Project 1: Algorithm Development Strategies

A Study on 'The Change-Making Problem'

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Abstract **– It is widely known that many computationally-demanding problems require more efficient solutions than to simply search exaustively for the answers. This is particularly true for tasks that execute repeated operations in considerable amounts of times. In this report, I focus on a famous challenge called 'The Change-Making Problem' and present a study on the computational performance of dynamic programming algorithms for solving such problem. The implemented algorithms were 3, and they all follow the same strategy to tackle the issue so that a high fidelity comparison could be analysed.**

Keywords **– Algorithm Complexity, 'The Change-Making Problem', Dynamic Programming, Memoization, Scalable Algorithms.**

1. Problem Contextualization

This report was written for the course of 'Advanced Algorithms', taught by professor Joaquim Madeira for the master's in Informatics Engineering at DETI UA. It describes the work done for the first assignment of the course [*1*], aimed to elaborate a study on the complexity and performance of algorithms that solve one of the proposed challenges. The chosen hypothesis for the study was 'B - Dynamic Programming', and the chosen problem was 'The Change-Making Problem' (hereafter referred to by tCMP).

The basic idea behind tCMP is to find the minimum number of coins, from a given set of coins of different values, that add up to a given amount of money. To simplify the problem, it is usually assumed that for each possible coin value there is an infinite amount of coins available to return the change. This challenge quickly becomes computationally-demanding when the amount to return as change increases, considering a limited number of available coin alternatives.

TCMP is considered a special case of 'The Knapsack Problem' [*2*], if the change amount is mapped to the knapsack's maximum weight and the chosen coins mapped to the weights of the items to be inserted into the knapsack. In this specialization, the chosen coins must add up to exactly the knapsack's maximum weight.

The problem's applications are wider than just currency; another famous example is 'The 9 Dard Finish Problem', where the aim is to find out how can one dart player get a perfect leg using the fewest amount of darts possible (which is 9) to sum a total of 501 points [*3*]. However,to keep my study focused, I do not consider any other application.

TCMP is also the most common variation of 'The Coin Change Problem' [*4*], where the aim is to find out the number of possible ways of making a change for a given amount, without considering the coins order and considering the infinite set of coin alternatives previously mentioned.

2. Implementations

In this chapter I present the mathematical definition of the problem and a description of the three chosen algorithms for solving it.

To simplify the mathematical operations, I only consider positive integer values for the coins and change amounts. With this in mind, the coin values can be modeled by a set of n distinct values, arranged in increasing order as a_1 through a_n, and the problem by: given an amount A, how to find a set of integers $\{x_1, x_2, ..., x_n\}$, with each x_j representing how often the coin with value a_j is used, which minimize the total number of coins $f(A)$, given by equation 1, subject to equation 2.

$$f(A) = \sum_{j=1}^{n} x_j \qquad (1) \qquad\qquad A = \sum_{j=1}^{n} a_j x_j \qquad (2)$$

When planning the implementation of the algorithms to solve tCMP, I looked for returning not only the minimum number of coins needed to add up to a given amount, but also the list of those same coins. This would force me to develop a more complete solution and would help me manually check for the validity of the results during the development phase and better compare results between algorithms.

The first implemented algorithm resorted to a recursive function. This function, *tCMP_recursive()*, accepts as arguments the amount to be given as change and a list containing the currency available, i.e. the coin alternatives to use when giving the change. The way it works is by iterating over the available coins (from the end to the beginning, i.e. from the highest coin to the lowest), comparing the results of using the current coin or using the next one and returning the best alternative. The function keeps track of the currently used coin with the help of an auxiliar function that receives the current index of the currency list as an argument; it is this function *tCMP_recursive_getMinCurrency()* that actually does the recursion process.

The second algorithm is an adaptation of the recursive, with the help of an optimization technique called memoization (or memoisation). This technique was ment to speed up the function by storing the results of expensive function calls and returning the cached result when the same inputs occur again. What this means exactly is that every time a computation receives input values that may repeat themselves, *tCMP_memoization()* checks if the result of such computation is stored in a global matrix (in memory) and, if not, it proceeds with the computation (usually envolving a recursive call), but if so, it saves time by retrieving the result from memory and avoiding expensive function calls.

The last algorithm implemented can be considered the classic dynamic programming solution, where results frequently calculated are also stored, but this time the function is iterative, unlike the previous solutions. Although *tCMP_dynamic()* works a little bit differently from *tCMP_memoization()* (the recursive algorithm with the use of memoization), this is not usually the case. Unfortunately, due to the structure of the developed implementations, it became tricky to translate the recursive solution with memoization to an iterative one. For this reason, some adaptations to the memory structure had to be done resulting in a slightly different implementation strategy for the third algorithm. As we will see further ahead, the characteristics of the solution remained the same, and the only noticeable divergence lies in the number of basic operations of the last two algorithms. The way *tCMP_dynamic()* works is, instead of keeping one possible solution per coin of the currency array per value between 0 and the desired amount (like in the recursive function with memoization) and only comparing the possibilities per amount when needed, it frequently makes this comparison needing only to store the possible solution per value between 0 and the desired amount.

All functions are written in Python version 3 and all of them return a list in the same format, to make post-processing easier. This list contains 4 elements, being the first the minimum number of coins required to give the amount as change, the second the list of the actual coins that added up are equal to the amount, the third the number of basic operations executed (which is given by the number of results comparisons), and the fourth and final the execution time of the function, calculated with the help of the Python library Time [*5*].

```
Amount: 187
Currency: [1, 2, 5, 10, 20, 50, 100]
Recursive Solution:
[6, [100, 50, 20, 10, 5, 2], 2046405, 2.46101]
Recursive Solution w/ Memoization:
[6, [100, 50, 20, 10, 5, 2], 445, 0.00083]
Dynamic Programming Solution:
[6, [100, 50, 20, 10, 5, 2], 328, 0.00036]
```

Console Output 1: algorithms execution example.

3. Algorithm Complexities

Let us now take a closer look at the theoretical behaviour of these functions. The complexity of the algorithms' execution depends on 3 major factors, with the following characteristics, derived from experimental observations during development:

1. The amount to be returned as change - the larger it is, the more coins will have to be selected and the more combinations of coins can be chosen.
2. The length of the currency array - the larger it is, the more coins available to be used as change have to be tested.
3. The content of the currency array - depending on the available coins, the algorithms might complete the task slower for some amounts.

Taking this in consideration, it is easy to infer that finding an exact formula that determines the algorithms' complexities is far from trivial. Nevertheless, with a formal analysis of the implementations, it seamed reasonable that all solutions would have something similar to exponential growths in the number of basic operations as well as in execution times, although varying in the "speed" of growth. But these were mere estimations and needed to be considered in greater depth.

For this reason, I resorted to an online tool that processes a set of values given as input and returns a list of function approximations (through regression analysis) that attempt to represent the behaviour of the input values. Planetcalc [*6*] received some coin amounts as X values and the execution times and number of basic operations of each algorithm as Y values (once at a time) and estimated the regression functions that best fitted them. The tested functions ranged from linear and quadratic regressions to hyperbolic, logarithmic and exponential regressions.

The first conclusions drawned were that, for single-choice currencies (i.e. a currency array containing only the atomic unit), all algorithms behaved exactly the same, with the number of basic occurrences given by $y = x$ and the execution times approximately given by $y = ax$ for all algorithms, varying only the value of 'a' in very small portions (in the scale of 10^{-6}). This means that, for such unique type of currency array, all algorithms have a complexity of $O(n)$.

Now, for all other currencies with more than one coin alternative, the behaviours diverged significantly. While the algorithms with the use of stored values remained with a steady growth, the recursive solution could clearly no longer be approximated with a linear regression and fitted best with an exponential one given by Planetcalc. By passing to the online tool the amounts from 1 to 150 and the respective results relative to *tCMP_recursive()*'s number of basic operations, and after tweaking with the exponential regression's function approximation with the help of Desmos [7] (an online graphical calculator) to fit it exactly with the practical results, I was able to reach the refined equation 3:

$$y = e^{7.00+0.0429x} \tag{3}$$

Note that y corresponds to the number of basic operations of the recursive algorithm, with x as the change amount and a fixed currency of [1, 2, 5, 10, 20, 50, 100] (simulating the european currency, with the cents and the large bills ignored). The currency was fixed in order to simplify the calculations; this currency was used for the remaining calculations as well. For different currency array sizes, the exponential approximation remained the best alternative, with larger arrays having a sharper exponential curve. The same was true for same size arrays with different coin alternatives.

I then managed to estimate the complexity in Big-Oh notation by removing constants from the formula and taking advantage of some mathematical properties of powers, as seen in equation 4. The result was a complexity of $O(1.04^n)$.

$$e^{7.00+0.0429x} \approx e^{0.0429x} = e^{0.0429^x} \approx 1.0438^x \tag{4}$$

Regarding the recursive with memoization and the dynamic programming algorithms, the approach had to be different as, for the used amounts, the linear regression remained the best approximation for both.

Having an auxiliar matrix with the results of computations already calculated slows down the growth in execution time and total number of basic operations and helps these algorithms be more efficient, but their internal functioning remains the same, so I knew that their evolution was not linear. So I ran the algorithms for larger amounts, up to 800 (since more than that started reaching the maximum recursion depth of the Python language); then, I used only a significant portion of the larger results on Planetcalc and analysed the returned regressions.

After comparing individual representative values with the function approximations and removing negative constants (that would render the functions' y values lower than those constants meaningless), I reached the formulas that best fit *tCMP_memoization()* and *tCMP_dynamic()* respectively:

$$y = 0.00012x^2 + 2.315x \tag{5}$$

$$y = 0.00014x^2 + 1.655x \tag{6}$$

I understood that the algorithms' complexities grew according to the amount (lets say 'n') and the currency array size ('m') (also according to the array's content, but its influence was small, so I ignored it for simplicity purposes) - this would result in a complexity of $O(nm)$. I also had in mind that any value for these two dimensions should be considered possible, so it was safe to consider 'n' as the largest of the two and derive from it a complexity of $O(n^2)$. Such complexity is typical of a quadratic function, so when choosing the best function approximations for the algorithms, I looked for a regression of that nature. The outcome was equations 5 and 6.

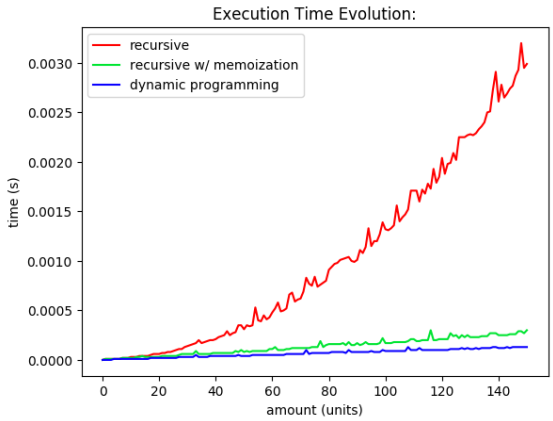
4. Performance Analysis

In this chapter I explain the practical studies applied to the previously described algorithms. Here, I execute each algorithm for different amounts and currencies. I also describe the post-processing operations applied to the results to achieve more valuable conclusions.

The first thing done was presenting the results of running the algorithms for different amounts. So, for each algorithm, several executions were done with amounts ranging from 0 to 150 (the later chosen for testing purposes, as larger numbers began to take too long for the recursive solution). This process was repeated for the following currencies: [[1], [1,2], [1,5], [1,2,5], [1,3,8,15,74,129], [1,2,5,10,20,50,100]]. These were chosen to evaluate the consequences of: increasing the size of the currency array, varying the coin alternatives for same size arrays, and choosing coin alternatives with no direct relations. The results of each execution were stored in a CSV file and printed in a formatted form similar to the one presented in Table 2 (see last page).

The next step was to visualize the results in order to better understand the performance for different inputs. To do so, I used the library Matplotlib [8], a Python 2D library that produces figures in a variety of formats, and focused on the line plot. The function responsible for generating the plots and storing them was *tCMP_generatePlot()*; it receives as arguments two lists, one containing the amounts tested, other containing the execution times of each algorithm for each amount, it also receives the title of the figure, a smoothing factor and a logarithmic base.

The logarithmic base passed to *tCMP_generatePlot()*, if greater or equal to 2, is used to convert the time values into their logarithmic form. The idea behind this was to allow the representation of all algorithms in the same plot, since the recursive grows a lot faster than the remainder and makes their visual analysis difficult, as seen in Figure 3. However, applying this transformation did not influence much so it was not used in the studies considered in the remainder of this document.



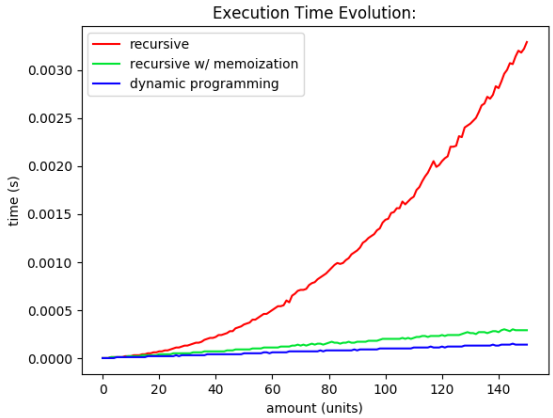


Fig. 1, 2: Execution times of the 3 algorithms (single sequence and time averages of several sequences).

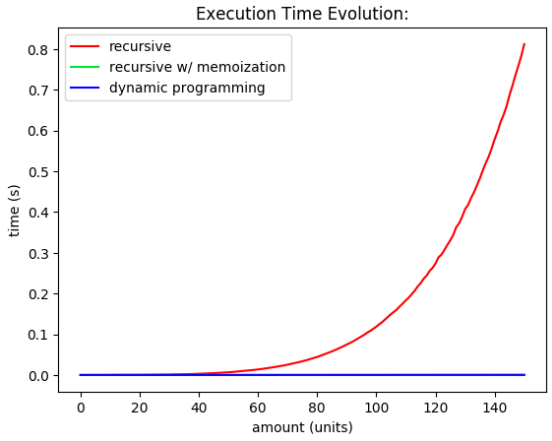
The smoothing factor, if larger than 1, is passed as argument to an auxiliar function *smooth_results()* responsible for, as the name states, smoothing the results. The way this works is by collecting groups of 'x' values from the execution times list and calculating the average value between them. Here, 'x' is given by the smoothing factor passed as argument. The sampling set is then reduced 'x' times, making the plot quicker, with less irregularities but with less detail as well. This smoothing process was also not used, as a better alternative was found.

These irregularities, or noise, correspond to small time rises during execution that negatively influence times spent in calculations in a way that does not represent the reality of the algorithms. What I mean is that, for external reasons other than the actual functioning of the developed functions, occasional time peaks occur for different executions. This was detected as these peaks occur in different places when the functions are executed several times with the same input. This noise is visible in Figure 1.

The solution developed that tackles this issue better than the smoothing process was to run the set of input combinations several times and calculate the average execution time of each combination. Figure 2 shows the plot of the execution of the algorithms for the same input values as of Figure 1, but doing so 50 times and calculating the averages, hence removing most of the noise. This value was chosen by trial and error, as beyond 50 the amount of noise removed did not compensate the additional amount of time required to execute the study.

The two plots correspond to executing the algorithms for values from 0 to 150, with a currency array of only two coins [1,5]. But what if this array increases? Figures 3 and 4 show what happens when we execute the algorithms for a currency of [1,2,5,10,20,50,100].

Both figures present the average times of running the procedure 50 times like we saw in Figure 2, and correspond to the exact same run, differing only in the fact that in the second plot the recursive values are not presented so that we can better analyse the others.



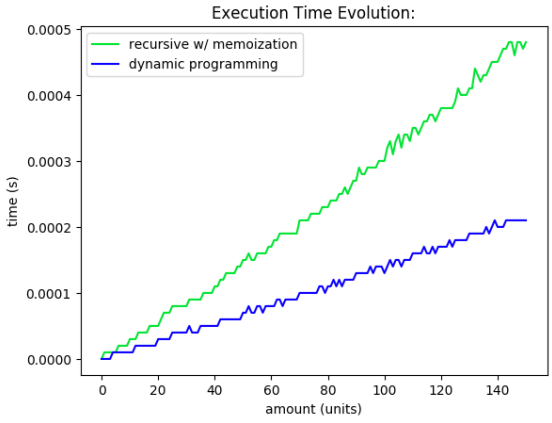


Fig. 3, 4: Execution times averages of the 3 algorithms (for a larger currency array).

Through the plots, one can conclude that the function approximations seem adequate to the behaviours of all three algorithms. It is also visible the fact that the larger the number of available coins, the sharper is the growth curve of the recursive algorithm, while the others remain very much stable.

Testing the algorithms for currencies with coins without clear patterns (e.g. having coins that are multiples of each other) proved to have no noticeable effect on the results.

5. Algorithm Comparisons & Additional Considerations

In this final chapter, I compare the three developed algorithms in terms of complexity, execution time, memory usage and readability and present the gathered conclusions. Finally I present a reasoned prediction of the algorithms behaviour for significantly larger amounts, with the help of regressions, and mention a few additional considerations worth stating.

It is needless to say that, of the three algorithms, the slowest and less adviseable is the recursive without memoization. This algorithm, although elegant as most recursive functions are, is not scaleable at all as it is constantly repeating computations it has already made before. As for the other two, from the analysis previously made, it is concluded that the recursive with memoization is usually twice as slow as the dynamic. This is due to the fact that recursive calls are inherently slower than an iterative loop. But the small implementation differences might also have an impact that benefits the dynamic programming solution.
In terms of readability, *tCMP_dynamic()* has the advantage as well, since it contains less verifications. The only aspect in which *tCMP_recursive()* is better than the others is in terms of reduced memory usage, obviously, as it does not keep track of any variable other than the ones strictly necessary.

For a final and hypothetical exercise, I present an estimation of the execution times of the three algorithms for amounts equal to 10^3, 10^6 and 10^9 . This is ment to illustrate how big can be the consequences of not searching for efficient and scalable solutions, and resorts to the regressions presented in chapter 3 (see equations 3, 5 and 6) to provide with the estimations. Note that these equations return an estimation of the total number of basic operations of each algorithm given an amount; in order to use them, we need to convert the results into execution times; this is easily done by finding the relation between basic operations and execution times. Once again, Planetcalc helps in finding these relations. For the recursive solution, the execution time is 1.1767 x 10^{-6}th of the total number of basic operations; for the recursive with memoization, it is 1.3546 x 10^{-6}th; and for the dynamic programming it's 0.8164 x 10^{-6}th. The results are visible in Table 1.

	Execution Time Estimations		
Amount	Recursive	Recursive w/ Memo.	Dynamic Progr.
10^3	175046645 years	0.0033 sec.	0.0015 sec.
10^6	-	2 min. and 46 sec.	1 min. and 56 sec.
10^9	-	5 years and 2 months	3 years and 7 months

Table 1: Execution time estimations derived from the function approximations.

The estimations greater than 10^3 are not presented for the recursive algorithm as their values were ridiculously big, tending to infinity. The other two proved to be very robust algorithms, given that, even for an amount of a billion to return as change, they could complete the task in less than 10 minutes.

I end this report with a few final considerations. Regarding shortcomings of my work, I have two aspects to mention. The fact that the recursive algorithms quickly reached Python's maximum recursion depth was unfortunate, but my efforts of avoiding such limitation were not as effective as would be desired. Also, the differences between the matrices of *tCMP_memoization()* and *tCMP_dynamic()* and the way they are manipulated was an aspect to overcome in the event of continuing the work, as it would probably demand for changes in all three algorithms. I would like to mention that, although the algorithm implementations were developed by me, most of the reasoning behind was taken from several sources combined, in order to attempt to deliver the code as best optimized as possible.
The entire code is present in the file *TheChangeMakingProblem.py*, with a few execution examples in the end. I make available a *requirements.txt* file containing the dependencies for the execution of the code. The functions and some important code lines have comments to help guide myself during development and a potential reader henceforward. Along with this document and the source code, I also deliver a results folder containing all of the plots presented here and a few more with different variable combinations.

References

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		Recursive		Recursive w/ Memoization		Dynamic Programming	
Currency	Amount	# of Basic Op	Exec Time	# of Basic Op	Exec Time	# of Basic Op	Exec Time
[1]	0	0	0.0	0	0.0	0	0.0
[1]	1	1	1e-05	1	1e-05	1	0.0
[...]							
[1]	22	22	2e-05	22	3e-05	22	2e-05
[1]	23	23	3e-05	23	3e-05	23	2e-05
[1]	24	24	4e-05	24	6e-05	24	3e-05
[1]	25	25	6e-05	25	7e-05	25	5e-05
[1]	26	26	7e-05	26	8e-05	26	2e-05
[...]							
[1]	40	40	9e-05	40	0.00011	40	6e-05
[1]	41	41	0.0001	41	0.00011	41	6e-05
[1]	42	42	0.0001	42	0.00013	42	7e-05
[...]							
[1]	145	145	0.00021	145	0.00023	145	0.00014
[1]	146	146	0.00022	146	0.00023	146	0.00014
[1]	147	147	0.00022	147	0.00025	147	0.00015
[1]	148	148	0.00023	148	0.00023	148	0.00014
[1]	149	149	0.00023	149	0.00023	149	0.00015
[1]	150	150	0.00024	150	0.00023	150	0.00014
[1, 2]	0	0	0.0	0	0.0	0	0.0
[1, 2]	1	1	1e-05	1	0.0	1	0.0
[...]							
[1, 2]	150	5775	0.00846	225	0.00038	225	0.00018
[1, 5]	0	0	0.0	0	0.0	0	0.0
[1, 5]	1	1	1e-05	1	0.0	1	0.0
[...]							
[1, 5]	150	2355	0.00306	180	0.00027	180	0.00013
[1, 2, 5]	0	0	0.0	0	0.0	0	0.0
[1, 2, 5]	1	1	1e-05	1	1e-05	1	0.0
[...]							
[1, 2, 5]	150	61445	0.07672	327	0.00046	240	0.00016
[1, 3, 8, 15, 74, 129]	0	0	0.0	0	0.0	0	0.0
[1, 3, 8, 15, 74, 129]	1	1	1e-05	1	1e-05	1	0.0
[...]							
[1, 3, 8, 15, 74, 129]	150	90868	0.11205	408	0.00055	210	0.00019
[1, 2, 5, 10, 20, 50, 100]	0	0	0.0	0	0.0	0	0.0
[1, 2, 5, 10, 20, 50, 100]	1	1	1e-05	1	1e-05	1	0.0
[...]							
[1, 2, 5, 10, 20, 50, 100]	150	684716	0.81894	358	0.0005	265	0.00021

Table 2: Portion of the output of a simple study of the algorithms.