



CZECH TECHNICAL UNIVERSITY IN PRAGUE  
Faculty of Nuclear Sciences and Physical Engineering



# **Real Options Valuation: A Dynamic Programming Approach**

## **Oceňování projektů metodou reálných opcí z pohledu dynamického programování**

Master's Thesis

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# ZADÁNÍ DIPLOMOVÉ PRÁCE

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|-------------------------|---|
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## Pokyny pro vypracování:

1. Seznamte se s tradičním přístupem k analýze reálných opcí obvyklým ve finanční analýze.
2. Formulujte analýzu reálných opcí jako úlohu stochastického řízení.
3. Navrhněte vhodnou metodu numerické aproximace dynamického programování.
4. Implementujte algoritmus oceňování ve Vámi zvoleném výpočetním nástroji a demonstруйте jeho chování na ilustrativní aplikaci a simulovaných datech.
5. Analyzujte přínosy teorie stochastického řízení pro analýzu reálných opcí. Identifikujte případná omezení a otevřené otázky.





Doporučená literatura:

1. Copeland, Thomas E., and Vladimir Antikarov, Real Options: A Practitioner's Guide. Revised ed. New York: Texere, 2003.
2. Guthrie, Graeme, Real Options in Theory and Practice. Oxford, England: Oxford University Press, 2009.
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vedoucí katedry

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děkan





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*Author's declaration:*

I declare that this Master's thesis is entirely my own work and I have listed all the used sources in the bibliography.

Prague, June 21, 2020

Filip Rolenec



*Název práce:*

## **Oceňování projektů metodou reálných opcí z pohledu dynamického programování**

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*Druh práce:* Diplomová práce

*Vedoucí práce:* Ing. Rudolf Kulhavý, DrSc.

*Abstrakt:* Oceňování převážné většiny investičních příležitostí se dnes stále určuje metodou *diskontovaných peněžních toků* (DCF) ???. DCF metoda je přímočará a pro jednoduché projekty dává velmi přesné výsledky. Složitější projekty, které pro tuto práci definujeme jako projekty s vysokou mírou vnitřní neurčitosti a existencí manažerských rozhodnutí značně ovlivňujících strukturu projektu (reálné opce), jsou dnes oceňovány metodou *real option analysis* (ROA). Metoda ROA přiznává hodnotu možnostem změny projektového plánu, v důsledku čehož je ohodnocení ROA vyšší než DCF.

Tato práce má za cíl interpretovat řízení projektů a s ním související ocenění, jako řízení stochastického systému. Rozsáhlá teorie stochastického řízení ??, ??, ??, dovoluje vybudovat novou teorii oceňování postavenou na základech ROA, která mimo jiné řeší omezení ROA na pouze jediný zdroj neurčitosti - například cenu těžebních minerálů na trhu [].

Nový originální přístup k oceňování projektů, který kombinuje dosavadní dosažené znalosti v ekonomické teorii (ROA) a teorii stochastického řízení systémů (SDT) je obecně definován v první části práce a prakticky otestován simulacemi na třídě problémů z oblasti <Area of study>.

Simulace ukazují, že kromě lepší interpretovatelnosti modelu a možnosti vložení subjektivních pravděpodobností je ocenění metodou ... přesnější o ... %. Výsledky vyjadřují naději na zvýšenou adopci metody ... v praxi.

*Klíčová slova:* Analýza reálných opcí, Diskontované peněžní toky, Oceňování projektů, Stochastické řízení, Těžební průmysl

*Title:*

## **Real Options Valuation: A Dynamic Programming Approach**

*Author:* Filip Rolenc

*Abstract:* The valuation of investment opportunities (projects) is nowadays still predominantly computed by the *discounted cash flow* (DCF) method ???. DCF is straightforward and gives solid results for simple projects. For more complicated projects, which are in this thesis defined as projects with substantial degree of inner uncertainty and with an existence of ability to change the course of the project by direct managerial actions, the *real option analysis* (ROA) is now being used. ROA recognizes the value in the possibility of projects' plan change, which usually results in loss limitation, increasing the project's overall expected value.

This thesis aims to interpret project execution and its valuation as a problem of stochastic decision control. The extensive stochastic decision control theory (SDT) allows, based on ROA, to design new valuation theory, that for example solves the limitation of ROA on only one source of uncertainty - such as the volatile price of the mined materials on the market.

The new and original approach to project valuation, that combines both the economical theory (ROA) and the stochastic control theory is defined in the first part of the thesis and then tested by simulations on a class of valuation problems - valuation of projects in <Industry>.

Simulations show that in addition to better model's interpretability and a possibility to include a personal probability distributions, the valuation by <method> is <percentage> more precise than best current strategies. The results hint a possible increase in adoption of this method in practice.

*Keywords:* Discounted cash flow, Mining industry, Project valuation, Real option analysis, Stochastic decision control





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# Chapter 1

## Introduction

The ability to systematically and reliably value projects is the core of investment decision making. According to the economical theory of [] <sup>1</sup> an investment that is able to generate larger profits is better for the people that are influenced by such an investment than its alternative with lower profits. The ultimate metric of value added to the participants of market is according to [] the price they are willing to pay. Higher profit margins, and thus profits themselves indicate, in free market, the will of customers to pay extra for the additional value in their lives coming from the purchased product. To correctly value a project that is subject to some investment decision making, is to improve the lives (on average) <sup>2</sup> of the participants in that market.

The correct valuation technique enables companies to increase their profits and as [] says: the main and actually only goal of manager is to increase the wealth of stakeholders. It is rather fortunate that in free market, increasing the value for shareholders means to create profits, which are a symbol of adding value to market participants. One could thus extrapolate and say, that in free market, the goal of a manager is to improve lives of other participants in the market.

The current state of capital investment valuation techniques is very diverse. Majority of investors relies predominantly on the standard Net Present Value (NPV) technique or its slight generalizations in form of NPV scenarios, risk adjusted discount rate (RADR) hurdles (usually set to an arbitrary percentage), internal rates of return (IRR) and other metrics like ROIC, ENPV,....

<Talk about simple valuation techniques>

<Talk about ROA, its advantages and usage in real life> <sup>3</sup>

<Talk about it not being applied in reality and its reasons>

<Decide which problems will be solved in this thesis>

<Talk about how this all is actually decision making and that it would be interesting to look at it as a SDT problem>

<Say a brief history of SDT and how it fits naturally>

<Say what is missing in SDT and what more does it offer, arbitrage vs utility thinking and Bayes>

### 1.1 The rest of the chapter is kept as an inspiration for the future writing.

The current state of capital investment valuation techniques is a mess. Majority of investors still rely on the standard Net Present Value (NPV) technique or its slight generalizations in form of optimal/average/bad scenarios, risk adjusted discount rate (RADR) hurdles (usually set to an arbitrary per-

---

<sup>1</sup>That guy from Duke university

<sup>2</sup>one participant can pay loads of money or many can pay a lot... How to compare this is hard.

<sup>3</sup>Do not talk about DTA in the complicated sense.

centage), internal rate of return (IRR) metric (which is essentially the same as RADR hurdles) and other metrics like,... ROIC, ENPV.

All the listed generalizations try to cope with one of two main problems of NPV, which is inability to incorporate uncertainty (ENPV, scenario approach) or intra-investment comparison (IRR(?), ROIC).

None of these valuation techniques acknowledges the managerial ability to take action and improve the course of an investment. ENPV can be interpreted in such way but that is abusing of its notation.

The decision tree analysis (DTA) is the simplest approach for valuing an investment with acknowledging decision nodes as an inherent trait of an investment process. It should not be a surprise that being able to make a decision through the lifetime of an investment has value. Also, it is quite clear that the more uncertain the investment and its parts are, the more valuable the ability to act is. To value an ability to make a decision and change the path which the investment follows is not a complicated task. All it takes is to compare a value of a project with that ability and without it (both with DTA), and declare that the value of such "decision option" is the difference between the two.

Many authors from the financial world <sup>4</sup> use the term real option valuation when talking about this difference between DTA trees with or without some branches that represent the ability to make a decision and act upon it. I think that is rather unfortunate, since the terminology clashes with the real option valuation, which this thesis is about.

The true real option analysis (ROA) is about being able to value investment options in the same way financial options are valued. The Black-Scholes-Merton option pricing model that is widely used in financial world is an exceptionally elegant valuation tool. All that it needs to know about financial option to value it is four parameters - the strike price and expiration date of an option itself, the volatility and current price of the underlying asset upon which the option is written. <sup>5</sup>. This Nobel-Prize winning elegance led mathematical and financial experts to try to interpret investment decision making in terms of "real options", that can be valued in the same way the financial options are, by the BSM model.

The attempts to interpret each of four important parameters of a financial option in terms of investment parameters are successful for simple cases (Dealership in BERK), however they fail for more complex ones. The ability to use real option valuation technique depends heavily on the ability to find a relevant company, stock or other publicly traded proxy variable to the examined investment. One finds plausible to construct a replicating portfolio for an investment in oil mining facility or a food production line, however this assumption is hard to believe for highly innovative products in new fields, such as software development or R&D investments.

The ROA valuation approach in investment decision making has cyclical nature. The first wave of the real ROA approach can be traced to ... who presented his ideas shortly after the initial publication about financial option pricing [3]. Next wave comes with ... and finally the most recent publications trying to make practical use of real option analysis are [] and [].

Nearly all publications of real options published in the last <number> years talk about small penetrability of this approach to the actual usage by managers in large and small investment companies. The stated reasons are always the same,

- Vollert says that its complexity, but that is because he uses stochastic models, Ito lemma and computes PDEs, all the time. That has nothing to do with replicating portfolios and BSM model.
- Mr. Kulhavy says that the problem is in finding the replicating portfolio and even if you find it you have to persuade the manager to see it too.
- <Problem with adoption>

---

<sup>4</sup>See *this* and *that*

<sup>5</sup>probably some more assumptions

- <Another problem with adoption>

I believe that the current usage of ROA by investors makes sense. The actual ROA approach is based on finding a replicating portfolio that behaves in the same way as the investment. If a manager believes that he has found a replicating portfolio, then it does not make sense for him to value the project differently than the replicating portfolio. However, there is still the issue of finding such replicating portfolio.

The usage of ROA in sense of different DTA trees is well posed. The ability to hold production in refinery in times of large oil prices can surely increase its value. In this sense the ROA has the biggest impact on the valuation in investments that are highly uncertain and in which there are actions that can alter the course of future cash flow. This also means that in the opposite case, when there is high degree of certainty about the execution and cash flow of the project, or if there are no actions to be taken, the ROA does not create additional value.

The power of classic ROA is to seemly eliminate the probabilistic nature from the picture. This can be done by the power of all-wise market, that forbids the existence of arbitrage. If a tradeable <sup>6</sup>replicating portfolio is found, then I should not care if I make my investment or if I use the money to buy the replicating portfolio.

For the mathematicians from the field of stochastic decision making the problem of valuation is just another decision making under uncertainty. Define state and action space, define rewards (free cash flow) in each epoch and get prior probability density as expected transition probability from one state to another upon undertaking certain action.

The difference between the SDT and ROA is that the former relies on the prior density functions, that are the core of Bayesian statistic, while the latter, coming from economics, relies on the power of the market and managers ability to find a replicating portfolio.

In this thesis the state-of-the art theories of ROA valuation and SDT are presented. Their strengths and weaknesses are discussed and a hybrid approach from the cherry-picked parts of the corresponding theories is presented.

The power of the new theory is shown on a class of problems called ... The current literature copes with them in much simpler way and their assumptions are much larger. My <sup>7</sup> approach also potentiates adoption in practical usage, as it puts the decision making manager in charge. The manager decides how to create the prior distributions, he is the one who needs to determine all the possible actions and scenarios that can happen. The algorithm that is presented in this thesis can and should be used in real world as a managerial tool to maximize the profits of divisions or companies as a whole.

The goal to maximize wealth in financial world is approached in a really narrow matter. Money represents value, which in models decreases in time by a percentage given by some authority, most of the time central banks. In respected financial publications [2], [] or [], there is no notion of utility. It seems like for financial institutions, it does not matter to get 1M in cash now or to gamble for 2M in a fair coin flip. On the other hand, in SDT, the concept of utility is taken very seriously. Majority of people would prefer even a very small amount, such as 200k against a fair gamble between 0 and 2M. This is due to the concave utility function people tend to have when very high amounts are discussed and the question is more about what would make them happier.

Maximization of wealth either in the sense of investment company or individual is natural. Almost everybody would agree that to have more wealth is better than have less. Better investment decisions mean higher accumulation of wealth, which together with assumption of free market results in an increase of utility of each participant in the economy <sup>8</sup>. In other words, when a company makes higher profits it should mean that the part of the population that participated in the wealth creation should be happier.

---

<sup>6</sup>This is a valid word, Cambridge dictionary say so...

<sup>7</sup>Or the presented approach, should I avoid pronouns in the text?

<sup>8</sup>Find that reference... Duke university lectures...



## Chapter 2

# Preliminaries

..... Introduction to preliminaries .....

To properly understand a mathematical text it is important to first define the used notions and symbolism. Since this thesis is based on many different authors, from both financial and mathematical world, a short unifying overview of the used theory is important.

The notation used in this thesis comes predominantly from the most influential authors in the respective fields of study:

- general economy [2];
- real options [5];
- stochastic decision theory [1].

The pure mathematical symbolism comes from the author's studying experience at FNSPE CTU and its applicability is proven in his previous works [7] and [8].<sup>1</sup>

## 2.1 Used mathematical symbolism

..... Sets, random variables, what do I mean by 'probability' .....

In the whole thesis, bold capital letters, such as  $\mathbf{B}$ , represent a set of all elements  $b \in \mathbf{B}$  as in [7]. The cardinality of a set  $\mathbf{B}$  is denoted with two vertical lines as  $|\mathbf{B}|$ . Random variables, understood in a sense of the standard Kolmogorov's probability theory [6]<sup>2</sup>, are represented with a tilde above the variable, i.e.  $\tilde{s}_t$ . Realizations of a random variables are denoted by the same letters as the random variable without the tilde, i.e.  $s_t$ .

Probability of a realization  $s_t$  of random variable  $\tilde{s}_t$  is denoted as  $p(\tilde{s}_t = s_t) = p(s_t) = x$ , where  $x \in [0, 1]$ . This probability is understood as both subjective [] or frequentist [] and both discrete valued and as a probability density, depending on context.

The conditional probability of a realization of a random variable  $\tilde{x} = x$  after a realization of another random variable  $\tilde{y} = y$  is expressed as  $p(x|y) = \frac{p(x,y)}{p(y)}$ , which again represents both discrete values or probability density depending on the context.

<Probably some other definitions that will be needed in the following chapters>

.....

---

<sup>1</sup>Filip == Author or Filip == I/Me

<sup>2</sup>Does this citation make sense?

## 2.2 General economics

..... Introduction and basic financial concepts. ....

This thesis is built on two main theoretical pillars, the theory of corporate finance [2] and stochastic decision theory (SDT) [1]. A basic review of corporate finance terminology and procedures is presented in this section with a focus on project valuation techniques.

**Definition 2.1** (Project). *A project is defined as a piece of planned work or an activity that is finished over a period of time and intended to achieve a particular purpose, mainly a wealth increase of a company or an individual.*<sup>3</sup>

**Definition 2.2** (Value). *The amount of money that can be received for something.*<sup>4 5</sup>

A value of a project is naturally a function of realized elementary monetary transactions within the project and the potential selling price of remaining assets. To track and model each transaction of a project in detail is in principle possible, but such approach would be way too complex for practical usage. Furthermore, its benefits would most likely not be significant enough to defend the extra effort of decision makers.

For purposes of project valuation an aggregation of elementary monetary transactions - a concept of cash flow (CF) and free cash flow (FCF) are used in the financial world.

**Definition 2.3.** *(Cash flow) Cash flow is the net amount of cash and cash-equivalents being transferred into and out of a business (project).*<sup>6</sup>

**Definition 2.4.** *(Free cash flow) The incremental effect of a project on the firm's available cash is the project's free cash flow [2]:*

$$FCF = OperatingCashFlow - CapitalExpenditures. \quad (2.1)$$

Cash flows are in their detail nature discrete, each transaction within the project changes the global cash flow. The moments in which transactions legally take place could be taken as individual time instants.

However, it is easy to imagine understanding the cash flow as a continuous stream of money per time. By the nature of corporate management, one could also expect, that in majority of non-extreme applications, this would not result in a major distortion of cash flow reality.<sup>7</sup>

All of this is true also for free cash flow.

In this thesis we will further work only with the FCF, so from now on, the term *cash flow* will only mean free cash flow.<sup>8</sup>

.....

### 2.2.1 Standard valuation metrics

..... NPV, DTA, IRR, WACC and others.....

<sup>3</sup>First part comes from Cambridge dictionary. Is that ok just to cite it?

<sup>4</sup>Again cambridge dictionary...

<sup>5</sup>This definition is added to say that by value in economics we mean money that we can obtain from the asset.

<sup>6</sup>Investopedia, economical books do not define it.

<sup>7</sup>Should I discuss here more. I mean that corporate management does not care about each transaction in each store, it cares about daily or even monthly revenues.

<sup>8</sup>Maybe confusing, consider to define only FCF and call it cash flow...

Cash flows capture information about value added to a project in given periods. Due to time value of money and different lifespans of projects, one has to come up with algorithms for consistent and systematic valuation of projects, according to their cash flows.

Valuation techniques are attempts to aggregate cash flow vectors in one number to enable decision makers to help to choose an investment with highest expected profit.

**Net present value** First valuation technique, net present value, is arguably the most used valuation technique in capital budgeting [9]. Its computation is simple and it can be described as a sum of cash flows discounted for the time value of money:

$$NPV = \sum_{t \in T} \frac{C_t}{r^t}, \quad (2.2)$$

where  $T = \{0, 1, \dots, |T|\}$  is a set of time periods in which the cash flows  $C_t$  are obtained. These periods are usually years or months, but they can effectively have any granularity the decision maker wants them to. Discounting factor  $r$  expresses the time value of money and is usually derived from the current risk-free interest rate given by the central bank of a nation.<sup>10</sup>

The NPV valuation technique is simple to use, assuming we know the discount rate, which is constant through the project's duration, and free cash flows, that are assumed to be certain.<sup>11</sup>

A more advanced approach that acknowledges the variability in both cash flows and risk-free interest rate is called expected NPV (ENPV):<sup>12 13</sup>

$$ENPV = E \left[ \sum_{t \in T} \frac{\tilde{C}_t}{\tilde{r}_t^t} \right], \quad (2.3)$$

where both the distribution cash flow and interest rate is expected to be known.

**Decision tree analysis** The simplest valuation technique that acknowledges the importance of further management of investments is called decision tree analysis [9]. In addition to time value discounting of the cash flows it offers a framework that can incorporate active management of a project, potentially increasing its overall profits.

The ability to make actions in projects is sometimes interpreted as having *Real Options* [1]. This confusing terminology might result in misunderstandings. That is why it needs to be said, that valuation that recognizes the ability of a manager to act but ignores the law-of-one-price will be addressed as DTA.

**Definition 2.5** (Law of one price). *<Definition of Law of one price>*

The decision tree analysis is usually used only for valuation of projects with very simple scenarios. However, its structure could be potentially used for more complex problems.

The criticism of DTA coming from Vollert [9] and others [1] is that DTA uses a single discount rate for different branches of the project. This is contradictory to the standard rule in the economics that riskier projects should be discounted more [1].

<sup>9</sup>Is this a strong enough statement that I need to cite somebody?

<sup>10</sup>The discussion about negative interest rates and thus  $r < 1$  and also the correct discounting rate is left to my valuation in chapter 3.

<sup>11</sup>Should I put an example here or would that be too trivial? Cash flow is  $[-400, 100, 100, 200, 200]$ , interest rate 5%  $\rightarrow$  NPV is...

<sup>12</sup>Seems like literature does not use this and I made it up. In the eyes of economics, the cash flows seem to be expected values all the times, the distribution is not considered.

<sup>13</sup>Also there is ENPV as "Expanded NPV" NPV+options value used by Vollert, Pindyck and others.

However, this criticism could be countered with upgraded DTA with variable discount rates in different branches rather easily. Furthermore, both Vollert [] and ... [] do not address this problem and use a constant discount rates in their final valuation algorithms as well.

The discussion about risk and its role in valuation algorithm is more discussed in ??.

**Internal rate of return** The internal rate of return serves usually as a basic threshold for enterprises entering new projects. Each company would have their own internal threshold of IRR, based on their confidence in ability to make more or less returns overall. A startup would most likely have an IRR higher than a long time established bank.

**Definition 2.6 (IRR).** *The internal rate of return is the interest rate that sets the net present value of the cash flows equal to zero [2]. This means that IRR for cash flows  $C_t$  needs to satisfy the following equation:*

$$0 = \sum_{t=0}^T \frac{C_t}{(1 + IRR)^t} \quad (2.4)$$

The problems with IRR are that additional assumptions on cash flow vector have to be satisfied so that the result from equation ?? is unambiguous.<sup>14</sup>

**Weighted average cost of capital** <This paragraph is not true, needs to be redone. The WACC is a combination of a premium that investors want in order to hold the shares of the company and a rate for which the company can borrow money. The risk free rate does not play any clear role in WACC. >

The term time value of money represent the truth that the value of money today is not the same as next year. Due to the modern economical theory, inflation is a pursued phenomenon, keeping the economy running.

In economy, it is expected that there is a risk-free interest rate, usually determined by the rates on bonds issued by the central banks of given nations.

In capital budgeting the time value of money has a close connection to cost of capital. The cost of capital can have two forms, based on the money source for your investment. In the first case, you have some cash reserves and by investing in a project the only cost of capital is that you forego the opportunity to invest in bonds with risk-free rate of return, "paying" the opportunity cost. In second case, if company's funds are not sufficient, you need to borrow money on the market for an interest rate, which results in additional "cost" of capital.

The interest rates in second case are naturally higher, since lenders carry additional risk of borrower's default.

The term weighted average cost of capital combines the costs of capital coming from the two sources, further incorporating the notion of corporate tax reliefs.

**Definition 2.7 (WACC).** *The firm's effective after-tax cost of capital is called the weighted-average cost of capital (WACC):*

$$WACC = \frac{E}{E + D} \cdot r_E + \frac{D}{E + D} \cdot r_D(1 - \tau_C), \quad (2.5)$$

where  $E$  is the value of equity,  $D$  is the value of debt,  $r_E$  the equity cost of capital,  $r_D$  the debt cost of capital and  $\tau_C$  is the corporate tax rate.

<sup>14</sup>Or maybe there are no clear assumptions, but I know that there can be no IRR or two or more satisfying the equation.



## 2.3 Real option analysis

.....What is ROA, introduction and hint of larger future discussion in the next chapter.....

An option in financial world means having a right to buy (or sell) an asset in future for a fixed price (strike price) [2]. Option trading has origins in commodity markets (for example corn or oil), where participants want to in some sense insure themselves against the negative movement of a price on the market. Options that are being traded today on the derivative market span almost every tradeable asset that can be thought of <sup>15</sup>.

The value of an option naturally depends on its time to maturity, current price of the asset and a strike price. A proper valuation method for options came with the Scholz-Merton model [3], which in addition requires only the volatility of the underlying asset and the assumption of effective markets <sup>16</sup>.

<Sch-Merton model in more detail>

The established and well received technique for financial option valuation spawned the idea of real option analysis. The option to buy an asset for a given price is similar to the ability to buy an expected future cash flow, i.e. invest in a project.

<Origins of real options> The first economist, pioneer of the term *Real option analysis*, ... .. treats investments as a complete analogy of trading with options. To be able to delay an investment has a value and.....

The usage of the phrase real option analysis in literature is rather fuzzy. It is used by many authors such as [], [] and [], however their usage of the term differs. Three different usage classes of the term real option analysis were identified by the author. They differ by the level of analogy to the financial options.

First, there is a class of complete analogy. All five parameters of financial options needed for their valuation with BSM model are identified with the parameters of an investment opportunity. An example of a valuation of a car dealership can be found in [2]<sup>17</sup> where the following identification is made:

<Table with the identification>

Another example can be found in [] ...

This class of authors focuses on the clear analogy and thus its scope of possible actions of a manager is limited basically only to time options. A decision whether to invest in the project now or later.

The second class of authors uses only the core property of the financial analogy and that is the law-of-one-price. With the help of this assumption, the authors, eg. [5] usually derive risk-neutral probabilities which are then used for modeling of some internal variable of the cash flow functions.

This class of authors is the most numerous and most mathematically rigorous. The core of the works in this class is usually solving stochastic differential equations, eg. [9] whereas the role of the assumption about law-of-one-price is used mainly as the reasoning for obtaining one of the missing parameters of the stochastic model.

Third class of authors does not use the law-of-one-price and is thus the farthest away from the original idea of ....<sup>18</sup>. This class of authors understands the term real option analysis as a useful lens for looking at the project valuation. They accentuate the value of further managerial decision, but the valuation structure and ideas do not differ from the DTA approach.

Thus, this class of authors is declared as a misuse of terminology and not further considered.

In the next part of the thesis, namely ?? the core message behind the term real option analysis will be thoroughly discussed.

.....

<sup>15</sup>Find some citation or do not use this sentence.

<sup>16</sup>Maybe has a different name

<sup>17</sup>Should I reference page numbers?

<sup>18</sup>The pioneer name

## **2.4 Stochastic decision theory**

<Standard SDT as a framework, states, actions, inputs, outputs, rewards, probability distributions>

### **2.4.1 Bayesian statistics**

<Why is it important here? Prior posterior, Bayes equation, not very detailed, it is not a focus of this thesis>

### **2.4.2 Dynamic programming**

<Dynamic programming in a nutshell, its main message, bellman equation and say that it has practical problems with higher dimensions>

### **2.4.3 Approximate dynamic programming**

<Probably will be needed because of problems with applicability of standard DP. Talk about what problems does it solve, what are the commonly used types.>

## Chapter 3

# Real options in the language of SDT

..... Why is ROA in the language of SDT studied? .....

When an individual scholar studies stochastic decision theory and then educates himself in the economical theory of project valuation, it is hard to avoid the feeling of similarities in the respective fields. The SDT guides a decision maker in uncertain environment to make actions that give him maximal reward possible based on his (usually adaptive) knowledge. The same is true for the ROA. In an uncertain economical environment, the decision maker (manager) needs to decide which project to invest in, and, in this setting more importantly, how to "control" the project when it is already running.

I would argue that it is natural to look at the valuation of real option analysis with the optics of stochastic decision theory. The reason why this relation between ROA and SDT was studied only marginally by authors like Vollert [] and ... has in my opinion two parts. <sup>1</sup>

First is that Real Option Analysis is not precisely defined, as discussed in the previous chapter. There is no formal definition of ROA and its usage is mostly in showing that the ability to change parameters of a project in the future, an active management, can result in a significant increase in value of the project. Without a proper definition of the phrase Real option analysis a deeper look at its meaning is more complicated.

Second unfortunate reason is the relative closedness of scientific communities. Stochastic decision control theory focuses mostly on applications in physics [] or [], game theory, ... in their established communities. As far as I know, the stochastic decision theory was not used to value project in correspondence with the economical theory - respecting arbitrage principle, usual discounting, etc. <sup>2</sup> The community of economists and stochastic decision theory scholars probably does not have an intersection large enough to span the, in my opinion, natural idea of interpretation of ROA in terms of SDT.

.....

The core message of this thesis is to interpret real option analysis (ROA) in the terminology of stochastic decision theory (SDT). To fairly discuss advantages and disadvantages of both theories, to take the class of problems that are being solved by ROA and empower the decision makers with decades of SDT theory knowledge.

The new valuation approach should reflect the economical features of a project valuation, while using the core of SDT. In other words it should:

- Capture uncertainty in a project

---

<sup>1</sup>Or why nobody has ever done this? I do believe it is a natural thing to do...

<sup>2</sup>Really nobody valued project with a SDT before??? Without a replicating portfolio, there is no arbitrage principle, and the ROA is effectively only the ability to change the course of a project... Nobody has done this?

- Allow managers to implement their own approach to risk.
- Enable to rigorously handle a time devaluation of money according to the profile of the company making the investment.
- Allow to systematically compare projects in a portfolio to find the best candidates for an actual investment.
- <Add all other qualities the new valuation technique should have>

### 3.1 ROA vs SDT comparison

<Discussion over their definitions from the preliminaries chapter, where they are introduced>

### 3.2 New valuation technique

<sup>3</sup> Now, with all the information about the current state of valuation techniques and after a proper discussion of differences between ROA and SDT approaches, lets start to construct a new valuation technique that is a combination of the best properties of both for project valuation.

First, lets clarify the important properties of a general valuation formula. It needs to address uncertainty which is effectively a part of each project. It needs to capture options, the ability of a manager to make choices that significantly change the future cash flows of a project as it has been proven that they add value. And lastly it needs to give values by which projects can be compared now, the valuation needs to acknowledge the time value of money.

An intuitive mathematical expression of a valuation that fulfills the requirements set above is given by the following general equation:

<Say that this is a formula coming from SDT, that has only economical terminology>

<Discuss why the formulation is discrete>

<Repeat or introduce the notion of partial strategy  $\pi_t$  and use it in this complete chapter>

<I need to distinguish between a random variable  $s_t$  and its realization>

$$V = \max_{\pi \in \Pi} E \left[ \sum_{t \in \mathbf{T}} DC_t(s_t(\pi)) \middle| \pi \right], \quad (3.1)$$

where  $\pi$  is a strategy, a sequence of decisions that the manager is able to do to change the course of the project. Time evolution of the process is represented by a variable  $t$  which iterates over range of time epochs, so far finite or infinite countable  $\mathbf{T}$ .

<Talk about evolution of state and what is policy, what is input, output and reward possibly>

Discounted cash flow  $DC_t$  is assumed to be obtained at the end of an epoch  $t$  and depends on a state of the environment  $s_t$ , which is (most of the time only partially) a random function of a strategy  $\pi$ .

The general formula 3.1 is not hard to come up with or understand what it means. The actual challenge is to rigorously define its parts up to the moment, where a real project-specific values can be substituted and the actual value can be computed. The goal of the next text is to dive in the detail meaning of the equation 3.1, rigorously and justifiably describe each substitution in correspondence with the current theory and obtain actually practically useful equation.

---

<sup>3</sup>Come up with a clever name for it.

### 3.2.1 Discounted cash flow

Since details of set of time epochs  $t \in \mathbf{T}$ , policies  $\pi \in \mathbf{\Pi}$ , maximization over them and expected value  $E[\cdot]$  were already discussed in the preliminaries, the only unknown part of the general equation 3.1 is the discounted cash flow  $DC_t(s_t(\pi))$ .

Discounted cash flow  $DC_t(s_t(\pi))$  is defined as a value of free cash flow, discounted by factor  $r_t$  that reflects the time value of money for the investor of a project:

$$DC_t(s_t(\pi)) = \frac{C_t(s_t(\pi))}{r_t}. \quad (3.2)$$

Even though the adapted economical terminology requires  $r_t > 1$ , suggesting that  $DC_t < C_t$ , in this thesis only  $r_t > 0$  is required. The enlargement of the  $r_t$  possible range comes from the recent unprecedented behavior of central banks, where negative interest rates are being more and more commonplace. The requirement of positive discount factor  $r_t$  represents only a clear logic, that future profits, cannot be viewed as current losses.

The usual approach to obtaining  $r_t$  has the form ??, which discounts each time epoch with the same rate.

In this thesis the equation ?? is used only to determine boundaries of  $r_t$  with different constant discount factors  $r_l$ , the interest rate that the company would get from a bank to invest the project and  $r_b$  which is the classical risk-free interest rate considered in the financial textbooks, usually given by an interest on government bonds (in stable countries).

<The actual discussion about what discount rate means, how similar is it to utility function and other CAPM model, WACC, ....>

... which concludes the discussion about the actual values of  $r_t$ .

### 3.2.2 Cash flow

The ability to substitute for  $DC_t$  with equation ?? enables us to express the general equation ?? in a new form:

$$V = \max_{\pi \in \mathbf{\Pi}} E \left[ \sum_{t \in \mathbf{T}} \frac{C_t(s_t(\pi))}{r_t} \middle| \pi \right]. \quad (3.3)$$

The cash flow  $C_t$  depends on a state of the environment  $s_t$ , which is generally influenced by former actions of the chosen policy  $\pi$ .

Evaluation of a cash flow is in a broad sense simple, it is the difference between sales and costs in a given time period <sup>4</sup>

$$C_T(s_t(\pi)) = S a_t(s_t(\pi)) - C o_t(s_t(\pi)), \quad (3.4)$$

where  $S a_t$  are sales and  $C o_t$  represents costs in period  $t$ . This equation can be further rewritten with the concept of unit costs of inputs  $C i_t$  and outputs  $C o_t$  as:

$$C_T(s_t(\pi)) = C o_t(s_t(\pi)) N o_t(s_t(\pi)) - C i_t(s_t(\pi)) N i_t(s_t(\pi)) \quad (3.5)$$

where  $N o_t$  is the unit output of a product <sup>5</sup> and  $N i_t$  is the number of input units consumed by the project in time period  $t$ . <Possibly talk about how one can introduce the input/output ratio>

<sup>4</sup>Improve this definition or cite something.

<sup>5</sup>It should be understood in broader sense already...

Now, it is important to emphasize the different levels of dependency of the four parameters that influence cash flows on the policy  $\pi$ . For this purpose the usual notation of a state  $s_t$  that is used in control theory (?) is split to two. In this thesis I call them, the "branch state"  $b_t$  and the uninfluenceable state  $u_t$ <sup>6</sup>. In mathematical notation

$$s_t = \text{union of vectors } u_t \text{ and } b_t \text{ somehow.} \quad (3.6)$$

<Write this as proper definitions>

As the names suggests, the uninfluenceable state  $u_t$  cannot be influenced by the actions of the manager, concentrated in the strategy  $\pi$  and the branch state  $b_t$  represent the branch of a project that it has evolved to.

**Assumption 1.** *Costs of inputs and outputs in the production process<sup>7</sup> are random functions of a uninfluenceable part of the environment state.*

**Assumption 2.** *Numbers of input and output units in the production process(?) are random functions of branch part of the environment state.*

With the definition of branch and uninfluenceable states and the assumptions ?? and ??, we can rewrite the equation ?? as<sup>8</sup>:

$$C_T(s_t(\pi)) = C_{O_t}(u_t)N_{O_t}(b_t(\pi)) - C_{I_t}(u_t)N_{I_t}(b_t(\pi)) \quad (3.7)$$

Another reasonable assumption is that there exists a transformation parameter, that determines what amount of input units are needed to get one unit of output. This parameter usually represents the effectiveness of the used machinery or employees, which can be naturally changed, within some limits, by the actions of a manager. Lets define this parameter as input-output factor in time  $iof_t$ .

**Assumption 3.** *The input-output factor is a random<sup>9</sup> variable and it is a function of a branch state  $b_t$ .*

The equation ??, then becomes:

$$C_t(s_t(\pi)) = C_{O_t}(u_t)N_{O_t}(b_t(\pi)) - C_{I_t}(u_t)N_{O_t}(b_t(\pi))iof_t(b_t(\pi)). \quad (3.8)$$

To get even more detailed description of a cash flow, both the prices of input and output are modeled as a function of supply and demand. This allows to make the cash flow dependent basically only on one variable in a given time period<sup>10</sup> and that is the required number of output units, naturally limited by the branch state.

**Definition 3.1.** <Supply and demand curves definition, parametrized families>

Thus the last form of the equation for evaluation of  $C_t$  has a form:

$$C_t = sp(N_{O_t}|\theta(s_t)) * N_{O_t} - sd(N_{O_t}iof_t|\theta(s_t)) * N_{O_t} * iof_t(b_t) \quad (3.9)$$

<sup>6</sup>This is not a good notation, find out how the control theory does that.

<sup>7</sup>probably a better name for it exists

<sup>8</sup>These assumptions are a problem with examples where the model of demand starts at a time determined by a strategy itself (tolls and bridge)

<sup>9</sup>Maybe not everything needs to be random

<sup>10</sup>Cannot be seen with this notation

### 3.2.3 Addressing uncertainty

The last element of the general equation 3.1 that is to be addressed is the uncertainty of the random variable  $\tilde{s}_t$  represented by the expectation function  $E[\cdot]$ .

Using the linear property of expectation  $E$  on the equation ?? we obtain:

$$V = \max_{\pi \in \Pi} \sum_{t \in \mathbf{T}} \frac{E[C_t(\tilde{s}_t(\pi_t))]}{r_t}, \quad (3.10)$$

**Assumption 4.** Random variable  $\tilde{s}_t$  has a finite or countable set of outcomes  $\mathbf{S}_t, \forall t \in \mathbf{T}$ .

This assumption is important only for simpler discussion later. Uncountable  $\mathbf{S}_t$  could probably work too, but we would need to perform integration instead of summation<sup>11</sup>. A direct consequence of this assumption is that the set of outcomes for both branch states  $\mathbf{B}_t$  and unfluenceable states  $\mathbf{U}_t$  are finite or at most countable.

**Assumption 5.** The state unfluenceable state  $u_t$  and the branch state  $b_t$  are stochastically independent random variables.

Using the definition of expected value we get:

$$V = \max_{\pi_t} \in \Pi \sum_{t \in \mathbf{T}} \frac{\sum_{u_t \in \mathbf{U}_t} \sum_{b_t \in \mathbf{B}_t(\pi_{t-1})} p(u_t)p(b_t)C_t(u_t, b_t(\pi))}{r_t}, \quad (3.11)$$

and by substituting for  $C_t$  with the equation ?? we get the final equation for the project valuation:

$$V = \max_{\pi_t} \in \Pi \sum_{t \in \mathbf{T}} \frac{\sum_{u_t \in \mathbf{U}_t} \sum_{b_t \in \mathbf{B}_t(\pi_{t-1})} p(u_t)p(b_t)[sp(No_t|\theta(s_t)) * No_t - sd(No_tiof_t|\theta(s_t)) * No_t * io f_t(b_t)]}{r_t}. \quad (3.12)$$

Now, there are several questions about our ability to actually compute this valuation:

- Where do the sets  $\mathbf{T}, \mathbf{U}_t, \mathbf{B}_t$  come from?
- How to determine the discount factor  $r_t$ .
- Where to get the family of supply and demand curves ?
- How do these curves depend on the unfluenceable state  $u_t$ ?
- How large is the  $io f_t$  parameter for different branches of the project?
- What are we maximizing over? What is the set of possible strategies  $\pi \in \Pi$ ?
- How to obtain the probabilities  $p(s_t), p(b_t(\pi_{t-1}))$ ?

All of these questions will be answered in the next section.

<sup>11</sup>The verb integrate probably has a different meaning.

### 3.3 Origins of variables in the valuation equation

The equation ?? is a final framework that is used for project valuation in this thesis. It fulfills all the required properties given in ... and together with the assumptions x-y presented along the way of its derivation, limits the set of project, that can be valued by it.

Now each part of this equation will be described in detail enabling its usage in real-world applications.

#### 3.3.1 Sets $\mathbf{T}$ , $\mathbf{S}_t$ , $\mathbf{B}_t$

All of the sets  $\mathbf{T}$ ,  $\mathbf{U}_t$ ,  $\mathbf{B}_t$  come from the model of a valuation problem.

The number and frequency of time periods is determined by  $\mathbf{T}$ , when the most usual approach is to have  $\mathbf{T}$  as a finite number of yearly periods. This can represent either a duration of project rights (for example in mining) or rights for a cash flow share (for example from bridge tolls), or such fuzzy future, where nobody really knows what will happen with the project 50 years from now.

The set of possible branches  $\mathbf{B}_t$  comes from the project manager. He needs to identify the project alteration possibilities in each time period  $t \in \mathbf{T}$ . The set  $\mathbf{B}_t$  is small most of the time, however in cases, when for example a start of a variable that is modeled by binomial tree is dependent on the strategy, the set  $\mathbf{B}_t$  can grow rather quickly. The example is again a bridge project, where the number of toll-paying customers in time  $t$  depends on a time period in which the bridge was finished.

And finally, the values of an environment state  $\mathbf{U}_t$  are determined by a model of the outer world. The set  $\mathbf{U}_t$  should be able to capture the uncertainty about all important parameters for the project while remaining as simple as possible. A classical example is a binomial tree of input prices for our process (oil, software engineer's salary).

#### 3.3.2 Discount factor $r_t$

<Detail discussion about risk approach, upper and lower limits, its correspondence with utility and CAPM and all of that>

#### 3.3.3 Supply and demand curves

<Present models of supply and demand curves>

<Define parameterized families and defend their choice>

#### 3.3.4 Set of strategies

#### 3.3.5 Probabilities of states



## **Chapter 4**

# **Application of the new valuation technique.**

The class of .... valuation problems is ideal for demonstration of the usability of the newly developed valuation technique. Due to the inherit uncertainty in this field and many actions that can be undertaken, the additional value assigned to the investment opportunity can be significant.

We start with a rigorous mathematical definition of this class of valuation techniques in terms of SDT. Then we pick one example and promptly show that the number of possible states is exponential with respect to... Approximate dynamic programming techniques like Q-learning or SARSA were developed in order to solve exactly these types of problems. Due to their strengths and only a minor flaws their usage is justified.

### **4.1 Approximate dynamic programming**

<Proposal of a fitting method to the class of problems defined above>

### **4.2 Valuation example**

#### **4.2.1 NPV valuation**

#### **4.2.2 DTA valuation**

#### **4.2.3 Classic ROA valuation**

#### **4.2.4 New approach**



## **Chapter 5**

### **Discussion**

The new approach is better because it solves the current problems with ... Also the applicability of the new approach is in my opinion broader since it can address multiple sources of uncertainty. Furthermore the power of the decision making process is kept in the hands of the decision maker through creation of prior distributions. The manager is guided through the world of utility functions and priors, which both can be created from a set of simple questions about gambles and beliefs of the manager. The creation and usage of the utility and prior density functions are fool-proof in a sense of mathematical coherence.



## **Chapter 6**

# **Conclusions**

In my master's thesis I have rigorously compared the state-of-the-art valuation techniques used in present investment companies. I have shown the advantages and disadvantages of real option analysis and stochastic decision theory. The combination of these, which I call ..., yields a new view on the world of risky investments that empowers the decision maker and thus allows for better adoption in the rigid environment of investing.



## Chapter 7

# Inspiration

### 7.1 Black-Scholes Merton model

Only four parameters and one assumption is needed to determine a value of an option according to BSM model for option pricing. Assume that the market is complete, and thus the law of one price holds []. Then to value a option you need to know only its time to maturity, its strike price, the current price of the underlying stock and its volatility as follows [2]:

$$C = S N(d_1) - PV(K)N(d_2), \quad (7.1)$$

where  $S$  is the strike price,  $PV(K)$  is a price of a bond paying  $K$  on the expiration day of the option and  $N(d)$  is a cumulative normal distribution, probability that a normally distributed variable is less than  $d$ . Value of  $d_1$  and  $d_2$  is then defined as:

$$d_1 = \frac{\ln(S/PV(K))}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2} d_2 = d_1 - \sigma \sqrt{T} \quad (7.2)$$

The dependency of the price of an option is positive in case of volatility and time to maturity. Increasing these parameters leads to a higher option price. On the contrary the rise in current stock price or strike price of the options lowers the value of an option.

### 7.2 Statistical decision theory

The second pillar upon which this thesis stands is the statistical decision theory. An area of applied mathematics with broad history. The class of problems that can be solved by this approach is very wide and greatly standardized.

The STDs main focus is to determine the optimal decision to act upon in dynamic and uncertain environment. A classical structure of a decision making problem is: ...

In SDT the uncertainty is modeled by the transition probability function which is either assumed to be known, or it is being estimated by either classical statistic methods or Bayesian methods.

The optimal decision strategy (sequence of actions) is the one that gives the maximal cumulative reward. This maximization can be total (in finite or discounted cases) or per period. We will focus on the total cumulative reward in finite processes (?)

To compute value function in all decision nodes of the grid of statistical decision theory a smart idea of backward induction, called dynamic programming is used. The narrative that a sequence of actions is optimal if and only if the last action of the sequence is clear, but powerful when used in this context.

Instead of going through <something like  $S^{A^B}$  possible paths and computing expected cumulative reward for all of them, one can optimize his actions from the horizon, lowering the number of computations needed significantly to an order of <something like  $S^A * N \dots$ >. For example if the scale of the problem is ... the change is from  $10^{100}$  to  $10^{20}$ .

To obtain a value function in each decision node of the stochastic decision process a Bellman equation is used.

$$TODO - BELLMANEQ \quad (7.3)$$

When valuing an investment both statistical approaches to estimation are used, but in different cases. The statistical theory allows to estimate parameters when a lot of data is present (for example to determine a volatility of a stock). On the other hand, when the data is scarce and the decision making is undertaken by an expert, the Bayesian theory enables to import his subjective idea into the process.

### 7.2.1 Bayesian statistics

<Bayesian statics allow to include subjective opinion of the uncertainty> <Bayesian statistics are based on randomization of a parameter> <There is a model for the data in the same way that in classical statistics> <Not even classical statistics is objective, since you always have to choose a model> <Bayes equation, updating on data, certainty equivalence,...>

### 7.2.2 Utility

When rewards are not valued linearly by the decision maker the concept of utility comes in. One of the simplest examples to demonstrate the usage of utility is given by [1]. Imagine you are given a choice, either get 500\$ right away or gamble for 1000 \$ in a fair coin toss. A rational decision maker driven only by the expected value given should be indifferent to these choices, but the majority of people tend to take the certain option []. The decision making is clearer as the amount of money rises, there is a little difference for an average human to obtain 10M and 20M, the change in his life will be almost the same with both results. However one is certain and the other is not.

Another interesting example of utility is the St. Petersburg paradox <find reference> where rational decision maker should be willing to pay unlimited amount of money to be able to play the game, but people are seldom willing to pay more than ...

There is also an interesting asymmetry between incurring a loss and getting a profit. This can be seen from picture ... [1].

Each person has their own utility function with respect to money and it can be sketched essentially by a questionnaire. More questions mean more precision, and it also keeps your answers coherent.

## 7.3 Project and cash flow

**Definition 7.1.** *A project is defined as a piece of planned work or an activity that is finished over a period of time and intended to achieve a particular purpose, mainly an increase of company's or individual's wealth<sup>1</sup>.*

Examples of a project are:

- developing a cooper mine;

---

<sup>1</sup>First part comes from Cambridge dictionary.



- innovation of chemical processes in an oil refinery;
- upgrade of current machinery in a production line;
- changing the form of software development philosophy towards agile practices.

When examining the expression  $\max E\{NPV(\text{Options})\}$  four observations come to mind.

First, the maximization is over some set of control strategies. This set can be generally large, even uncountable. Due to the class of the problems that are addressed in this thesis, the actions made by a manager are not expected to be continuous, not even very frequent. This would result in an assumption of small control strategy space, at least for now.

**Assumption 6.** *The strategy space is small, certainly finite.*

Second, there is an expectation  $E$ , which means that there is uncertainty for a single fixed strategy. This uncertainty comes from the nature of free cash flow in each period and originates in both project's inner development (Will I be able to change the production line to produce the goods I want it to?) or outer, most of the time in a form of market price uncertainty of input or output goods.

**Definition 7.2.** *Free cash flow is the cash a company produces through its operations, less the cost of expenditures on assets.*

Third, net present value (NPV), that is defined in all publications that I have seen as:

$$NPV = \sum_{t \in T} \frac{C_t}{r^t}, \quad (7.4)$$

where  $C_t$  is free cash flow and  $r$  is a discount rate that represents devaluation of money in time.

The last observation is that NPV is a function of options. There has to exist some opportunity to change future cash flow in the light of partial or full uncertainty resolution.

### 7.3.1 How is a similar task solved by different authors

Statistical decision theory is in my opinion captured the best in a book from Bacci and Chiandotto [1] and summarized by Fig. 4.1.

The project (SDT authors call it a *process*) is a sequence of states and actions (and sometimes decisions). To obtain the best performance out of the project a measure of utility instead of simple monetary reward can be introduced. The uncertainty in a project is addressed by a parameterized model with prior distribution that concentrates all knowledge relevant to the project.

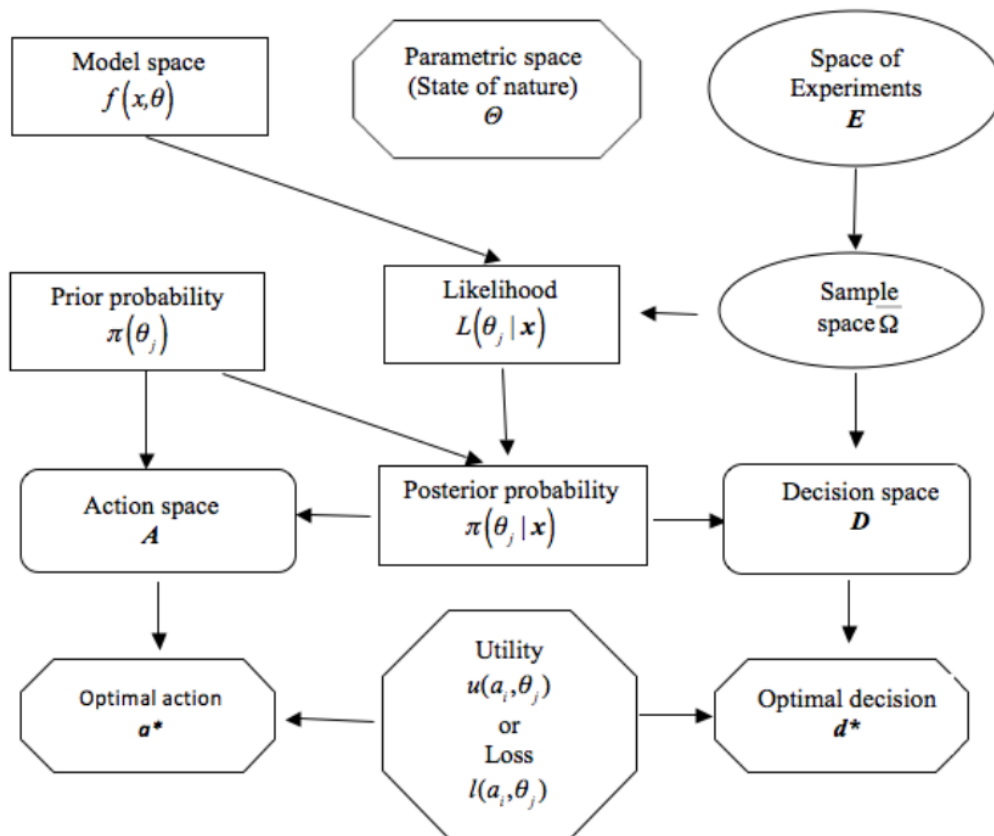
The best action strategy is the one that maximizes cumulative utility and it is usually obtained by a method of backward induction called dynamic programming.

**Definition 7.3.** *TODO - a proper definition of dynamic programming.*

Not sure about what is the difference between stochastic decision theory and stochastic control theory. They are probably synonyms.

Dynamic programming is a tool used for determining the optimal strategy and a value of a project (*process*). Its core is the Bellman equation, which represents the backward induction of value function computation.

Real option analysis can be understood in multiple ways. If a replicating portfolio exists, then a value of some expected cash flow is identical to the current value of a portfolio. The support for this statement



**FIGURE 5.1**  
Conceptual frame of a decision-making process.

**Fig. 7.1.** SDT Schema.

comes from the law of one price from economics. The assumption of replicating portfolio existence becomes hard to defend as the project that is to be valued becomes more abstract and/or complex.

For more complex projects, real option valuation uses binomial models with risk-neutral probabilities coming from the law of one price (Guthrie [5]), or stochastic models of an expected behavior of the state variable determining the future cash flow (Vollert [9]).

## 7.4 My valuation

This sub-chapter is my attempt to best describe a real option valuation understood in a sense of an opportunity to make change in future project's cash flows.

First I start with the intuitive equation:

$$V = \max_{\pi \in \Pi} E \left[ \sum_{t \in \mathbf{T}} DC_t(s_t, b_t(\pi)) \right], \quad (7.5)$$

where  $\pi$  is a strategy, a sequence of decisions that the manager is able to do to change the course of the project. Time evolution of the process is represented by a variable  $t$  which iterates over range of time epochs, so far finite or infinite countable  $\mathbf{T}$ . Discounted cash flow  $DC_t$  assumed to be obtained at the end of an epoch  $t$  is derived from the free cash flow  $C_t$  and a constant discount rate  $r > 1$  as:

$$DC_t = \frac{C_t}{r^t}. \quad (7.6)$$

**Assumption 7.** *Discount parameter is constant in time.*

We can see that  $DC_t$  depends on two variables, state variable  $s_t$ , which represents the state of an environment (outer world, like prices of oil, price of skilled software engineers, etc.) and  $b_t$ , project "branch", which represents that state of the project itself (inner property of the project, for example if a copper mine is active or idle, or if the decision was to produce chocolate instead of vanilla ice cream).

**Assumption 8.** *State variable  $s_t$  is a random variable and does not depend on actions of the project's management.*

Assumption 8 can be violated for example in duopolistic markets or others, where a single company decision has non-negligible influence on the market.

**Assumption 9.** *A "branch" upon which the project is in the moment is assumed to be a random variable and its value is determined only by actions in a policy  $\pi$  and its random nature.*

Assumption 9 is reasonable for majority of projects. Management makes a decision to open or close a mine, to produce chocolate or vanilla. However, for example in some R&D projects the outcome of a manager's decision is uncertain.

### 7.4.1 Addressing the uncertainty

In this subsection the meaning of the expected value will be examined. We follow on the equation 3.1<sup>2</sup>. Using the linear property of expectation  $E$  and substituting for  $DC_t$  with equation ?? we obtain:

$$V = \max_{\pi \in \Pi} \sum_{t \in \mathbf{T}} \frac{E[C_t(s_t, b_t(\pi))]}{r^t}. \quad (7.7)$$

---

<sup>2</sup>not sure about the expression

**Assumption 10.** Environment  $s_t$  can take on values from a finite or countable set  $\mathbf{S}_t, \forall t \in \mathbf{T}$ .

This assumption is important only for simpler discussion later. Uncountable  $\mathbf{S}_t$  could probably work too, but we would need to perform integration instead of summation <sup>3</sup>.

**Assumption 11.** The actual state of  $b_t \in \mathbf{B}_t$  depends only on actions in policy  $\pi$  that happened before the time period  $t \in \mathbf{T}$ .

As a result of assumption 11 a term *partial strategy* is defined.

**Definition 7.4.** A partial strategy  $\pi_t$  for a full strategy  $\pi$  is defined as the first  $t$  elements of its vector,  $\pi_t = (a_1, a_2, \dots, a_t)$ <sup>4</sup>.

**Assumption 12.** For a given partial strategy  $\pi_t$ , the project "branch" can take on values from a finite set  $\mathbf{B}_t, \forall t \in \mathbf{T}$ .

In most cases the "branch" of a project is a deterministic function of a managerial decision, however  $b_t$  also covers variables whose evolution is triggered by a managerial decision. For example building a bridge with toll payments a year later also postpones the "adoption curve" of the new road.

**Assumption 13.** The state of an environment  $s_t$  and the branch state  $b_t$  are stochastically independent random variables.

Using assumptions 11, 13 and definition 4.4 we arrive at the final valuation formula:

$$V = \max_{\pi_{t-1} \in \Pi} \sum_{t \in \mathbf{T}} \frac{\sum_{s_t \in \mathbf{S}_t} \sum_{b_t \in \mathbf{B}_t(\pi_{t-1})} p(s_t) p(b_t(\pi_{t-1})) C_t(s_t, b_t(\pi_{t-1}))}{r^t}. \quad (7.8)$$

Now, there are several questions about our ability to actually compute this valuation.

- Where do the sets  $\mathbf{T}, \mathbf{S}_t, \mathbf{B}_t$  come from?
- How to determine the discount factor  $r$ .
- How are we able to obtain the cash flow function  $C_t(s_t, b_t)$ ?
- What are we maximizing over? What is the set of possible strategies  $\pi \in \Pi$ ?
- How to obtain the probabilities  $p(s_t), p(b_t(\pi_{t-1}))$ ?

All of these questions will be answered in the next section.

## 7.4.2 Actual value computation

In this section we will be discussing, how to actually value a project with the equation 4.8.

<sup>3</sup>The verb integrate probably has a different meaning.

<sup>4</sup>It can be noticed that  $\pi_{|\mathbf{T}=\pi}$

#### 7.4.2.1 Sets $\mathbf{T}$ , $\mathbf{S}_t$ , $\mathbf{B}_t$

All of the sets  $\mathbf{T}$ ,  $\mathbf{S}_t$ ,  $\mathbf{B}_t$  come from the model of a valuation problem.

The number and frequency of time periods is determined by  $\mathbf{T}$ , when the most usual approach is to have  $\mathbf{T}$  as a finite number of yearly periods. This can represent either a duration of project rights (for example in mining) or rights for a cash flow share (for example from bridge tolls), or such fuzzy future, where nobody really knows what will happen with the project 50 years from now.

The set of possible branches  $\mathbf{B}_t$  comes from the project manager. He needs to identify the project alteration possibilities in each time period  $t \in \mathbf{T}$ . The set  $\mathbf{B}_t$  is small most of the time, however in cases, when for example a start of a variable that is modeled by binomial tree is dependent on the strategy, the set  $\mathbf{B}_t$  can grow rather quickly. The example is again a bridge project, where the number of toll-paying customers in time  $t$  depends on a time period in which the bridge was finished.

And finally, the values of an environment state  $\mathbf{S}_t$  are determined by a model of the outer world. The set  $\mathbf{S}_t$  should be able to capture the uncertainty about all important parameters for the project while remaining as simple as possible. A classical example is a binomial tree of input prices for our process (oil, software engineer's salary).

#### 7.4.2.2 Discount factor $r$

The discount factor, representing time value of money plays a big role in the financial world. It is expected to be  $r > 1$  because today's economy depends on non-negative inflation.

**Assumption 14.** *The discount factor  $r$  is expected to be larger than one and constant through the project's lifetime.*

The assumption 14 serves as a simplification. To study time variant discount factor in the light of real option valuation is not interesting. However, one could simply model the discount rate as  $r_t$  with some probabilities  $p(r_t)$  and compute an expected discount rate.

#### 7.4.2.3 Cash flow function

The core of each project valuation is its cash flow function  $C_t(s_t, b_t)$ , which represents the additional monetary value created by it over some time period. Usually, this function is simply defined as the difference between the total price of inputs (usually given by  $s_t$ ) and outputs (usually given by  $b_t$  or  $s_t$ ) for a given period. In more complex models, the inputs can also represent depreciation of machinery, expenses on marketing campaigns or additional costs to acquiring new workforce (not only their salary).

An example of zero cash flow is when a project is deferred and waits to be undertaken or not. A simple example of non-zero cash flow is a project of running ammonia factory, where the cash flow is:

$$C_t = \frac{n_g}{c_{g \rightarrow a}} p_a - n_g * p_g, \quad (7.9)$$

where  $n_g$  is the volume of processed gas,  $p_g, p_a$  are prices of gas and ammonia on the market and  $c_{g \rightarrow a}$  is an inner company coefficient, that says how many units of gas are needed to produce one unit of ammonia.

#### 7.4.2.4 Strategies $\pi$

As was pointed out by many articles <sup>5</sup> the ability to change the course of a project has value. The sequence of actions (in some papers called decisions) form a strategy  $\pi = (a_1, a_2, \dots, a_{|\mathbf{T}|})$ .

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<sup>5</sup>See for example [], [], or [].

**Assumption 15.** Action  $a_t$  is expected to happen in the end of the corresponding time period  $t \in \mathbf{T}$ .

The set of possible strategies  $\Pi$  needs to be given by a manager responsible for the project or his team. He should know best, what are the possible changes in the project, that could improve its cash flow and increase its value.

The set of possible strategies is usually well defined by a real options understood in a sense of an ability to act.

In an example from [5], an operator of a gas power plant can choose to produce electricity or not to, based on the current price of gas and power. The strategies in this example can be represented as binary vectors:

$$\Pi = \{(a_1, \dots, a_{|\mathbf{T}|}) | a_i \in \{1, 0\}, \forall i \in \mathbf{T}\}, \quad (7.10)$$

where  $a_i = 1$  means that the plant is running in the next time period  $i + 1$  and  $a_i = 0$  means that the power plant will be in an idle state.

#### 7.4.2.5 Probabilities

Finally, the most complex problem from the list in the end of section 4.4.1 is to determine what to substitute all  $p(s_t)$  and  $p(b_t)$  in the equation 4.8 for.

The difference in the ROA, SDT and for example DTA techniques is mainly in the difference in which they cope with the derivation of these values.

For DTA, both  $p(s_t)$  and  $p(b_t)$  are assumed to be known in each time period. This is because problems solved with DTA are usually simple and they need to declare only small number of probabilities.

For ROA the probabilities  $p(s_t)$  arise as the so-called risk-free probabilities that come from the idea of replicating portfolio and the validity of the law of one price. For example Guthrie in [5] uses the capital asset pricing model (CAPM), that provides risk-neutral probabilities of up and down movements,  $p_u, p_d$ . All other probabilities that could be defined as  $p(s_t)$  are in [5] determined by a binomial pricing model as a product of  $p_u$  and  $p_d$ .

Another approach to ROA can be seen in the work of Vollert [9], who models the replicating portfolios as some stochastic Ito process, which gives the probabilities of  $s_t$ .

The branch probabilities  $p(b_t)$  are in ROA not understood as a separate category, thus they do not need to be further discussed<sup>6</sup>.

Finally, in SDT, the value of both  $p(s_t)$  and  $p(b_t)$  comes from the knowledge about the underlying problem expressed as prior (or posterior, when updated for new data) probability distribution. The prior probability distribution is constructed from the inner knowledge of experts on the subject, former relevant data, principle of insufficient reasons or their combinations. In SDT it is usually expected that  $p(s_t)$  comes from some parameterized family of distributions:

$$\mathcal{F} = \{p(s_t|\theta) | \theta \in \Theta\}. \quad (7.11)$$

An example is a binomial distribution  $Bi(t, \theta)$ :

$$p(s_t = k|\theta) = \binom{t}{k} \theta^k (1 - \theta)^{t-k}. \quad (7.12)$$

I believe that this approach is the most general, because one can interpret the prior distribution in many possible ways. The SDT approach also allows for combination of many different sources of knowledge about the project [].

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<sup>6</sup>Not sure about this, check it.

The problems of ROA approach are the inability to find a replicating portfolio to any more complex project and unclear and undiscussed origins of "actual" probabilities in CAPM model and other variables in it used in [5].

The STD approach is protected against these problems, as even when there is no prior, there is always a principle of insufficient reasons, which needs only a model, a requirement of all techniques that try to determine  $p(s_t)$  and  $p(b_t)$ .

### 7.4.3 The maximization process

TODO after supervisor's review. In a nutshell it is a dynamic programming and after realization that the complexity is large, it is approximate dynamic programming.

## 7.5 Bachelor's thesis parts that could be useful for TeX styling

**Definition 7.5.** Let  $\mathcal{M} = (T, S, A, P, R)$  be a MDP and  $\pi_t \in \Pi_t$  a policy, then *value functions* of  $\mathcal{M}$ ,  $\varphi_t^{\pi_t} : S \rightarrow \mathbb{R}$  are defined  $\forall \pi_t \in \Pi_t, \forall t \in T \setminus \{N\}$  as:

$$\varphi_t^{\pi_t}(s) = E \left[ \sum_{\tau > t, \tau \in T} R(s_\tau, a_{\tau-1}, s_{\tau-1}) | s_t = s \right], \quad (7.13)$$

**Theorem 1.** Let  $\mathcal{M} = (T, S, A, P, R)$  be a MDP, then the optimal value function  $\varphi_t^o(s) = \max_{\pi_t \in \Pi_t} \varphi_t^{\pi_t}$  can be computed recursively  $\forall t \in T \setminus \{N\}$  through the equation:

$$\varphi_t^o(s) = \max_{a_t \in A} E \left[ R(s_{t+1}, a, s) + \varphi_{t+1}^o(s_{t+1}) | a_t, s \right], \quad (7.14)$$

considering that  $\varphi_N^o = 0$  as Definition 8.1 implies. Furthermore the optimal policy  $\pi_t^o$ , as the argument of maxima, is concentrated on maximising arguments in Equation (8.2).

*Proof.* This theorem will be proven via a finite backward mathematical induction.

At first, let us take an action  $a_{N-1}^o$  in the time epoch  $N - 1$  defined as:

$$a_{N-1}^o(s) = \text{Arg max}_{a \in A} E[R(s_N, a, s) | a, s]. \quad (7.15)$$

By the definition of maxima the inequality

□

We can now describe the dynamic programming algorithm in detail.

**Example 1.** A real life example of this MDP could be again a biased coin tossing. Imagine a game when you are sitting at the table on which there are multiple coins with generally different bias.

It is important to mention that you are able to see the other coins "on the table" so that the position state  $s^p$  is known in each time epoch.

<There was table here. That should be easy to find again>

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**Algorithm 1** Finding the optimal policy for a *single system* MDP with known  $P$

---

**Require:**  $\mathcal{M} = (\mathbf{T}, \mathbf{S}, \mathbf{A}, P, R)$

- 1:  $\varphi_N^o(s) \leftarrow 0, \forall s \in \mathbf{S}$  ▷ Based on Definition 8.1.
  - 2:  $t \leftarrow N$
  - 3: **while**  $t \neq 0$  **do**
  - 4:     **for** each  $s \in \{1, 2, \dots, |\mathbf{S}|\}$  **do**
  - 5:          $\varphi_{t-1}^o(s) \leftarrow \text{Equation (8.2)}$  ▷ With known  $P$  and  $\varphi_t^o$
  - 6:      $t \leftarrow t - 1$
  - 7:  $\pi_0^o(s) \leftarrow \text{argmax } \varphi_0^o(s) \forall s \in \mathbf{S},$  ▷ Deriving the optimal policy
  - 8: **return**  $\pi_0^o$
-



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