Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Real Options Valuation: A Dynamic Programming Approach

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Master's Thesis

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ZADÁNÍ DIPLOMOVÉ PRÁCE

Student:	Bc. Filip Rolenec
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Název práce (anglicky):	Real Options Valuation: A Dynamic Programming Approach

Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstrujte jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.

Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.

2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.

3. Powell, Warren B, Approximate Dynamic Programming: Solving the Curses of Dimensionality. 2nd ed. Hoboken, NJ: Wiley, 2011.

4. Puterman, Martin L., Markov Decision Processes: Discrete Stochastic Dynamic Programming. Hoboken, NJ: Wiley, 2005.

5. Vollert, Alexander. A Stochastic Control Framework for Real Options in Strategic Valuation. Boston, MA: Birkhäuser, 2003.

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V Praze dne 11. října 2019

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I would like to thank my supervisor Ing. Rudolf Kulhavý, DrSc. for his professional guidance and all the advice given while creating this thesis.

Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, July 5, 2020 Filip Rolenec

Název práce:

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Autor: Filip Rolenec

Obor: Matematické inženýrství

Druh práce: Diplomová práce

Vedoucí práce: Ing. Rudolf Kulhavý, DrSc.

Abstrakt: Investiční příležitosti jsou v současné době oceňovány pomocí řady algoritmů a metrik vzešlých z ekonomické teorie. Nejčastěji používaná metoda *diskontovaných peněžních toků* (DCF) zohledňuje časovou hodnotu peněz a pro jednoduché projekty dává investorům velmi dobré odhady s minimálními požadavky na matematické znalosti. Složitější projekty, které v této práci chápeme jako projekty s vysokou mírou neurčitosti a existencí následných manažerských rozhodnutí, je možné oceňovat pomocí teorie *reálných opcí* (ROA). Metoda ROA vychází z nedokonalé analogie oceňování finančních opcí a přiznává hodnotu možnostem změny projektového plánu.

Tato práce má za cíl představit nový rámec pro oceňování investičních příležitostí, jejichž řízení je chápáno jako stochastický rozhodovací problém. Tento rámec, umožňující využití desítek let výzkumu v oblasti stochastické rozhodovací teorie, pokrývá metody DCF a ROA, přičemž zjemňuje jejich předpoklady. Hlavní přínosy nového oceňovacího rámce jsou: možnost zodhlednění více zdrojů neurčitosti, modelování neurčitosti libovolnou distribucí, přímočaré začlenění bayesovského učení, modelování přístupu k riziku rozhodovacího subjektu a libovolný počet i druh povolených manažerských akcí.

Nový rámec znatelně rozšiřuje třídu projektů ocenitelných s velkou přesností a lze ho chápat jako sjednocující zobecnění technik oceňování v podnikovém řízení.

Klíčová slova: Analýza reálných opcí, Black-Scholes model, Diskontované peněžní toky, Dynamické programování, Energetika, Oceňování projektů, Stochastické řízení

Title:

Real Options Valuation: A Dynamic Programming Approach

Author: Filip Rolenec

Abstract: The valuation of investment opportunities is currently done via metrics and algorithms formed by the economical theory. The majority of investors and companies still values projects with a method of *discounted cash flows* (DCF), which takes into account the time value of money and gives solid results for simple projects with minimal requirement on mathematical skills. More complicated projects, which are in this thesis thought of as projects with a substantial degree of inner uncertainty and with an existence of further managerial decisions, can be valued by the *real options analysis* (ROA). This method comes from an imperfect analogy to financial option valuation and it recognizes the value of the ability to change the course of a given project.

This thesis presents a new valuation framework for projects, which are understood as problems of optimal stochastic decision control. This framework incorporates the DCF and ROA methods, simplifies the requested assumptions and allows for decades of research in the field of *stochastic decision theory* (SDT) to be used. The main contributions of the new framework are: ability to incorporate multiple sources of uncertainty, usage of any distribution for modeling the uncertainty, ability to conveniently incorporate Bayesian learning, ability to model user's approach to risk and ability to model any type and scale of managerial actions.

The new framework significantly expands the class of projects that can be reasonably valued and it can be understood as a unification of project valuation in business management.

Keywords: Black-Scholes model, Discounted cash flow, Dynamic programming, Power industry, Project valuation, Real option analysis, Stochastic decision control

Contents

Chapter 1

Introduction

The ability to systematically and reliably value projects is the core of investment decision making. According to the economical theory of [] [1] an investment that is able to generate larger profits is better for the people that are influenced by such an investment than its alternative with lower profits. The ultimate metric of value added to the participants of market is according to [] the price they are willing to pay. Higher profit margins, and thus profits themselves indicate, in free market, the will of customers to pay extra for the additional value in their lives coming from the purchased product. To correctly value a project that is subject to some investment decision making, is to improve the lives (on average) [2] of the participants in that market.

The correct valuation technique enables companies to increase their profits and as [] says: the main and actually only goal of manager is to increase the wealth of stakeholders. It is rather fortunate that in free market, increasing the value for shareholders means to create profits, which are a symbol of adding value to market participants. One could thus extrapolate and say, that in free market, the goal of a manager is to improve lives of other participants in the market.

The current state of capital investment valuation techniques is very diverse. Majority of investors relies predominantly on the standard Net Present Value (NPV) technique or its slight generalizations in form of NPV scenarios, risk adjusted discount rate (RADR) hurdles (usually set to an arbitrary percentage), internal rates of return (IRR) and other metrics like ROIC, ENPV,....

<Talk about simple valuation techniques>

<Talk about ROA, its advantages and usage in real life> [3]

<Talk about it not being applied in reality and its reasons>

<Decide which problems will be solved in this thesis>

<Talk about how this all is actually decision making and that it would be interesting to look at it as a SDT problem>

<Say a brief history of SDT and how it fits naturally>

<Say what is missing in SDT and what more does it offer, arbitrage vs utility thinking and Bayes>

························· **The remaining text is left as inspiration** ·························

The current state of capital investment valuation techniques is a mess. Majority of investors still rely on the standard Net Present Value (NPV) technique or its slight generalizations in form of optimal/average/bad scenarios, risk adjusted discount rate (RADR) hurdles (usually set to an arbitrary per-

[1] That guy from Duke university

[2] one participant can pay loads of money or many can pay a lot... How to compare this is hard.

[3] Do not talk about DTA in the complicated sense.

centage), internal rate of return (IRR) metric (which is essentially the same as RADR hurdles) and other metrics like,... ROIC, ENPV.

All the listed generalizations try to cope with one of two main problems of NPV, which is inability to incorporate uncertainty (ENPV, scenario approach) or intra-investment comparison (IRR(?), ROIC).

None of these valuation techniques acknowledges the managerial ability to take action and improve the course of an investment. ENPV can be interpreted is such way but that is abusing of its notation.

The decision tree analysis (DTA) is the simplest approach for valuing an investment with acknowledging decision nodes as an inherit trait of an investment process. It should not be a surprise that being able to make a decision through the lifetime of an investment has value. Also, it is quite clear that the more uncertain the investment and its parts are, the more valuable the ability to act is. To value an ability to make a decision and change the path which the investment follows is not a complicated task. All it takes is to compare a value of a project with that ability and without it (both with DTA), and declare that the value of such "decision option" is the difference between the two.

Many authors from the financial world [4] use the term real option valuation when talking about this difference between DTA trees with or without some branches that represent the ability to make a decision and act upon it. I think that is rather unfortunate, since the terminology clashes with the real option valuation, which this thesis is about.

The true real option analysis (ROA) is about being able to value investment options in the same way financial options are valued. The Black-Scholes-Merton option pricing model that is widely used in financial world is an exceptionally elegant valuation tool. All that it needs to know about financial option to valuate it is four parameters - the strike price and expiration date of an option itself, the volatility and current price of the underlying asset upon which the option is written. [5]. This Nobel-Prize winning elegance led mathematical and financial experts to try to interpret investment decision making in terms of "real options", that can be valued in the same way the financial options are, by the BSM model.

The attempts to interpret each of four important parameters of a financial option in terms of investment parameters are successful for simple cases (Dealership in BERK), however they fail for more complex ones. The ability to use real option valuation technique depends heavily on the ability to find a relevant company, stock or other publicly traded proxy variable to the examined investment. One finds plausible to construct a replicating portfolio for an investment in oil mining facility or a food production line, however this assumption is hard to believe for highly innovative products in new fields, such as software development or R&D investments.

The ROA valuation approach in investment decision making has cyclical nature. The first wave of the real ROA approach can be traced to ... who presented his ideas shortly after the initial publication about financial option pricing [5]. Next wave comes with ... and finally the most recent publications trying to make practical use of real option analysis are [] and [].

Nearly all publications of real options published in the last <number> years talk about small penetrability of this approach to the actual usage by managers in large and small investment companies. The stated reasons are always the same,

- Vollert says that its complexity, but that is because he uses stochastic models, Ito lemma and computes PDEs, all the time. That has nothing to do with replicating portfolios and BSM model.
- Mr. Kulhavy says that the problem is in finding the replicating portfolio and even if you find it you have to persuade the manager to see it too.
- <Problem with adoption>

[4]See *this* and *that*

[5]probably some more assumptions

- <Another problem with adoption>

I believe that the current usage of ROA by investors makes sense. The actual ROA approach is based on finding a replicating portfolio that behaves in the same way as the investment. If a manager beliefs that he has found a replicating portfolio, then it does not make sense for him to value the project differently than the replicating portfolio. However, there is still the issue of finding such replicating portfolio.

The usage of ROA in sense of different DTA trees is well posed. The ability to hold production in refinery in times of large oil prices can surely increase its value. In this sense the ROA has the biggest impact on the valuation in investments that are highly uncertain and in which there are actions that can alter the course of future cash flow. This also means that in the opposite case, when there is high degree of certainty about the execution and cash flow of the project, or if there are no actions to be taken, the ROA does not create additional value.

The power of classic ROA is to seemly eliminate the probabilistic nature from the picture. This can be done by the power of all-wise market, that forbids the existence of arbitrage. If a tradeable [6]replicating portfolio is found, then I should not care if I make my investment or if I use the money to buy the replicating portfolio.

For the mathematicians from the field of stochastic decision making the problem of valuation is just another decision making under uncertainty. Define state and action space, define rewards (free cash flow) in each epoch and get prior probability density as expected transition probability from one state to another upon undertaking certain action.

The difference between the SDT and ROA is that the former relies on the prior density functions, that are the core of Bayesian statistic, while the latter, coming from economics, relies on the power of the market and managers ability to find a replicating portfolio.

In this thesis the state-of-the art theories of ROA valuation and SDT are presented. Their strengths and weaknesses are discussed and a hybrid approach from the cherry-picked parts of the corresponding theories is presented.

The power of the new theory is shown on a class of problems called ... The current literature copes with them in much simpler way and their assumptions are much larger. My [7] approach also potentiates adoption in practical usage, as it puts the decision making manager in charge. The manager decides how to create the prior distributions, he is the one who needs to determine all the possible actions and scenarios that can happen. The algorithm that is presented in this thesis can and should be used in real world as a managerial tool to maximize the profits of divisions or companies as a whole.

The goal to maximize wealth in financial world is approached in a really narrow matter. Money represents value, which in models decreases in time by a percentage given by some authority, most of the time central banks. In respected financial publications [3], [] or [], there is no notion of utility. It seems like for financial institutions, it does not matter to get 1M in cash now or to gamble for 2M in a fair coin flip. On the other hand, in SDT, the concept of utility is taken very seriously. Majority of people would prefer even a very small amount, such as 200k against a fair gamble between 0 and 2M. This is due to the concave utility function people tend to have when very high amounts are discussed and the question is more about what would make them happier.

Maximization of wealth either in the sense of investment company or individual is natural. Almost everybody would agree that to have more wealth is better than have less. Better investment decisions mean higher accumulation of wealth, which together with assumption of free market results in an increase of utility of each participant in the economy [8]. In other words, when a company makes higher profits it should mean that the part of the population that participated in the wealth creation should be happier.

[6]This is a valid word, Cambridge dictionary say so...
[7]Or the presented approach, should I avoid pronouns in the text?
[8]Find that reference... Duke university lectures...

. . . .

Chapter 2

Preliminaries

································ Introduction to preliminaries ································

To properly understand a mathematical text it is important to first define the used notions and symbolism. Since this thesis is based on many different authors, from both financial and mathematical world, a short unifying overview of the used theory is important.

The notation used in this thesis comes predominantly from the most influential authors in the respective fields of study:

- general economy [3];
- real options [9];
- stochastic decision theory [1].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU and its applicability is proven in his previous works [19] and [20]. [1]

··

2.1 Used mathematical symbolism

···················· Sets, random variables, what do I mean by 'probability' ····················

In the whole thesis, bold capital letters, such as $\mathbf{X}$, represent a set of all elements $x \in \mathbf{X}$ as in [19]. The cardinality of a set $\mathbf{X}$ is denoted with two vertical lines as $|\mathbf{X}|$. Random variables, understood in a sense of the standard Kolmogorov's probability theory [13][2], are represented with a tilde above the variable, i.e. $\tilde{x}$. Realizations of random variables are denoted by the same letters as the random variable without the tilde, i.e. x.

Definition 2.1. *(Probability) Let $\tilde{x}$ be a random discrete variable. Then $P(x)$ denotes a probability that the realization of $\tilde{x} = x$. Similarly if $\tilde{x}$ is a continuous random variable, then $p(x)$ denotes a probability density of the realization $\tilde{x} = x$.*

Remark. *To rigorously unify the notation and simplify the formulas a Radon-Nikodým (RN) density [18] is introduced with the notation $p(x)$ and the name "probability density". The dominating measure of this RN density is either the counting measure (in discrete case) or a the Lebesgue measure (in continuous case). The notation $P(X)$ is reserved only for the cases when the discreteness of the argument needs to be emphasized.*

[1] Filip == Author or Filip == I/Me

[2] Does this citation make sense?

The last general definition is the definition of well known concept of conditional probability [10].

Definition 2.2. *(Conditional probability) Let, depending on the context, symbol $p(x|y)$ represents either the conditional probability on discrete variables or the conditional probability density on continuous variables. Then the $p(x|y)$ is defined as:*

$$p(x|y) = \frac{p(x, y)}{p(y)}, \tag{2.1}$$

where $p(x, y)$ is a joint probability density of x and y.

Remark. *The definition of conditional probability expressed by the equation (2.1) corresponds with the classic definitions of the conditional probability and conditional probability density in both the discrete and continuous case.*

<Probably some other definitions that will be needed in the following chapters>

2.2 General economics

Introduction and basic financial concepts.

This thesis is built on two main theoretical pillars, the theory of corporate finance [3] and stochastic decision theory (SDT) [1]. A basic review of corporate finance terminology and procedures is presented in this section with a focus on project valuation techniques.

Definition 2.3 (Project)**.** *A project is defined as a piece of planned work or an activity that is finished over a period of time and intended to achieve a particular purpose, mainly a wealth increase of a company or an individual.* [3]

Definition 2.4 (Value)**.** *The amount of money that can be received for something.* [4] [5]

A value of a project is naturally a function of realized elementary monetary transactions within the project and the potential selling price of remaining assets. To track and model each transaction of a project in detail is in principle possible, but such approach would be way too complex for practical usage. Furthermore, its benefits would most likely not be significant enough to defend the extra effort of decision makers.

For purposes of project valuation an aggregation of elementary monetary transactions - a concept of cash flow (CF) and free cash flow (FCF) are used in the world of corporate finance.

Definition 2.5. *(Cash flow) Cash flow is the net amount of cash and cash-equivalents being transferred into and out of a business (project).* [6]

Definition 2.6. *(Free cash flow) The incremental effect of a project on the firm's available cash is the project's free cash flow [3]:*

$$FCF = OCF - CapitalExpenditures, \tag{2.2}$$

where OCF is the operating cash flow and CE are the capital expenditures.

[3] First part comes from Cambridge dictionary. Is that ok just to cite it?

[4] Again cambridge dictionary...

[5] This definition is added to say that by value in economics we mean money that we can obtain from the asset.

[6] Investopedia, economical books do not define it.

Cash flows are in their detail nature discrete, each transaction within the project changes the global cash flow. The moments in which transactions legally take place could be taken as individual time instants.

However, it is easy to imagine understanding the cash flow as a continuous stream of money per time. By the nature of corporate management, one could also expect, that in majority of non-extreme applications, this would not result in a major distortion of cash flow reality. [7]

All of this is true also for free cash flow.

In this thesis we will further work only with the FCF, so from now on, the term *cash flow* will only mean free cash flow. [8]

. . . .

2.2.1 Standard valuation metrics

. . . . NPV, DTA, IRR, WACC and others . . . .

Cash flows capture information about value added to a project in given periods. Due to time value of money and different lifespans of projects, one has to come up with algorithms for their consistent and systematic valuation, according to their cash flows.

Valuation techniques are attempts to aggregate cash flow vectors in one meaningful number to enable decision makers to choose the best investment.

Net resent value First valuation technique, net present value, is arguably the most used valuation technique in capital budgeting [] [9]. Its computation is simple and it can be described as a sum of cash flows discounted for the time value of money:

$$NPV = \sum_{t \in \mathbf{T}} \frac{C_t}{r^t}, \tag{2.3}$$

where $\mathbf{T} = \{0, 1, ..., |\mathbf{T}|\}$ is a set of time periods in which the cash flows C_t are obtained. These periods are usually years or months, but they can effectively have any granularity the decision maker wants them to. Discounting factor r expresses the time value of money and is usually derived from the current risk-free interest rate given by the central bank of a nation. [10]

The NPV valuation technique is simple to use, assuming we know the discount rate, which is constant through the project's duration, and free cash flows, that are assumed to be certain. [11]

A more advanced approach that acknowledges the variability in both cash flows and risk-free interest rate is called expected NPV(ENPV) :[12] [13]

$$ENPV = E\left[\sum_{t \in \mathbf{T}} \frac{\tilde{C}_t}{\tilde{r}_t^{\,t}}\right], \tag{2.4}$$

where both cash flow and interest rate distributions are expected to be known.

[7] Should I discuss here more. I mean that corporate management does not care about each transaction in each store, it cares about daily or even monthly revenues.

[8] Maybe confusing, consider to define only FCF and call it cash flow...

[9] Is this a strong enough statement that I need to cite somebody?

[10] The discussion about negative interest rates and thus r<1 and also the correct discounting rate is left to my valuation in chapter 3.

[11] Should I put an example here or would that be too trivial? Cash flow is [-400, 100,100,200,200], interest rate 5% -> NPV is...

[12] Seems like literature does not use this and I made it up. In the eyes of economics, the cash flows seem to be expected values all the times, the distribution is not considered.

[13] Also there is ENPV as "Expanded NPV" NPV+options value used by Vollert, Pindyck and others.

Decision tree analysis The simplest valuation technique that acknowledges the importance of further management of investments is called decision tree analysis (DTA) [21]. In addition to time value discounting of cash flows it offers a framework that can incorporate active management of a project, potentially increasing its overall profits.

The ability to make actions in projects is sometimes interpreted as having *Real Options* [8] or [21].[14] This confusing terminology might result in misunderstandings. That is why it needs to be emphasized that, in this thesis, valuation that recognizes the ability of a manager to act but ignores the *law of one price* will be called DTA.

Definition 2.7 (Law of one price)**.** *<Definition of Law of one price>*

The decision tree analysis is usually used only for valuation of projects with very simple scenarios. However, its structure could be potentially used for much more complex problems.

The criticism of DTA coming from Vollert [21] and others [] is that DTA uses a single discount rate for different branches of the project. This is contradictory to the standard rule in the economics that riskier projects should be discounted more [].

However, this criticism could be countered with upgraded DTA in which one would allow variable discount rates in different branches rather easily. Furthermore, both Vollert [21] and ... [] do not address the problem of the constant discounting rates and use them in their final valuation algorithms as well. [15]

The discussion about risk and its role in a valuation algorithm is deferred to the section **??**.

Internal rate of return The internal rate of return serves usually as a basic threshold for enterprises entering new projects. Each company would have their own internal threshold of IRR, based on their confidence in ability to make more or less returns on their investments. A startup would most likely have an IRR higher than a long time established bank.

Definition 2.8 (IRR)**.** *The internal rate of return is the interest rate that sets the net present value of the cash flows equal to zero [3]. This means that IRR for cash flows C_t needs to satisfy the following equation:*

$$0 = \sum_{t=0}^{T} \frac{C_t}{(1 + IRR)^t} \tag{2.5}$$

The problems with IRR are that additional assumptions on cash flow vector have to be satisfied so that there exists only one unambiguous result of the equation 2.5. [16]

Weighted average cost of capital **<This paragraph is not true, needs to be redone. The WACC is a combination of a premium that investors want in order to hold the shares of the company and a rate for which the company can borrow money. The risk free rate does not play any direct role in WACC. >**

The term time value of money represent the truth that the value of money today is not the same as next year. Due to the modern economical theory, inflation is a pursued phenomenon, keeping the economy running.

The existence of a risk-free interest rate in an economy is assumed and it is usually determined by the rates of bonds issued by the central banks of given nations.

[14] Check if Vollert uses ROA as a lens only...

[15] Needs to be confirmed

[16] Or maybe there are no clear assumptions, but I know that the result of the equation might not exist or it might have multiple results.

In capital budgeting the time value of money has a close connection to cost of capital. The cost of capital can have two forms, based on the money source for your investment. In the first case, you have some cash reserves and by investing in a project the only cost of capital is that you forego the opportunity to invest in bonds with risk-free rate of return, "paying" the opportunity cost. In second case, if company's funds are not sufficient, you need to borrow money on the market for an interest rate, which results in additional "cost" of capital.

The interest rates in second case are naturally higher, since lenders carry additional risk of borrower's default.

The term weighted average cost of capital combines the costs of capital coming from the two sources, further incorporating the notion of corporate tax reliefs.

Definition 2.9 (WACC)**.** *The firm's effective after-tax cost of capital is called the weighted-average cost of capital (WACC):*

$$WACC = \frac{E}{E+D} \cdot r_E + \frac{D}{E+D} \cdot r_D(1 - \tau_C), \tag{2.6}$$

where E is the value of equity, D is the value of debt, r_E the equity cost of capital, r_D the debt cost of capital and τ_C is the corporate tax rate.

. . . .

2.3 Real option analysis

· · · · · · · · · · What is ROA, introduction and hint of larger future discussion in the next chapter · · · · · · · · · ·

An option in financial world means having a right to buy (or sell) an asset in future for a fixed price (strike price) [3]. Option trading has origins in commodity markets (for example corn or oil), where participants want to in some sense insure themselves against the negative movement of a price on the market. Options that are being traded today on the derivative market span almost every tradeable asset that can be though of [17].

The value of an option naturally depends on its time to maturity, current price of the asset and a strike price. A proper valuation method for European options with no dividends came with the Black-Scholes-Merton model [5], which in addition requires only the volatility of the underlying asset and the following assumptions:

- Effective markets ...
- Log-normal distribution of the asset price
- There are no transaction costs in buying the option
- The risk free rate is known and constant.

This established and well received technique for financial option valuation spawned the idea of real option analysis. The option to buy an asset for a given price is similar to the ability to buy an expected future cash flow, i.e. invest in a project.

<Origins of real options> The first economist, pioneer of the term *Real option analysis*, ... treats investments as a complete analogy of trading with options. To be able to delay an investment has a value and.....

[17] Find some citation or do not use this sentence.

The usage of the phrase real option analysis in literature is rather fuzzy. It is used by many authors such as [3], [9] and [8], however their usage of the term differs. Three different usage classes of the term real option analysis were identified by the author. They differ by the level of analogy to the valuation of financial options.

Complete analogy First, there is a class of complete analogy. All five parameters of financial options needed for their valuation with BSM model are identified with the parameters of an investment opportunity. An example of a valuation of a car dealership can be found in [3][18] where the following identification is made:

Financial option	Real option
Stock price	Current market value of asset
Strike price	Upfront investment required
Expiration date	Final decision date
Risk-free rate	Risk-free rate
Volatility of stock	Volatility of asset value
Dividend	FCF lost from delay

Table 2.1: Identification of parameters for real options with respect to the financial option [3].

Another example can be found in [17] where a telecommunication company is being valued by the complete analogy valuation technique.

This class of authors focuses on the clear analogy and thus the acknowledged scope of possible manager's actions is limited basically only to timing options. The only decision is to invest to a project now or later.

Partial analogy The second class of authors uses only the core property of the financial analogy and that is the law-of-one-price. With the help of this assumption, the authors, eg. [9] usually derive risk-neutral probabilities which are then used for modeling of some internal variable of the cash flow functions.

This class of authors is the most numerous and most mathematically rigorous. The core of publications in this class is usually in solving stochastic differential equations, eg. [21] whereas the role of the assumption about law-of-one-price is used mainly as the ground for obtaining one of the missing parameters of the stochastic model.

No analogy Third class of authors does not use the law-of-one-price and is thus the farthest away from the original idea of ....[19]. This class of authors, i.e. [11] and [8], understands the term real option analysis as a useful lens for looking at the project valuation. They accentuate the value of further managerial decision, but the valuation structure and algorithms do not differ from the DTA approach as defined earlier.

Thus, this class of authors is declared as a misuse of terminology and not further considered.

In the next part of the thesis, namely **??** the core message behind the term real option analysis will be thoroughly discussed.

. . . .

[18] Should I reference page numbers? Will anybody actually look up the example?

[19] The pioneer name

2.4 Statistical decision theory

$\cdots$ Standard SDT as a framework, states, actions, inputs, outputs, rewards, probability distributions $\cdots$

The second pillar upon which this thesis stands is the statistical decision theory (SDT). An area of applied mathematics that formalizes and studies optimal decision making of agents. As decision making in its broadest sense encapsulates a vast amount of human behavior, the class of problems it is able to solve is quite large.

The SDT's main focus is to determine the optimal strategy (a sequence of decisions) to act upon, generally in dynamic and uncertain environment. A classical structure of a decision making problem consists of five building blocks

- Set of time epochs - $\mathbf{T}$;
- Set of environment states in those epochs - $\mathbf{S}$;[20]
- Set of actions in those states - $\mathbf{A}$;
- Reward function of transition from one state to another - $r(s_t|a_t, s_{t-1})$;
- Transition probabilities governing the transitions from one state to another $p(s_t|a_t, s_{t-1})$.

The set of time epochs, states, actions is usually known, defined by the structure of the decision problem that is being solved. Reward and transition functions tend to be unknown in solving these problems and they need to be often somehow estimated.

Usually, the biggest task in SDT is to correctly approach the uncertainty about transition probabilities between the different states of a project. There are two approaches to parameter estimation in statistics, classical approach and a Bayesian approach. Since the Bayesian approach seems to fit the format of decision making better - allowing for smooth updating on newly observed data - it is used in this thesis.

The goal of SDT is to find the optimal strategy - sequence of actions. The optimality of such strategy is defined as it having the maximal expected cumulative reward among all eligible strategies

$$\pi^* = \arg\max_{\pi\in\mathbf{\Pi}} E\left[\sum_{t\in\mathbf{T}} r(s_t|a_t, s_{t-1})|\pi\right]. \tag{2.7}$$

This maximization can be in total absolute values (in finite or discounted cases) or per time period (mostly in infinite non-discount cases). Due to the economic nature of this thesis we will focus on the total cumulative reward of a finite process (?).

2.4.1 Dynamic programming

To maximize over all possible strategies by computing the expected cumulative reward for each one of them is a very demanding task even for low-dimensional decision problems.

Thus, a clever idea of backward induction called dynamic programming is used. A function, called the value function is defined on the set of all possible states $\mathbf{S}$. This function represents the expected cumulative reward to be obtained from the given state onwards. The idea of backward induction is based on the truth that a sequence of actions is optimal if and only if the last action is optimal.

This clever computation of value functions from the problem horizon backwards through all the possible states of the problem decreases the complexity from exponential to polynomial. Instead of

[20]Possibly different S_t in different times.

maximizing over $|\mathbf{A}|^{|\mathbf{S}|^{|\mathbf{T}|}}$ possible strategies at once, one needs to compute significantly less demanding complexity of $|\mathbf{A}| \cdot |\mathbf{T}| \cdot |\mathbf{S}|$. [21]

The formula representing the backward induction is called the Bellman equation:

$$V(s_{t-1}) = \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1})[r(s_t|a_t, s_{t-1}) + V(s_t)]. \tag{2.8}$$

By defining the value function on the horizon, we can compute value functions of states with lower and lower time indexes, until we get to the time 0, which represents the present. Not only that we have the expected value of the optimal decision making, but we have also derived the optimal strategy for every possible path through the state space.

The backward induction reduces the computation complexity significantly. However, for even a moderate-dimensional decision problems, the number of computations is still extremely large.

The problem of computational complexity of dynamic programming is called "three curses of dimensionality" [16] and various solutions have been proposed. These solutions are as a group referenced as approximate dynamic programming.

2.4.2 Approximate dynamic programming

The computational complexity of dynamic programming for moderate and high-dimensional decision making problems is so demanding that results cannot be obtained in a reasonable amount of time.

The response to this problem comes in a form of approximate dynamic programming, a section of decision making under uncertainty, that is represented by a number of algorithms that are trying to obtain quasi-optimal strategies with more reasonable demand for computation power.

There are many different algorithms, that try to obtain approximate results of the precise dynamic programming represented by the bellman equation. In this thesis the ADP algorithm called <Q-learning, SARSA...> is used because of its high performance in ..., while being still relatively easy to implement. A longer discussion of its choice is left to its corresponding chapter **??**.

<Q-Learning, SARSA,..> <Detailed description of the chosen ADP algorithm>

2.4.3 Bayesian statistics

The field of mathematical statistics can be divided into two branches, classical (also called frequentist) and Bayesian. The philosophies of each one are fundamentally different, however in principle, they can serve for revealing new truths of the measured data in a similar fashion.

Mathematical statistics is a very broad topic, not possible to summarize it in one paragraph. The use of Bayesian statistics in this thesis is only as a tool, no broader discussions about the internal philosophy of different approaches are presented.

In general, statistical theory is used to determine a distribution from which the observed data come from. In majority of cases, it is assumed that the data are realizations of a random variable with a distribution from some parameterized class - normal, log-normal, poisson, etc. The goal is then to determine, with some level of confidence, the parameters that fit the observed data in some sense the best. [22]

The main difference between the Bayesian and classical statistics is how the parameters of a distribution are perceived by the statistician. In the classical theory, it is assumed that observed data come

[21] Check this

[22] Large simplification, statistics can be used in many different ways.

from some distribution with some firm but unknown parameters Θ. In contrast, the Bayesian view on the parameters is such that they are perceived as random variables $\tilde{\Theta}$.

This terminology twist can be a source of initial confusion for frequentist statisticians, but it allows a simple and elegant update of parameter estimates with the Bayes formula.

$$p(\Theta|d) = \frac{p(d|\Theta)p(\Theta)}{p(d)}, \tag{2.9}$$

where Θ is generally a multivariate parameter and d are observed data. [23]

The interpretation of Bayes formula, is that the distribution of parameter $p(\Theta)$ called the prior distribution, is updated for the newly observed data d, providing new, posterior, distribution $p(\Theta|d)$.

This update can be understood as learning about the "true value" of a parameter, which is very useful structure for dynamic decision problems.

Since the Bayesian theory tells us only how to update an already existing distribution, a prior distribution needs to be given, even though no data were measured yet.

This problem is in Bayesian statistics understood as an advantage, since one can use his knowledge about the problem that is being solved and incorporate it to the prior distribution, which is then updated on the measured data.

The task of consistent creation of prior distribution is a complicated topic and can be found in more detail in [2]. Furthermore the prior information always exists, as Peterka [15] puts it: "No prior information is a fallacy: an ignorant has no problems to solve".

2.4.4 Utility

The concept of utility instead of monetary or other globally measurable gain comes in when the gains are valued non-linearly.

Multiple studies show [24], that the majority of people are risk-averse, meaning that the value of uncertain monetary gain is not equal to its expected value.

One of the simplest example to demonstrate the usage of utility is given by [1]. Imagine an individual is given a choice, either to get 500$ right away or to gamble for 1000$ in a fair coin toss. A rational decision maker driven only by the expected value of his actions would be indifferent to the two choices. However, the majority of people tend to take the certain amount instead of gambling.

This example can be reformulated as follows: How much money would the decision maker need to obtain for certain so that he would be indifferent to gamble for a 1000$. In other words, how much the risk-averse person values that gamble.

The non-linearity of utility obtained from large amounts of money is only more understandable for very large sums of money. There is a little difference for an average human in obtaining 10M USD and 20M USD. The change in the person's life will be almost the same and presumably positive. However one result is certain and the other one has only a probability of 1/2.

Another interesting example of the risk-aversion of people is the famous St. Petersburg paradox first formulated by Bernoulli in 1738, [4]. A risk-neutral [25] decision maker would be willing to pay any amount of money to be able to play a game defined by the paradox. However it is shown that people seldom value the game more than 25 USD, which corresponds to a case that the initiator of the bet does not have an infinite amount of money, rather only 16,5M USD []. [26]

[23] The p(d) in denominator needs to be rewritten as integral if this formula is really to be used.

[24] Find citations (?) or omit this formulation

[25] Define risk-neutral (?)

[26] This is from wikipedia, find more cool sources. Interesting, but does not have to be in the thesis

Regarding to utility there is also an interesting asymmetry in human psychology about obtaining gains and incurring losses. The graphical expression of this asymmetry can be found in [1]. [27]

The utility function of each decision maker is different and an approximation of its shape can be obtained by an algorithm based on a questionnaire, which also ensures the consistency of responses of a given individual.

[27] Put the picture here, or cite the exact page?

Chapter 3

Real options in the language of SDT

· Why is ROA in the language of SDT studied? ·

When an individual scholar studies stochastic decision theory and then educates himself in the economical theory of project valuation, it is hard to avoid the feeling of similarities in the respective fields. The SDT guides a decision maker in uncertain environment to make actions that give him maximal reward possible based on his (usually adaptive) knowledge. The same is true for the ROA. In an uncertain economical environment, the decision maker (manager) needs to decide which project to invest in, and, in this setting more importantly, how to "control" the project when it is already running.

I would argue that it is natural to look at the valuation of real option analysis with the optics of stochastic decision theory. The reason why this relation between ROA and SDT was studied only marginally by authors like Vollert [21] and ... has in my opinion two parts. [1]

First is that ROA is not precisely defined, as discussed in the section 2.3. There is no formal definition of ROA and its usage is mostly focused on showing that further management of projects can lead to significant increase in project's value, which should be recognized in the decision making of undergoing the initial investment. Without a proper definition of the phrase *Real Option Analysis* a deeper look at its meaning is more complicated.

Second unfortunate reason is the relative closedness of scientific communities. Stochastic decision control theory focuses mostly on applications in engineering [14] or [7], game theory, ... in their established communities. As far as I know, the stochastic decision theory was not used to value project in correspondence with the economical theory - respecting arbitrage principle, usual discounting, etc. [2] The communities of economists and stochastic decision theory scholars probably does not have a large enough intersection to spawn the, in my opinion, natural idea of interpretation of ROA in terms of SDT.

· ·

The core message of this thesis is to interpret real option analysis (ROA) in the terminology of stochastic decision theory (SDT). To fairly discuss advantages and disadvantages of both theories, to take the class of problems that are being solved by ROA and empower the decision makers with decades of SDT theory knowledge.

The new valuation approach should incorporate the advantages of ROA into the stable framework of SDT, improving the performance of each approach individually, namely it should:

- Capture uncertainty of a project

[1] Or why nobody has ever done this? I do believe it is a natural thing to do...

[2] Really nobody valued project with a SDT before??? Without a replicating portfolio, there is no arbitrage principle, and the ROA is effectively only the ability to change the course of a project... Nobody has done this?

- Allow managers to implement their own approach to risk.
- Enable to rigorously handle a time devaluation of money according to the profile of the company making the investment.
- Allow to systematically compare projects in a portfolio to find the best candidates for an actual investment.
- <Add all other qualities the new valuation technique should have> [3]

3.1 The core of ROA

The Real Option Analysis is not clearly defined project valuation framework. Coming from non-optimal analogy with valuation of financial options, namely the BSM model, it brings a new way of looking at the valuation of a project.

To be able to talk about the real option analysis, we should first talk about the essence of the BSM financial option valuation model.

When the BSM model is deconstructed, it can be seen that it only describes the price of the underlying asset by one specific distribution and that the final valuation is the expected profit of the option holder.

The BSM model expects the logarithm of the underlying asset's price to follow a wiener process, a limit of random walk. This means that in each given time instant the asset price is expected to be distributed log-normally, i.e. coming from the class of log-normal distributions, which has the following probability density:

$$p(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(-\frac{(ln(x)-\mu)^2}{2\sigma^2}\right) \tag{3.1}$$

where $\sigma > 0, \mu \in \mathbb{R}$ are parameters of the distribution and $x \in (0, +\infty)$ is a variable.

The first discussed assumption of ROA thus means that, given the current price of an asset x the distribution of price in any given future time t can be described by only two parameters σ and μ.

The second assumption for BSM model is rather simple. The variation of the asset price is assumed to be known. Its actual derivation is usually obtained from the abundant historical data about the asset price as can be seen in [9].

The next assumption of the BSM model is in my opinion the core of the whole ROA approach and it is certainly worth to be incorporated into the SDT framework. This assumption is about effective markets and the law of one price. The law of one price says, that there cannot exist an arbitrage, a way to get more than the risk-free interest rate with no additional risk.

This assumption effectively results in obtaining the second parameter in modeling the price of the underlying asset as log-normally distributed, μ. Due to the non-existence of arbitrage, the value of option, its replicating portfolio and given amount of bonds have the same expected returns,the risk-free interest rate. [4]

The last assumption about risk-free rate being constant is not of major interest, it merely serves as a reminder of the time-value of money, which is taken in the financial world very seriously.

[3] For example allow for Bayesian learning, cope with a high-dimensional problems, multiple uncertainties,...

[4] Already confirmed on a binomial model, further, needs to be confirmed on the continuous case also.

To conclude, the origins of ROA in BSM model are giving us the distribution of the future price in any given time through two variables, σ obtained from the past observations of the price movement [5] and μ from the assumption about arbitrage-free world. [6]

3.2 Approach to risk

The usage of the term risk varies widely in the literature. In the spoken English the meaning of risk is the possibility of something bad happening. In parts of SDT, namely [6], the meaning of risk means the minimal expected loss. In parts of business administration [7] [12] risk means uncertainty that cannot be modeled.

The literature uses the term risk in many different meanings. In the spoken English the meaning of risk is the possibility of something bad happening. In parts of SDT, namely [6], the meaning of risk means the minimal expected loss. In parts of the <Podnikova ekonomika> [] risk means the uncertainty, that cannot be modeled. In economical books [3] or [], the risk is understood as ...

ENDED HERE, THINK MORE ABOUT RISK AND CONTINUE

The term risk is widely used not only in the economic literature. In a person that is not biased by the terminology by one of

In this chapter the term risk and its different usages in the literature is addressed. In different

<Talk about approach to risk in a sense of uncertain results> In correspondence to project valuation

3.3 Time value of money

<Talk about time value of money, what does literature say, lower, upper bounds, finding the correct discount value>

3.4 New valuation technique

[8] Now, with all the information about the current state of valuation techniques and after a proper discussion of differences between ROA and SDT approaches, lets start to construct a new valuation technique that is a combination of the best properties of both for project valuation.

First, lets clarify the important properties of a general valuation formula. It needs to address uncertainty which is effectively a part of each project. It needs to capture options, the ability of a manager to make choices that significantly change the future cash flows of a project as it has been proven that they add value. And lastly it needs to give values by which projects can be compared now, the valuation needs to acknowledge the time value of money.

An intuitive mathematical expression of a valuation that fulfills the requirements set above is given by the following general equation:

<Say that this is a formula coming from SDT, that has only economical terminology>

<Discuss why the formulation is discrete>

<Repeat or introduce the notion of partial strategy π_t and use it in this complete chapter>

<I need to distinguish between a random variable s_t and its realization>

[5] Time scale adjusted

[6] Arbitrage free world gives us the μ when there is a replicating portfolio, it cannot be used all the time...

[7] Podnikova ekonomika

[8] Come up with a clever name for it.

$$V = \max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} DC_t(s_t(\pi)) \middle| \pi\right], \tag{3.2}$$

where π is a strategy, a sequence of decisions that the manager is able to do to change the course of the project. Time evolution of the process is represented by a variable t which iterates over range of time epochs, so far finite or infinite countable $\mathbf{T}$.

<Talk about evolution of state and what is policy, what is input, output and reward possibly>

Discounted cash flow DC_t is assumed to be obtained at the end of an epoch t and depends on a state of the environment s_t, which is (most of the time only partially) a random function of a strategy π.

The general formula 3.2 is not hard to come up with or understand what it means. The actual challenge is to rigorously define its parts up to the moment, where a real project-specific values can be substituted and the actual value can be computed. The goal of the next text is to dive in the detail meaning of the equation 3.2, rigorously and justifiably describe each substitution in correspondence with the current theory and obtain actually practically useful equation.

3.4.1 Discounted cash flow

Since details of set of time epochs $t \in \mathbf{T}$, policies $\pi \in \mathbf{\Pi}$, maximization over them and expected value $E[]$ were already discussed in the preliminaries, the only unknown part of the general equation 3.2 is the discounted cash flow $DC_t(s_t(\pi))$.

Discounted cash flow $DC_t(s_t(\pi))$ is defined as a value of free cash flow, discounted by factor r_t that reflects the time value of money for the investor of a project:

$$DC_t(s_t(\pi)) = \frac{C_t(s_t(\pi))}{r_t}. \tag{3.3}$$

Even thought the adapted economical terminology requires $r_t > 1$, suggesting that $DC_t < C_t$, in this thesis only $r_t > 0$ is required. The enlargement of the r_t possible range comes from the recent unprecedented behavior of central banks, where negative interest rates are being more and more commonplace. The requirement of positive discount factor r_t represents only a clear logic, that future profits, cannot be viewed as current losses.

The usual approach to obtaining r_t has the form **??**, which discounts each time epoch with the same rate.

In this thesis the equation **??** is used only to determine boundaries of r_t with different constant discount factors r_l, the interest rate that the company would get from a bank to invest the project and r_b which is the classical risk-free interest rate considered in the financial textbooks, usually given by an interest on government bonds (in stable countries).

<The actual discussion about what discount rate means, how similar is it to utility function and other CAPM model, WACC, ...>

... which concludes the discussion about the actual values of r_t.

3.4.2 Cash flow

The ability to substitute for DC_t with equation **??** enables us to express the general equation **??** in a new form:

$$V = \max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} \frac{C_t(s_t(\pi))}{r_t} \middle| \pi\right]. \tag{3.4}$$

The cash flow C_t depends on a state of the environment s_t, which is generally influenced by former actions of the chosen policy π.

Evaluation of a cash flow is in a broad sense simple, it is the difference between sales and costs in a given time period [9]

$$C_T(s_t(\pi)) = Sa_t(s_t(\pi)) - Co_t(s_t(\pi)), \tag{3.5}$$

where Sa_t are sales and Co_t represents costs in period t. This equation can be further rewritten with the concept of unit costs of inputs Ci_t and outputs Co_t as:

$$C_T(s_t(\pi)) = Co_t(s_t(\pi))No_t(s_t(\pi)) - Ci_t(s_t(\pi))Ni_t(s_t(\pi)) \tag{3.6}$$

where No_t is the unit output of a product [10] and Ni_t is the number of input units consumed by the project in time period t. <Possibly talk about how one can introduce the input/output ratio>

Now, it is important to emphasize the different levels of dependency of the four parameters that influence cash flows on the policy π. For this purpose the usual notation of a state s_t that is used in control theory (?) is split to two. In this thesis I call them, the "branch state" b_t and the unifluenceable state u_t [11]. In mathematical notation

$$s_t = union of vectors u_t and b_t somehow. \tag{3.7}$$

<Write this as proper definitions>

As the names suggests, the unifluececable state u_t cannot be influenced by the actions of the manager, concentrated in the strategy π and the branch state b_t represent the branch of a project that it has evolved to.

Assumption 1. *Costs of inputs and outputs in the production process* [12] *are random functions of a unifluencable part of the environment state.*

Assumption 2. *Numbers of input and output units in the production process(?) are random functions of branch part of the environment state.*

With the definition of branch and uninfluencable states and the assumptions **??** and **??**, we can rewrite the equation **??** as [13]:

$$C_T(s_t(\pi)) = Co_t(u_t)No_t(b_t(\pi)) - Ci_t(u_t)Ni_t(b_t(\pi)) \tag{3.8}$$

Another reasonable assumption is that there exists a transformation parameter, that determines what amount of input units are needed to get one unit of output. This parameter usually represents the effectiveness of the used machinery or employees, which can be naturally changed, within some limits, by the actions of a manager. Lets define this parameter as input-output factor in time iof_t.

Assumption 3. *The input-output factor is a random* [14] *variable and it is a function of a branch state b_t.*

[9] Improve this definition or cite something.
[10] It should be understood in broader sense already...
[11] This is not a good notation, find out how the control theory does that.
[12] probably a better name for it exists
[13] These assumptions are a problem with examples where the model of demand starts at a time determined by a strategy itself (tolls and bridge)
[14] Maybe not everything needs to be random

The equation **??**, then becomes:

$$C_t(s_t(\pi)) = Co_t(u_t)No_t(b_t(\pi)) - Ci_t(u_t)No_t(b_t(\pi))iof_t(b_t(\pi)). \tag{3.9}$$

To get even more detailed description of a cash flow, both the prices of input and output are modeled as a function of supply and demand. This allows to make the cash flow dependent basically only on one variable in a given time period [15] and that is the required number of output units, naturally limited by the branch state.

Definition 3.1. *<Supply and demand curves definition, parametrized families>*

Thus the last form of the equation for evaluation of C_t has a form:

$$C_t = sp(No_t|\theta(s_t)) * No_t - sd(No_t iof_t|\theta(s_t)) * No_t * iof_t(b_t) \tag{3.10}$$

3.4.3 Addressing uncertainty

The last element of the general equation 3.2 that is to be addressed is the uncertainty of the random variable $\tilde{s}_t$ represented by the expectation function $E[]$.

Using the linear property of expectation E on the equation **??** we obtain:

$$V = \max_{\pi\in\mathbf{\Pi}} \sum_{t\in\mathbf{T}} \frac{E[C_t(\tilde{s}_t(\pi_t))]}{r_t}, \tag{3.11}$$

Assumption 4. *Random variable $\tilde{s}_t$ has a finite or countable set of outcomes $\mathbf{S_t}$, $\forall t \in \mathbf{T}$.*

This assumption is important only for simpler discussion later. Uncountable $\mathbf{S_t}$ could probably work too, but we would need to perform integration instead of summation [16]. A direct consequence of this assumption is that the set of outcomes for both branch states $\mathbf{B_t}$ and unifluenceable states $\mathbf{U_t}$ are finite or at most countable.

Assumption 5. *The state unifluecable state u_t and the branch state b_t are stochastically independent random variables.*

Using the definition of expected value we get:

$$V = \max_{\pi_t} \in \mathbf{\Pi} \sum_{t\in\mathbf{T}} \frac{\sum_{u_t\in\mathbf{U_t}} \sum_{b_t\in\mathbf{B_t}(\pi_{t-1})} p(u_t)p(b_t)C_t(u_t, b_t(\pi))}{r_t}, \tag{3.12}$$

and by substituting for C_t with the equation **??** we get the final equation for the project valuation:

$$V = \max_{\pi_t} \in \mathbf{\Pi} \sum_{t\in\mathbf{T}} \frac{\sum_{u_t\in\mathbf{U_t}} \sum_{b_t\in\mathbf{B_t}(\pi_{t-1})} p(u_t)p(b_t)[sp(No_t|\theta(s_t)) * No_t - sd(No_t iof_t|\theta(s_t)) * No_t * iof_t(b_t)]}{r_t}. \tag{3.13}$$

Now, there are several questions about our ability to actually compute this valuation:

- Where do the sets $\mathbf{T}, \mathbf{U_t}, \mathbf{B_t}$ come from?

[15] Cannot be seen with this notation

[16] The verb integrate probably has a different meaning.

- How to determine the discount factor r_t.
- Where to get the family of supply and demand curves ?
- How do these curves depend on the uniflueceable state u_t?
- How large is the iof_t parameter for different branches of the project?
- What are we maximizing over? What is the set of possible strategies $\pi \in \Pi$?
- How to obtain the probabilities $p(s_t)$, $p(b_t(\pi_{t-1}))$?

All of these questions will be answered in the next section.

3.5 Origins of variables in the valuation equation

The equation **??** is a final framework that is used for project valuation in this thesis. It fulfills all the required properties given in ... and together with the assumptions x-y presented along the way of its derivation, limits the set of project, that can be valued by it.

Now each part of this equation will be described in detail enabling its usage in real-world applications.

3.5.1 Sets $\mathbf{T}, \mathbf{S_t}, \mathbf{B_t}$

All of the sets $\mathbf{T}, \mathbf{U_t}, \mathbf{B_t}$ come from the model of a valuation problem.

The number and frequency of time periods is determined by $\mathbf{T}$, when the most usual approach is to have $\mathbf{T}$ as a finite number of yearly periods. This can represent either a duration of project rights (for example in mining) or rights for a cash flow share (for example from bridge tolls), or such fuzzy future, where nobody really knows what will happen with the project 50 years from now.

The set of possible branches $\mathbf{B_t}$ comes from the project manager. He needs to identify the project alteration possibilities in each time period $t \in \mathbf{T}$. The set $\mathbf{B_t}$ is small most of the time, however in cases, when for example a start of a variable that is modeled by binomial tree is dependent on the strategy, the set $\mathbf{B_t}$ can grow rather quickly. The example is again a bridge project, where the number of toll-paying customers in time t depends on a time period in which the bridge was finished.

And finally, the values of an environment state $\mathbf{U_t}$ are determined by a model of the outer world. The set $\mathbf{U_t}$ should be able to capture the uncertainty about all important parameters for the project while remaining as simple as possible. A classical example is a binomial tree of input prices for our process (oil, software engineer's salary).

3.5.2 Discount factor r_t

<Detail discussion about risk approach, upper and lower limits, its correspondence with utility and CAPM and all of that>

3.5.3 Supply and demand curves

<Present models of supply and demand curves>

<Define parameterized families and defend their choice>

3.5.4 Set of strategies

3.5.5 Probabilities of states

Chapter 4

Application of the new valuation technique.

The class of .... valuation problems is ideal for demonstration of the usability of the newly developed valuation technique. Due to the inherit uncertainty in this field and many actions that can be undertaken, the additional value assigned to the investment opportunity can be significant.

We start with a rigorous mathematical definition of this class of valuation techniques in terms of SDT. Then we pick one example and promptly show that the number of possible states is exponential with respect to... Approximate dynamic programming techniques like Q-learning or SARSA were developed in order to solve exactly these types of problems. Due to their strengths and only a minor flaws their usage is justified.

4.1 Approximate dynamic programming

<Proposal of a fitting method to the class of problems defined above>

4.2 Valuation example

4.2.1 NPV valuation

4.2.2 DTA valuation

4.2.3 Classic ROA valuation

4.2.4 New approach

Chapter 5

Discussion

The new approach is better because it solves the current problems with ... Also the applicability of the new approach is in my opinion broader since it can address multiple sources of uncertainty. Furthermore the power of the decision making process is kept in the hands of the decision maker through creation of prior distributions. The manager is guided through the world of utility functions and priors, which both can be created from a set of simple questions about gambles and beliefs of the manager. The creation and usage of the utility and prior density functions are fool-proof in a sense of mathematical coherence.

Chapter 6

Conclusions

In my master's thesis I have rigorously compared the state-of-the-art valuation techniques used in present investment companies. I have shown the advantages and disadvantages of real option analysis and stochastic decision theory. The combination of these, which I call ..., yields a new view on the world of risky investments that empowers the decision maker and thus allows for better adoption in the rigid environment of investing.

Chapter 7

Inspiration

7.1 Black-Scholes Merton model

Only four parameters and one assumption is needed to determine a value of an option according to BSM model for option pricing. Assume that the market is complete, and thus the law of one price holds []. Then to value a option you need to know only its time to maturity, its strike price, the current price of the underlying stock and its volatility as follows [3]:

$$C = SN(d_1) - PV(K)N(d_2), \tag{7.1}$$

where S is the strike price, $PV(K)$ is a price of a bond paying K on the expiration day of the option and $N(d)$ is a cumulative normal distribution, probability that a normally distributed variable is less than d. Value of d_1 and d_2 is then defined as:

$$d_1 = \frac{ln(S/PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} d_2 = d_1 - \sigma\sqrt{T} \tag{7.2}$$

The dependency of the price of an option is positive in case of volatility and time to maturity Increasing these parameters leads to a higher option price. On the contrary the rise in current stock price or strike price of the options lowers the value of an option.

7.2 Statistical decision theory

The second pillar upon which this thesis stands is the statistical decision theory. An area of applied mathematics with broad history. The class of problems that can be solved by this approach is very wide and greatly standardized.

The STDs main focus is to determine the optimal decision to act upon in dynamic and uncertain environment. A classical structure of a decision making problem is: ...

In SDT the uncertainty is modeled by the transition probability function which is either assumed to be known, or it is being estimated by either classical statistic methods or Bayesian methods.

The optimal decision strategy (sequence of actions) is the one that gives the maximal cumulative reward. This maximization can be total (in finite or discounted cases) or per period. We will focus on the total cumulative reward in finite processes (?)

To compute value function in all decision nodes of the grid of statistical decision theory a smart idea of backward induction, called dynamic programming is used. The narrative that a sequence of actions is optimal if and only if the last action of the sequence is clear, but powerful when used in this context.

Instead of going through <something like S^{A^B} possible paths and computing expected cumulative reward for all of them, one can optimize his actions from the horizon, lowering the number of computations needed significantly to an order of <something like $S^A * N$ ...>. For example if the scale of the problem is ... the change is from $10^{1}00$ to $10^{2}0$.

To obtain a value function in each decision node of the stochastic decision process a Bellman equation is used.

$$TODO - BELLMANEQ \tag{7.3}$$

When valuing an investment both statistical approaches to estimation are used, but in different cases. The statistical theory allows to estimate parameters when a lot of data is present (for example to determine a volatility of a stock). On the other hand, when the data is scarce and the decision making is undertaken by an expert, the Bayesian theory enables to import his subjective idea into the process.

7.2.1 Bayesian statistics

<Bayesian statics allow to include subjective opinion of the uncertainty> <Bayesian statistics are based on randomization of a parameter> <There is a model for the data in the same way that in classical statistics> <Not even classical statistics is objective, since you always have to choose a model> <Bayes equation, updating on data, certainty equivalence,...>

7.2.2 Utility

When rewards are not valued linearly by the decision maker the concept of utility comes in. One of the simplest examples to demonstrate the usage of utility is given by [1]. Imagine you are given a choice, either get 500$ right away or gamble for 1000 $ in a fair coin toss. A rational decision maker driven only by the expected value given should be indifferent to these choices, but the majority of people tend to take the certain option []. The decision making is clearer as the amount of money rises, there is a little difference for an average human to obtain 10M and 20M, the change in his life will be almost the same with both results. However one is certain and the other is not.

Another interesting example of utility is the St. Petersburg paradox <find reference> where rational decision maker should be willing to pay unlimited amount of money to be able to play the game, but people are seldom willing to pay more than ...

There is also an interesting asymmetry between incurring a loss and getting a profit. This can be seen from picture ... [1].

Each person has their own utility function with respect to money and it can be sketched essentially by a questionaire. More questions mean more precision, and it also keeps your answers coherent.

7.3 Project and cash flow

Definition 7.1. *A project is defined as a piece of planned work or an activity that is finished over a period of time and intended to achieve a particular purpose, mainly an increase of company's or individual's wealth*[1].

Examples of a project are:

- developing a cooper mine;

[1] First part comes from Cambridge dictionary.

- innovation of chemical processes in an oil refinery;
- upgrade of current machinery in a production line;
- changing the form of software development philosophy towards agile practices.

When examining the expression max E{NPV(Options)} four observations come to mind.

First, the maximization is over some set of control strategies. This set can be generally large, even uncountable. Due to the class of the problems that are addressed in this thesis, the actions made by a manager are not expected to be continuous, not even very frequent. This would result in an assumption of small control strategy space, at least for now.

Assumption 6. *The strategy space is small, certainly finite.*

Second, there is an expectation E, which means that there is uncertainty for a single fixed strategy. This uncertainty comes from the nature of free cash flow in each period and originates in both project's inner development (Will I be able to change the production line to produce the goods I want it to?) or outer, most of the time in a form of market price uncertainty of input or output goods.

Definition 7.2. *Free cash flow is the cash a company produces through its operations, less the cost of expenditures on assets.*

Third, net present value (NPV), that is defined in all publications that I have seen as:

$$NPV = \sum_{t \in \mathbf{T}} \frac{C_t}{r^t}, \tag{7.4}$$

where C_t is free cash flow and r is a discount rate that represents devaluation of money in time.

The last observation is that NPV is a function of options. There has to exist some opportunity to change future cash flow in the light of partial or full uncertainty resolution.

7.3.1 How is a similar task solved by different authors

Statistical decision theory is in my opinion captured the best in a book from Bacci and Chiandotto [1] and summarized by Fig. 7.1.

The project (SDT authors call it a *process*) is a sequence of states and actions (and sometimes decisions). To obtain the best performance out of the project a measure of utility instead of simple monetary reward can be introduced. The uncertainty in a project is addressed by a parameterized model with prior distribution that concentrates all knowledge relevant to the project.

The best action strategy is the one that maximizes cumulative utility and it is usually obtained by a method of backward induction called dynamic programming.

Definition 7.3. *TODO - a proper definition of dynamic programming.*

Not sure about what is the difference between stochastic decision theory and stochastic control theory. They are probably synonyms.

Dynamic programming is a tool used for determining the optimal strategy and a value of a project (*process*). Its core is the Bellman equation, which represents the backward induction of value function computation.

Real option analysis can be understood in multiple ways. If a replicating portfolio exists, then a value of some expected cash flow is identical to the current value of a portfolio. The support for this statement

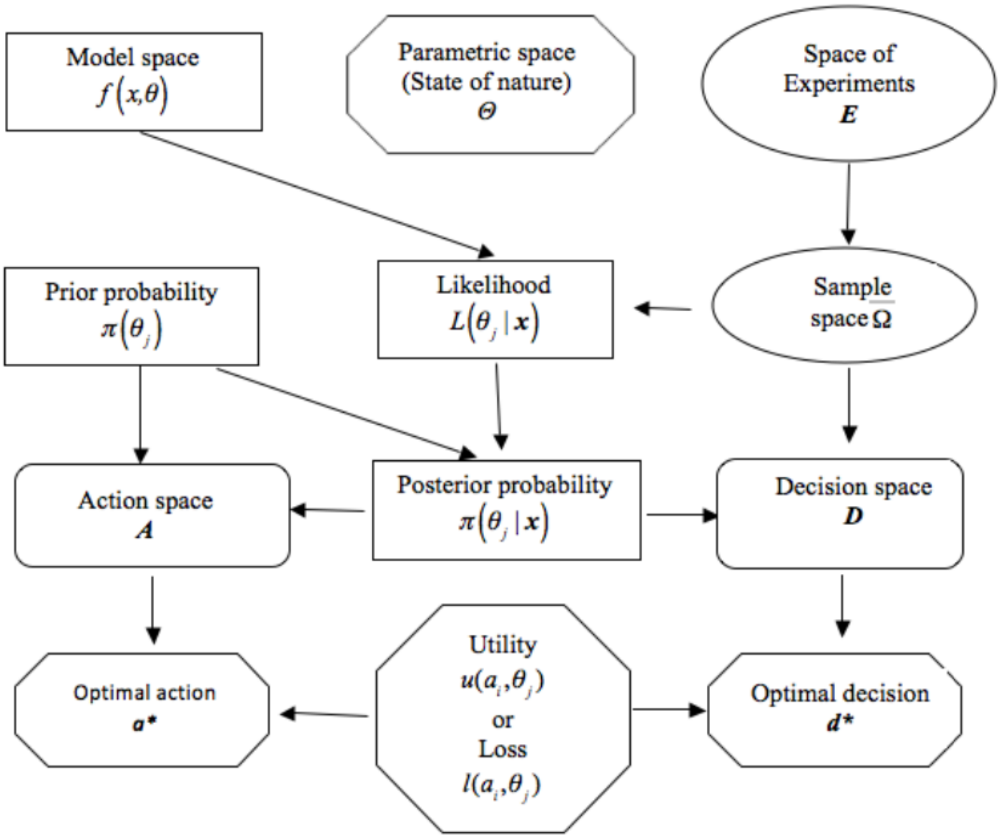

FIGURE 5.1
Conceptual frame of a decision-making process.

Fig. 7.1. SDT Schema.

comes from the law of one price from economics. The assumption of replicating portfolio existence becomes hard to defend as the project that is to be valued becomes more abstract and/or complex.

For more complex projects, real option valuation uses binomial models with risk-neutral probabilities coming from the law of one price (Guthrie [9]), or stochastic models of an expected behavior of the state variable determining the future cash flow (Vollert [21]).

7.4 My valuation

This sub-chapter is my attempt to best describe a real option valuation understood in a sense of an opportunity to make change in future project's cash flows.

First I start with the intuitive equation:

$$V = \max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} DC_t(s_t, b_t(\pi))\right], \tag{7.5}$$

where π is a strategy, a sequence of decisions that the manager is able to do to change the course of the project. Time evolution of the process is represented by a variable t which iterates over range of time epochs, so far finite or infinite countable $\mathbf{T}$. Discounted cash flow DC_t assumed to be obtained at the end of an epoch t is derived from the free cash flow C_t and a constant discount rate $r > 1$ as:

$$DC_t = \frac{C_t}{r^t}. \tag{7.6}$$

Assumption 7. *Discount parameter is constant in time.*

We can see that DC_t depends on two variables, state variable s_t, which represents the state of an environment (outer world, like prices of oil, price of skilled software engineers, etc.) and b_t, project "branch", which represents that state of the project itself (inner property of the project, for example if a copper mine is active or idle, or if the decision was to produce chocolate instead of vanilla ice cream).

Assumption 8. *State variable s_t is a random variable and does not depend on actions of the project's management.*

Assumption 8 can be violated for example in duopolistic markets or others, where a single company decision has non-negligible influence on the market.

Assumption 9. *A "branch" upon which the project is in the moment is assumed to be a random variable and its value is determined only by actions in a policy π and its random nature.*

Assumption 9 is reasonable for majority of projects. Management makes a decision to open or close a mine, to produce chocolate or vanilla. However, for example in some R&D projects the outcome of a manager's decision is uncertain.

7.4.1 Addressing the uncertainty

In this subsection the meaning of the expected value will be examined. We follow on the equation 3.2 [2]. Using the linear property of expectation E and substituting for DC_t with equation **??** we obtain:

$$V = \max_{\pi \in \mathbf{\Pi}} \sum_{t \in \mathbf{T}} \frac{E[C_t(s_t, b_t(\pi))]}{r^t}. \tag{7.7}$$

[2]not sure about the expression

Assumption 10. *Environment s_t can take on values from a finite or countable set $\mathbf{S_t}, \forall t \in \mathbf{T}$.*

This assumption is important only for simpler discussion later. Uncountable $\mathbf{S_t}$ could probably work too, but we would need to perform integration instead of summation [3].

Assumption 11. *The actual state of $b_t \in \mathbf{B_t}$ depends only on actions in policy π that happened before the time period $t \in \mathbf{T}$.*

As a result of assumption 11 a term *partial strategy* is defined.

Definition 7.4. *A partial strategy π_t for a full strategy π is defined as the first t elements of its vector, $\pi_t = (a_1, a_2, ..., a_t)$[4].*

Assumption 12. *For a given partial strategy π_t, the project "branch" can take on values from a finite set $\mathbf{B_t}, \forall t \in \mathbf{T}$.*

In most cases the "branch" of a project is a deterministic function of a managerial decision, however b_t also covers variables whose evolution is triggered by a managerial decision. For example building a bridge with toll payments a year later also postpones the "adoption curve" of the new road.

Assumption 13. *The state of an environment s_t and the branch state b_t are stochastically independent random variables.*

Using assumptions 11, 13 and definition 7.4 we arrive at the final valuation formula:

$$V = \max_{\pi_{t-1} \in \mathbf{\Pi}} \sum_{t \in \mathbf{T}} \frac{\sum_{s_t \in \mathbf{S_t}} \sum_{b_t \in \mathbf{B_t}(\pi_{t-1})} p(s_t) p(b_t(\pi_{t-1})) C_t(s_t, b_t(\pi_{t-1}))}{r^t}. \tag{7.8}$$

Now, there are several questions about our ability to actually compute this valuation.

- Where do the sets $\mathbf{T}, \mathbf{S_t}, \mathbf{B_t}$ come from?
- How to determine the discount factor r.
- How are we able to obtain the cash flow function $C_t(s_t, b_t)$?
- What are we maximizing over? What is the set of possible strategies $\pi \in \Pi$?
- How to obtain the probabilities $p(s_t)$, $p(b_t(\pi_{t-1}))$?

All of these questions will be answered in the next section.

7.4.2 Actual value computation

In this section we will be discussing, how to actually value a project with the equation 7.8.

[3] The verb integrate probably has a different meaning.

[4] It can be noticed that $\pi_{|\mathbf{T}=\pi}$

7.4.2.1 Sets $\mathbf{T}, \mathbf{S_t}, \mathbf{B_t}$

All of the sets $\mathbf{T}, \mathbf{S_t}, \mathbf{B_t}$ come from the model of a valuation problem.

The number and frequency of time periods is determined by $\mathbf{T}$, when the most usual approach is to have $\mathbf{T}$ as a finite number of yearly periods. This can represent either a duration of project rights (for example in mining) or rights for a cash flow share (for example from bridge tolls), or such fuzzy future, where nobody really knows what will happen with the project 50 years from now.

The set of possible branches $\mathbf{B_t}$ comes from the project manager. He needs to identify the project alteration possibilities in each time period $t \in \mathbf{T}$. The set $\mathbf{B_t}$ is small most of the time, however in cases, when for example a start of a variable that is modeled by binomial tree is dependent on the strategy, the set $\mathbf{B_t}$ can grow rather quickly. The example is again a bridge project, where the number of toll-paying customers in time t depends on a time period in which the bridge was finished.

And finally, the values of an environment state $\mathbf{S_t}$ are determined by a model of the outer world. The set $\mathbf{S_t}$ should be able to capture the uncertainty about all important parameters for the project while remaining as simple as possible. A classical example is a binomial tree of input prices for our process (oil, software engineer's salary).

7.4.2.2 Discount factor r

The discount factor, representing time value of money plays a big role in the financial world. It is expected to be $r > 1$ because today's economy depends on non-negative inflation.

Assumption 14. *The discount factor r is expected to be larger than one and constant through the project's lifetime.*

The assumption 14 serves as a simplification. To study time variant discount factor in the light of real option valuation is not interesting. However, one could simply model the discount rate as r_t with some probabilities $p(r_t)$ and compute an expected discount rate.

7.4.2.3 Cash flow function

The core of each project valuation is its cash flow function $C_t(s_t, b_t)$, which represents the additional monetary value created by it over some time period. Usually, this function is simply defined as the difference between the total price of inputs (usually given by s_t) and outputs (usually given by b_t or s_t) for a given period. In more complex models, the inputs can also represent depreciation of machinery, expenses on marketing campaigns or additional costs to acquiring new workforce (not only their salary).

An example of zero cash flow is when a project is deferred and waits to be undertaken or not. A simple example of non-zero cash flow is a project of running ammonia factory, where the cash flow is:

$$C_t = \frac{n_g}{c_{g \to a}} p_a - n_g * p_g, \tag{7.9}$$

where n_g is the volume of processed gas, p_g, p_a are prices of gas and ammonia on the market and $c_{g \to a}$ is an inner company coefficient, that says how many units of gas are needed to produce one unit of ammonia.

7.4.2.4 Strategies π

As was pointed out by many articles [5] the ability to change the course of a project has value. The sequence of actions (in some papers called decisions) form a strategy $\pi = (a_1, a_2, ..., a_{|\mathbf{T}|})$.

[5] See for example [], [], or [].

Assumption 15. *Action a_t is expected to happen in the end of the corresponding time period $t \in \mathbf{T}$.*

The set of possible strategies $\mathbf{\Pi}$ needs to be given by a manger responsible for the project or his team. He should know best, what are the possible changes in the project, that could improve its cash flow and increase its value.

The set of possible strategies is usually well defined by a real options understood in a sense of an ability to act.

In an example from [9], an operator of a gas power plant can choose to produce electricity or not to, based on the current price of gas and power. The strategies in this example can be represented as binary vectors:

$$\mathbf{\Pi} = \{(a_1, ...a_{|\mathbf{T}|})|a_i \in 1, 0, \forall i \in \mathbf{T}\}, \tag{7.10}$$

where $a_i = 1$ means that the plant is running in the next time period $i+1$ and $a_i = 0$ means that the power plant will be in an idle state.

7.4.2.5 Probabilities

Finally, the most complex problem from the list in the end of section 7.4.1 is to determine what to substitute all $p(s_t)$ and $p(b_t)$ in the equation 7.8 for.

The difference in the ROA, SDT and for example DTA techniques is mainly in the difference in which they cope with the derivation of these values.

For DTA, both $p(s_t)$ and $p(b_t)$ are assumed to be known in each time period. This is because problems solved with DTA are usually simple and they need to declare only small number of probabilities.

For ROA the probabilities $p(s_t)$ arise as the so-called risk-free probabilities that come from the idea of replicating portfolio and the validity of the law of one price. For example Guthrie in [9] uses the capital asset pricing model (CAPM), that provides risk-neutral probabilities of up and down movements, p_u, p_d. All other probabilities that could be defined as $p(s_t)$ are in [9] determined by a binomial pricing model as a product of p_u and p_d.

Another approach to ROA can be seen in the work of Vollert [21], who models the replicating portfolios as some stochastic Ito process, which gives the probabilities of s_t.

The branch probabilities $p(b_t)$ are in ROA not understood as a separate category, thus they do not need to be further discussed[6].

Finally, in SDT, the value of both $p(s_t)$ and $p(b_t)$ comes from the knowledge about the underlying problem expressed as prior (or posterior, when updated for new data) probability distribution. The prior probability distribution is constructed from the inner knowledge of experts on the subject, former relevant data, principle of insufficient reasons or their combinations. In SDT it is usually expected that $p(s_t)$ comes from some parameterized family of distributions:

$$\mathcal{F} = \{p(s_t|\theta)|\theta \in \mathbf{\Theta}\}. \tag{7.11}$$

An example is a binomial distribution $Bi(t, \theta)$:

$$p(s_t = k|\theta) = \binom{t}{k}\theta^k(1-\theta)^{t-k}. \tag{7.12}$$

I believe that this approach is the most general, because one can interpret the prior distribution in many possible ways. The SDT approach also allows for combination of many different sources of knowledge about the project [].

[6] Not sure about this, check it.

The problems of ROA approach are the inability to find a replicating portfolio to any more complex project and unclear and undiscussed origins of "actual" probabilities in CAPM model and other variables in it used in [9].

The STD approach is protected against these problems, as even when there is no prior, there is always a principle of insufficient reasons, which needs only a model, a requirement of all techniques that try to determine $p(s_t)$ and $p(b_t)$.

7.4.3 The maximization process

TODO after supervisor's review. In a nutshell it is a dynamic programming and after realization that the complexity is large, it is approximate dynamic programming.

7.5 Bachelor's thesis parts that could be useful for TeX styling

Definition 7.5. *Let $\mathcal{M} = (\boldsymbol{T}, \boldsymbol{S}, \boldsymbol{A}, P, R)$ be a MDP and $\pi_t \in \mathbf{\Pi_t}$ a policy, then **value functions** of $\mathcal{M}$, $\varphi_t^{\pi_t} : \mathbf{S} \to \mathbb{R}$ are defined $\forall \pi_t \in \mathbf{\Pi_t}$, $\forall t \in \mathbf{T} \setminus \{N\}$ as:*

$$\varphi_t^{\pi_t}(s) = E\Big[\sum_{\tau > t, \tau \in \mathbf{T}} R(s_\tau, a_{\tau-1}, s_{\tau-1}) \Big| s_t = s \Big], \tag{7.13}$$

Theorem 1. *Let $\mathcal{M} = (\boldsymbol{T}, \boldsymbol{S}, \boldsymbol{A}, P, R)$ be a MDP, then the optimal value function $\varphi_t^o(s) = \max_{\pi_t \in \mathbf{\Pi_t}} \varphi_t^{\pi_t}$ can be computed recursively $\forall t \in \mathbf{T} \setminus \{N\}$ through the equation:*

$$\varphi_t^o(s) = \max_{a_t \in \mathbf{A}} E\Big[R(s_{t+1}, a, s) + \varphi_{t+1}^o(s_{t+1}) | a_t, s \Big], \tag{7.14}$$

considering that $\varphi_N^o = 0$ as Definition 7.5 implies. Furthermore the optimal policy π_t^o, as the argument of maxima, is concentrated on maximising arguments in Equation (7.14).

Proof. This theorem will be proven via a finite backward mathematical induction.

At first, let us take an action a_{N-1}^o in the time epoch $N - 1$ defined as:

$$a_{N-1}^o(s) = \operatorname{Arg} \max_{a \in \mathbf{A}} E[R(s_N, a, s) | a, s]. \tag{7.15}$$

By the definition of maxima the inequality

□

We can now describe the dynamic programming algorithm in detail.

Example 1. *A real life example of this MDP could be again a biased coin tossing. Imagine a game when you are sitting at the table on which there are multiple coins with generally different bias.*

It is important to mention that you are able to see the other coins "on the table" so that the position state s^p is known in each time epoch.

<There was table here. That should be easy to find again>

Algorithm 1 Finding the optimal policy for a *single system* MDP with known P

Require: $\mathcal{M} = (\mathbf{T}, \mathbf{S}, \mathbf{A}, P, R)$

```
φ_N^o(s) ← 0, ∀s ∈ S                          ▷ Based on Definition 7.5.
t ← N
while t ≠ 0 do
    for each s ∈ {1, 2, ..., |S|} do
        φ_{t−1}^o(s) ← Equation (7.14)         ▷ With known P and φ_t^o
    t ← t − 1
π_0^o(s) ← argmax φ_0^o(s) ∀s ∈ S,             ▷ Deriving the optimal policy
return π_0^o
```

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