

Ornstein Uhlenbeck Process Statistical Arbitrage and Cointegration on Equity Markets

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A Mathematical Exploration for IB Math: Applications & Interpretations HL

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1 Brownian Motion and the Wiener Process

Consider a stochastic process, where per each iteration, we add a Δ to the previous value, this Δ is distributed by a normal distribution with a μ of zero, and these are independent stochastic events this is called the Wiener process. In discrete time, we can define the Wiener process with the following equation

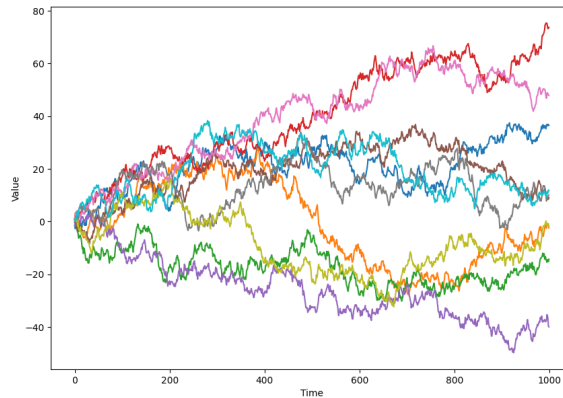
$$W_{t+1} = W_t + \mathcal{N}(0, \sigma^2)$$

Where W_{t+1} is the next value of the process (under discrete time), $\mathcal{N}(0, \sigma^2)$ is a normally distributed drift variable with the variance σ^2 .

Intuitively, the expected value, i.e. the limit of this process as it approaches infinity, should be zero, since the normal distribution is symmetric. (maybe prove this? Using a limits on the $\mathbb{E}[W]$ and the normal dist)

The spread of the Wiener process grows at a rate of $\sigma_t = \sigma\sqrt{t}$, due to each timestep adding additional compounded uncertainty.

Here's an example of a Wiener process with $W_0 = 0$ and $\sigma^2 = 1$:



In the diagram, we can clearly observe that the the variance of the Wiener process gets larger as the time progresses.

The wiener process is crucial and has many financial applications, for example, equity prices under the assumption of the efficient market hypothesis follow a purely-stochastic Wiener process. This although disregards stochastic drift, which applies directional pressure on equity

prices, i.e., the Wiener process is market-agnostic, and applies to equity which do not follow the upward trend of the market due to the positive risk-free rate.

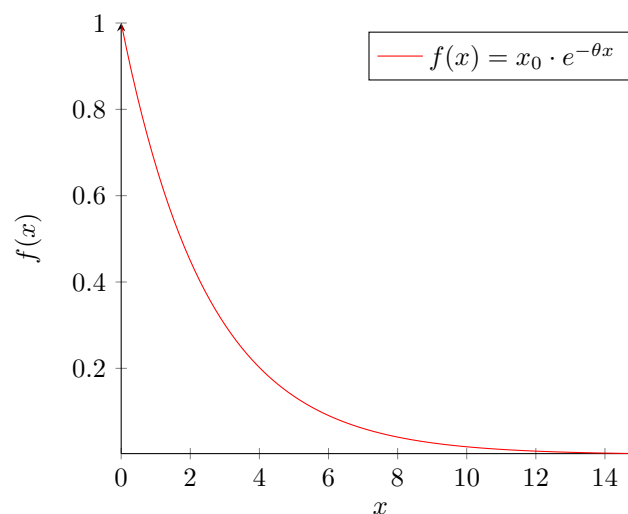
2 Ornstein-Uhlenbeck Process

2.1 Deriving a Mean Reversion Mechanism

But what if we don't want the variance of the process to get larger as time progresses? We call these processes mean-reverting, as in they tend to revert their values back towards the mean μ . This has numerous financial applications (not to mention its applications outside of finance), for example: statistical arbitrage, interest rate models, heston model, and mean reverting equities.

So how do we implement mean reversion? We can start by thinking about what mean reversion actually is? Mean reversion is the tendency of a variable (or usually a stochastic process) to revert back to its long-term mean. If we consider the roots of brownian motion - physics -, we can consider a mean-reverting term to be similar to a force pushing a particle back towards its mean; note that this doesn't happen instantaneously, it happens over a period of time, where the steadiness of this decay is dictated by the rate of mean-reversion, denoted as θ .

We know that an exponential function with a negative coefficient applied to the exponent will produce a function that will slowly decay.

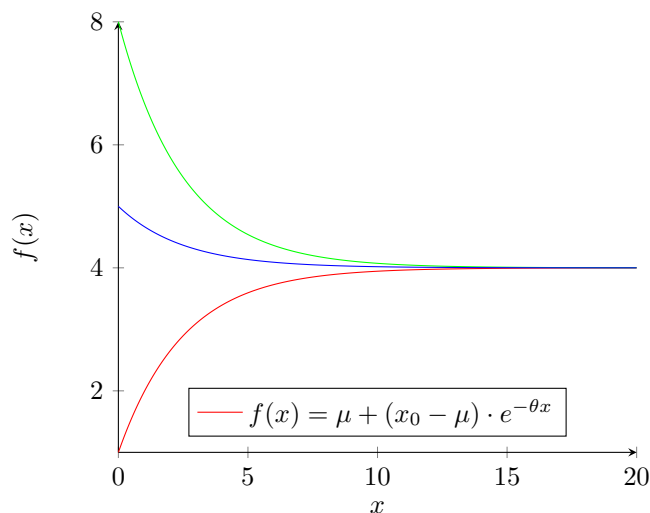


But this is not exactly mean reverting, we need to adjust the function to not only have the

tendency to approach zero, but rather, the function must approach the mean. And the function must be applied from any starting point x_0 . Considering that $f(0) = x_0$, we can adjust the function and introduce a base term: $f(x) = x_0 e^{-\theta x}$, this ensures the initial value is x_0 , since $e^0 = 1$. Now we have to ensure that the function converges at the mean, not at zero. Initially, we can try to add the mean as a constant term, such that $f(x) = \mu + x_0 e^{-\theta x}$, but when graphing this function, we notice that we break our first condition - the initial value of the function $f(0)$ is not equal to x_0 . We have to adjust for this in the coefficient of the exponential term. We notice that the difference between x_0 and $f(0)$ is equal to the mean, thus modifying the function to the following will solve our problem:

$$f(x) = \mu + (x_0 - \mu)e^{-\theta x}$$

The function produces the following graph, assuming $\mu = 4$, $\theta = 0.4$ and varying the initial value x_0 to be $x_0 \in \{8, 5, -1\}$ for each of the separate lines respectively:



2.2 Differentiating for the Mean-reverting Force

We can now think about creating a stochastic process that will take the Wiener process as a basis, but add mean-reverting properties in order to ensure the variance stays within a certain range. Just adding a term for the brownian motion will not be enough, this would be misinterpreting the

mean-reverting effect and would be mostly useless for our purposes, thus we can't do something like:

$$X_t = \mu + (X_0 - \mu) \cdot e^{-\theta t} + \sigma W_t$$

We must now calculate the tug or the actual amount by which the function changes per unit t . This will be necessary for our stochastic process, as they are inherently defined in differential form (in discrete time), where the next value depends on the previous value and we simply add a modifying term, e.g. $M_{t+1} = M_t + \sigma$. We can do this by taking a simple derivative of the function. This derivative can then be thought of as the per-unit t amount of tug applied for each value above or below the mean.

$$\begin{aligned} \frac{d}{dt} (\mu + (X_0 - \mu) \cdot e^{-\theta t}) \\ \frac{d}{dt} \mu + \frac{d}{dt} (X_0 - \mu) e^{-\theta t} \end{aligned}$$

We know that the derivative of the constant μ is zero, thus we can eliminate the term. We are left with the derivative of a product, for which we can use the product rule of differentiation: $\frac{d}{dt} x \cdot y = x' y + x y'$. In our case, the two terms are $(X_0 - \mu)$ and $e^{-\theta t}$. The first term is a constant that does not change with respect to t , thus it will be zero. The second term can be differentiated using the chain rule.

We need to define an outer and an inner function in order to apply the chain rule, the inner function will be e^x and the outer function will be $-\theta t$. The $\frac{d}{dx} e^x = e^x$ is a standard differential. And the $\frac{d}{dx} -\theta x$ will simply collapse down to $-\theta$, since it's a constant that is applied to our differentiating variable. The chain rule states:

$$\begin{aligned} \frac{d}{dx} f(g(x)) &= f'(g(x)) \cdot g'(x) \\ \frac{d}{dt} e^{-\theta x} &= e^{-\theta x} \cdot (-\theta) = -\theta e^{-\theta x} \end{aligned}$$

We can now revisit the product rule and apply it, we know that the derivative of our first term

is zero, thus we can ignore the first term of the product rule. Only leaving the following: $\frac{d}{dt}x \cdot y = xy'$, we have already calculated $y' = -\theta e^{-\theta x}$ in the previous step. Thus we can Just multiply $-\theta e^{-\theta x}$ with $(X_0 - \mu)$, and we get the following result for our whole derivative:

$$\frac{d}{dt} (\mu + (X_0 - \mu) \cdot e^{-\theta t}) = (-\theta e^{-\theta t}) \cdot (X_0 - \mu)$$

We can rearrange this to:

$$\frac{d}{dt} (\mu + (X_0 - \mu) \cdot e^{-\theta t}) = -\theta(X_0 - \mu)e^{-\theta t}$$

This differential describes the mean-reverting tug applied to our brownian motion model.

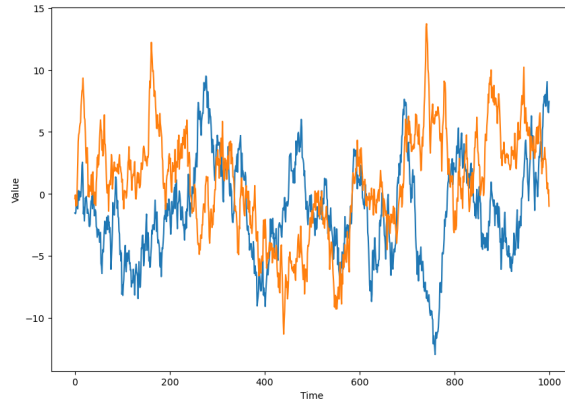
2.3 Full Naive Derivation of the OU Process

We can now try to perform a few experiments on a naively-formulated OU (Ornstein Uhlenbeck) process. We can begin by simply adding a brownian motion to our stochastic process, and assuming (naive step) $X_0 = X_t$, since at each point, the differential will get us tug, this will mean that our t is now zero. We are left with the following stochastic process:

$$X_{t+1} = -\theta(X_t - \mu)e^{-\theta \cdot 0} + W_t$$

$$X_{t+1} = -\theta(X_t - \mu)e + \mathcal{N}(0, \sigma^2)$$

We can not create stochastic paths along the process and they should be mean reverting. A stochastic path (more formally called a sample path of a stochastic process) is simply a possible outcome of running the process, essentially a random path that conforms to the process.



Here we can see that the stochastic process is much more contained within a range, and is thus mean-reverting.

2.4 Formal Derivation of the OU Process

The Ornstein Uhlenbeck process is formally defined as a stochastic differential equation (SDE). An SDE is just a differential equation with a stochastic term.

We begin with adding a brownian motion component. A standard wiener process in its differential form is: $dX_t = dW_t$, this is pretty straight forward if you think about it, the change in X_t is just the change in W_t , this being a stochastic component, dW_t will always be different, and for the sake of simplicity, we can visualize this as if the term follows the normal distribution $dW_t \sim \mathcal{N}(0, \sigma^2)$, this isn't entire true because the variance of the normal distribution itself depends on dt , but we will get to that later.

We now add the drift term (which we derived in a previous subchapter) $\theta(\mu - X_t)$ (we derived it as $-\theta(X_t - \mu)$, but these are functionally equivalent), we can say that this drift happens - continuously - over time t , thus can be written as $\theta(\mu - X_t)dt$, recall that dt here represents an infinitesimal (very small) change in time t , thus the whole term represents the change in the process value X_t as we change time t by an infinitesimal amount and can thus be written as $\frac{dX_t}{dt} = \theta(\mu - X_t)$, but this is just basic differential equations. We end up with:

$$dX_t = \theta(\mu - X_t)dt + dW_t$$

We also add a variable σ to our process, not to be confused with the standard deviation, in our context, θ represents the intensity of the brownian motion (stochastic) component. We end up with the final formal definition of the OU process:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

3 Long-term Equilibrium Cointegration in Equity Markets

We perform the **Two Step Engle-Granger Test** for cointegration of time series to get the cointegration coefficient. This method is widely accepted in quantitative finance and econometrics.

But what is cointegration? Cointegration represents the level to which a linear combination of two (or more) time series $X_t = \beta Y_t$ follows a stationary time series $I(0)$. Or in other words, whether the linear combination has a zero long-term mean. We can take any linear combination, such as the linear regression of X_t on Y_t , in the next sub-chapter, we form this regression and add an error term, the error term is essentially formed from a linear combination of the two time series. A high cointegration coefficient means that the two time series move in tandem with each other and that deviations in their spread revert to the long-term mean (since the mean of the spread is zero), divergence of spreads resulting in a zero long-term mean spread is impossible because the function is continuous.

3.1 Static Regression of Equity Time Series

First, we need to determine the residuals $\hat{\epsilon}_t$ (similar to the error) of a regression between the two time series of the prices of the equities we're analyzing. We define a linear regression $Y_t = \alpha + \beta X_t + \epsilon_t$. Where Y_t and X_t are the t -th elements of the time series of the equity prices of X and Y . And where α is the intercept term (which is the mean level of Y_t where X_t is zero) and where β is mean slope coefficient, i.e. the change in the value of Y_t per unit X_t . We define

the following intuitively or from the definition of the linear regression:

$$\beta = \frac{\sum_{i=0}^n (X_i - \mu_X) \cdot (Y_i - \mu_Y)}{\sum_{x \in X} (x - \mu_X)^2}$$

$$\alpha = \mu_Y - \beta \cdot \mu_X$$

Now we calculate the residuals $\hat{\epsilon}_t$ using the formula $\hat{\epsilon}_t = Y_t - (\alpha + \beta X_t)$, which we obtain from rearranging the terms of the linear regression equation with residuals. We can interpret these residuals as the difference (for each timestep t) between the actual value X_t and the predicted value - the value of the regression solution for the parameter t .

3.2 Dickey Fuller Test

The Dickey Fuller Test or (DF Test for short) is a statistical test that tests for the stationarity of a time series, or in other words, it tests for if the order of integration of the time series is $I(0)$. The non-augmented version of the Dickey Fuller Test is only applicable for an auto-regressive AR(1) process. Equity spreads are widely considered to be AR(1) processes. We can define an AR(1) time series as $y_t = \alpha + \phi y_{t-1} + \epsilon_t$, where α is the constant shift term, ϕy_{t-1} is the autoregressive term where ϕ is a coefficient denoting how strongly the previous value of the time series affects the current value at time t , and ϵ_t (not to be confused with the residuals from our regression model) denotes a random error term, which is assumed to be white noise $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$. Now we can define the hypotheses for the DF Test.

$$H_0 : \phi = 1$$

$$H_1 : \phi < 1$$

Where the null hypothesis denotes that the process is non-stationary and the alternative hypothesis indicates that the process is stationary. We could simply use a t-test to see if a $\hat{\phi}$ estimator is statistically significant, and based on that reject or fail to reject the null hypothesis of the DF Test. But the t-test is inapplicable in models where the central limit theorem doesn't

hold true (because the ϵ values of the process are not independent and identically distributed), thus we need to use the DF test. First, we transform our original time series expression in as a differencing equation. Note that we also rewrite the $y_t - y_{t-1}$ as Δy_t .

$$y_t = \alpha + \phi y_{t-1} + \epsilon_t$$

$$y_t - y_{t-1} = \alpha + \phi y_{t-1} - y_{t-1} + \epsilon_t$$

$$\Delta y_t = \alpha + y_{t-1}(\phi - 1) + \epsilon_t$$

For the sake of simplicity, we introduce δ instead of $(\phi - 1)$. We can simplify our express to $\Delta y_t = \alpha + \delta y_{t-1} + \epsilon_t$, which is the standard form you'll find in most literature on the DF Test. Our hypotheses remain and can be rewritten in terms of δ instead of ϕ .

$$H_0 : \delta = 0$$

$$H_1 : \delta < 0$$

We apply an Ordinary Least Squares linear regression to find the estimator $\hat{\delta}$, this is the p-value for our test and we compare it against our hypotheses. But in our case, we this won't be rigorous enough since we didn't account for autocorrelation (In our case, this means that previous values have some sort of effect on the current value y_t in our time series) properties of financial time series, to remedy this and correctly account for autocorrelation, we use the Augmented Dickey Fuller Test or ADF for short.

3.3 Augmented Dickey Fuller Test

This is an extension of the Dickey Fuller Test that correctly accounts for autocorrelation between consecutive values of the time series and is also extended to the autoregressive AR(p) process; it tests whether a time series of the order of integration $I(p)$ is stationary and for the presence of a unit root (which is a prerequisite of non-stationarity). In the ADF test, our hypotheses stay

the same.

$$H_0 : \delta = 0 \rightarrow \text{non-stationary, unit root}$$

$$H_1 : \delta < 0 \rightarrow \text{stationary, no unit root}$$

We need to add lags to our expression to account for autocorrelation. We can model autocorrelation as some coefficient applied on some previous value of the time series also being accounted in the value of the time series at t . Considering we add two lags to account for autocorrelation between the current value y_t and values y_{t-1} and y_{t-2} , our expression will intuitively become $\Delta y_t = \alpha + y_{t-1}(\phi - 1) + \beta_1(\Delta y_{t-1}) + \beta_2(\Delta y_{t-2}) + \epsilon_t$. Where the betas β_1 are our coefficients on the lagged differences. We also define p to be the number of lags we include, this value is automatically optimized in various statistical application programming interfaces (APIs) and one can use various Bayesian estimation methods to get the optimal p value, or one can perform a t-test (or an F-test when testing for the significance of multiple variables) and keep adding lagged differences (thus increasing p) until the addition is no longer statistically significant. Our expression can be generalized in the following form:

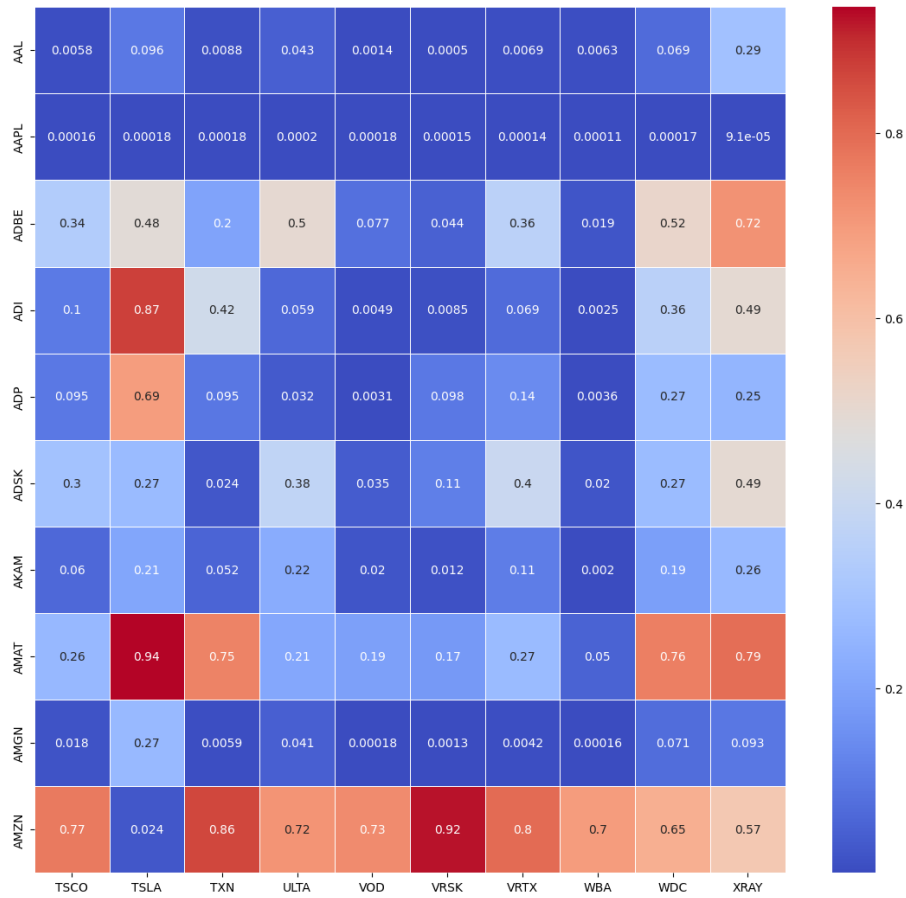
$$\Delta y_t = \alpha \delta y_{t-1} + \sum_{i=1}^p \beta_i \Delta y_{t-i} + \epsilon_t$$

Now that we have this equation, we can do an Ordinary Least Squares regression (in its multiple linear regression form, since we have multiple regressors) to find $\alpha, \delta, \beta_1, \beta_2, \dots, \beta_p$, we then use the δ value as the final p-value for the Augmented Dickey Fuller Test.

3.4 Performing the Two-Step Engle Granger Test on Equity Prices

We now perform the two-step engle granger test on 78 mature stocks from the selected from NASDAQ-100 index from January 1st 2014 to January 1st 2024. We pair every stock with every other stock resulting in $78^2 = 6084$ pairs, for each of these pairs we perform an OLS regression and get the residual values (as described in subsection 1) and then perform the Augmented

Dickey Fuller Test using the 'statsmodels' library. We extract the p-values of the test and end up with the following heatmap:



Only 100 pairs are displayed as the complete 78x78 heatmap is too large to present here.

We now select 10 of the most cointegrated (lowest p-values) equity pairs to further examine, the ten most cointegrated pairs were, in descending order of cointegration, we observe that most of the equity pairs with very heavy cointegration have AAPL (Apple Inc.) as the base stock, this is interesting as AAPL has heavily outperformed even the growth-heavy NASDAQ, thus this should create a larger spread between most other stocks, suggesting limitations of our test.

Stock 1 Ticker	Stock 2 Ticker	ADF P-Value
AAPL	XRAY	9.104296439449678e-05
AAPL	WBA	0.00011254543577701014
AAPL	VRTX	0.00014280647279306093
AAPL	VRSK	0.00015429781044445213
AMGN	WBA	0.00016469559752946386
AAPL	TSCO	0.00016489122300362627
AAPL	WDC	0.0001652537550668745
AAPL	TSLA	0.00017722866444703876
AMGN	VOD	0.00018081206670114813
AAPL	TXN	0.00018223283943511726

For reference, we can plot the residuals of the initial regression of a select heavily cointegrated pair and a non-cointegrated pair to better visualize the idea of cointegration.

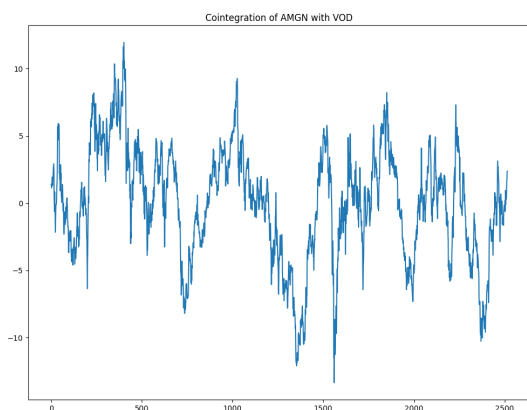


Figure 1: AMGN, VOD, p-value: 0.000181

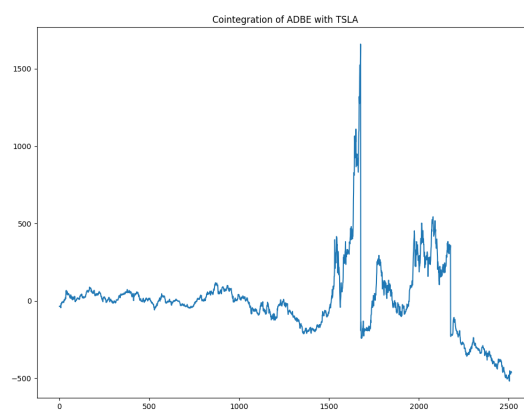


Figure 2: ADBE, TSLA, p-value: 0.484251

3.5 Engle Granger Residuals vs Equity Spreads

If we plot the residuals of AMGN and VOD, we can observe that the resultant time series clearly follows an OU process (or a time-discrete AR(1) time series), as is also indicated by the test we performed, but if we plot the equity spreads (values of stock 1 minus values of stock 2), we

observe a different pattern, indicating that our model is still incomplete and that OLS residuals might not give us the full picture. Let's examine the difference between the graph of the residuals and the graph of the spreads:

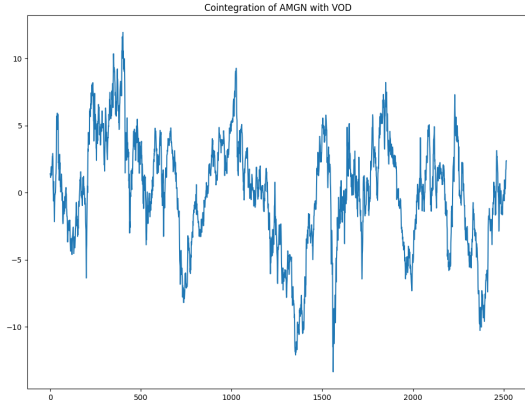


Figure 3: AMGN-VOD cointegration regression residuals

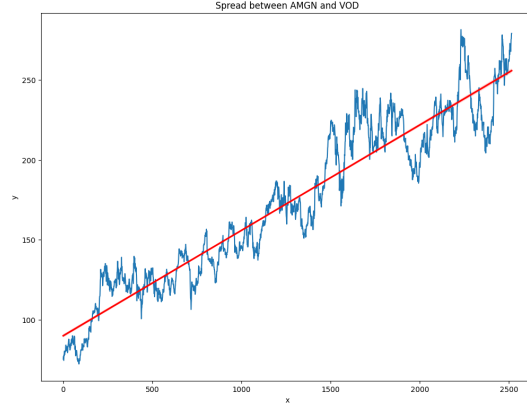


Figure 4: AMGN-VOD spread with line of best fit

We can see that the cointegration essentially disregards deterministic drift in the spreads (deterministic drift is essentially the non-stochastic trend). This makes sense if you think about our test methodology, we defined an OLS linear regression between two stock price time series and then we used to residuals to perform the stationarity test, our graph just shows the residuals at each time step. A residual is just the difference between the what the regression model predicts (using a standard linear equation model $y = \beta x + \alpha$) and the actual value of the time series.

Our Ornstein Uhlenbeck model resembles the residuals and not the actual spreads, it doesn't yet account for deterministic drift.

4 Fitting the Ornstein Uhlenbeck Process to our Data

There are two ways you can calibrate an Ornstein Uhlenbeck process for a particular dataset. Either using a simple OLS regression for the OU parameters (θ, σ, μ) , or you can use a much more rigorous technique that yields better results, called: Maximum Likelihood Estimation (MLE), while if this weren't a Math IA, I would use MLE (due to there already being pre-

existing implementation and it being fairly easy to understand, just requires basic multivariable calculus), we do not have the capacity to explain MLE in this IA, additionally, OLS parameter estimation is frequently used in Quant Finance, despite it's obvious limitations. The first step is obtaining a formal solution to the OU stochastic differential equation (SDE).

4.1 Formal Analytical Solution to the OU SDE

We start with the OU SDE we derived in a previously, and we expand the deterministic drift term.

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

$$dX_t = \theta\mu dt - \theta X_t dt + \sigma dW_t$$

Looking at the deterministic part of our SDE $dX_t = \theta\mu dt - \theta X_t dt$, we notice that we need to simplify this ordinary differential equation (ODE) and find an integrating factor. An integrating factor is given by the following: $I(x) = e^{\int P(x) dx}$, where $P(x)$ is defined by the standard form of a first-order ODE: $\frac{dx}{dy} + P(x)y = Q(x)$, which we can fit to our deterministic ODE:

$$\begin{aligned}\frac{dX_t}{dt} &= \theta\mu - \theta X_t \\ \frac{dX_t}{dt} + \theta X_t &= \theta\mu\end{aligned}$$

where $P(x) = \theta$ and $Q(x) = \theta\mu$, we can now use the above-mentioned formula to find the integrating factor, $I(t) = e^{\int P(t) dx}$ (notice that we switch to t since dt is our differential and not dx). Since θ is a constant, we can easily evaluate the integral $\int \theta dt = \theta t + C$, thus our integrating factor becomes: $I(t) = e^{\theta t}$, notice that we deliberately ignore the constant term C . Now, going back to our SDE, we can apply our integrating factor.

$$dX_t = \theta\mu dt - \theta X_t dt + \sigma dW_t$$

$$e^{\theta t} dX_t = e^{\theta t} \theta \mu dt - e^{\theta t} \theta X_t dt + e^{\theta t} \sigma dW_t$$

We now notice that we can use the product rule of differentiation to simplify the above SDE. If we rearrange the SDE, we notice that $e^{\theta t}dX_t + e^{\theta t}\theta X_t dt$ conforms to the product rule, that states $(u \cdot v)' = u \cdot v' + u' \cdot v$. We aim to express our two terms as a single stochastic derivative, we can't simply use a regular derivative since X_t is a stochastic process, thus we introduce the Itô's derivative, which is a derivative on a stochastic process, and in our case $d(\cdot)$ represents an infinitesimal change in the stochastic process changes with respect to time t , this can be expanded with the use of the Itô's Lemma. Going back to solving our SDE, we work backwards through the application of the product rule and see that $u = e^{\theta t}$ and $v = X_t$, our terms will simplify to:

$$d(e^{\theta t} \cdot X_t) = e^{\theta t}dX_t + e^{\theta t}\theta X_t dt$$

And we can apply this to the SDE, like so:

$$e^{\theta t}dX_t = e^{\theta t}\theta \mu dt - e^{\theta t}\theta X_t dt + e^{\theta t}\sigma dW_t$$

$$e^{\theta t}dX_t + e^{\theta t}\theta X_t dt = e^{\theta t}\theta \mu dt + e^{\theta t}\sigma dW_t$$

$$d(e^{\theta t} \cdot X_t) = e^{\theta t}\theta \mu dt + e^{\theta t}\sigma dW_t$$

Now we can integrate to find the general solution to the linear stochastic differential equation. Note that for our stochastic components, we can't perform the regular Riemann Integral, but rather, we have to use the Itô Integral. We define an Itô Integral as an integral on a stochastic process, the result of the Itô Integral is another stochastic process, that essentially defines the "sum" of the process up until a certain time t , it can again, be solved by application of the Itô's Lemma. We define a general Itô Integral as:

$$\int_0^t f(s)dX_s$$

Where $f(s)$ is a deterministic process, which is generated from a filtration $\mathcal{F}(s)$, which is just a collection of all known information about the process up until a time s . You can think of $f(s)$ as just a deterministic process that describes how the stochastic process behaves over time s . And

dX_t is, of course, is an infinitesimal change in the stochastic process X_t . Let's take our brownian motion term as an example, $\int_0^t e^{\theta t} \sigma dW_s$, our deterministic process generated by the filtration $\mathcal{F}(s)$ would be $f(s) = e^{\theta t} \sigma$, note that this is constant and not dependent on time s . We can now apply the Itô Integral on the left-hand side of the SDE and on the stochastic Wiener Process term.

$$d(e^{\theta t} \cdot X_t) = e^{\theta t} \theta \mu dt + e^{\theta t} \sigma dW_t$$

$$\int_0^t d(e^{\theta s} \cdot X_s) = \int_0^t e^{\theta s} \theta \mu ds + \int_0^t e^{\theta s} \sigma dW_s$$

We can now evaluate each of these integrals. We begin by evaluating the deterministic integral in the middle of the SDE. We notice that $\theta \mu$ is a constants term, thus a part of it can be factored out of the integral. We also recall the rule for integrating exponential functions $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$ and apply it.

$$\int_0^t e^{\theta s} \theta \mu ds = \mu \int_0^t e^{\theta s} \theta ds = \mu \left| \theta \frac{1}{\theta} e^{\theta t} \right|_0^t = \mu \left| e^{\theta t} \right|_0^t$$

Next we evaluate the integral from 0 to time s , using the standard procedure for evaluating definite integrals.

$$\mu \left| e^{\theta t} \right|_0^t = \mu (e^{\theta t} - e^{\theta \cdot 0}) = \mu (e^{\theta t} - 1)$$

Next, we need to solve two of our Itô Integrals. Let's solve the left-hand side of our SDE first. We will apply the fundamental theorem of calculus adapted to Itô integrals, which states that for a well-behaved process Y_t , the Itô Integral of the differential of that process dY_t is described by: $\int_0^t dY_s = Y_t - Y_0$, we recognize that $Y_s = e^{\theta s} X_s$, thus we can apply it to our left-hand side:

$$\int_0^t d(e^{\theta s} \cdot X_s) = e^{\theta t} X_t - e^{\theta \cdot 0} X_0 = e^{\theta t} X_t - X_0$$

We cannot further simplify our stochastic integral over the Wiener Process, thus we leave it as is - which is fine, in our algorithmic trading strategy this will be fairly simple to compute and correctly account for. We combine everything and get the following:

$$\begin{aligned}\int_0^t d(e^{\theta s} \cdot X_s) &= \int_0^t e^{\theta s} \theta \mu ds + \int_0^t e^{\theta s} \sigma dW_s \\ e^{\theta t} X_t - X_0 &= \mu(e^{\theta t} - 1) + \int_0^t e^{\theta s} \sigma dW_s\end{aligned}$$

Next, we express the term X_t which represents the value of the Ornstein Uhlenbeck process at time t , which is the final solution to the SDE.

$$\begin{aligned}e^{\theta t} X_t &= X_0 + \mu(e^{\theta t} - 1) + \int_0^t e^{\theta s} \sigma dW_s \\ X_t &= \frac{1}{e^{\theta t}} X_0 + \frac{1}{e^{\theta t}} \mu(e^{\theta t} - 1) + \frac{1}{e^{\theta t}} \int_0^t e^{\theta s} \sigma dW_s\end{aligned}$$

We can now simplify each of these terms. $\frac{1}{e^{\theta t}} X_0$ will become $X_0 e^{-\theta t}$. We continue simplifying, $\frac{1}{e^{\theta t}} \mu(e^{\theta t} - 1) = \frac{1}{e^{\theta t}} (\mu e^{\theta t} - \mu) = (\mu - \mu e^{-\theta t}) = \mu(1 - e^{-\theta t})$. We recall the rule about factoring constants out of integrals, we can also do the reverse and factor a constant into an integral, while this is contraproductive most of the time, it can be used to simplify certain terms. Thus, $\frac{1}{e^{\theta t}} \int_0^t e^{\theta s} \sigma dW_s = \int_0^t \frac{1}{e^{\theta t}} e^{\theta s} \sigma dW_s = \int_0^t e^{-\theta t} e^{\theta s} \sigma dW_s = \int_0^t e^{\theta s - \theta t} \sigma dW_s = \int_0^t e^{-\theta(t-s)} \sigma dW_s$. Now that all of our terms have been simplified, we can combine them to form the final formal solution to the stochastic differential equation governing the Ornstein Uhlenbeck Process.

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t e^{-\theta(t-s)} \sigma dW_s$$

4.2 Adjusting OU Deterministic Drift for Trend

Recall the graph of the sample paths generated by the Ornstein Uhlenbeck process, you can notice that the mean is static, the process always reverts back to a constant mean. But Looking at the equity spreads between most of our cointegration pairs, we notice that the spread is actually growing, while still adhering to the Ornstein Uhlenbeck Process, we know this because we checked

for the stationarity of regression residuals of the spreads. What if we simply introduce a function of time representing a changing mean? We can represent this with the following function $\mu(t)$, and we can directly substitute it to the SDE, but we still have to do some work to properly find a general solution.

$$dX_t = \theta (\mu(t) - X_t) dt + \sigma dW_t$$

We can't use the general solution we found in the previous subchapter directly, because we assumed that μ is a constant, which is not true. Previously, we defined an integral for finding the general solution to the deterministic part of the SDE and we simplified it. We can no longer do this since we would like to keep things general and assume that $\mu(t)$ is some function and we don't have any further information about it. Thus, we can change our integral over the deterministic term:

$$\int_0^t e^{\theta s} \theta \mu ds \implies \int_0^t e^{\theta s} \theta \mu(s) ds$$

Recall that we also had to divide by our integrating factor $e^{-\theta t}$, we also have to apply the operation here.

$$e^{-\theta t} \int_0^t e^{\theta s} \theta \mu(s) ds = \int_0^t e^{\theta s} e^{-\theta t} \theta \mu(s) ds = \int_0^t e^{\theta(s-t)} \theta \mu(s) ds$$

We can now substitute this new integral in place of the deterministic part of the general solution to the SDE.

$$X_t = X_0 e^{-\theta t} + \int_0^t e^{\theta(s-t)} \theta \mu(s) ds + \int_0^t e^{-\theta(t-s)} \sigma dW_s$$

4.3 Time Discretization Using the Euler Maruyama Method

We can define any Itô Process using the following generalization of a stochastic differential equation:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$

Where $f(t, X_t)$ represents the deterministic drift term and $g(t, X_t)$ represents the stochastic diffusion term, which you can think of as a modifier on the brownian motion. Looking at the SDE describing the Ornstein Uhlenbeck Process with a non-constant time-dependent mean $dX_t = \theta (\mu(t) - X_t) dt + \sigma dW_t$, we notice that we can formulate a $f(t, X_t)$ function and a $g(t, X_t)$ function such that the SDE conforms to the before-mentioned generalization.

$$f(t, X_t) = \theta (\mu(t) - X_t)$$

$$g(t, X_t) = \sigma$$

Applying the Euler Maruyama Method for discretization, we get the following time-discrete approximation of the OU SDE:

$$X_{t+1} = X_t + \tau f(t, X_t) + g(t, X_t)\Delta W_n$$

$$X_{t+1} = X_t + \tau \theta (\mu(t) - X_t) + \sigma \Delta W_n$$

We define τ to be a small increment (essentially the scale of our time series X_t), we also define ΔW_t as an increment of the Wiener process over a discrete time-increment interval $[t, t+1]$, such that $\Delta W_t = W_{t+1} - W_t \sim \mathcal{N}(0, \tau)$. In our model, we consider $\tau = 1$, so we can disregard it. We end up with the following discretization:

$$X_{t+1} = X_t + \theta (\mu(t) - X_t) + \sigma \Delta W_n$$

4.4 Calibrating OU Process Parameters using OLS

We start by defining a time series X_t , which contains the underlying dataset we're trying to fit the OU process to, in our case, this would be the spread between two equities. We need to estimate the parameters θ - mean reverting force, $\mu(t)$ - time-dependent mean function, and σ - volatility of the noise.

We can easily estimate the mean function $\hat{\mu}(t) = \beta t + \alpha$ by running a linear regression on our dataset. The slope of the result of this regression is β and the intercept is α .

Next, to estimate the other parameters, we rearrange our time-discrete solution in terms of the changes between consecutive elements of the time series.

$$X_{t+1} = X_t + \theta (\mu(t) - X_t) + \sigma \Delta W_n$$

$$X_{t+1} - X_t = \theta (\mu(t) - X_t) + \sigma \Delta W_n$$

$$\Delta X_t = \theta (\mu(t) - X_t) + \sigma \Delta W_n$$

We need to fit our equation to the form $Y_t = \beta X_t + \alpha + \epsilon_t$, where ϵ_t represents the residuals of the OLS, this is the standard form of an OLS regression which we can then feed into our algorithm. We start by expanding the bracket to get the following: $\Delta X_t = \theta \mu(t) - \theta X_t + \sigma \Delta W_n$, notice how this is already starting to look like a standard OLS form, we just need to define Y_t . If we move the term $\theta \mu(t)$ to the other side (since we can pre-compute it), we can define $Y_t = \Delta X_t - \theta \mu(t)$, thus leaving us with: $Y_t = -\theta X_t + \sigma \Delta W_n$. Notice how there's no intercept term α , in a perfect dataset, we would expect this intercept term to be zero, but since our data cannot be fully fit to the OU model, this would represent a type of error. We can now run the OLS regression.

5 Unoptimized OU-based Trading Strategy

6 Optimal Execution Region and OU Parameters