Disclaimer

This summary is part of the lecture “Electrodynamics” by Prof. Dr. L. Novotny (FS19). It is based on the lecture.

Please report errors to huettern@student.ethz.ch such that others can benefit as well.

The upstream repository can be found at https://github.com/noah95/formulasheets

ETH Electrodynamics

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1 Conventions

V	Volume
dV	infinitesimal volume elements
A	surface
da	infinitesimal surface elements
ds	infinitesimal line element
∂V	closed surface of the volume V
∂A	circumference of area A
$\mathbf{n}$	unit vector normal to suface / circumference
ρ	Charge density
$\mathbf{j}$	Current density
$\mathbf{E}$	Electric field
$\mathbf{H}$	Magnetic field
$\mathbf{B}$	Magnetic flux density
$\mathbf{D}$	Displacement
$\mathbf{M}$	Magnetization
$\varphi(r)$	Electrostatic potential at point r

2 Mathematics

2.1 Linear Algebra

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}$$

Rotation

The cross product of the nabla operator and the vector field $\vec{F}$

$$\nabla\times\vec{F} = \det\begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{F}_x & \vec{F}_y & \vec{F}_z \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y}\vec{F}_z - \frac{\partial}{\partial z}\vec{F}_y \\ -\frac{\partial}{\partial x}\vec{F}_z + \frac{\partial}{\partial z}\vec{F}_x \\ \frac{\partial}{\partial x}\vec{F}_y - \frac{\partial}{\partial y}\vec{F}_x \end{bmatrix}$$

Divergence

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z = \operatorname{div}(\vec{F})$$

Combination

Always solve right to left.

$$\nabla\times\nabla\varphi(\vec{r}) = \nabla\times(\nabla\varphi(\vec{r})) = \cdots = 0$$

$$\nabla\cdot\nabla\times\vec{F} = \nabla\cdot(\nabla\times\vec{F}) = \cdots = 0$$

Rotation of rotation:

$$\nabla\times\nabla\times\vec{F} = \nabla\nabla\cdot\vec{F} - \nabla^2\vec{F}$$

2.2 Integrals

Line integral inside Vector Field

1. Parametrize curve with t. Split integral if necessary for differenc parametrizations

$$x : f(t) \qquad y : f(t) \qquad z : f(t)$$

2. Calcualate derivative of normal vector along curve

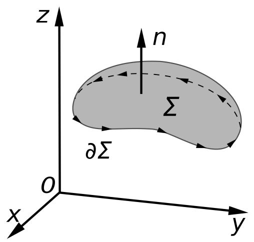
$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = x'(t)\vec{e}_x + y'(t)\vec{e}_y + z'(t)\vec{e}_z$$

$$\mathrm{d}\vec{s} = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}\mathrm{d}t$$

3. Solve integral

$$\int_{\partial A} F(\vec{r})\,\mathrm{d}\vec{s} = \int_a^b F(x(t), y(t), z(t)) \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}\mathrm{d}t$$

2.3 Stokes' Therorem

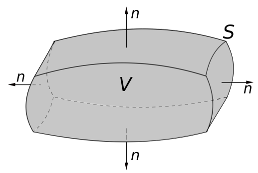


$$\int_{\partial A} F(\vec{r})\,\mathrm{d}\vec{s} = \int_A [\nabla\times F(\vec{r})]\cdot\vec{n}\,\mathrm{d}a$$

The sum of flux along the contour ∂A in contour direction is the same as the sum of curl $\nabla\times\vec{F}$ in normal direction $\vec{n}$ on the area A.

2.4 Gauss Therorem

Also known as divergence theorem.



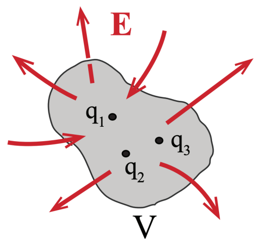
$$\int_{\partial V} \vec{F}(\vec{r})\cdot\vec{n}\,\mathrm{d}a = \int_V \nabla\vec{F}(\vec{r})\,\mathrm{d}V$$

Sum of flux across surface ∂A in normal direction $\vec{n}$ is the same as the sum of divergence inside the region V.

3 Pre-Maxwellian Electrodynamics

Gauss' Law

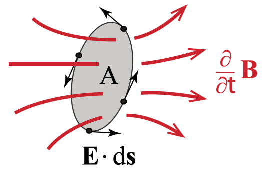
The net flux through a surface is equal to $1/\epsilon_0$ times the net electric charge within that surface.



$$\int_{\partial V}\mathbf{E}(\mathbf{r},t)\cdot\mathbf{n}\,da = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r},t)\,\mathrm{d}V$$

Faraday's Law

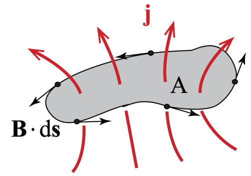
The electromotive force around a path is equal to the negative change in time of the magnetic flux enclosed by the path.



$$\int_{\partial A}\mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = -\frac{\partial}{\partial t}\int_A \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da$$

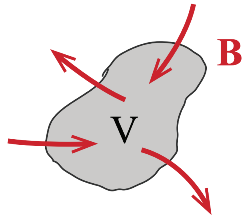
Ampere's Law

The magnetic field created by an electric current is proportional to the size of that electric current.



$$\int_{\partial A}\mathbf{B}(\mathbf{r},t)\cdot\,\mathrm{d}s = \mu_0\int_A \mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da$$

Non-existence of magnetic charges



$$\int_{\partial V}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,da = 0$$

Kirchhoff

Reducing Apere's law to any closed surfece states that the flux of current through any closed surface is zero: What flows in has to flow out.

$$\int_{\partial V}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,da = 0$$

From Faraday's law if no time-varying magnetic fields are present follows the second Kirchhoff law (Knotenregel):

$$\int_{\partial A}\mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = 0$$

The two Kirchhoff laws form the basis for circuit theory and electronic design.

4 Electrostatics

We modify Faraday's law by setting the change of magnetic flux to zero.

$$\int_{\partial A}\mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = 0$$

By applying the theorems of Gauss and Stokes we get the following identities:

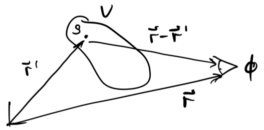
$$\nabla\cdot\mathbf{E}(\vec{r}) = \frac{1}{\epsilon_0\rho(\vec{r})} \qquad \nabla\times\mathbf{E}(\vec{r}) = 0$$

$$\mathbf{E} = -\nabla\varphi \qquad \nabla^2\varphi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})$$

Combining them we can write the poisson equation:

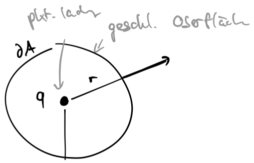
$$\nabla^2\Phi(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r}) \qquad \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r})}{|\vec{r}-\vec{r}'|}\,\mathrm{d}V'$$

Where $\varphi(\vec{r})$ is the electric potential at point $\vec{r}$.



4.1 Point Charge

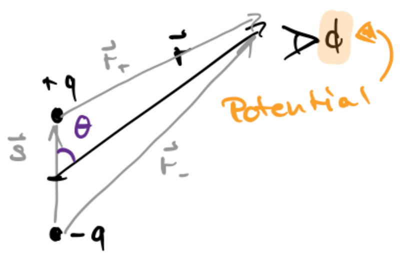
E-field of a point charge:



$$\int_{\partial A}\vec{E}\vec{n}\,\mathrm{d}a = E\cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\vec{n}_r \qquad \vec{E} = -\nabla\varphi \qquad \varphi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

4.2 Dipole: 2-Point Charge



The electrostatic potential at a given point $\vec{r}$ for the two point charges is:

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+}-\frac{1}{r_-}\right) = \frac{q}{4\pi\epsilon_0}\frac{s}{r^2}\cos\theta$$

$$\varphi(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{s}\cdot\vec{r}}{r^3}$$

Where $r = |\vec{r}|$.

Definition: Dipole

$$\vec{p} = \lim_{\substack{\vec{s}\to 0\\ \vec{q}\to\infty}}(q\vec{s}) \Longrightarrow \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

5 Maxwell's Equations

The pre-Maxwellian equations summarize the electromagnetism before Maxwell. In 1873 however, Maxwell introduced a critical modification.

5.1 Displacement Current

The law that the net flux through a closed surface is zero is flawed. For example: Identical charges released will speed out because of Coulomb repulsion and there will be a net outward current. Kirchhoffs first law has to be corrected as follows:

$$\int_{\partial V}\mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = -\frac{\partial}{\partial t}\int_V\rho(\mathbf{r},t)\,\mathrm{d}V$$

`continuity equation`: The outward current is balanced by the decrease of charge inside the surface.

Modify Ampere's law

To end up with the correct conservation law:

$$\int_{\partial A}\mathbf{B}(\mathbf{r},t)\,\mathrm{d}s = \mu_0\int_A\mathbf{j}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a + \frac{1}{c^2}\frac{\partial}{\partial t}\int_A\mathbf{E}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a$$

5.2 Interaction of Fields with Matter

E and **B** can interact with materials and generate induced charges and currents. These are then called secondary sources.

Induced by electrical field E

ρ_{pol} is the charge density induced in matter through the interaction with the electric field. ρ_{pol} is the polarization charge density. The net charge density inside the material remains zero.

$$\rho = \rho_0 + \rho_{\text{pol}}$$

To account for polarization charges we introduce the polarization **P**:

$$\int_{\partial V}\mathbf{P}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = \int_V\rho_{\text{pol}}(\mathbf{r},t)\,\mathrm{d}V$$

After inserting this into Gauss' law, we define the *electric displacement* **D** as:

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$$

Further we denote the change in polarization as the polarization current density:

$$\mathbf{j}_{\text{pol}} = \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)$$

Induced by magnetic field

According to Ampere's law, the interaction of matter with magnetic fields can induce magnetization currents. The magnetization current density is called $\mathbf{j}_{\text{mag}}$. The total current density can be written as

$$\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_{\text{pol}} + \mathbf{j}_{\text{mag}}$$

To relate the induced magnetization current to the **B**-field we define an analogy:

$$\int_{\partial A}\mathbf{M}\cdot\,\mathrm{d}s = \int_A\mathbf{j}_{\text{mag}}\cdot\mathbf{n}\,\mathrm{d}a$$

We can write the magnetic field **H**:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

5.3 Maxwell's Equations in Integral Form

$$\int_{\partial V}\mathbf{D}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = \int_V\rho_0(\mathbf{r},t)\,\mathrm{d}V$$

$$\int_{\partial A}\mathbf{E}(\mathbf{r},t)\cdot\,\mathrm{d}s = -\frac{\partial}{\partial t}\int_A\mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a$$

$$\int_{\partial A}\mathbf{H}(\mathbf{r},t)\cdot\,\mathrm{d}s = \int_A\left[\mathbf{j}_0(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t)\right]\cdot\mathbf{n}\,\mathrm{d}a$$

$$\int_{\partial V}\mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\,\mathrm{d}a = 0$$

The displacement D and the magnetic field H account for secondary sources through

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0\mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t) = \mu_0[\mathbf{H}(\mathbf{r},t) + \mathbf{M}(\mathbf{r},t)]$$

5.4 Maxwell's Equations in Differential Form

$$\nabla\cdot\mathbf{D}(\mathbf{r},t) = \rho_0(\mathbf{r},t)$$

$$\nabla\times\mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)$$

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) + \mathbf{j}_0(\mathbf{r},t)$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t) = 0$$

Continuity equation

$$\nabla\cdot\mathbf{j}_0(\mathbf{r},t) + \frac{\partial}{\partial t}\rho_0(\mathbf{r},t) = 0$$

6 Wave Equation

Arranging the first and second Maxwell equation yields the electro magnetic wave equation:

$$\nabla\times\nabla\times\mathbf{E} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = -\mu_0\frac{\partial}{\partial t}\left(\mathbf{j}_0 + \frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{M}\right)$$

$$\nabla\times\nabla\times\mathbf{H} + \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = \nabla\times\mathbf{j}_0 + \nabla\times\frac{\partial\mathbf{P}}{\partial t} - \frac{1}{c^2}\frac{\partial^2\mathbf{M}}{\partial t^2}$$

6.1 Homogeneous Solution in Free Space

In absence of matter and sources. In free space $\nabla\cdot\mathbf{E} = 0$ the equation becomes:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

For **H**-field it is equivalent. The 1D solution is solved by d'Alembert $E(x,t) = E(x-ct)$, which shows that the field propagates through space at constant velocity c.

For the general solution using separation of variables, the ansatz becomes:

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}$$

Where $\mathbf{E}(\mathbf{r})$ is complex. Inserting into the wave equation result in the Helmholtz equation ($\omega\in\mathbb{C}$):

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\mathbf{E}(\mathbf{r}) = 0$$

Plane waves

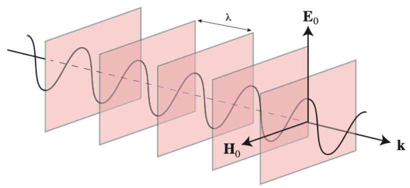
The solution is:

$$\vec{E}_0 e^{\pm i\vec{k}\vec{r}} \qquad \vec{k}\perp\vec{E}_0$$

With the dispersion relation:

$$k = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

Solutions with the $+$ sign are outgoing waves propagating in direction of $\vec{k}$. The corresponding magnetic field is found by using Maxwell's equation $\nabla\times\vec{E} = i\omega\mu_0\vec{H}$ we find $\vec{H}_0 = (\omega\mu_0)^{-1}(\vec{k}\times\vec{E}_0)$, that is, the magnetic field vector is perpendicular to the electric field vector and the wavevector.



Superposition of two waves

Superposition of two plane waves of the form:

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{(\vec{E}_1 + \vec{E}_2)e^{i(k_z - \omega t)}\right\}$$

1. Linear polarized

$$\vec{E}_1, \vec{E}_2 \in \mathrm{R}$$

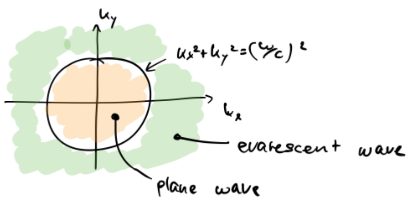
2. Circular polarized

$$\vec{E}_2 = i\vec{E}_1$$

3. Elliptic polarized

$$\vec{E}_1, \vec{E}_2 \in \mathrm{C}$$

6.2 Evanescent Waves



Exponentially decaying wave. Rewriting dispersion relation:

$$k_z\sqrt{(\omega^2/c^2) - (k_x^2 + k_y^2)}$$

If $(k_x^2 + k_y^2)$ becomes larger than $k^2 = \omega^2/c^2$ then k_z becomes imaginary. The solution then turns into

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\vec{E}_0 e^{\pm i(k_x x + k_y y) - i\omega t}\right\} e^{\mp|k_z|z}$$

This wave exponentially decays in the direction of z.

7 Mathematic Appendix

7.1 Trigonometry

- Trigonometric Powers

$$\sin^2(x) = \frac{1}{2}[1 - \cos(x)]$$
$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$
$$\sin^3(x) = \frac{1}{4}[3\sin(x) - \sin(3x)]$$
$$\cos^3(x) = \frac{1}{4}[3\cos(x) + \cos(3x)]$$

- Trigonometric Products

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$
$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y)] + \cos(x+y)]$$
$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$$
$$\sin(x)\cos(y) = \frac{1}{2}\sin(2x)$$

- Trigonometric Sumstest

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\cos(\frac{x-y}{2})\right)$$
$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2})\sin(\frac{x-y}{2}\right)$$
$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\cos(\frac{x-y}{2})\right)$$
$$\cos(x) - \cos(y) = 2\sin\left(\frac{x+y}{2})\sin(\frac{x-y}{2}\right)$$

- Double angle

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$

- Addition theorems

$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

$\sin(x+y)\sin(x-y) = \cos^2(y) - \cos^2(x) = \sin^2(x) - \sin^2(y)$

$\cos(x+y)\cos(x-y) = \cos^2(y) - \sin^2(x) = \cos^2(x) - \sin^2(y)$

7.2 Poisson Equation

$$\nabla^2 f(\vec{r}) = -g(\vec{r})$$

Solution:

$$f(\vec{r}) = \int_V \frac{g(\vec{r})}{4\pi|\vec{r} - \vec{r}'|}\,\mathrm{d}V'$$

7.3 Vector operations

Operation		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, φ, z)	Spherical Coordinates (r, θ, φ)
Vector field	A	$A_x\vec{n}_x + A_y\vec{n}_y + A_y\vec{n}_y$	$A_r\vec{n}_r + A_\varphi\vec{n}_\varphi + A_z\vec{n}_z$	$A_r\vec{n}_r + A_\theta\vec{n}_\theta + A_\varphi z\vec{n}_\varphi z$
Gradient	∇f	$\frac{\partial f}{\partial x}\vec{n}_x + \frac{\partial f}{\partial y}\vec{n}_y + \frac{\partial f}{\partial z}\vec{n}_z$	$\frac{\partial f}{\partial r}\vec{n}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{n}_\varphi + \frac{\partial f}{\partial z}\vec{n}_z$	$\frac{\partial f}{\partial r}\vec{n}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{n}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{n}_\varphi$
Divergence	$\nabla \cdot A$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \varphi}A_\varphi + \frac{\partial}{\partial z}A_z$	$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}A_\varphi$
Curl	$\nabla \times A$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{n}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{n}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial z}\right)\vec{n}_z$	$\left(\frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{n}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{n}_\varphi + \frac{1}{r}\left(\frac{\partial rA_\varphi}{\partial r} - \frac{\partial A_r}{\partial \varphi}\right)\vec{n}_z$	$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\varphi\sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right)\vec{n}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\vec{n}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\vec{n}_\varphi$