Time dependent NS

Recall incompressible NS: for $I = (0,T)$

$$\begin{cases} u_t - \nu \Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega \times I \\ \nabla \cdot u = 0 & \text{in } \Omega \times I \end{cases}$$

+ i.c. and b.c. :

$u(x,0) = u_0(x)$ with $\nabla\cdot u_0 = 0$

$u(x,t) = g(x,t)$ in $\partial\Omega \times I$
(parabolic boundary)

Weak form

It is a nonlinear parabolic eq.

By usual procedure ...: let $V = [H_0^1(\Omega)]^d$; $Q = L_0^2(\Omega)$.

Find $u \in L^2(I; [H^1(\Omega)]^d) \cap C^0(\bar{I}; [L^2(\Omega)]^d)$, $p \in L^2(I; Q)$

$$\begin{cases} \underbrace{\int_\Omega u_t \cdot v \, dx}_{m(u_t, v)} - \underbrace{\int_\Omega \nu \nabla u \cdot \nabla v}_{a(u,v)} + \underbrace{\int_\Omega (u\cdot\nabla)u\cdot v}_{c(u;u,v)} - \underbrace{\int_\Omega \nabla p \cdot v}_{b(v,p)} = \int_\Omega f\cdot v & \forall v \in V \\ -\int_\Omega q \, \nabla\cdot u = 0 & \forall q \in Q \\ \text{a.e. in } t \text{ with } u(x,0) = u_0. \end{cases}$$

(Quarteroni, p. 463)

For u sol. of problem in div-free space, if Ω is Lipschitz, $\exists!$ p sol of the above

Time discretisation

New issue is discretisation in time.

For V_h, Q_h appropriate FEM spaces, semi-discrete in time problem reads: $(g \equiv 0)$

Find $(u_h(t), p_h(t)) \in V_h \times Q_h$:

$$\begin{cases} m(\partial_t u_h, v_h) + a(u_h, v_h) + c(u_h(t); u_h(t), v) + b(v_h, p_h(t)) = (f_h(t), v_h) & \forall v_h \in V_h \\ b(u_h(t), q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

yielding nonlinear system of ODEs

$$\begin{cases} M \dfrac{du(t)}{dt} + A u(t) + C(u(t)) u(t) + B^T p(t) = F(t) \\ B u(t) = 0 \end{cases}$$

which needs to be discretised in time...

1) FD methods. E.g.

ϑ-method $\quad C_\vartheta(u^{k+1,k}) u^{k+1,k} = \vartheta C(u^{k+1}) u^{k+1} + (1-\vartheta) C(u^k) u^k$

$\tau = t^{k+1} - t^k$

- $\vartheta = 0 \rightarrow$ Explicit Euler $\rightarrow$ overdetermined syst. $\begin{cases} M u^{k+1} = H(u^k, p^k, f^k) \\ B u^{k+1} = 0 \end{cases}$

but can modify using p^{n+1} instead (semi-implicit scheme)

If $\operatorname{Ker} B^T = 0$ $\begin{cases} \frac{1}{\Delta t} M U^{k+1} + B^T p^{k+1} = G \\ B U^{k+1} = 0 \end{cases} \Rightarrow \begin{cases} U^{k+1} = \Delta t M^{-1} (G - B^T p^{k+1}) \\ \underbrace{B M^{-1} B^T}_{\text{non-singular}} p^{k+1} = B M^{-1} G \end{cases}$

stable if $\Delta t \leq C \min \left(\frac{h^2}{\nu}, \frac{h}{\max_x |u^k(x)|} \right)$

$\vartheta > 0 \rightarrow$ Implicit

popular method: Newton-Krylov (GMRES, BiCGStab)

Note that Schur complement approach is limited to low Reynolds number case. Indeed, e.g. if you consider the time-dependent Stokes problem (NS without convective term), e.g. by implicit Euler or other, yields Schur complement with condition number proportional to Δt^{-2}

A possibility is IMEX or semi-implicit schemes with

($\vartheta = 1$)
- linear terms $\rightarrow$ implicit
- nonlinear term $\rightarrow$ explicit

stable if $\Delta t \leq C \frac{h}{\max_x |u^k(x)|}$

Fractional step or projection methods (Chorin-Temam)

General idea: given a linear evolution equation:

$$\begin{cases} u' + L u = 0 & I \times \Omega \\ u(0) = u_0 \end{cases}$$

a discretisation in time, eg by Explicit Euler, gives (1st order in this case)

(same with implicit or higher order time-stepping)

$$\begin{cases} u^0 = u_0 \\ \dfrac{u^{k+1} - u^k}{\tau} + L u^k = 0 \end{cases}$$

The splitting method is based on the idea of splitting L:

$$L = \sum_{i=1}^{q} L_i$$

and define recursively at each time-step $u^{k+i/q}$, $i = 1, \dots, q$

$$\frac{u^{k+i/q} - u^{k+(i-1)/q}}{\tau} + L_i u^{k+(i-1)/q} = 0$$

Any step requires inversion of $I + \tau L_i$, $i = 1, \dots, q$.

So, splitting is useful only if $I + \tau L_i$ easier to invert then $I + \tau L$.

Example: 1st order projection method (splitting + proj) for NS

$q = 2$ and $L_1 = -\nu \Delta u + (u \cdot \nabla) u - f$; $L_2 = \nabla p$

idea: separate 2 main difficulties: nonlinear term and incompressibility constraint

From $\underbrace{\frac{u^{k+1}-u^{k}}{\tau}}_{\frac{u^{k+1}-u^{*}+u^{*}-u^{k}}{\tau}} - \nu \Delta u^{k} + (u^{k}\cdot\nabla)u^{k} + \nabla p^{k+1} = f^{k}$ $\quad (*)$

$(*)$ equiv. to operator splitting

1) $\dfrac{u^{*}-u^{k}}{\tau} = \nu \Delta u^{k} + (u^{k}\cdot\nabla)u^{k} + f^{k}$

2) $\dfrac{u^{k+1}-u^{*}}{\tau} = -\nabla p^{k+1}$

We can further modify the 2nd equation by "projection":

- take divergence: $\nabla\cdot \dfrac{u^{k+1}-u^{*}}{\tau} = -\nabla\cdot(\nabla p^{k+1})$
- Impose incompressibility constraint $\nabla\cdot u^{k+1} = 0$

$$\Downarrow$$

$$\underbrace{-\Delta p^{k+1} = \frac{1}{\tau}\nabla\cdot u^{*}}_{\text{Poisson problem}}$$

b.c. are a problem... see discussion in tutorial step_35

Method in step_35 is 2nd order semi-implicit version based on
- extrapolation (BDF2)
- split $(u\cdot\nabla)u$ in skew-symmetric form (consistent for $\nabla\cdot u=0$):

$$(u\cdot\nabla)u = (u\cdot\nabla)u + \frac{1}{2}(\nabla\cdot u)u \quad \text{1/2 ?}$$

$$\sim (u^{*}\cdot\nabla)u^{k+1} + \frac{1}{2}(\nabla\cdot u^{*})u^{k+1} \quad \text{semi-implicit}$$