$$\begin{cases} K^{-1}\underline{u} + \nabla p = 0 & \text{in } \Omega \\ -\operatorname{div} \underline{u} = -f & \text{in } \Omega \\ p = g & \text{on } \partial\Omega \end{cases}$$

$$\underset{\sim}{u} \in RT_k \,, \quad p \in DG_k$$

$$\begin{array}{c c} \underline{U}\,\{ & \\ \underline{P}\,\{ & \end{array} \underbrace{\begin{bmatrix} M & B \\ B^T & 0 \end{bmatrix}}_{A} \begin{bmatrix} \underset{\sim}{U} \\ \underline{P} \end{bmatrix} = \begin{bmatrix} \underline{F} \\ \underline{G} \end{bmatrix}$$

- A is symmetric
- A is indefinite (i.e. it has eigenvalues >0, <0)
  - Benzi, Simoncini 2006
- There is a 0 diagonal block $\rightarrow$ hard to precondition

Solutions:

1) Use a direct solver $\rightarrow$ inefficient for large problems
2) Use the Schur complement

SCHUR COMPLEMENT

$$\begin{cases} M\underline{U} + B\underline{P} = \underline{F} \longrightarrow \underline{U} = M^{-1}(\underline{F} - B\underline{P}) \\ B^T\underline{U} = \underline{G} \end{cases}$$

$$B^T M^{-1}\underline{F} - B^T M^{-1} B \underline{P} = \underline{G}$$

1) $\underbrace{B^T M^{-1} B}_{S} \underline{P} = -\underline{G} + B^T M^{-1} \underline{F}$

S: sym. and pos. def.

2) $\underline{U} = M^{-1}(\underline{F} - B\underline{P})$

P1) How to precondition $\underline{S}$?

$$\tilde{S}^{-1} = \left[B^T \left(\underline{\mathrm{diag}(M)^{-1}}\right) B\right]^{-1}$$

P2) In S, how to compute M^{-1}? → You don't!

Iterative solvers don't require S but only the "action of S" on a vector

$$S\underline{v} = B^T M^{-1} B v$$

1) $\underline{w} = B\underline{v}$

2) $\underline{y} = M^{-1}\underline{w} \longrightarrow \overset{\text{s.p.d.}}{M}\,\underline{y} = \underline{w}$ (solve with CG)

3) $B^T \underline{y}$

In deal.II, M, B, B^T, S, M^{-1}, S^{-1} are represented as "linear operators", i.e. we only represent their action on a vector.