Ficha 5

21

a) $3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$

$$3^n = (2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 2^k$$

b)

$$k^n = (\underbrace{1+1+1+\dots+1}_{k \text{ termos}})^n = \sum_{n_1+n_2+n_3+\dots+n_k} \binom{n}{n_1, n_2, n_3, \dots, n_k} 1^{n_1} \cdot 1^{n_2} \cdot 1^{n_3} \cdot \dots \cdot 1^{n_k}$$

$$= \sum_{n_1+n_2+\dots+n_k=n}^{\infty} \binom{n}{n_1, n_2, \dots, n_k}$$

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a)

$$\left(x^2 + \frac{y}{x} + 2z\right)^6 = (x^2 + y^1 \cdot x^{-1} + 2z)^6$$

$$= \sum_{n_1+n_2+n_3=6} \binom{6}{n_1, n_2, n_3} x^{2n_1} \cdot (yx^{-1})^{n_2} \cdot (2z)^{n_3}$$

$$= \sum_{n_1+n_2+n_3=6} \binom{6}{n_1, n_2, n_3} x^{2n_1-n_2} \cdot y^{n_2} \cdot 2^{n_3} \cdot z^{n_3}$$

$$xy^3z = x^{2n_1-n_2} \cdot y^{n_2} \cdot z^{n_3}$$

$$\Leftrightarrow \begin{cases} 2n_1 - n_2 = 1 \\ n_2 = 3 \\ n_3 = 1 \end{cases} \Leftrightarrow \begin{cases} 2n_1 = 1+3 \\ - \\ \end{cases} \Leftrightarrow \begin{cases} n_1 = 2 \\ n_2 = 3 \\ n_3 = 1 \end{cases}$$

$$\binom{6}{1, 3, 2} x^{2\cdot 2-3} \cdot y^3 \cdot (2)^1 \cdot z^1 \Rightarrow \frac{6!}{1! \, 3! \, \times 2!} \times 2 = 6 \times 5 \times 4 = 120$$

b) $(a+b)^4 = \sum_{k=0}^{4} \binom{4}{k} a^k b^{4-k} = C_0 a^0 b^4 + C_1 a^1 b^3 + C_2 a^2 b^2 + C_3 a^3 b^1 + C_4 a^4 b^0$

$$(a+b)^4 = 5^4 = C_0 4^0 + C_1 4^1 + C_2 4^2 + C_3 4^3 + C_4 4^4$$

$a = 4$
$b = 1$

$C_0 = \binom{4}{0} = 1 \quad C_2 = \binom{4}{2} = 6 \quad C_4 = \binom{4}{4} = 1$

$C_1 = \binom{4}{1} = 4 \quad C_3 = \binom{4}{3} = 4$

c) $(x^3 + \sqrt{x})^n = \sum_{k=0}^{n} \binom{n}{k} x^{3k} \cdot x^{\frac{1}{2}(n-k)} = \sum_{k=0}^{5} \binom{5}{k} \cdot x^{3k + \left(\frac{1}{2}(5-k)\right)}$

$$\sum_{k=0}^{n} \binom{n}{k} = 32 \Leftrightarrow 2^n = 32 \Leftrightarrow 2^n = 2^5 \Leftrightarrow n = 5$$

$$C \cdot x^{10} = \binom{5}{k} \cdot x^{3k + \frac{1}{2}(5-k)}$$

$$10 = 3k + \tfrac{1}{2}(5-k) \Leftrightarrow 6k + 5 - k = 20 \Leftrightarrow 5k = 15 \Leftrightarrow k = 3$$

$$C = \binom{5}{k} \Leftrightarrow C = \binom{5}{3} = 10$$

23

$(x_1 - x_2 + 2x_3 - 8x_4)^8$

$$= \sum_{n_1+n_2+n_3+n_4=8} \binom{8}{n_1, n_2, n_3, n_4} x_1^{n_1} \cdot (-x_2)^{n_2} \cdot (2x_3)^{n_3} \cdot (-8x_4)^{n_4}$$

$$x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \cdot x_4^{n_4} = x_1^2 x_2^3 x_3^1 x_4^2$$

$n_1 = 2$
$n_2 = 3$
$n_3 = 1$
$n_4 = 2$

$$\binom{8}{2,3,1,2} \cdot x^2 \cdot (-x_2)^3 \cdot (2x_3)^1 \cdot (-8x_4)^2$$

$$= \frac{8!}{\cancel{2}!\,3!\,1!\,2!} x_1^2 \cdot (-1)\, x_2^3 \cdot \cancel{2} x_3 \cdot 64\, x_4$$

$$= -64 \times \frac{8 \times \cancel{7} \times 6 \times 5 \times \overset{2}{\cancel{4}}}{2} \times \ldots$$

$=$

24. $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$\sum_{k=0}^{n} \binom{n}{k} = 256$$

$2^n = 256$
$n = 8$

$$\left(\frac{n}{2}\right)! = \left(\frac{8}{2}\right)! = 4! = 24$$

25

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Para $k=2$, o termo é $\binom{n}{2} a^2 b^{n-2}$

De acordo c/ o enunciado, $\binom{n}{2} = 28$

$$\frac{n!}{2!\,(n-2)!} = 28 \Leftrightarrow n(n-1) = 56$$

$\Leftrightarrow n^2 - n - 56 = 0 \Leftrightarrow n = \frac{1 \pm \sqrt{1+224}}{2} \Leftrightarrow n = \frac{1 \pm \sqrt{225}}{2} \Leftrightarrow n = \frac{1 \pm 15}{2} \wedge n > 0$

$\Leftrightarrow n = 8$

~~Para $k=3$, o termo é $\binom{n}{3} a^3$~~

Para $k = n-3$, o termo é $\binom{n}{n-3} a^{n-3} b^3$

$$C_8 = \binom{n}{n-3} = \binom{8}{5} = \frac{8 \times 7 \times 6}{3!} = 56$$

$$\left(x-\frac{1}{x}\right)^{100}=\left(x-x^{-1}\right)^{100}=\sum_{k=0}^{100}\binom{100}{k}x^{k}\cdot(-x)^{100-k}$$

$$=\sum_{k=0}^{100}\binom{100}{k}(-1)^{100-k}\cdot x^{k-(100-k)}$$

$$=\sum_{k=0}^{100}\binom{100}{k}(-1)^{100-k}x^{2k-100}$$

Para calcular Cx^{a}, $a\in\mathbb{Z}$

$$x^{a}=x^{2k-100}$$

$$a=2k-100$$

$$\frac{a+100}{2}=k \Leftrightarrow k=\frac{a}{2}+50$$

$$C=\binom{100}{50+\frac{k}{2}}\cdot(-1)^{100-50-\frac{k}{2}}=\binom{100}{50+\frac{k}{2}}(-1)^{50-\frac{k}{2}}$$

7.27

a) $$\sum_{k=0}^{n}k\binom{n}{k}=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\times k=\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}$$

$$=\sum_{a=0}^{n}\frac{n!}{a!(n-a-1)!}=n\cdot\sum_{a=0}^{n}\frac{(n-1)!}{a!((n-1)-a)!}=n\cdot 2^{n-1}$$

$a=k-1$

$k=1$

$a=0$