes>

Action set 1 : <We are adding these types of actions to the model>

<The reward function changes in this way>

<The probability function changes in this way>

<Results>

Action set 2 <The reward function changes in this way>

<The probability function changes in this way>

<Results>

Action set 3 <The reward function changes in this way>

<The probability function changes in this way>

<Results>

4.3 Sensitivity toward time epoch granularity

<Now we will study the sensitivity of the valuation for the length of the time epoch>

<The structure for this model is the same as in example ... above, because <reason»

<We will change the time set to <Mathematical expression»

<Results>

With this result we finish the chapter of experiments. Its results and implications will be discussed in the next chapter.

Chapter 5

Discussion

In this chapter we will discuss both the results of the 6 variations of a single project from chapter 3 and its theoretical background from the Chapter 2.

Comparing the results of different granularity of ... we can state that ...

Looking back at the formulation ... there is a potential for improvement in ...

With the newly obtained knowledge we can state that the new valuation technique helps with ... and is more general than the techniques used nowadays. All this while preserving ... and hopefully not exceeding the mathematical capabilities of the potential users.

Chapter 6

Conclusions

The core message of this thesis is to interpret the problem of project valuation in the form of stochastic decision making. The contributions of the newly presented valuation algorithm in contrast to already existing techniques are:

- Usage of general distributions
- Theoretically any number and type of actions
- ...

Furthermore, the thesis copes with the problem of computational complexity, arising as a result of high-dimensional problems, with identifying a <ADP theory> as the best fitting algorithm from the class of ADP for the problem of project valuation.

The new approach to project valuation is demonstrated on six variations of one project type, which show its real applicability in real world. First three examples confirm the expected sensitivity of the project's value on the level of possible managerial actions, endorsing the idea that projects with higher degree of managerial action space have more value. The second trio of experiments shows how is the valuation sensitive on the choice of SDT framework. We conclude that … <probably not much>

The limitations of this approach are:

Finally, through the time of writing this thesis I have identified the following directions for further research as:

- ...

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