Cálculo I

Fórmulas

Trigonometria

$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\sin(a+b) = \sin a \cos b + \sin b \cos a$
$\cos(a+b) = \cos a \cos b - \sin a \sin b$
$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

$\cdot \sin x = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$; $\cdot \cos x = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$; $\cdot \tan x = \frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}$
$\Rightarrow$ Para Primitivas:
$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, \tan x = \frac{2t}{1-t^2}, \cot x = \frac{1-t^2}{2t}$

Polinómios

$(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$

Limites

Limites Notáveis

$\lim_{x\to\infty}(1+\frac{1}{x})^x = e$
$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$
$\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$
$\lim_{x\to +\infty} \frac{\ln(x)}{x} = 0$
$\lim_{x\to +\infty} \frac{e^x}{x^p} = +\infty, (p \in \mathbb{R})$

Limites Gerais

$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
Indeterminações $\frac{0}{0}$ ou $\frac{\pm\infty}{\pm\infty}$:
$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$
Indeterminações 1^∞:
[redacted]

Derivadas

Derivadas Básicas

$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$
$(c\ f)' = c\ f'(x)$
$(x^n)' = nx^{n-1}$
$(f \pm g)' = f'(x) \pm g'(x)$
$(fg)' = f'g + fg'$
$(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$
$(f \circ g)' = f'(g(x)) \cdot g'(x)$

$(f^{-1})' = \frac{1}{f'(f^{-1})}$
$(f^g)' = (e^{g \ln(f)})' = f^g(f'\frac{g}{f} + g' \ln(f))$

Derivadas Logarítmicas e Exponênciais

$(a^x)' = a^x \ln a$
$(\ln x)' = \frac{1}{x},\ x > 0$
$(\log_a x)' = \frac{1}{x \ln a},\ x > 0$

$(x^x)' = x^x(1 + \ln x)$
$(e^{f(x)})' = f'(x)e^{f(x)}$
$([f(x)]^n)' = n[f(x)]^{n-1} f'(x)$
$(\ln[f(x)])' = \frac{f'(x)}{f(x)},\ f(x) > 0$

Derivadas Trigonométricas

$(\sin x)' = \cos x$
$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
$(\sec x)' = \sec x \tan x$
$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}}$
$(\cos x)' = -\sin x$
$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\csc x)' = -\csc x \cot x$
$(\operatorname{arccsc} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$
$(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x} = 1 + \tan^2 x$
$(\arctan x)' = \frac{1}{1+x^2}$
$(\cot x)' = -\csc^2 x = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x$
$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$

$(\sin[f(x)])' = f'(x)\cos[f(x)]$
$(\cos[f(x)])' = -f'(x)\sin[f(x)]$
$(\tan[f(x)])' = f'(x)\sec^2[f(x)] = \frac{f'(x)}{\cos^2[f(x)]}$

Integrais

Integrais Básicos

$\int c \cdot f(x)\ dx = c\int f(x)\ dx, c \in \mathbb{R}$
$\int f(x) \pm g(x)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
$\int_a^b f(x)\ dx = F(x)|_a^b = F(b) - F(a),\ F(x) = \int f(x)\ dx$
$[F(\varphi(x))]' = F'(\varphi(x))\varphi'(x)$

Integrais Polinomiais

$\int dx = x + C,\ C \in \mathbb{R}$
$\int k\ dx = kx + C,\ C \in \mathbb{R}$
$\int \frac{1}{x}\ dx = \ln|x| + C,\ C \in \mathbb{R}$
$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$ e $C \in \mathbb{R}$
$\int \frac{1}{ax+b}\ dx = \frac{1}{a}\ln|ax+b| + C,\ C \in \mathbb{R}$

$\int \frac{\varphi'(x)}{\varphi(x)}\ dx = \ln[\varphi(x)] + C,\ C \in \mathbb{R}$
$\int \varphi'(x)[\varphi(x)]^a\ dx = \frac{\varphi^{a+1}(x)}{a+1} + C,\ a \neq -1$ e $C \in \mathbb{R}$

Integrais Trigonométricos

$\int \cos x\ dx = \sin x + C,\ C \in \mathbb{R}$
$\int \sin x\ dx = -\cos x + C,\ C \in \mathbb{R}$
$\int \sec^2 x\ dx = \int \frac{1}{\cos^2 x} = \tan x + C,\ C \in \mathbb{R}$
$\int \csc^2 x\ dx = \int \frac{1}{\sin^2 x} = -\cot x + C,\ C \in \mathbb{R}$
$\int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x + C,\ C \in \mathbb{R}$
$\int \frac{1}{\sqrt{1-x^2}}\ dx = -\arccos x + C,\ C \in \mathbb{R}$
$\int \frac{1}{1+x^2}\ dx = \arctan x + C,\ C \in \mathbb{R}$
$\int \frac{1}{1+x^2}\ dx = -\operatorname{arccot} x + C,\ C \in \mathbb{R}$

$\int \varphi'(x)\cos[\varphi(x)]\ dx = \sin\varphi(x) + C,\ C \in \mathbb{R}$
$\int \varphi'(x)\sin[\varphi(x)]\ dx = -\cos\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{\cos^2\varphi(x)}\ dx = \tan\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{\sin^2\varphi(x)}\ dx = -\cot\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{\sqrt{1-\varphi(x)^2}}\ dx = \arcsin\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{\sqrt{1-\varphi(x)^2}}\ dx = -\arccos\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{1+\varphi(x)^2}\ dx = \arctan\varphi(x) + C,\ C \in \mathbb{R}$
$\int \frac{\varphi'(x)}{1+\varphi(x)^2}\ dx = -\operatorname{arccot}\varphi(x) + C,\ C \in \mathbb{R}$

Integrais Exponênciais/Logarítmicos

$\int e^x\ dx = e^x + C,\ C \in \mathbb{R}$
$\int a^x\ dx = \frac{a^x}{\ln a} + C,\ C \in \mathbb{R}$

$\int \varphi'(x)e^{\varphi(x)}\ dx = e^{\varphi(x)} + C,\ C \in \mathbb{R}$

Primitivação por partes

$\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)\ dx$
Nota: $d\varphi(x) = \varphi'(x)\ dx$
$\to \int f(x)g'(x)\ dx = \int v\ du = uv - \int u\ dv$

[redacted]

I

$$\left.\begin{array}{l} \int P_k(x)\sin(bx)\ dx \\ \int P_k(x)\cos(bx)\ dx \\ \int P_k(x)e^{ax} \end{array}\right\} \quad \begin{array}{l} u = P_k(x) \\ v' = \begin{cases} \cdot \sin(bx) \\ \cdot \cos(bx) \\ \cdot\ e^{ax} \end{cases} \end{array}$$

II

$$\left.\begin{array}{l} \int P_k(x)\ln(bx)\ dx \\ \int P_k(x)\arcsin x\ dx \\ \int P_k(x)\arccos x\ dx \\ \int P_k(x)\arctan x\ dx \\ \int P_k(x)\operatorname{arccot} x\ dx \end{array}\right\} \quad \begin{array}{l} u = \begin{cases} \cdot \ln(bx) \\ \cdot \arcsin x \\ \cdot \arccos x \\ \cdot \arctan x \\ \cdot \operatorname{arccot} x \end{cases} \\ v' = P_k(x) \end{array}$$

III - <u>2 vezes por partes</u>

$$\left.\begin{array}{l} \int e^{ax}\sin(bx)\ dx \\ \int e^{ax}\cos(bx)\ dx \end{array}\right\} \quad \begin{array}{l} \text{Hipótese 1} \\ u = e^a x,\ v' = \begin{cases} \cdot \sin(bx) & (1) \\ \cdot \cos(bx) & (2) \end{cases} \\ \text{Hipótese 2} \\ u = \begin{cases} \cdot \sin(bx) \\ \cdot \cos(bx) \end{cases},\ v' = e^a\ x \quad \begin{array}{l} (1) \\ (2) \end{array} \end{array}$$

Primitivação de Funções Racionais (por decomposição)

Função Racional: $\frac{P(x)}{Q(x)}$, P e Q polinómios de coeficientes reais.
·Função Racional **Própria** $\rightarrow \text{gr}(P(x)) < \text{gr}(Q(x))$
·Função Racional **Imprópria** $\rightarrow \text{gr}(P(x)) > \text{gr}(Q(x))$

0. Função Racional Imprópria $\rightarrow$ Polinómio+F. R. Própria
1. Resolver $Q(x) = 0$, decompondo-se $Q(x)$ em :
   - Constantes (a)
   - $(x-R)^l,\ l \in \mathbb{N} \rightarrow l-$Mult. de Raizes Reais
   - $(x^2+px+q)^k,\ k \in \mathbb{N} \rightarrow k-$Mult. de Raizes $\alpha \pm i\beta$

   $Q(x) = a(x-R_1)^{l_1} \cdot (x-R_2)^{l_2} \cdot ... \cdot (x^2+p_1x+q_1)^{k_1} \cdot (x^2+p_2x+q_2)^{k_2} \cdot ...$
2. $\frac{P(x)}{Q(x)} = \frac{P(x)}{a(x-R_1)^{l_1}\cdot ... \cdot (x^2+p_1x+q_1)^{k_1} \cdot ...}$
   - Determinar:
     $\frac{A_1}{x-R_1} + \frac{A_2}{(x-R_1)^2} + \frac{A_3}{(x-R_1)^3} + ... + \frac{A_{l1}}{(x-R_1)^{l1}}$
     n^{o} de parcelas $= l1$ (multiplicidade)
   - Determinar:
     $\frac{E_1+D_1x}{x^2+p_1x+q} + \frac{E_2+D_2x}{(x^2+p_1x+q_1)^2} + ... + \frac{E_{k1}+D_{k1}x}{(x^2+p_1x+q_1)^{k1}}$
     n^{o} de parcelas $= k1$ (multiplicidade)

   $\Longrightarrow \frac{P(x)}{Q(x)} =$ soma de todas as parcelas
3. Calcular valores $A_1, ..., A_{l1}$ e $E_1, D_1, ..., E_{k1}, D_{k1}$ através do método dos coeficientes indeterminados.

Primitivas de Funções Racionais

$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\arctan(\frac{x}{a}) + C,\ C \in \mathbb{R}$
$\int \frac{1}{(x\pm b)^2+a^2}\ dx = \frac{1}{a}\arctan(\frac{x\pm b}{a}) + C,\ C \in \mathbb{R}$
Se $\exists \int f(x)\ dx = F(x) + C$
$\Rightarrow \int f(ax+b)\ dx = \frac{1}{a}F(ax+b) + C,\ C \in \mathbb{R}$

$\int \frac{1}{[\varphi(x)]^2-a^2}\varphi'(x)\ dx = \frac{1}{2}\ln|\frac{\varphi(x)-a}{\varphi(x)+a}| + C,\ C \in \mathbb{R}$
$\int x(x+a)^n\ dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} + C,\ C \in \mathbb{R}$
$\int \frac{x}{a^2+x^2}\ dx = \frac{1}{2}\ln|a^2+x^2| + C,\ C \in \mathbb{R}$
$\int \frac{1}{ax^2+bx+c}\ dx = \frac{2}{\sqrt{4ac-b^2}}\arctan(\frac{2ax+b}{\sqrt{4ac-b^2}}) + C,\ C \in \mathbb{R}$
$\int \frac{1}{(x+a)(x+b)}\ dx = \frac{1}{b-a}\ln(\frac{a+x}{b+x}) + C,\ a \neq b$ e $C \in \mathbb{R}$

Primitivação por Mudança de Variável

Seja f uma função contínuma em $[a,b]$ e $x = \varphi(t)$ uma aplicação com derivada contínua e que não anula:
$P_x(f(x)) = P_t(f(\varphi(t))) \cdot \varphi'(t)|_{t=\varphi^{-1}(x)}$
$\rightarrow \int f(x)\ dx = \int f(\varphi(t))\ d\varphi(t) = \int f(\varphi(t)) \cdot \varphi'(t)\ dt|_{t=\varphi^{-1}(x)}$

Substituições

Primitivas	Substituição
$\int f(e^x)\ dx$	$t = e^x \Rightarrow x = \ln t$
$\int f(\ln x)\ dx$	$t = \ln x \Rightarrow x = e^t$
$\int f(x, x^{\frac{p}{q}}, x^{\frac{r}{s}}, ...)\ dx$	$t = x^{\frac{1}{m}} \Rightarrow x = t^m$, com $m = m.m.c.(q,s,...)$
$\int f(x, (ax+b)^{\frac{p}{q}}, (ax+b)^{\frac{r}{s}}, ...)\ dx$	$t = (ax+b)^{\frac{1}{m}} \Rightarrow ax+b = t^m$, com $m = m.m.c.(q,s,...)$
$\int f(x, \sqrt{ax^2+bx+c})$, $a > 0$	$\sqrt{ax^2+bx+c} = t + x\sqrt{a}$
$\int f(x, \sqrt{ax^2+bx+c})$, $c > 0$	$\sqrt{ax^2+bx+c} = tx + \sqrt{c}$
$\int f(x, \sqrt{ax^2+bx+c})$, $b^2-4ac > 0$	$\sqrt{ax^2+bx+c} = (x-\alpha)t$, α é raíz de ax^2+bx+c

Primitivação de Funções Trigonométricas

1. $\int f(\sin^k x, \cos^m x)\ dx$
   (a) $k-$par, $m-$ímpar
   substituição $t = \sin x$
   (b) $k-$ímpar, $m-$par
   substituição $t = \cos x$
   (c) $k,\ m-$ímpares
   substituição $t = \sin x$ ou $t = \cos x$
   + fórm.: $\sin x \cos x = \frac{1}{2}\sin(2x)$
   (d) $k,\ m-$pares
   "baixar" ordem de $\sin x$ e $\cos x$:
   $\sin^2 x = \frac{1}{2}(1-\cos(2x))$
   $\cos^2 x = \frac{1}{2}(1+\cos(2x))$
2. $\int f(\tan^k x)\ dx$ ou $\int f(\cot^k x)\ dx$
   Fórmulas:
   - $\tan^2 x = \frac{1}{\cos^2 x} - 1$
   - $\cot^2 x = \frac{1}{\sin^2 x} - 1$
3. $\int f(\sin x,\ \cos x)\ dx,\ \int f(\tan x)\ dx,\ \int f(\cot x)\ dx$
   Substituição "Universal":
   $t = \tan(\frac{x}{2}) \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2}{1+t^2}\ dt$

Primitivação de Funções Irracionais

$\rightarrow$ Substituir usando Fórmulas Trigonométricas

1. $\int f(\sqrt{a^2-b^2x^2})\ dx$
   $\sqrt{a^2-b^2x^2} = \sqrt{a^2(1-(\frac{b}{a}x)^2)} = a\sqrt{1-(\frac{b}{a}x)^2}$
   Subst.: $\frac{b}{a}x = \sin t \Rightarrow dx = \frac{a}{b}\cos t\ dt$
   $\int f(\sqrt{a^2-b^2x^2})\ dx = \int f(a\sqrt{1-\sin^2 t}) \cdot \frac{a}{b} \cdot \cos t\ dt$
   $\Longrightarrow \int f(a \cdot \cos t) \cdot \frac{a}{b}\cos t\ dt + C,\ C \in \mathbb{R}$
2. $\int f(\sqrt{a^2+b^2x^2})\ dx$
   $\sqrt{a^2+b^2x^2} = \sqrt{a^2(1+(\frac{b}{a}x)^2)} = a\sqrt{1+(\frac{b}{a}x)^2}$
   Subst.: $\frac{b}{a}x = \tan t \Rightarrow dx = \frac{a}{b}\frac{1}{\cos^2 t}\ dt$
   $\int f(\sqrt{a^2+b^2x^2})\ dx = \int f(a\sqrt{1+\tan^2 t}) \cdot \frac{a}{b} \cdot \frac{1}{\cos^2 t}\ dt$
   $\Longrightarrow \int f(a \cdot \frac{1}{\cos t}) \cdot \frac{a}{b} \cdot \frac{1}{\cos^2 t}\ dt + C,\ C \in \mathbb{R}$
3. $\int f(\sqrt{a^2x^2-b^2})\ dx$
   $\sqrt{a^2x^2-b^2} = \sqrt{b^2((\frac{a}{b}x)^2-1)} = b\sqrt{(\frac{a}{b}x)^2-1}$
   Subs.: $\frac{a}{b}x = \frac{1}{\cos t} \Rightarrow dx = \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t}\ dt$
   $\int f(\sqrt{a^2x^2-b^2})\ dx = \int f(b\sqrt{(\frac{1}{\cos t})^2-1}) \cdot \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t}\ dt$
   $\Longrightarrow \int f(b\tan t) \cdot \frac{b}{a} \cdot \frac{\sin t}{\cos^2 t}\ dt + C,\ C \in \mathbb{R}$

Integrais de Riemann

Integral de Riemann é o limite da soma de Riemman.
Soma de Riemann:
$\text{Sf}(P,C) = \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i,\ \Delta x_i = x_i - x_{i-1}$
$\Longrightarrow$ Integral de Riemann $= \lim_{x\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta x_i = I$
$I = \int_a^b f(x)\ dx = \lim_{\Delta P \to 0} \text{Sf}(P,C)$

$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

Geometria de integral de Riemann

$f : [a,b] \rightarrow \mathbb{R} -$ integrável, $a < b$

1. $f(x) > 0,\ \forall x \in [a,b]$
   $I = \int_a^b f(x)\ dx$
2. $c \in]a,b[$: $f(c) = 0$
   $I = \int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
3. $f, g : [a,b] - \mathbb{R} -$ integráveis, e $f > g,\ \forall x \in [a,b]$
   $I = \int_a^b |f(x) - g(x)|\ dx$

Propriedades

$f, g : [a,b] \rightarrow \mathbb{R} -$ integráveis e $\alpha, \beta \in \mathbb{R}$

- $\int_a^b[\alpha f(x) + \beta g(x)]\ dx = \alpha\int_a^b f(x)\ dx + \beta\int_a^b g(x)\ dx$
- Se $a < c < b \Rightarrow \int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$
- Se $f(x) \geq 0,\ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\ dx \geq 0$
- Se $f(x) \geq g(x),\ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx$
- Se $m \leq f(x) \leq M,\ \forall x \in [a,b]$
  $\Rightarrow m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$
- $|\int_a^b f(x)\ dx| \leq \int_a^b |f(x)|\ dx$

Critérios de Integrabilidade

Se $f : [a,b] \rightarrow \mathbb{R}$ é integrável (no sentido de Riemann)
$\Longrightarrow f$ é limitada em $[a,b]$
$*$*Importante*$*$ $\Rightarrow$ Se f não é limitada em $[a,b] \Rightarrow$
f não é integrável em $[a,b]$ (no sentido de Riemann)