Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

Real Options Valuation: A Dynamic Programming Approach

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Master's Thesis

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Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstrujte jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.

Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.
2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.
3. Powell, Warren B, Approximate Dynamic Programming: Solving the Curses of Dimensionality. 2nd ed. Hoboken, NJ: Wiley, 2011.
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5. Vollert, Alexander. A Stochastic Control Framework for Real Options in Strategic Valuation. Boston, MA: Birkhäuser, 2003.

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Author's declaration:
I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, November 29, 2020 Bc. Filip Rolenec

Název práce:

Oceňování projektů metodou reálných opcí z pohledu dynamického progamování

Autor: Bc. Filip Rolenec

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Klíčová slova: Analýza reálných opcí, Blackův-Scholesův model, Čistá současná hodnota, Dynamické programování, Energetika, Oceňování projektů, Stochastická rozhodovací teorie

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Keywords: Black-Scholes model, Dynamic programming, Net present value, Power industry, Project valuation, Real option analysis, Stochastic decision control

Contents

Chapter 1

Historical background and research motivation

The foundations of financial derivatives date back to the origins of commerce in Mesopotamia in the fourth millennium BC [42]. The derivative market consisted mainly of forward contracts [1] and it was introduced to the European continent through Spain in Roman times. After the expulsion of derivative trading in Spain the center of this type of commerce for Europe were the Low Lands. There, in the end of the 17th century, the first ideas about options [2] and option trading are published by La Vega [20].

The first attempts of a mathematical option pricing come from Bachelier (1900) [8] and Bronzin (1908) [17]. Based on their work the boom of option pricing methods in the 1970's culminated by Nobel-price-winning Black-Scholes model [13], which is today's standard in the option pricing theory[43].

The publicity and wide adoption of the BSM model most likely inspired an expert on capital budgeting, Stewart Myers, to introduce the term "Real Options" [29], which is one of two main pillars of this thesis. Myers builds on the idea that part of the project's value is hidden in the form of real options - ability to change the course of the project in the future. Myers' approach to the real options is mostly philosophical in a sense that he stresses the importance of thinking about the additional value options bring, while he does not present any computational tool for the said value.

The idea of Real Option Analysis (ROA) as a valuation tool for projects was further developed by several influential authors in the following decades, for example Guthrie [23], Vollert [41] Pindyck [21] and Kulatilaka [6].

The valuation of project's free cash flows with the theory of ROA is in the economical world understood as very advanced and its adoption in practice is slow [5]. It is argued that this slow adoption is caused mainly by misunderstanding the more difficult mathematical concept of ROA [38] and the low adoption rate of a competition: "Why should our company use a new tool that no one else is using?" [1].

Through the study of the ROA state of the art, we have identified that there is a discrepancy in understanding what ROA actually is. As will be illustrated in depth in section ... we identify three classes of ROA authors based on the level of analogy to the BSM model.

In this thesis we focus on the second class of ROA authors represented by brilliant publications of Graeme Guthrie [23] and Alexander Vollert [41].[3]

[1] A forward contract is a contract to purchase an asset at a fixed price on a particular date in the future. [11]

[2] A financial option is the **ability** to buy (call option) or sell (put option) a defined volume of an asset for a defined amount of money in a given future time instant. [11]

[3] Second class utilizes only the no-arbitrage principle to determine probabilities of the models.

Through my studies at the FNSPE CTU, I have specialized in the theory of dynamic decision making under uncertainty [34], [35]. When studying ROA it came only natural to understand the project valuation as an optimal decision making problem.

The goal of this thesis is to take the project valuation problem structure as is understood in ROA and look at it from the perspective of stochastic decision theory (SDT). One of the challenges of this task is to implement domain specific economical truths about the behavior of investors and the way they perceive value. Two main addressed concepts are: the time value of money and the risk aversion of investors.

The goal of this thesis is to provide an SDT-based valuation algorithm for projects, whose value is understood as a maximal possible current cash equivalent of the uncertain future cash flows. This valuation algorithm will cover the classes of problems now solved by ROA and allow for new classes.

The new SDT based valuation algorithm will enable:

- seamless integration of multiple uncertainty sources;
- integration of theoretically any probability distribution as a model of uncertain variables;
- usage of high number of possible actions, regardless of their nature;
- utilization of approximate dynamic programming tools for high-dimensional problems;
- preservation the economical truths as time value of money and the risk aversion of investors.

To illustrate the usage of the new SDT-based algorithm a valuation of a project from a selected class is performed. This class is denoted as *facilities with simple input-output process models*. It covers all projects whose cycle time is equal zero and the input-output transformation rate is constant. This class is a generalization motivated by an example of gas power plant valuation presented by Guthrie in [23].

The thesis is structured in 5 chapters. Chapter 1 reminds the reader of the most important mathematical and economical concepts used in this thesis. Beginning with the declaration of mathematical notation the chapter continues with key concepts of economical theory and deeper description of the BSM model. The first chapter continues with a summary of the last decades in ROA research with the focus on a specific level of analogy represented by Vollert [41] and Guthrie [23]. The reminder of the first chapter is reserved for key parts of stochastic decision theory.

Chapter 2 represents the core of this thesis. First, we define what will be understood as a problem of ROA project valuation. We state the key features that define a project and we limit this features accordingly (?). Then we focus on the interpretation of the problem by a general SDT framework. We illustrate the identification of ROA project features in SDT. The reminder of the second chapter is reserved for resolving the economical nuances that need to be accounted for in the SDT framework in order to make the valuation procedure consistent with the economical reality of investors' behavior.

Chapter 3 illustrates the new valuation algorithm from chapter 2 on an example of valuation of a facility with a simple input-output process model. A valuation of a gas power plant is chosen as the representative of this class. The first half of the chapter focuses on the value sensitivity with regard to the volume of available managerial actions in the project. The second half presents a value comparison of project alternations in different granularity of the model structure and it is aimed to demonstrate the existence of trade-of between computational complexity and precision of computed results.

Chapter 4 discusses the new findings both theoretical and observed from the performed experiments.

Chapter 5 summarizes the thesis - reminds the motivation, underlines the main message and lists all contributions of this thesis. Furthermore, it outlines many possible future research paths in this field as it is, to our best knowledge, the first available publication on this topic.

Chapter 2

Preliminaries

To properly understand a mathematical text it is important to first define the used notions and symbolism. Since this thesis is based on many different authors, from both financial and mathematical world, a short unifying overview of the used theory is important.

The notation used in this thesis comes predominantly from the most influential authors in the respective fields of study:

- general economy [11];
- real options [23];
- stochastic decision theory [7].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU and its proven applicability in his previous publications [34] and [35].

2.1 Used mathematical symbolism

In the whole thesis, bold capital letters, such as $\mathbf{X}$, represent a set of all elements $x \in \mathbf{X}$ as in [34]. The cardinality of a set $\mathbf{X}$ is denoted with two vertical lines as $|\mathbf{X}|$. Random variables, understood in a sense of the standard Kolmogorov's probability theory [26][1], are represented with a tilde above the variable, i.e. $\tilde{x}$. Realizations of random variables are denoted by the same letters as the random variable without the tilde, i.e. x.

Definition 2.1. *(Probability) Let $\tilde{x}$ be a random discrete variable. Then $P(x)$ denotes a probability that the realization of $\tilde{x} = x$. Similarly if $\tilde{x}$ is a continuous random variable, then $p(x)$ denotes a probability density of the realization $\tilde{x} = x$.*

Remark. *To rigorously unify the notation and simplify the formulas a Radon-Nikodým (RN) density [33] is introduced with the notation $p(x)$ and the name "probability density". The dominating measure of this RN density is either the counting measure (in discrete case) or the Lebesgue measure (in continuous case). The notation $P(X)$ is reserved only for the cases when the discreteness of the argument needs to be emphasized.*

The last general definition is the definition of well known concept of conditional probability [24].

[1] Does this citation make sense?

Definition 2.2. *(Conditional probability) Let, depending on the context, symbol $p(x|y)$ represent the conditional probability density of a random variable variables. Then the $p(x|y)$ is defined as:*

$$p(x|y) = \frac{p(x, y)}{p(y)}, \tag{2.1}$$

where $p(x, y)$ is a joint probability density of $\tilde{x}$ and $\tilde{y}$.

Remark. *The definition of conditional probability expressed by the equation (2.1) corresponds with the classic definitions of the conditional probability and conditional probability density in both the discrete and continuous case.*

Definition 2.3. *(Expected value) Expected value of a random variable $\tilde{x}$ is defined as:*

$$\int_{\mathbf{X}} p(x) x d\mu, \tag{2.2}$$

where μ is the dominating measure of the RN density and $\mathbf{X}$ is the set of all possible realizations of $\tilde{x}$.

<Probably some other definitions that will be needed in the following chapters >

2.2 General economics

This thesis is built on two main theoretical pillars, the theory of corporate finance [11] and stochastic decision theory (SDT) [7]. A basic review of corporate finance terminology is presented in this section with a focus on project valuation techniques.

Definition 2.4. *(**Process**) A process is understood as the production of goods, purchase and trade of goods or services, driven by the supply of inputs and demand for outputs.* [2]

Definition 2.5. *(**Project**) A project is defined as an sequence of actions that serve as an implementation or a innovation of a process, purposefully allocating existing sources to increase the economical value of given project.*

Definition 2.6. *(**Free cash flow**) The incremental effect of a project on the firm's available cash is the project's free cash flow [11].*

Definition 2.7. *(**Economical value**) An economical value of a project is understood to be in the future free cash flows. In this thesis the economical value of a project is the amount of cash to which is the investor logically indifferent to having in comparison to the future cash flow vector.*

Remark. *The indifference and time value of money will be further discussed in section ... The theory of net present value (NPV) [11] can be used as a simplification.*

Definition 2.8. *(**Optimal Project**) The goal of each project is to increase the economical value of a process. A project, if such exists, is called optimal project if the additional economical value is maximal given the set of possible projects.*

[2] How to cite Mr. Kulhavy?

2.2.1 Net present value and other valuation metrics (?)

<Say that this section is for illustration of current state of thinking about investments>

<Say that NPV is the standard in this field>

<Define NPV and say it is for illustration purposes>

<A short paragraph about other metrics used in the decision making about a project, DTA, IRR, WACC>

2.2.2 Risk averse investors

According to the observations made in [11] there is a positive correlation between the volatility of an investment and its average profit. This correlation is being explained by the risk aversion of investors, where investors are happy to hold onto a low-volatile investment even though the average returns are lower in the long run.

The phenomenon of risk aversion is well documented also in psychological publications, i.e. [2], where we observe that people do not like uncertainty and they do value uncertain rewards lower than their expected value.

2.2.3 Option valuation - Black-Scholes-Merton model

As outlined in the first chapter, the motivation for option valuation technique came with the increased adoption of derivative trading after the WWII. The famous 1973 article from Black and Scholes [13] presented today's standard in the option valuation - the BSM model.

Remark. *The M in BSM model stands for Merton, as the publication of Black and Scholes "relied on earlier work by Robert Merton."* [3]

In what follows we will present the BSM model in a form of a theorem [14]. In addition we will present an opinion that should summarize the idea of BSM model in few sentences.

Theorem 1. *(BSM model) The Black-Scholes-Merton option valuation model says that if the following list of assumption is satisfied:*

- *risk-free interest rate and volatility of the underlying asset are constant;*
- *the underlying asset pays no dividends and its price is continuous;*
- *the asset price evolves according to a log-normal process;*
- *the markets are efficient - the no-arbitrage principle holds;*
- *the option has European style;*
- *there are no commission or transaction costs;*
- *market is perfectly liquid;*

[3] I want to talk about it because I want to explain why BSM and not BS model, where I do not want to use BS model for obvious reasons.

then based only on the knowledge of time to maturity T, option's strike price K, the current price of underlying asset S and its volatility σ the value of a call option can be computed as:

$$C = SN(d_1) - PV(K)N(d_2), \tag{2.3}$$

where $PV(K)$ is the present value of a strike price K [4] and $N(d)$ is a cumulative normal distribution, probability that a normally distributed variable is less than d. Value of d_1 and d_2 is then defined as:

$$d_1 = \frac{ln(S/PV(K))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} \quad d_2 = d_1 - \sigma\sqrt{T} \tag{2.4}$$

Remark. *The dependency of the price of an option is positive in case of volatility and time to maturity Increasing these parameters leads to a higher option price. On the contrary the rise in current stock price or strike price of the options lowers the value of an option.*

Now, we would like to present our opinion about what the BSM model actually says.

Remark. *The core of BSM model, assuming that "smooth" conditions hold, represents a way to derive a parameter in a log-normal model of the underlying asset. Based on the no-arbitrage principle and assumption of known volatility σ, only the parameter μ of log-normal process is missing. Building on this, the value of an option is the expected value of the maximum of difference between strike price and the realized price of the asset and zero [5], discounted by the risk-free interest rate.*

2.3 Real option analysis

As outlined in the first chapter, the theory of real option analysis (ROA) was born after the boom in publications about valuation of financial options in the 1970's. The first ideas represented by Myers [29] are of a philosophical nature - options (ability to make project changes) adds value to the project.

Many publications were published on the ROA topic since. Through our studies of the state of the art, we have identified three classes of authors which differ by the level of analogy with the BSM model.

No analogy The first class is the class of the ROA founder, Myers. This class understands the term real option analysis as a useful lens for looking at the project valuation. Authors like [25] and [22] accentuate the value of further managerial decisions, but the valuation strategy they use is NPV with scenarios (so called decision tree analysis DTA).

Partial analogy The second class of authors takes advantage of the core property of the financial option valuation and that is the no-arbitrage principle. Based on this principle and further assumption of replication portfolio existence, this class of authors, e.g. [23], [39] and [36], derives so called risk-neutral probabilities, which are then used for modeling of some project's internal variable of the cash flow functions.

Because we find this type of approach to real options as the most appropriate one and because we build on and respect the work of Guthrie [23], **this approach is the one considered as representative of the term ROA.**

Another author in this class whose work is notable is publication [41] from Vollert, who goes deep into detail with modeling framework implementing complex conditional options. Vollert's publication is very advanced, using i.e. stochastic differential equations, which might be an obstacle for practitioners and real world applications.

[4] Price of a bond paying K on the expiration day of the option

[5] Since the option does not have to be realized, no further loss will occur.

Full analogy The final class of the authors understands project valuation with real options as a complete analogy to thee BSM model for valuation of financial options. This class of authors is predominantly represented by voluminous economical textbooks, e.g. [11], [15] or [19][6].

A complete analogy means to identify all parameters of financial option with a parameter of investment. For example in [11] the following identification table is presented:

Financial option	Real option
Stock price	Current market value of asset
Strike price	Upfront investment required
Expiration date	Final decision date
Risk-free rate	Risk-free rate
Volatility of stock	Volatility of asset value
Dividend[7]	FCF lost from delay

Table 2.1: Identification of parameters for real options with respect to the financial option [11].

Another example can be found in [32], where a telecommunication company is being valued by the same one-to-one identification of BSM model parameters.

By focusing on the complete analogy, the authors of this class strictly limit the application scope of the ROA valuation technique (as they understand it). One of the problematic assumptions (that is in the partial-analogy class solved by the CAMP model [8]) is that there exists a market tradeable replicating portfolio of the asset we want to valuate. Another limitation is that this approach considers only one decision, which is usually to invest in the project now, or later [9].

In what follows, our understanding of ROA will be based on the one presented by Guthrie. This decision is based on the exceptionality of his publication [23]. A rigorous definition of ROA as we will understand it will be presented in the beginning of the following chapter.

2.4 Statistical decision theory

The second pillar upon which this thesis stands is the statistical decision theory (SDT). An area of applied mathematics that formalizes and studies optimal decision making of agents. As decision making under uncertainty in its broadest sense encapsulates the majority of human behavior, the class of problems it is able to solve (at least theoretically) is quite large.

The SDT's main focus is to determine the optimal strategy (a sequence of decisions) to act upon, generally in dynamic and uncertain environment. In this thesis we will be modeling the decision making problems by the standard framework of Markov Decision Process (MDP).

Definition 2.9. *(Markov decision process)*

Markov decision process is defined by its five building blocks:

- *Set of time epochs* - $\mathbf{T}$*;*
- *Set of states in those epochs* - $\mathbf{S_t}$, $t \in \mathbf{T}$*;*

[6]Crundwell also discusses the partial analogy approach in detail.

[7]For the BSM model with dividends

[8]Capital Asset Pricing model - for details please see [23]

[9]Timing option in Guthrie's terminology

- *Set of actions in those states -* $\mathbf{A_{s_t}}$*,* $s_t \in \mathbf{S_t}$*,* $t \in \mathbf{T}$*;*
- *Reward function of transition from one state to another -* $r(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*
- *Transition probabilities governing the transitions from one state to another* $p(s_t|a_t, s_{t-1})$*, where* $s_t, \in \mathbf{S_t}$*,* $s_{t-1}, \in \mathbf{S_{t-1}}$*, and* $a_t \in \mathbf{A_{s_t}}$*;*

Remark. *The set of time epochs, states, actions is usually known, defined by the structure of the decision problem that is being solved. Reward and transition functions tend to be unknown in solving these problems and they need to be often somehow estimated.*

Remark. *For further simplification of the text, we define* $\mathbf{S} = \bigcup_{t \in \mathbf{T}} \mathbf{S_t}$ *and* $\mathbf{A} = \bigcup_{s \in \mathbf{S}} \mathbf{A_s}$*.*

Usually, the biggest task in SDT is to correctly approach the uncertainty about transition probabilities between the different states of a decision making problem. There are two approaches to parameter estimation in statistics, classical approach and a Bayesian approach. Since the Bayesian approach seems to fit the format of decision making better - allowing for notion of prior probabilities, incorporating experts knowledge and possibility for smooth updating on newly observed data - it is used in this thesis.

As outlined above, the goal of SDT is to find the optimal strategy - sequence of actions. The optimality of such strategy is defined as it having the maximal expected cumulative reward among all eligible strategies $\mathbf{\Pi}$:

$$\pi^* = \arg\max_{\pi \in \mathbf{\Pi}} E\left[\sum_{t \in \mathbf{T}} r(s_t|a_t, s_{t-1})|\pi\right]. \tag{2.5}$$

Remark. *This definition of optimal strategy is used mainly in finite decision problems or problems with exponential discounting of future rewards. Alternative definitions of optimality, for example maximal average reward per period, exist.*

Due to the nature of project valuation, where projects are considered to be finite or their cash flow exponentially discounted, this thesis will focus on the total cumulative reward.

2.4.1 Dynamic programming

Finding the optimal policy by computing the expected reward for all policies $\pi \in \mathbf{\Pi}$ is due to the cardinality of $\mathbf{\Pi}$:

$$|\mathbf{\Pi}| = \prod_{t \in \mathbf{T}} \prod_{s_t \in \mathbf{S_t}} |\mathbf{A_{s_t}}| \tag{2.6}$$

very demanding task even for low-dimensional problems.

To cope with such computational complexity a clever idea of backward induction, called dynamic programming, is used. The core of dynamic programming is to define so called value function $V(s)$ on each of the possible states $s \in \mathbf{S}$. Each of the values is computed via the Bellman equation:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A_{s_{t-1}}}} \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1})[r(s_t|a_t, s_{t-1}) + V(s_t)]. \tag{2.7}$$

Value function represents the expected cumulative reward from given state onward. The idea of computing this value through the backward induction is based on the truth that a sequence of actions is optimal if and only if the last action is optimal. Optimal strategy comes together with the value function seemingly as a byproduct, where in each state, the optimal action is the argmax of the expression in equation 2.7.

This clever approach significantly reduces the computational complexity to

$$\sum_{t \in \mathbf{T}} \sum_{s_t \in \mathbf{S_t}} |\mathbf{A_{s_t}}| \tag{2.8}$$

expected values.

Remark. *For constant number of states and actions in each time epoch, this reduction is from* $|\mathbf{T}|^{|\mathbf{S_t}|^{|\mathbf{A_{s_t}}|}}$ *to* $|\mathbf{T}| \cdot |\mathbf{S_t}| \cdot |\mathbf{A_{s_t}}|$.

The reduction of computational complexity with the DP algorithm is significant. However, for the majority of real world applications this reduction is not enough. In reality, due to the structure of decision making problems and their formulation, the cardinality of state space can explode even for fairly simple decision problems.

The problem of remaining computational complexity of DP algorithm is in literature addressed as "three curses of dimensionality" [31] and various solutions under the label of approximate dynamic programming (ADP) were proposed.

2.4.2 Approximate dynamic programming

The computational complexity of dynamic programming for middle and high-dimensional decision making problems is so demanding that its results cannot be obtained in a reasonable amount of time.

To cope with this problem a relevant topic to look at is the section of SDT called approximate dynamic programming (ADP). The ADP label can be understood as a unifying name for a number of algorithms[10] that are trying to obtain quasi-optimal strategies for decision making problems with reasonable demands for computational power.

ADP algorithms can be divided into two main classes, policy and value iteration algorithms. In this thesis we will be focusing on the value iteration class, since we believe that due to the structure of project valuation problems (rather large $|\mathbf{S}|$ and small $|\mathbf{A}|$) it is a better fit.

The idea of value iteration is to have some initial heuristic value function approximation, which is being updated based on the sample of possible paths.

In this thesis, we will focus on the most simple approximation model of the value function, that enables to parameterize the value function with only a small number of parameters[11]. We will model the value function in each time epoch v_t as a linear function of basis functions ϕ_i and parameters $\theta_{i,t}$:

$$v_t(s) = \sum_i \theta_{i,t} \cdot \phi_i(s), \tag{2.9}$$

where each of the basis functions ϕ_i represents some heuristically important feature of each state $s \in \mathbf{S}$. A project valuation example of such basis function might be a difference between state elements representing prices of inputs and outputs or indicator function of state element representing running production.

The update of v_t then unfolds as follows:

- Sample of states $\mathcal{S}$ in time $t \in \mathbf{T}$ is generated[12]

[10] for details see Powell [31].

[11] A value function is in classical DP similar to a look-up table, there is no simple relationship between states and the values of value functions.

[12] Usually based on the model of state distribution in time t.

- In each $s_t \in \mathcal{S}$ the optimal action a_t^* is determined as argument maxima of the Bellman equation, where the value function is understood as its last approximation.
- In each $s_t \in \mathcal{S}$ undertaking the action a_t^* and transition to the following state s_{t+1} is simulated. The reward-state pair $(s_t, r(s_t, a_t^*, s_{t+1}))$ is saved.
- Based on all state-reward pairs a new linear model is constructed, resulting in new parameters $\theta_{i,t}$[13].

By updating the parameters $\theta_{i,t}$ in all $t \in \mathbf{T}$ (in this thesis we will be updating v_t from the horizon) the value function approximation is getting more and more precise[14]. The algorithm ends when a predefined stopping rule is met.

An example of such rule can be that the sum of parameter changes is lower than some predefined threshold.

After the stopping rule is met, we are in possession all parameters $\theta_{i,t}$ with which we are able to determine our best approximation of the expected value of each individual state. One particularly relevant state is the one that describes the current state of the world and thus its value is our estimate of the project's valuation.

2.4.3 Bayesian statistics (?)

The field of mathematical statistics can be divided into two branches, classical (also called frequentist) and Bayesian. The philosophies of each one are fundamentally different and they can be used with a various level of success in different applications.

In this thesis, we are not trying to broadly discuss the internal philosophies of the Bayesian and classical approach as this is not a short and clear discussion. Instead, our approach is to simply use the Bayesian statistics, because it is a better fit for the narrative of learning and decision making under uncertainty.

In general, statistical theory is used to determine a distribution from which the observed data come from. In majority of cases, it is assumed that the data are realizations of a random variable with a distribution from some parameterized class - normal, log-normal, Poisson, etc. The goal is then to determine, with some level of confidence, the parameters that fit the observed data in some sense the best. [15]

The main difference between the Bayesian and classical statistics is how the parameters of a distribution are perceived by the statistician. In the classical theory, it is assumed that observed data come from some distribution with some firm but unknown parameters Θ. In contrast, the Bayesian view on the parameters is such that they are perceived as random variables $\tilde{\Theta}$.

This terminology twist can be a source of initial confusion for frequentist statisticians, but it allows a simple and elegant update of parameter estimates with the Bayesian formula:

$$p(\Theta|d) = \frac{p(d|\Theta)p(\Theta)}{p(d)}, \tag{2.10}$$

where Θ is generally a multivariate parameter and d are observed data. [16]

[13] We could also introduce learning here, meaning that new parameters are a weighted average of the last ones and the newly determined.

[14] Really? I tried to look in Powell, but did not find any convergence theorems to real vf, or at least the optimal parameters of the approximation

[15] Large simplification, statistics can be used in many different ways.

[16] The p(d) in denominator needs to be rewritten as integral if this formula is really to be used.

The interpretation of Bayes formula, is that the distribution of parameter $p(\Theta)$ called the prior distribution, is updated for the newly observed data d, providing new, posterior, distribution $p(\Theta|d)$.

This update can be understood as learning about the "true value" of a parameter, which is very useful structure for dynamic decision problems.

Since the Bayesian theory tells us only how to update an already existing distribution, a prior distribution needs to be given, even though no data were measured yet.

This problem is in Bayesian statistics understood as an advantage, since one can use his knowledge about the problem that is being solved and incorporate it to the prior distribution, which is then updated on the measured data.

The task of consistent creation of prior distribution is a complicated topic and can be found in more detail in [10]. Furthermore, the prior information always exists, as Peterka [30] puts it: “No prior information is a fallacy: an ignorant has no problems to solve“.

2.4.4 Utility

In many decision making situations, rational decision makers do not behave in a way that their decisions would maximize the expected nominal monetary value.

One of the simplest examples used to demonstrate this behavior is given by Bacci and Chiandotto [7]. Imagine an individual is given a choice, either to get 500 $ right away or to gamble for 1000$ in a fair coin toss. A rational decision maker, driven only by the expected value of his actions would be indifferent to the two choices. However, the majority of people ten to take the certain amount of 500$, suggesting that the perceived value of the gamble is lower than 500$.

This effect and its implications becomes more understandable for very large sums of money. There is a little difference for an average human in obtaining 10M USD and 20M USD in a fair coin toss. The change in his quality of life will be almost the same and presumably positive. However one result is certain and the other one has only a 50% probability.

Another interesting example of the non-linear gain perception of individuals is the famous St. Petersburg paradox first formulated by Bernoulli in 1738, [12]. An expected value of the proposed game is infinite, however it is shown that people would seldom pay more than 25 USD to play it. It is interesting that this amount corresponds with the assumption that the counterparty does not posses infinite amount of money, but rather a more reasonable amount of 16.5M USD []. [17]

By these two examples we demonstrate that real decision makers must, in some cases, decide based on something different than the expected value. Building on the extension of the first example, we say that decision makers maximize their well-being measured in utility of the given monetary rewards.

This relation between the perceived utility and monetary value if formalized by the **utility function**.

Based on the shape of utility function we are able to define three classes of decision makers:

- **Risk-averse** decision makers (majority of the population) have concave utility functions. They value uncertain monetary gain lower than its expected value.
- **Risk-neutral** linear utility function. They value uncertain monetary gain exactly as its expected value.
- **Risk-seeking** convex utility function. They value uncertain monetary gain more than its expected value.

[17]This is from wikipedia, find more cool sources. Interesting, but does not have to be in the thesis

The examples above illustrate that the majority of people are risk-averse. This statement is supported by many publications (for example [2]). An individuals utility function can be obtained from a questionnaire by an algorithmic approach that ensures the consistency of responses given by the individual [7].

Another relevant fact with the perception of utility is the asymmetry in human response to gains and losses. The graphical expression of this asymmetry can be seen in fig. 2.1

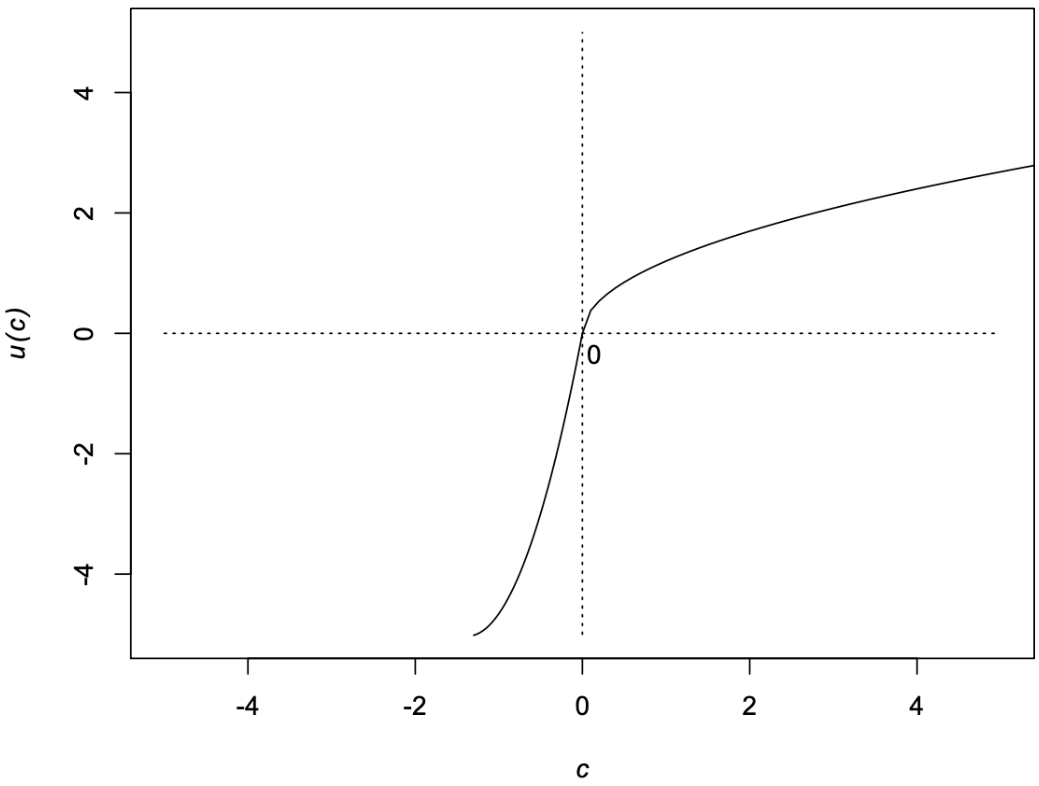

Fig. 2.1. An example of utility function [7].

Chapter 3

Project valuation as stochastic decision problem

In this chapter we develop the core idea of this thesis. The idea is to take the valuation problem as defined in ROA and solve it with SDT, while preserving the economical truths about project valuation, such as time value of money and risk aversion of investors.

We begin by clarifying the meaning of the term project as it is used in the publications in the field of ROA. We define project valuation problem as a collection of mathematical constructs **(?)** and identify the main limitations of this approach.

Next, we focus on the identification of the project valuation in terms of SDT framework. We define all the relevant sets and functions to be able to talk about project valuation as a structured problem of decision making under uncertainty.

The reminder of this chapter is reserved for the incorporation of the economical truths to the model, namely the time value of money and the risk aversion of investors.

3.1 Project valuation - problem definition

To be able to rigorously talk about the project valuation we need to define what a *project* is and what do we mean by its *valuation* in ROA. The inspiration for these definitions comes from examples and used rhetoric in the ROA publications, namely [23], [41] and [6].

None of the books that we have studied goes in detail to define a project as a collection of mathematical constructs, as for example SDT does with MDPs. Guthrie in [23] opens with a three initial examples of a project and in each chapter adds new real investment opportunity. This investment opportunity is presented in such way, that it is clear what the project is, what are its parameters and what metric is optimized by the investor.

In other books like Vollert's [41] and Kulatilaka's [6] a definition of a project is also not given, rather, it is assumed that the used terms *project*, *capital investment* or *investment opportunity* are clear.

It is worth noting that a definition of project *valuation* is also not deeply discussed in the ROA books. We feel like the term of valuation is assumed to be clear and is always represented by expected net present value (NPV), which as we discussed before is easily understandable, but not very robust metric.

As outlined above, nothing like a clear mathematical definition of a project valuation is presented in the ROA books. However, the used rhetoric is similar and we strongly believe that the project valuation can be understood as: "An amount of value that I am able to create with actions that can be considered as a part of one project[1], measured by the metric of expected net present value with a special non-axiomatic determination of the discount rate." **(???)**

Now, that the position of ROA to project valuation is clearer, we can follow with its interpretation in the SDT framework.

3.2 Project valuation in the stochastic decision theory framework

Trying to solve the project valuation task as a stochastic decision problem means first and foremost to identify all the necessary parts of the SDT framework in the ROA formulations. This in not a particularly hard task given the rather loose definition given above.

After this identification the standard tool of SDT, dynamic programming (or potentially approximate dynamic programming), can be used to solve the valuation problem.

Solving a valuation problem in SDT means to define it as MDP, which consists of two parts. First there are three sets: time set $\mathbf{T}$, state set $\mathbf{S}$[2] and action set $\mathbf{A}$[3], which describe the structure of the decision making problem. The second part consists of two functions: transition probability function p and reward function r, where p is responsible for describing the stochastic evolution of the project and r for informing about the value gains in each time epoch.

In the following sections we will focus on each of these five important building blocks in detail. To better illustrate each of the building blocks, and to prepare our ground for the experiment, an example concerning the valuation of a gas power plant is presented.

3.2.1 Time set

Even though the SDT theory is capable of handling infinite time horizons and continuous time modeling, these sophisticated formats are not needed for valuation of real life projects. The time dimension of a project can be reasonably described by a discrete set with known finite horizon, which is true for two reasons.

First is that observing new information and making impactful decisions by project's manager is not done continuously at all times but discretely after some practical time intervals. No manager changes the course of a project 10 times a day[4].

Second reason is that managers do not think about projects as ever-lasting. Potential profits after certain time threshold are neglected. This is given either by the finite-lifespan nature of the projects (gas power plant lifespan) or the extreme uncertainty in modeling of cash flows (and their equivalent is present values) in the far future. To have a good model of project's cash flow in 100 years is a wishful thinking.

Time intervals in our model reflect the frequency of influential management meetings at which the course of the project can be changed significantly, e.g. week, month or quarterly intervals. This notion is further supported by its usage in the ROA publications.

Example 1. *Monthly decision time intervals in a duration of gas power plant lifespan, say 25 years. The time set is then* $\mathbf{T} = \{0, 1, 2, ..., 300\}$.

[1]Defined as in preliminaries

[2]This global state set is a union of state set in each time $\mathbf{S_t}$

[3]This global action set is a union of action set in each state $\mathbf{A_{s_t}}$

[4]This thesis does not focus on the individual management of the internal processes of the project. This thesis focuses on managerial decisions that modify the project in a major way.

3.2.2 State set

Defining the state set $\mathbf{S}$ in a project valuation problem means to find a list of relevant measurable parameters of both the project and its environment. A state $s \in \mathbf{S}$ is then a vector of elementary states of such individual parameters.

The state set $\mathbf{S}$ can be constant, meaning that in each time epoch, the same parameters are measured. However, it might also be useful to think about dynamic state sets in time $\mathbf{S_t}$, $t \in \mathbf{T}$, where for some particular reasons the structure of a problem changes in time[5].

It is worth noting that there are usually some elementary states that are influenceable by the managerial actions and some that are not. This classification is not reflected in our notion.

In our models, each elementary state is understood as a random variable, which probability distribution is conditioned on the previous state and the last action taken. This probability is described with the transition probability function p, which is discussed in detail below.

Example 2. *Relevant features for a gas power plant might be for example: price of gas, price of CO2 allowances, price of power, installed capacity of the plant, debt to be repayed or the remaining lifespan of its blocks. The first four elementary states would then be considered as uninfluenceable by our future actions, while the last three would not.*

3.2.3 Action set

In SDT structure the action set $\mathbf{A}$ is usually understood as an actual set, however in the format of project valuation, we find it better to represent it as an action function, whose parameter is a given state $s_t \in \mathbf{S_t}$ and output is a set of possible actions $\mathbf{A}(\mathbf{s_t}) = a(s_t)$.

The reason for this is that possible managerial actions are most of the time strictly conditioned on the current status of the project itself. Only a small subset of all possible actions might be actually taken in a given state.

In ROA publications the term options is used to describe possible managerial actions both current and future. Even though we believe that this terminology helps with understanding that possibility of future managerial action has value, we do not embrace it in this thesis, where the standard SDT terminology is used.

The advantage of the SDT approach in contrast to ROA is that there is no theoretical complication in adding an arbitrary amount of actions of any type (as classified in ROA by Guthrie [23] for example) possibly even conditioned on one another. The only concern that needs to be reminded is that of computational complexity, where large number of possible managerial actions decrease the ability to compute the valuation in practice.

Example 3. *Actions in gas power plant project might be to: build new block of the plant, run the plant if it has some installed capacity, mothball or sell the power plant for salvage value.*

We believe that this example clearly shows the dependency of possible actions on given state and why is it thus better to use the function notion from now on.

With the definition of action set, now function, we have defined the general structure of a project, boundaries within which the project will evolve. Now we will study the rules that guide the evolution and metrics that measure the value created.

[5]There are more or less relevant parameters to measure.

3.2.4 Transition probability function

Given the nature of projects, the evolution from one state to another is stochastic. In this thesis we want to model the project as MDP and thus the probability distribution of the next state, described by so called transition probability function, is conditioned only on the previous state and the last action taken. In mathematical notation:

$$p(s_t|a_{t-1}, s_{t-1}). \tag{3.1}$$

The probabilities in this thesis are understood as both discrete probabilities and probability distributions in case of continuous variables. Furthermore, as we will model each elementary state by a different distribution we need to be able to compute the overall probability of the future state given the elementary distributions.

To make things simple, we assume individual elementary states to be represented by independent variables and thus the probability (or probability distribution) of a next state is a product of the elementary probabilities[6]:

$$p(s_t|a_t, s_{t-1}) = \prod_i p(s_t^i|a_t, s_{t-1}), \tag{3.2}$$

where s_t^i is the i-th elementary state of s_t.

Remark. *It is possible that some elementary states are fully determined by the managerial action a_t. Such corner case does not create a problem for the probabilistic notion above, the new elementary state s_t^i is realized with probability $p(s_t^i|a_t, s_{t-1}) = 1$.*

It is clear that in the majority of real-life projects, determining, or estimating, this transition probability function is a very hard, but crucial task. Decisions will be made based on its values increasing or decreasing the value of a project.

The approach of ROA authors to modeling of these probabilities varies a lot. Some authors like Guthrie [23], or Amram [6] use the arbitrage principle to determine the probabilities of their binomial models. Some authors, like Kulatilaka [27], use the principle of insufficient reasons[7], where they assign 50% probability to movements in both directions of their binomial models. Some authors, like [23] and [41], go also deeper in statistical modeling of the probabilities.

Some details of how does SDT approach the estimation of $p(\cdot)$ will be discussed later in section 3.3.1, however we must note that correct estimation of these probabilities is not the focus of this thesis.

3.2.5 Reward function

The final part of modeling the project valuation as MDP is the reward function. Its purpose is to assign a numerical value to the state realization given the previous state and last managerial action, mathematically:

$$r(s_t|s_{t-1}, a_{t-1}). \tag{3.3}$$

As discussed in preliminaries, the notion of "value" is complicated. In our case of project valuation, the first approximation of the entity that is to be maximized is the free cash flow (FCF). FCF usually consists of expenses, which are a result of immediate managerial actions, and income, which tends to be

[6] Combination of discrete and continuous variable as elementary states results in continuous global variable (and thus probability distribution).

[7] Even though they do not call it that way.

a result of the environment (supply and demand for manufactured products or services) conditioned on a previous action or action sequence.

Usually, the goal of MDP is to find a strategy, which maximizes the expected reward. However, as economical theory guides us, there is a clear preference in having money now instead of later (time value of money) and that investors do not value uncertain rewards the same as their expected values (risk aversion of investors). Both of these phenomena lead us to build a generalization of the reward function, that reflects them and its optimization defines the optimal strategy in projects defined as MDPs.

The incorporation details of these phenomena will be discussed deeply in the following sections 3.3.2 and 3.3.3. For now, lets declare that the generalized reward function will be based on FCF, adjusted for individual risk preferences and borrowing and risk-free investment opportunities of the investor.

Example 4. *Reward function of a gas power plant is driven by its ability to make money by transforming the gas and CO2 allowances into the electrical power. The initial expenses for building the individual blocks result in extreme negative rewards (driven by action of building), while the profits are made as a multiple of installed capacity and the difference of input costs plus static costs and the revenue from selling the electricity (conditioned on action of running the plant).*

This paragraph concludes the basic identification of sets required by the SDT framework. In the next section, we will focus on the solution to the project valuation problem in detail. We will discuss the sources of transition probability function, the actual incorporation of time value of money and the risk aversion of investors into the model.

3.3 Solution of the project valuation as SDT problem

In the previous chapter, we focused on the basic structure of a project valuation understood as MDP. In this chapter we go deeper and we focus on the details of the actual solution of such valuation problem.

We begin this section by looking in detail at the estimation of the transition probability function p. We outline, how the SDT can not only incorporate the ideas of ROA, but also help with more advanced estimation techniques.

Then, we pursue with incorporation of the economical truths - time value of money and risk aversion of investors in a form of utility maximization principle and the notion of PCE. We focus on implementation details with accent on applicability by real-life managers.

Finally, the last part of this section addresses the computational complexity problems of classical dynamic programming, proposing an algorithm from the ADP class, identified as the best fit for a project-valuation-style MDPs.

3.3.1 Probability

In this thesis we focus on real-life projects. Such projects are from their nature stochastic and except for some edge cases, the laws guiding the evolution of the relevant parameters are unknown and complex. Our search for the optimal strategy is based on the assumption that we have some model estimating the future paths of the project states and we act as if this model was the reality. In our case the model of this evolution is materialized in a form of transition probability function p, where for example one of the assumptions is the Markovian property of the states.

There are many ways how to model p. Let's discuss the techniques used in ROA publications and how they can be translated into SDT terminology. Furthermore, lets also outline the more advanced techniques that SDT can offer in this section.

Risk neutral probabilities The idea of risk neutral probabilities is the major modeling force in the ROA publications. We observe two levels of its usage, both of which are based on the non-existence of arbitrage.

First, simpler approach, used by [36] or [3], adjusts only for the time value of money represented by the risk-free rate r_f, where the probability of up move is computed as:

$$\pi_u = \frac{Xr_f - X_d}{X_u - X_d}, \tag{3.4}$$

where X, X_u, X_d are the values of the asset now, after one up move and after one down move. This equation represents the idea that the probability of up move of an asset is such that its expected appreciation is only the risk-free interest rate.

The more complex equation, adjusting also for the specificity of the risk of the field, wee invest in, called risk-premium comes from the capital asset pricing model (CAPM). This model is used by the frequently mentioned Guthrie in [23], but also in other publications, like [28].

This techniques is used for binomial models, but it is easy to imagine its usage for its limit case which is a variable with the Poisson distribution.

Insufficient reasons The second widely observed modeling style in ROA publications, observed again mostly with the binomial models, is to assign 50% to both up and down move. This technique can be used on different levels of the model, as the final model [**?**] or for example as a helping distribution modeling a particular non-final distribution as in [23].

It needs to be emphasized that the ROA authors do not use this terminology themselves.

More complex models In more mathematical publications we observe more complex models of the future state outcomes. For example the normal process with dynamic parameters [9] or [4], mean-reverting process with Poisson jumps [37] or the modeling by a general Îto process [40].

It seems that these models come from authors with more mathematical than economic background and their unifying feature is the rigorous usage of random variables and continuous distributions.

SDT interpretation Now we would like to discuss the interpretation of the ROA techniques in SDT. The notion of risk-neutral probabilities can be approached with the framework of ‘expert knowledge“ where the first expert is the one (usually the economist using the CAPM formula) who determines the individual variables like the risk-premium, or expected market growth. In accordance with the philosophy of experts, the second expert is the market behaving by the non-arbitrage principle, giving us the equations for risk-neutral probabilities.

The term of insufficient reasons comes from the SDT itself and thus it was already interpreted above.

The class of more advanced approach that we have discussed above is easily covered with the SDT too because of its structure. The outputs in terms of distributions, coming either from the data or again the “expert knowledge“ are easily incorporated into the SDT framework as prior (static or dynamic) probabilities.

SDT innovation The portfolio of SDT estimation techniques is much wider than it was discussed so far. Because probability estimation is not the main focus of this thesis, we will only outline two of the most interesting techniques.

First is the Bayesian modeling, where each time new data are being observed, our probability model is updated by the Bayesian formula. This very influential modeling technique in SDT was not seen in the ROA publications, even though there is certainly a space for it.

The second modeling strategy that we want to talk about is the consistent way of information fusion from different sources. This niche part of the SDT, lead by Karny [], allows for incorporation of multiple sources of prior probabilities, for example multiple experts, data sources, and more **(?).**

Summary To conclude, we advice to use one of two approaches to the problem of p estimation. First, when a lot of information about the project is known, we have a strong case for the parameters behaving according to our smooth distributions, and we believe the market to compactly reflect the expert knowledge, we prefer Bayesian updated risk-neutral probabilities.

On the other hand, if the project is truly innovative and there is very little data to base our model on, we prefer to use a combination of expert knowledge and principle of insufficient reasons.

In the end, we leave the decision of the actual modeling to the framework user, where we express the sympathy for simple models, where more "unclear" models in spite of their better precision, might not be accepted by the board making the investment decision. Clearly, we do not advice to use advanced SDT modeling techniques like probability distribution fusion.

3.3.2 Time value of money

As outlined in preliminaries, money does not have the same value through time. This economical truth is one of the most important ones in project valuation and capital budgeting. The approach of the economical theory to this problem is to exponentially discount the future cash with so called risk-free interest rate.

In the studied ROA publications, the problem of different borrowing and risk-free investment rate is not addressed. The ROA publications assume that the investor is able to borrow at the risk-free interest rate, which is in reality not the case.

In our modeling, the first approximation of the optimized entity was the expected sum of future cash flows. Now, we can present the second approximation of the optimized entity, which is somehow new for both the ROA and SDT[8] world.

This newly defined entity is called present cash equivalent (PCE) and it represents the amount of money that an investor is logically indifferent to having instead of a vector of future cash flows. As argued in preliminaries, this value is unique and for natural borrowing and risk-free rates always defined.

The second approximation of the MDP's optimized entity is thus the expected present cash equivalent of the free cash flow vector of the project, mathematically:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A_{s_{t-1}}}} \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1}) PCE_{t-1} \left(r(s_t|a_t, s_{t-1}) + V(s_t), s^b_{t-1} \right), \tag{3.5}$$

where s^b_{t-1} is the project's balance state in time $t-1$.

Remark. *It needs to be clarified that if we want to embrace the notion of ROA publications, where we assume the simple discounting of the FCF, we are able to do it with this framework too. This notion is only a corner case of our framework, where the borrowing rate is equal to the risk-free rate.*

3.3.3 Risk aversion of investors

As discussed above, the nature of real-life projects is stochastic. The uncertain evolution of states results in the uncertain FCFs defined by the function r, which has implications for the decision making of investors.

[8]SDT theory uses exponential discounting for example in models with infinite time sets, however we have not been able to find a study of discounting conditioned on a state**(?)**.

As discussed in the preliminaries, section 2.2.2, the majority of investors is risk averse, meaning that they tend to value uncertain gains lower than is their expected value.

We believe that this characteristic of investors is important to consider in the valuation of a project. Fortunately, the SDT theory already has a framework for coping with such skewness in reward perception called utility theory.

The third and final approximation of the optimized entity by the Bellman equation is thus the expected utility of the present cash equivalent of individual cash flows. The third and final update of the original Bellman equation can be thus described as:

$$V(s_{t-1}) = \max_{a_{t-1} \in \mathbf{A_{s_{t-1}}}} \sum_{s_t \in \mathbf{S_t}} p(s_t|a_t, s_{t-1})\mu\left(PCE_{t-1}\left(r(s_t|a_t, s_{t-1}) + V(s_t), s^d_{t-1}\right)\right), \tag{3.6}$$

Remark. *Even though there are consistent methods for obtaining the utility function of the individual investors, it might be hard to get individual investor on board with the idea of utility*[9]. *This does not present a fundamental problem for our valuation technique, because we can always use the utility of the risk-neutral investor, which is unique and its usage supported by the lack of bias against uncertain outcomes.*

3.3.4 Approximate dynamic programming

As mentioned in the previous sections of this thesis, the classic DP algorithm can be used for computing the optimal strategy MDP's only with rather small cardinalities of the **A**, **S** and **T** sets. This known DP problem is in literature addressed as "three curses of dimensionality" [31].

Real-life projects, interpreted as MDPs are usually rather complex and hardly-ever fulfill this condition. For example, when even one measured parameter is modeled to come from a continuous distribution, the DP algorithm brakes down not only from the limitation of the actual computational complexity, but also theoretically.

It might be clear from the rhetoric of this thesis, that our goal is to use the developed valuation technique in practice. That is why we want to address this problem with the goal to make the solvable class of project valuation problems as large as possible. And we do so with the approximate dynamic programming (ADP) approach.

From all the possible ADP techniques, we have chosen the value iteration with parameter model approximation for two main reasons, both of which originate in the fundamental characteristics of projects.

First is that the real-life projects tend to have large state spaces (even uncountable), while on the other hand the action set is usually limited. We cannot ask a manager to make a choice between 10 000 actions for example. This supports the choice of value iteration over a policy iteration class of ADP.

Second is that we usually have a good intuition of what exactly in the given states "makes money" and we are able to identify and cluster the important parameters. This allows us to build a reasonable basis function set, which should produce a good approximation of the value function.

It needs to be clarified, that even though we believe that our approach is generally the best, regarding the mathematical complexity, its precision and clarity, there might be better ADP algorithms for individual projects the reader intends to value.

3.3.5 Summary

This subsection summarizes the core idea of this thesis.

[9] Investor might not have a time to answer the utility questionnaire as suggested by [7].

Firstly we have presented the interpretation of the project valuation problem as a MDP. We have defined all its important parts in detail and offered examples for clarity.

Secondly, we have adjusted the MDP theory to respect the economical truths with the notion of PCE respecting the time value of money and utility function interpretation of the investors' risk aversion.

Lastly, we have advised the best ADP algorithm that copes with the problems of computational complexity of the usual DP solving algorithm for non-trivial projects.

Chapter 4

Valuation of facilities with simple I/O process

In the previous chapter we have presented an algorithmic approach to a project valuation based on the SDT framework and its ideas. Now we want to illustrate the actual usage of this algorithmic approach on a chosen class of projects.

We have chosen the class of projects that can be labeled as investment in facilities with simple I/O process and further managerial guidance. This class is characterized by a large outflow of money in the beginning used for building the facility (or its first functional part) and a small but long-term positive future revenue driven by the difference in the price of inputs and outputs of the facility.

This class choice is supported by its appropriate level of complexity, where it allows to demonstrate the power of the new algorithm, while at the same time not being too complex. It is also a type of project that is substantially represented in the world of capital investment.

We could write this whole chapter using a general description of the chosen project class, however we believe that using only one specific representative will result in more clear picture of the situation.

Our choice of the representative - an investment into a gas power plant - is based on three grounds. First, there is certainly an influence by Guthrie's example of a similar valuation problem [23]. Second, as we will see, this valuation problem has reasonable dimensions, that allow for a good presentation of the valuation algorithm. Lastly, the author of this thesis has a legitimate domain knowledge based on his short, but intensive work experience in field of power trading.

In the first part of this chapter we will describe the valuation problem with the upmost precision.

In the second part we will focus on the dependency between the project valuation and the amount and type of available managerial decisions.

In the final part of this experiment, we will be comparing different setups of the project and their influence on the final valuation. This sensitivity analysis will be done in two parts. First, we will study the effects of variable time granularity (simulating more and less frequent managerial decision making). The second subject to a sensitivity analysis will be the different choice of parameters modeling the prices of inputs and outputs.

In the first half of the experiment we await a confirmation of the ROA conclusion, that options (actions) add significant value to the project. In the second half, we have no strong opinion about what the changes in project valuation will be.

In the end, the main aim of this experiment is to present the valuation algorithm, observe and describe its possible shortcoming and support its viability, not to prove any other theorems or ideas about project valuation.

4.1 General settings of the experiment

First, lets clarify what we mean by the phrase *investment in the gas power plant*. We are positioning ourselves in a role of an investment analyst of a large utilities company[1] whose task is to evaluate the value of building and managing new gas power plant.

For simplicity, we are considering building only one or two 200MW blocks with expected life-span of 25 years [18]. We assume, that the price of each block is 80M EUR, which is a rough estimate based on [16]. We assume that the efficiency of the power plant will be 0.7 **(?)**, meaning that 0.7 units of gas (therms) are needed to run one 200MW block at full capacity. Furthermore, we expect 0.2 **(?)** units of required CO2 allowances for the same capacity. Both of these transformation parameters come from author's experience and assumption of a gas power plant with reasonable efficiency.

In addition to the variable costs of CO2 and gas inputs we are counting with fixed costs of 3 EUR/MWh of installed capacity for maintenance and security. We are also allowing for a mothballed state, which cost we estimate for 500k EUR, with lowered maintenance and security costs to only 1 EUR/MWh.

The salvage price for a block is set to be 2000 EUR per installed MW per month.

Initially we assume that the power plant is being managed in a monthly pattern. Its power is being sold by monthly contracts in the beginning of each month and the needed gas and CO2 inputs are also being purchased at the same time. As was indicated earlier in this thesis, we do not want to go deep in the plant management and its internal processes.

In this example we would also like to account for the influence of the government subsidies for renewable power plants and its effect on the power market price volatility. This idea is elaborated in detail below.

Regarding the project's financing, we assume that the majority of the initial payment will be made by a flexible loan[2] with an interest rate of 6%. We define the risk-free interest rate as 2%. The reality of having non-zero initial funds is reflected in the existence of corresponding elementary state.

Now that we have clarified the project that we want to value, we can proceed with precise definitions of the MDP parts.

4.1.1 Time set

As outlined in the introduction of this chapter, we assume that the life-span of a gas power plant is 25 years and that the management to work in monthly patterns.

Because we are allowing to build the blocks only in the first 3 years from now, the final time set is:

$$\mathbf{T} = \{0, ...336\}. \tag{4.1}$$

4.1.2 State set

The state set needs to consist of the smallest number of relevant parameters that enable us to model the process of building, running and potentially selling the power plant and capturing the FCF and its derived metrics of the project.

As such, we identify nine elementary states as parameters describing:

- the price of electricity - s^1,
- the price of CO2 allowances - s^2,

[1] Companies that generate electric power usually provide also gas, water, sewage or other basic services.

[2] An ideally flexible loan where we can repay any amount at any time.

- the price of gas - s^3,
- government support for renewables - s^4,
- number of blocks built - s^5,
- indicator of mothballed power plant- s^6,
- lifespan left (in months) of the first block - s^7,
- lifespan left (in months) of the second block - s^8,
- cash balance of the project (models the debt) - s^9,

in the start of each time epoch, initially each month.

The states of the state set are defined as having a constant length, since there is no significant change in relevant parameters through the project.

Mathematically the state set $\mathbf{S}$ is defined as:

$$\mathbf{S} = \{(s^1, \ldots, s^9) | s^i \in \mathbf{S^i} \; i \in (1, ..., 9))\}, \tag{4.2}$$

where $\mathbf{S^i}$ represents the limitation of the individual states. These limitations will be in detail discussed now.

For the states representing prices, $s^i, i \in \{1, 2, 3\}$, we define:

$$\mathbf{S^i} = \mathbb{R}_0^+. \tag{4.3}$$

The next four elementary states are either reflecting the actual number of what the state represents, like the number of blocks built, s^5, or the remaining lifespans fo blocks s^7, s^8, or encoded information like the state of the power plant or the government support for renewables. In both cases possible states are represented by a sequence of natural numbers $\mathbb{N}_0$.

Regarding the government support we define:

$$\mathbf{S^4} = \{1, 2, 3, 4, 5\}, \tag{4.4}$$

where 1 represents the current support and 5 the highest possible support.

Then, we present the fifth and sixth elementary state which represents the influentiable states by our actions:

$$\mathbf{S^5} = \{0, 1, 2\}, \tag{4.5}$$

showing the number of blocks built and:

$$\mathbf{S^6} = \{0, 1\}, \tag{4.6}$$

which is only an indicator state of power plant being mothballed - 1 or not - 0.

The final two sets represented as a sequence of natural numbers are:

$$\mathbf{S^7} = \mathbf{S^8} = \{0, 1, \ldots, 300\}. \tag{4.7}$$

The final elementary state is then allowed to have any real value:

$$\mathbf{S^9} = \mathbb{R}. \tag{4.8}$$

These definitions represent the problem structure, the boundaries within which the simulation of initial investment and further managerial actions will take place.

Remark. *The government support variable enters the simulation in a very simple way. We expect that by increase of one level the volatility of power prices rises 20%. This value is not based on any data, and similarly to the choice of scale it is only illustrative.*

4.1.3 Action function

In this experiment we consider six managerial actions. They could be understood as two-dimensional, where the first dimension represents the act of running the installed capacity of the plant (if one exists), whereas the second manages more global actions of building, mothballing or selling the plant. However, because of the relations between the considered actions we decided for the following one-dimensional encoding:

- 0 - do not change the current state of the project,
- 1 - run the existing installed capacity,
- 2 - run the existing capacity and build new 200MW block,
- 3 - build new 200MW block,
- 4 - change the mothballed state (mothball or unmothball the plant),
- 5 - sell the plant for salvage value.

As discussed earlier in this thesis, certain actions are available only in certain states. The states that determine what actions are possible are exclusively the elementary states s^5 and s^6 describing the state of the project.

In what follows we define the possible action set is any given combination of these elementary states with a short explanation of its meaning.

- $a(s^5 = 0, s^6 = 0) = \{0, 3\}$ when nothing is built, we can build the first block or do nothing and wait.
- $a(s^5 = 1, s^6 = 0) = \{0, 1, 2, 3, 4, 5\}$ when only one block is built, all actions are possible.
- $a(s^5 = 2, s^6 = 0) = \{0, 1, 4, 5\}$ when two blocks are built, all but building actions are available.
- $a(s^5 = 1, s^6 = 1) = \{0, 3, 4, 5\}$ when the plant is mothballed we cannot run it, however with only one block, we can build the second one.
- $a(s^5 = 2, s^6 = 1) = \{0, 4, 5\}$ when the plant is mothballed we cannot run it and since we have both blocks built, we cannot build more.

It needs to be emphasized that state combination $(s^5 = 0, s^6 = 1)$ does not make sense and that the original set of actions might be limited in parts of the experiment, reducing correspondingly the outputs of the action function.

4.1.4 Transition probability function

The model of evolution from one state to another is fairly complex. The best way to describe it is thus to exploit the specific features of individual elementary state evolution patterns.

In our example, there are 9 elementary states, which can be clustered into different groups.

Deterministic evolution The first group of states is the simplest because the transition from one state to another is deterministic. This category is represented by the counters of remaining lifespan of the power plant blocks, elementary states s^7 and s^8, and the states describing the state of the power plant, installed capacity s^5 and the mothball state indicator s^6. The first two are purely deterministic, meaning:

$$p(s^i_{t+1} = y | s^i_t = x) = \begin{cases} 1 & \text{if } x > 0 \wedge y = x - 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

for $i \in \{7, 8\}$.

The project states are also deterministic, but conditioned on the previous action. First let's look at the state representing the number of blocks built, s^5:

$$p(s^5_{t+1} = y | s^5_t = x, a_t) = \begin{cases} 1 & \text{if } x = y \wedge a_t \in \{0, 1, 4\} \\ 1 & \text{if } x + 1 = y \wedge a_t \in \{2, 3\} \\ 1 & \text{if } y = 0 \wedge a_t = 5 \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

Similarly, the state indicating whether the plant is mothballed or not evolves deterministically according to:

$$p(s^6_{t+1} = y | s^6_t = x, a_t) = \begin{cases} 1 & \text{if } x = y \wedge a_t \neq 4 \\ 1 & \text{if } x \neq y \wedge a_t = 4 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

Stochastic evolution simple The second group of elementary states consists of the ones which evolve stochastically, but their value in the next epoch is conditioned only on the current value.

By this structure we model the prices of gas and CO2 allowances on the market as well as the government approach to renewables. In terms of our notation, states s^2, s^3 and s^4.

The prices of CO2 and gas are modeled by the log-normal stochastic process, which is a standard model in economics used in many publications, i.e. [] or [].

The probability [3] of the next state is thus conditioned only on the value of the previous state as:

$$p(s^i_{t+1} | s^i_t) = LogN(s^i_t, \sigma_i), \tag{4.12}$$

for $i \in \{2, 3\}$. To actually model the prices we need to supply the variances and the initial values s^2_0 and s^3_0.

The initial prices that we use in our model come from the state of the market on 1st December 2020, where the Intercontinental Exchange (ICE) is understood as market. The prices (with transformed units to "per MW") are: $s^2_0 = ...EUR$ (price of CO2) and $s^3_0 = ...EUR$ (price of gas).

Even though an extensive computation of the volatility of each market could be performed, the exact values are actually not important for this thesis. Thus, the volatility parameters come from an educated guess made by the author. The volatilities used for modeling are $\sigma_2 = ...$ and $\sigma_3 = ...$

The government approach to renewables, s^4 is modeled with a realizations of random variable with three outcomes: -1,0 and 1 representing decreased, stable and increased support. This variable is also bounded between 1 and 5.

$$p(s^4_{t+1} = y | s^4_t = x,) = \begin{cases} 0.04 & \text{if } x + 1 = y \wedge y \leq 5 \\ 0.02 & \text{if } x - 1 = y \wedge y \geq 1 \\ 0.94 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

[3]Define LogN in preliminaries

Power price The price of power is modeled similarly to other prices in our experiment, by a log-normal process. The initial price is now taken from the Czech power exchange pxe (part of German eex), which is 42.30 EUR for December 2020 [4]. The initial volatility is then expert guess $\sigma_0 = ...$

The difference in modeling of this variable is that we assume σ to be non-constant. We model the volatility as being on 5 possible levels based on the state of government support, mathematically:

$$\sigma(s^4) = \sigma_0(0.8 + 0.2 \cdot s^4), \tag{4.14}$$

which represents increase of 20% with each level of government support.

Thus, the transition probability of the power price, state s^1 is defined as:

$$p(s^1_{t+1}|s^1_t, s^4_t) = LogN(s^i_t, \sigma(s^4_t). \tag{4.15}$$

Balance The last state whose modeling we need to talk about is the one representing financial balance of the project. This state simulates the actual balance based on actions of "responsible manager" and the assumptions of completely continuous loans and possible risk free investment with constant interest.

Based on two types of monthly interest rates, the one for which the investor can borrow in a bank $r_b = \sqrt[12]{0.06}$ [5] and the one of risk-free interest rate $r_r = \sqrt[12]{0.02}$, we define the evolution of s^9 as:

$$p(s^9_{t+1} = y|s_t, s^9_t = x) = \begin{cases} 1 & \text{if } y = RM(FCF(s_t) + x, r_r, r_b) \\ 0 & \text{otherwise,} \end{cases} \tag{4.16}$$

which means that this elementary state is deterministic and its value s^9_{t+1} is computed as a result of responsible managerial actions with the previous balance adjusted for the FCF obtained in the previous state.

The actual computation of $FCF(s_t)$ will be revealed in the next section, whereas the $RM(\cdot, r_b, r_r)$ function can be found in preliminaries [6]. There can be an initial balance of the project if the investor will not be borrowing the full amount, but in this example, the initial balance $s^9_0 = 0$.

Reward function : The FCF model in our example, and actually in all projects in the class of facilities with simple I/O process, is fairly straightforward.

First we account for the fixed price of maintenance, in our case 3 EUR for each MW of installed capacity per hour when the plant is ready and 1 EUR when mothballed.

Then conditioned on the action of running or not running the facility, we account for input costs. Finally we add the profit from selling our product on the market and like that we have computed the FCF for the current time epoch, mathematically:

$$FCF(s_t, a_t) = (s^5_t * 200) * (3 - (s^6_t) \cdot (2)) + I_{\{1,2\}}(a_t)(s^1_t - (s^2_t + s^3_t))(s^5_t * 200) \cdot h_m, \tag{4.17}$$

where $h_m = 720$ is a constant of hours in month, not accounting for the changes in moth lengths.

As discussed before, the strategy is not being optimized for the expected cumulative FCF, but rather for the cumulative expected utility of its present cash equivalent.

[4] Tradeable on the last day of November at noon.

[5] We assume the yearly borrowing interest rate of 0.06%

[6] Not yet

4.2 Approximate dynamic programming

In this section we will describe the setup of ADP algorithm used in our models. We are using the value iteration algorithm with linear value function approximation, which was in detail discussed in preliminaries.

To ensure the reproducibility of our experiment we need to present the initial value function approximation and its basis functions. The initial value of value function was set to $v_t(s) = 0$.

The basis functions $\phi_i(s)$ represent the following logical units of our designed state.

- $\phi_1(s) = s^1 - s^2 - s^3$ - representing the different between prices of inputs and products;
- $\phi_2(s) = s^5$ - installed capacity of the power plant;
- $\phi_3(s) = I_{\mathbb{R}_0^+}(s^9)$ - indicator of positive or negative balance;
- $\phi_4(s) = s^4$ - representing the government support of renewables directly;
- $\phi_5(s) = s^6$ - basically an$I$ function of mothballed ferature.

The final algorithm that is used to determine the final parameters $\theta_{i,t}$ for each of the experiment setup can be described as:

Algorithm 1 Value function approximation algorithm

Require: $xxxx$
$xxxx$ ▷ yyyy
while $t \neq 0$ **do**
  for each $ainb$ **do**
    $ahoj$ ▷ ahoj
  $t \leftarrow t - 1$
$xxxx$, ▷ Deriving the optimal policy
return $\theta_{i,t}$

The final valuation of the project is then determined by the function:

$$v_0(s_0) = \sum_i \theta_{i,0} \cdot \phi_i(s_0), \tag{4.18}$$

4.3 Sensitivity towards potential action set

<In this section we will focus on the effects of adding managerial actions on the valuation, in both complexity of computation and results>

<We will add two levels of possible managerial actions and we will investigate the changes>

4.3.1 Action set 1

: <We are adding these types of actions to the model>

<The reward function changes in this way>

<The probability function changes in this way>

<Results>

4.3.2 Action set 2

<The reward function changes in this way>
<The probability function changes in this way>
<Results>

4.3.3 Action set 3

<The reward function changes in this way>
<The probability function changes in this way>
<Results>

4.4 Sensitivity toward time epoch granularity

<Now we will study the sensitivity of the valuation for the length of the time epoch>
<The structure for this model is the same as in example ... above, because <reason»
<We will change the time set to <Mathematical expression»
<Results>

With this result we finish the chapter of experiments. Its results and implications will be discussed in the next chapter.

Chapter 5

Discussion

In this chapter we will discuss both the results of the 6 variations of a single project from chapter 3 and its theoretical background from the Chapter 2.

Comparing the results of different granularity of … we can state that …

Looking back at the formulation … there is a potential for improvement in …

With the newly obtained knowledge we can state that the new valuation technique helps with … and is more general than the techniques used nowadays. All this while preserving … and hopefully not exceeding the mathematical capabilities of the potential users.

Chapter 6

Conclusions

The core message of this thesis is to interpret the problem of project valuation in the form of stochastic decision making. The contributions of the newly presented valuation algorithm in contrast to already existing techniques are:

- Usage of general distributions
- Theoretically any number and type of actions
- ...

Furthermore, the thesis copes with the problem of computational complexity, arising as a result of high-dimensional problems, with identifying a <ADP theory> as the best fitting algorithm from the class of ADP for the problem of project valuation.

The new approach to project valuation is demonstrated on six variations of one project type, which show its real applicability in real world. First three examples confirm the expected sensitivity of the project's value on the level of possible managerial actions, endorsing the idea that projects with higher degree of managerial action space have more value. The second trio of experiments shows how is the valuation sensitive on the choice of SDT framework. We conclude that … <probably not much>

The limitations of this approach are:

Finally, through the time of writing this thesis I have identified the following directions for further research as:

- ...

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[1]

[2]

[3]

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