Computer Vision Exercise 6

Structure from Motion & Model Fitting

Epipolar Geometry

You have two images I_1, I_2 of the same scene with camera poses $\mathbf{C}_1$, $\mathbf{C}_2$, respectively. The images contain two corresponding keypoints $\mathbf{x}_1 = (400, 200)^\intercal$ (in I_1) and $\mathbf{x}_2 = (480, 260)^\intercal$ (in I_2).

$$\mathbf{C}_1 = [I|0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_2 = [R|t] = \begin{bmatrix} 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

(given as $\boxed{\text{transformations from world space to camera space}}$)

The images are of size $(w, h) = (800, 600)$. For both images the focal length is 200 pixels and the principal point is at $(400, 300)$.

Epipolar Geometry

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question Compute the Essential Matrix E so that, for a perfect correspondence $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2)$, $\hat{\mathbf{x}}_2^\top E \hat{\mathbf{x}}_1 = 0$. Scale your solution so that the value in the bottom right corner is 1.

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answer

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$\begin{aligned} E &= \boxed{[t]_\times R} \\ &= \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & -1 \end{bmatrix} \\ &\equiv \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 1 \end{bmatrix} \end{aligned}$$

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$$K = \begin{pmatrix} f_x & s & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow K = \begin{bmatrix} 200 & 0 & 400 \\ 0 & 200 & 300 \\ 0 & 0 & 1 \end{bmatrix}$$

Epipolar Geometry

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$$\hat{\mathbf{x}}_2^\top E \hat{\mathbf{x}}_1 = 0 \quad \Longrightarrow \quad \mathbf{x}'^\top \underbrace{\boxed{\mathrm{K}'^{-\top} \mathrm{E} \mathrm{K}^{-1}}}_{F} \mathbf{x} = 0 \quad \Longrightarrow \quad F = K^{-\top} E K^{-1}$$

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$$K = \begin{bmatrix} 200 & 0 & 400 \\ 0 & 200 & 300 \\ 0 & 0 & 1 \end{bmatrix}$$

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answer

$$P_1 = K\mathbf{C}_1 = \begin{bmatrix} 200 & 0 & 400 & 0 \\ 0 & 200 & 300 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_2 = K\mathbf{C}_2 = \begin{bmatrix} -400 & 0 & 200 & 400 \\ -300 & 200 & 0 & 800 \\ -1 & 0 & 0 & 2 \end{bmatrix}$$

Epipolar Geometry

$$\mathbf{x}_1 = (400, 200)^\mathsf{T}$$
$$\mathbf{x}_2 = (480, 260)^\mathsf{T}$$

question Compute the residual of the Fundamental Matrix constraint for the correspondence $(\mathbf{x}_1, \mathbf{x}_2)$.

Hint: You don't need to explicitly assemble the Fundamental Matrix.

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1. The value is the same as for the essential matrix constraint with normalized keypoint coordinates

$$r = \mathbf{x}_2^\mathsf{T} F \mathbf{x}_1 = \mathbf{x}_2^\mathsf{T} K^{-\mathsf{T}} E K^{-1} \mathbf{x}_1 = \left(K^{-1}\mathbf{x}_2\right)^\mathsf{T} E \left(K^{-1}\mathbf{x}_1\right) = \bar{\mathbf{x}}_2^\mathsf{T} E \bar{\mathbf{x}}_1$$

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2. Compute the keypoint positions in the normalized image plane.

$$K^{-1} = \begin{bmatrix} \frac{1}{f_x} & 0 & -\frac{c_x}{f_x} \\ 0 & \frac{1}{f_y} & -\frac{c_y}{f_y} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.005 & 0 & -2 \\ 0 & 0.005 & -1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\mathbf{x}}_1 = K^{-1}\mathbf{x}_1 = (0, -0.5)$$
$$\bar{\mathbf{x}}_2 = K^{-1}\mathbf{x}_2 = (0.4, -0.2)$$

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3. Compute the residual

$$r = 0$$

Epipolar Geometry

question Assume you know that the 3D point $\mathbf{X}$ lies in the 3D plane $z = 3$ (i.e., $\mathbf{X}_z = 3$) and its observation $\mathbf{x}_1$ in image I_1 is noise-free. Compute the 3D point position and its 2D reprojection residual in pixels compared to $\mathbf{x}_2$ in image I_2. (You don't need to compute the norm)

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answer

$$\mathbf{X} = \boxed{homogeneous(\bar{\mathbf{x}}_1)} * 3 = (0, -1.5, 3)^\intercal$$

$$\bar{\mathbf{x}}_1 = K^{-1}\mathbf{x}_1 = (0, -0.5)$$

homogeneous

$$(0, -0.5, 1)$$

$$\mathbf{x}' = P_2\mathbf{X} = (500, 250)$$

$$res = \mathbf{x}_2' - \mathbf{x}_2 = (500, 250) - (480, 260) = (20, -10)$$

Model Fitting

question We use RANSAC to robustly fit a model to a set of observations. The model fitting requires a subset of size N. Given the inlier ratio e, write down the formula to compute the maximum number of iterations K such that with probability at least p (e.g., $p = 0.001$), there exist outlier(s) in all the random subsets sampled so far. You can suppose that the sampling is done with replacement. Explain all intermediate steps.

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answer The probability of sampling only inliers / outlier-free in a subset is given by $p_s = e^N$.

The probability of sampling at least one outlier in a subset is given by $(1 - p_s)$.

Thus, the probability of sampling only subsets with at least one outlier for K iterations is $p_K = (1 - p_s)^K$.

Next, since with probability at least p there exist outlier(s) in each random subset, we know $p_K \geq p$.

Finally, we get $(1 - e^N)^K \geq p$, $K \leq \frac{\log p}{\log(1-e^N)}$.

Model Fitting

question Based on the previous question, write down the main steps of the adaptive RANSAC algorithm that determines online the required number of iterations based on the best inlier ratio e found so far. **Hint**: The inlier ratio e is no longer known, to start with, we can simply initialize it as 0. The probability p and size of random subset N are the same as previous question. k is the number of the current iteration. Based on the previous question we compute online K to represent the maximum number of iterations such that with probability at least p, there exist outlier(s) in each random subset with size N sampled so far.

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How many samples?

Choose N so that, with probability p, at least one random sample is free from outliers. e.g. p=0.99

Adaptively determining the number of samples

e is often unknown a priori, so pick worst case, e.g. 50%, and adapt if more inliers are found, e.g. 80% would yield e=0.2

- $N=\infty$, *sample_count* =0
- While N >*sample_count* repeat
  - Choose a sample and count the number of inliers
  - Set e=1-(number of inliers)/(total number of points)
  - Recompute N from e
  - Increment the *sample_count* by 1
- Terminate

$$\left(N = \log(1-p)/\log\left(1-(1-e)^s\right)\right)$$

answer 1. Let $K = \infty$, $e = 0$, and $k = 0$.

2. While $K > k$ repeat 3-6.

3. Choose a random subset, and estimate parameters.

4. Determine the inlier ratio of this subset, e_{cur} = number of inliers/total number of points. Update e if $e_{\text{cur}} > e$.

5. Recompute $K = \frac{\log p}{\log(1-e^N)}$.

6. Increment k by 1.