Homework 7

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CS 161

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1. Prove the following:

   (a) Generalized product rule: $Pr(A, B|K) = Pr(A|B, K)Pr(B|K)$.

   Proof. $Pr(A, B|K) = Pr(A|B, K)Pr(B|K)$

   $$\frac{Pr(A, B, K)}{Pr(K)} = \frac{Pr(A, B, K)}{Pr(B, K)} \cdot \frac{Pr(B, K)}{Pr(K)}$$

   $$\frac{Pr(A, B, K)}{Pr(K)} = Pr(A, B, K)/Pr(K)$$ ∎

   (b) Generalized Bayes rule: $Pr(A|B, K) = Pr(B|A, K)Pr(A|K)/Pr(B|K)$.

   Proof. $Pr(A|B, K) = Pr(B|A, K)Pr(A|K)/Pr(B|K)$

   $$\frac{Pr(A, B|K)}{Pr(B|K)} = \frac{Pr(B, A, K)}{Pr(A, K)} \cdot \frac{Pr(A, K)}{Pr(K)} \cdot \frac{Pr(B, K)}{Pr(K)}$$

   $$\frac{Pr(A, B|K)}{Pr(B|K)} = \frac{Pr(B, A, K)}{Pr(K)} \cdot \frac{Pr(B, K)}{Pr(K)}$$

   $$\frac{Pr(A, B, K)}{Pr(K)} \cdot \frac{Pr(B, K)}{Pr(K)} = \frac{Pr(B, A, K)}{Pr(K)} \cdot \frac{Pr(B, K)}{Pr(K)}$$

   $$\frac{Pr(A, B, K)}{Pr(K)} = \frac{Pr(B, A, K)}{Pr(K)}$$

   $$\frac{Pr(A, B, K)}{Pr(K)} = \frac{Pr(A, B, K)}{Pr(K)}$$ ∎

2. An oil well may be drilled on Mr. Arthurs farm in Atlanta. Based on what has happened to similar farms, we judge the probability of oil being present to be 0.5, the probability of only natural gas being present to be 0.2, and the probability of neither being present to be 0.3. If oil is present, a geological test will give a positive result with probability 0.9; if only natural gas is present, it will give a positive result with probability 0.3; and if neither are present, the test will be positive with probability

0.1. Suppose the test comes back positive. What is the probability that oil is present?

Given probabilities:

$$Pr(Oil) = 0.5$$
$$Pr(Gas) = 0.2$$
$$Pr(Neither) = 0.3$$
$$Pr(Positive|Oil) = 0.9$$
$$Pr(Positive|Gas) = 0.3$$
$$Pr(Positive|Neither) = 0.1$$

We can show:

$$
\begin{aligned}
Pr(Positive) &= Pr(Positive|Oil) \cdot Pr(Oil) \\
&+ Pr(Positive|Gas) \cdot Pr(Gas) \\
&+ Pr(Positive|Neither) \cdot Pr(Neither) \\
&= 0.9 \cdot 0.5 + 0.3 \cdot 0.2 + 0.1 \cdot 0.3 \\
&= 0.54 \\
Pr(Oil|Positive) &= \frac{Pr(Oil)Pr(Positive|Oil)}{P(Positive)} \\
&= \frac{0.5 \cdot 0.9}{0.54} \\
&= 0.833333...
\end{aligned}
$$

3. Mr. Arthur picked up an object at random from the above set. We want to compute the probabilities of the following events:

- α_1: the object is yellow;
- α_2: the object is square;
- α_3: if the object is one or yellow, then it is also square.

Construct the joint probability distribution of this problem. Use it to compute the above probabilities by explicitly identifying the worlds at which each α_i holds. Identify two sets of sentences α, β, γ such that α is independent of β given γ with respect to the constructed distribution.

$$Pr(\alpha_1) = Pr(Yellow) = \frac{9}{13}$$

$$Pr(\alpha_2) = Pr(Square) = \frac{8}{13}$$

$$Pr(\alpha_3) = Pr(Square|One \vee Yellow) = \frac{Pr(Square \wedge (One \vee Yellow))}{Pr(One \vee Black)} = \frac{7}{11}$$

The resulting joint probability distribution table is as follows:

	Yellow		$\neg$Yellow	
	Square	$\neg$Square	Square	$\neg$Square
One	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$
$\neg$One	$\frac{4}{13}$	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{1}{13}$

Using this table:

- α_1: all worlds in the first two columns

- α_2: all worlds in the first and third columns

- α_3: all worlds in the first and third columns that are in the first row or the first two columns divided by all cells except the last two

Therefore:

$$Pr(\alpha_1) = \frac{2}{13} + \frac{4}{13} + \frac{1}{13} + \frac{2}{13} = \frac{9}{13}$$

$$Pr(\alpha_2) = \frac{2}{13} + \frac{4}{13} + \frac{1}{13} + \frac{1}{13} = \frac{8}{13}$$

$$Pr(\alpha_3) = \frac{\frac{2}{13} + \frac{4}{13} + \frac{1}{13}}{\frac{2}{13} + \frac{4}{13} + \frac{1}{13} + \frac{2}{13} + \frac{1}{13} + \frac{1}{13}} = \frac{7}{11}$$

Two sets of sentences that satisify the condition above are:

α	β	γ
Square	One	$\neg$Yellow
Square	$\neg$One	$\neg$Yellow

4. Consider the DAG in Figure 1:

   (a) List the Markovian assumptions asserted by the DAG.

      - $I(A, \varnothing, \{B, E\})$
      - $I(B, \varnothing, \{A, C\})$
      - $I(C, \{A\}, \{A, B, D, E\})$

- $I(D, \{A, B\}, \{A, B, C, E\})$
- $I(E, \{B\}, \{A, B, C, D\})$
- $I(F, \{C, D\}, \{A, B, C, D, E\})$
- $I(G, \{F\}, \{A, B, C, D, E, F, H\})$
- $I(H, \{E, F\}, \{A, C, B, D, E, F, G\})$

(b) True or false? Why?

- $d\_separated(A, BH, E)$
  False, A and E converge on H through F.
- $d\_separated(G, D, E)$
  True, D blocks the only diverging path between G and E at B
- $d\_separated(AB, F, GH)$
  False, the removal of F blocks the sequential path from A and B to G but through $B \rightarrow E \rightarrow H$ there still exists a sequential path.

(c) Express $Pr(a, b, c, d, e, f, g, h)$ in factored form using the chain rule for Bayesian networks

$$
\begin{aligned}
Pr(a, b, c, d, e, f, g, h) = {} & Pr(a|b, c, d, e, f, g, h) \cdot Pr(a|b, c, d, e, f, g, h) \\
& \cdot Pr(c|d, e, f, g, h) \cdot Pr(d|e, f, g, h) \cdot Pr(e|f, g, h) \\
& \cdot Pr(f|g, h) \cdot Pr(g|h) \cdot Pr(h)
\end{aligned}
$$

(d) Compute $Pr(A = 0, B = 0)$ and $Pr(E = 1|A = 1)$ using the CPTs below. Justify your answers.

As A and B are independent:

$Pr(A = 0, B = 0) = Pr(A = 0) \cdot Pr(B = 0) = 0.8 \cdot 0.3 = 0.24$

As A and E are independent:

$$Pr(E = 1 | A = 1) = \frac{Pr(E = 1, A = 1)}{Pr(A = 1)}$$
$$= \frac{Pr(E = 1) \cdot Pr(A = 1)}{Pr(A = 1)}$$
$$= Pr(E = 1)$$
$$= Pr(E = 1, B = 0) + Pr(E = 1, B = 1)$$
$$= Pr(E = 1 | B = 0) \cdot Pr(B = 0) + Pr(E = 1 | B = 1) \cdot Pr(B = 1)$$
$$= 0.9 \cdot 0.3 + 0.1 \cdot 0.7$$
$$= 0.27 + 0.07$$
$$= 0.34$$