Numerical methods for nonlocal conservation laws

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Preface

1 Introduction

Since many interesting conservation laws that arise in applications (e.g. traffic simulation) are nonlinear, it is impossible to obtain explicit solution formulas. Hence, we rely on designing numerical methods to approximate the solutions in the best possible way. When designing such methods it is natural to ask ourselves how reliable those methods could be. That is significant to study the stability and convergence of the numerical methods.

For the purpose of numerical experimentation we focus on the specific one dimensional traffic flow model which is modeled by a nonlocal conservation law. In other words the goal is to design an efficient numerical method and ultimately to prove that the method convergences to the correct solution.

2 Preliminaries

2.1 Scalar local and nonlocal conservation laws

Scalar conservation laws are partial differential equations that can be written on the form

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where $\partial_t = \frac{\partial}{\partial_t}$ is temporal differentiation, $\nabla\cdot$ is divergence operator, $u = u(x,t)$ is the unknown function of x and t and f is a given flux function.

If we now let $f(u) = uv(u)$ where $v(u) = 1 - u$, then Eq. (1) can be used for modelling traffic flow. For a detailed setup of this model, we refer to [3]. This model is however based only on local information. when a car driver usually decides his speed, it depends on the traffic information within a road segment of length $\epsilon > 0$ ahead of the car's current location. For managing that case we look for a better way to model the velocity, so we need a nonlocal model which is on the following form

$$\begin{aligned} \partial_t u + \nabla \cdot (uV(u)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2}$$

where now V is a nonlocal operator, say,

$$\begin{aligned} V(u)(x,t) &= v(u * \omega_\epsilon) \\ &= 1 - \int_0^\epsilon \omega_\epsilon(y) u(x+y,t)\, dy \end{aligned} \tag{3}$$

for some nonlocal kernel $\omega_\epsilon : [0,\epsilon] \to \mathbb{R}$ satisfying $\omega_\epsilon \geq 0$ and $\int_0^\epsilon \omega_\epsilon(y)\, dy = 1$. Formally speaking, $V(u) \to v(u)$ as $\epsilon \to 0$, but whether the corresponding solutions u_ϵ of (2) converge to a solution of (1), is the key question that has been studied in the literature (need a citation).

In conclusion, let us denote (1) and (2) the local and nonlocal model, respectively. We are then aiming to approximate solutions of the local model by solutions of the nonlocal model, where the solution $u(x,t)$ represents the density of cars which is the number of cars per square meter.

2.2 Solutions to the Riemann problem

For numerical experiments we need entropy solutions for different initial data. Hence, we briefly mention the notion of weak solution and entropy solution.

Definition 2.1 (Weak solutions) *A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution of (1) with initial data $u_0 \in L^\infty(\mathbb{R})$ if the following identity holds for all test functions $\phi \in C^1(\mathbb{R} \times \mathbb{R}_+)$*

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} u\phi_t + f(u)\phi_x \, dx\, dt + \int_{\mathbb{R}} u_0 \phi(x,0)\, dx = 0. \tag{4}$$

Since weak solutions are not unique, some additional conditions have to be imposed in order to pick a weak solution that describes the physical flow correctly. One such condition is entropy condition. Detailed characteristics of the entropy solution can be found in [4].

We now look at the explicit solutions for the Riemann problem. The Riemann problem is the initial value problem

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \end{aligned} \tag{5}$$

Solution of this equation is constructed as

$$u(x,t) = \begin{cases} u_l & \text{for } x \leq h'(u_l)t \\ (h')^{-1}(\frac{x}{t}) & \text{for } h'(u_l)t \leq x \leq h'(u_r)t \\ u_r & \text{for } x \geq h'(u_r)t \end{cases} \tag{6}$$

where $h(u)$ is defined by

$$h(u) = \begin{cases} \sup\{g(u) \mid g \leq f \text{ and } g \text{ is convex on } [u_l, u_r]\} & \text{if } u_l < u_r, \\ \inf\{g(u) \mid g \geq f \text{ and } g \text{ is concave on } [u_r, u_l]\} & \text{if } u_r < u_l. \end{cases} \tag{7}$$

3 Numerical method

3.1 Finite volume scheme for general f

Designing the finite difference method consists of four steps:

1. Discretizing the domain,
2. Satisfying the equation at discrete points,
3. Replacing derivatives by finite differences,
4. Solving the discretized problem.

If we take a look at step 3, it requires that solutions have to be smooth sufficiently. However, solutions of conservations laws may not be differentiable or even continuous. For this and other reasons, we have to adapt the steps above to design a reasonable numerical method. That is a numerical method appropriate to our problem which is called a finite volume scheme.

3.2 Discretizing the domain and control volumes

For simplicity, we consider a uniform discretization of both the one dimensional spatial domain $[x_L, x_R]$ and temporal domain $[0, T]$. We split the spatial domain

into $\mathcal{C}_j$ called control volumes, which are smaller and non-overlapping subdomains, such that

$$\begin{aligned}[x_L, x_R] &= \bigcup_{j=0}^{K} [x_L + j\Delta x, x_L + (j+1)\Delta x] \\ &= \bigcup_{j=0}^{K} [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \\ &= \bigcup_{j=0}^{K} \mathcal{C}_j \end{aligned} \tag{8}$$

where $\Delta x = \frac{x_R - x_L}{K+1}$.

We denote each midpoint of each $\mathcal{C}_j$ by $x_j = x_L + (j + \frac{1}{2})\Delta x$ for $j = 0, \cdots\cdots, K$.

Setting $\Delta t = \frac{T}{N+1}$, we discretize the temporal domain $[0, T]$ as $0 = t^0 < t^1 < \cdots < t^N < t^{N+1} = T$ where each $t^n = n\Delta t$ for $n = 0, \cdots\cdots, N+1$.

3.3 Cell averages and a finite volume scheme

As solutions of conservations laws may be discontinuous, pointwise evaluation does not make sense. Instead, at each time level t^n, we take cell averages:

$$u(x_j, t^n) = u_j^n \approx \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\, dx \tag{9}$$

Assuming now the cell averages are known at some time level t^n, we find the cell averages at next time level t^{n+1} as follows:
Taking an integral of (1) over the domain $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$

$$\int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t \, dx \, dt + \int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u)_x \, dx \, dt = 0$$

And applying the fundamental theorem of calculus yields

$$\begin{aligned} &\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^{n+1})\, dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^n)\, dx \\ &= -\int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t))\, dt + \int_{t^n}^{t^{n+1}} f(u(x_{j-\frac{1}{2}}, t))\, dt \end{aligned} \tag{10}$$

Defining the numerical fluxes

$$\bar{F}_{j+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{j+\frac{1}{2}}, t))\, dt \tag{11}$$

and dividing both sides of (10) by Δx, we obtain

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(\bar{F}_{j+\frac{1}{2}} - \bar{F}_{j-\frac{1}{2}}) \tag{12}$$

Note that the relation in (12) is not explicit, since $\bar{F}$ requires a priori knowledge of the exact solution. By introducing an appropriate approximation of the numerical flux $\bar{F}$ by F, we obtain the finite volume scheme for a general flux f:

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{j+\frac{1}{2}} - F_{j-\frac{1}{2}}) \tag{13}$$

3.4 Finite Volume Scheme for the Nonlocal Model

Let the flux be defined as $f = uV$, where V is the nonlocal operator given in (3). There are several ways to specify the nonlocal numerical fluxes. In this work, we focus on a Lax-Friedrichs type scheme; see the approaches in [1] and [2]. let us set

$$V_j^n = 1 - \Delta x \sum_{k=0}^{m-1} \omega_k u_{j+k}^n, \tag{14}$$

Where $\{\omega_k\}_{k=0}^{m-1}$ is a set of numerical quadrature weights that are used to approximate the nonlocal kernel ω_ϵ and $m = \lceil \frac{\epsilon}{\Delta x} \rceil$ is the number of cells involved in V.
We now take the numerical fluxes in (13) as following:

$$F_{j-\frac{1}{2}} = \frac{1}{2}u_{j-1}^n V_{j-1}^n + \frac{1}{2}u_j^n V_j^n + \frac{\alpha}{2}(u_{j-1}^n - u_j^n) \tag{15}$$

and

$$F_{j+\frac{1}{2}} = \frac{1}{2}u_j^n V_j^n + \frac{1}{2}u_{j+1}^n V_{j+1}^n + \frac{\alpha}{2}(u_j^n - u_{j+1}^n) \tag{16}$$

The numerical quadrature weights ω_k involved in (14) can be given by
left endpoints

$$\omega_k = \omega_\epsilon(k\Delta x)\Delta x \text{ for } k = 0, \cdots, m-1,$$

normalized left endpoint

$$\omega_k = \frac{\omega_\epsilon(k\Delta x)\Delta x}{\sum_{k=0}^{m-1} \omega_\epsilon(k\Delta x)\Delta} \text{ for } k = 0, \cdots, m-1$$

and exact quadrature

$$\omega_k = \omega_\epsilon(k\Delta x)\Delta x \text{ for } k = 0, \cdots, m-1,$$

One may ask whether the choice of numerical quadrature will result in different outcomes when implementing the finite volume scheme (13) together with (15) and (16). That is, whether the scheme converges for some numerical quadratures but fails to converge for others.
This question is addressed in [1] through numerical experiments and mathematical explanations. In the next section, we also provide an answer based on numerical experiments.

4 Numerical experiments

5 Convergence of the numerical method to entropy solution

References

[1] S. Blandin and P. Goatin. "Well-posedness of a conservation law with nonlocal flux arising in traffic flow modeling". In: *Numerische Mathematik* 132 (2016), pp. 217–241.

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[4] Ulrik Skre Fjordholm Siddhartha Mishra and Rémi Abgrall. "Numerical methods for conservation laws and related equations". In: - - (-), p. 29.