TIME-DEPENDENT N-S

For $I = (0,T)$

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } \Omega \times I \\ \nabla \cdot u = 0 & \text{in } \Omega \times I \end{cases}$$

+ i.c. and b.c.

$$u(x,0) = u_0(x), \quad \text{with } \nabla \cdot u_0 = 0$$

$$u(x,t) = g(x,t) \quad \text{in } \underbrace{\partial\Omega \times I}_{\text{parabolic boundary}}$$

WEAK FORM

N-S are non-linear parabolic eqs.

$$V = [H_0^1(\Omega)]^d, \quad Q = L_0^2(\Omega)$$

Find $u \in L^2(I; [H^1(\Omega)]^d) \cap C^0(\bar{I}; [L^2(\Omega)]^d)$, $p \in L^2(I; Q)$

1) $$\underbrace{\int_\Omega u_t v}_{m(u_t, v)} + \underbrace{\int_\Omega \nu \nabla u : \nabla v}_{a(u,v)} + \underbrace{\int_\Omega (u \cdot \nabla) u v}_{c(u,u,v)} - \underbrace{\int_\Omega p \operatorname{div} v}_{b(v,p)} = \underbrace{\int_\Omega f v}_{f(v)} \qquad \forall v \in V$$

2) $$-\int_\Omega q \operatorname{div} u = 0 \qquad \forall q \in Q$$

with $u(x,0) = u_0$

(Quarteroni, page 463)

For u sol. of 1)-2) in the div-free space, if Ω is Lipschitz

$\Rightarrow \exists!\, p$ sol. of the above problem

SEMI-DISCRETE (IN SPACE) DISCRETIZATION

$\boxed{g=0}$ FOR SIMPLICITY

V_h, Q_h FEM SPACES

Find $(u_h, p_h) \in V_h \times Q_h$:

$$\begin{cases} m(\partial_t u_h, v_h) + a(u_h, v_h) + c(u_h, u_h, v_h) + b(p_h, v_h) = (f_h, v_h) & \forall v_h \in V_h \\ b(u_h, q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

YIELDING TO A NON-LINEAR SYSTEM OF ODES:

$$\begin{cases} M \frac{du(t)}{dt} + A u(t) + \underbrace{C(u(t))\, u(t)} + B^T p(t) = F(t) \\ B u(t) = 0 \end{cases}$$

TIME DISCRETIZATION

1) FINITE DIFFERENCES

FOR INSTANCE θ-METHOD, BDF SCHEMES

$$\theta C(u^{k+1}) u^{k+1} + (1-\theta) C(u^k) u^k$$

- $\theta = 0 \Rightarrow$ EXPLICIT EULER : $\begin{cases} M u^{k+1} = H(u^k, p^k, f^k) \\ B u^{k+1} = 0 \end{cases}$

$\rightarrow$ MODIFY TO USE p^{k+1} and if $\operatorname{Ker}(B^T) = 0$:

(semi-implicit)

$$\begin{cases} \frac{1}{\Delta t} M U^{k+1} + B^T P^{k+1} = G \\ B U^{k+1} = 0 \end{cases} \Longrightarrow \begin{cases} U^{k+1} = \Delta t\, M^{-1} (G - B^T p^{k+1}) \\ \underbrace{B M^{-1} B^T}_{\text{NON-SINGULAR}} p^{k+1} = B M^{-1} G \end{cases}$$

Stable if $\Delta t \leq C \min \left(\frac{h^2}{D}, \frac{h}{\max_x |u^k(x)|} \right)$

- $\theta > 0 \rightarrow$ OK (IMPLICIT), Newton-Krylov (GMRES, BiCGStab)

- $\vartheta = 1 \rightsquigarrow$ FULLY IMPLICIT, stable if $\Delta t \le C \dfrac{h}{\max_x |u^k(x)|}$

2) FRACTIONAL STEPS OR PROJECTION METHODS (Chorin – Temam)

$$\begin{cases} u' + Lu = 0 & \Omega \times I \\ u(x,0) = u_0 \end{cases}$$

EXPL. EULER

$$u^0 = u_0$$

$$\frac{u^{k+1} - u^k}{\Delta t} + Lu^k = 0$$

IDEA: $L \simeq \sum_{i=1}^{q} L_i$ (SPLITTING)

$$\frac{u^{k+\frac{i}{q}} - u^{k+\frac{(i-1)}{q}}}{\Delta t} + L_i u^{k+\frac{i-1}{q}} = 0 \qquad \forall i = 1, \dots, q$$

OR $\frac{i}{q}$ (IMPLICIT)

(FRACTIONAL STEPPING)

$\longrightarrow$ IF IMPLICIT INVERT $I + \Delta t L_i$, HOPEFULLY EASIER THAN $I + \Delta t L$ (IMPL. EULER)

Ex. $q = 2$, $L_1 = -\nu \Delta u + (u \cdot \nabla) u - f$, $L_2 = \nabla p$.

$$\underbrace{\frac{u^{k+1} - u^k}{\Delta t}}_{\displaystyle \frac{u^{k+1} - u^*}{\Delta t} + \frac{u^* - u^k}{\Delta t}} - \nu \Delta u^k + (u^k \cdot \nabla) u^k + \nabla p^{k+1} = f^k$$

EQUIVALENT TO SPLITTING:

1) $$\frac{u^* - u^k}{\Delta t} = \nu \Delta u^k + (u^k \cdot \nabla) u^k + f^k$$

2) $$\frac{u^{k+1} - u^*}{\Delta t} = \nabla p^{k+1}$$

WE CAN MODIFY 2) BY "PROJECTION"

$$\nabla \cdot \left(\frac{u^{k+1} - u^*}{\Delta t} \right) \simeq \nabla \cdot \nabla p^{k+1}$$

$$\xrightarrow{\nabla \cdot u^{k+1} = 0} \quad -\Delta p^{k+1} = \frac{1}{\Delta t} \nabla \cdot u^*$$ POISSON PROBLEM FOR THE PRESSURE

Step-35 in deal. II :

- BDF2 (based on extrapolation)
- splitting $(u \cdot \nabla) u$ in its skew-symmetric form

$$(u \cdot \nabla) u \longrightarrow (u \cdot \nabla) u + \frac{1}{2} (\nabla \cdot u) u$$

$$\sim (u^* \cdot \nabla) u^{k+1} + \frac{1}{2} (\nabla \cdot u^*) u^{k+1}$$ |semi-implicit|

$\Longrightarrow$ STABLE