Numerical methods for nonlocal conservation laws

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October 2025

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Preface

1 Introduction

Since many interesting conservation laws that arise in applications are nonlinear, it is impossible to obtain explicit solution formula. Hence, we relay on designing numerical methods to approximate the solution in the best possible way. When designing numerical methods it is natural to ask ourselves how reliable a method could be. Therefore, we must study the stability and convergence of the numerical method.

For numerical experiment purpose we focus on the specific one dimensional traffic flow model which is modeled by a nonlocal conservation law. So, the goal is to design an efficient numerical method and ultimately to prove that the method convergence to the correct solution.

2 Preliminaries

2.1 Scalar local and nonlocal conservation laws

Scalar conservation laws are partial differential equations that can be written in the form

$$\begin{aligned} \partial u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where $\partial_t = \frac{\partial}{\partial_t}$ is time differentiation, $\nabla\cdot$ is divergence operator, $u = u(x,t)$ is the unknown and f is a given flux function.

If we now let $f(u) = u(1-u)$, then Eq. (1) can be used for modelling traffic. For detailed setup of this model, we refer to [1]. However, this model is based only on local information. Usually, when a car driver decides his speed, it depends on the traffic information within a road segment of length $\epsilon > 0$ ahead of the cars's current location. For this to take into account we look for a better way to model the velocity, hence we need a nonlocal model which is on the following form

$$\begin{aligned} \partial u + \nabla \cdot (uV(u)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2}$$

where now V is a nonlocal operator, say,

$$V(u)(x,t) = \int_0^\epsilon \omega_\epsilon(y) v(u(x+y,t))\, dy \tag{3}$$

for some nonlocal kernel $\omega_\epsilon : [0,\epsilon] \to \mathbb{R}$ satisfying $\omega_\epsilon \geq 0$ and $\int_0^\epsilon \omega_\epsilon(y)\, dy = 1$. formally speaking, $V(u) \to v(u)$ as $\epsilon \to 0$, but whether the corresponding solutions u_ϵ of (2) converge to a solution of (1), is the big question that has been studied.

To sum this up, let us denote (1) and (2) as local and nonlocal model, respectively. Then we are aiming to approximate solution of the local model by solutions of the nonlocal model, where the solution $u(x,t)$ denote the density of cars which is number of cars per square meter.

2.2 Solutions to the Riemann problem

For numerical experiments we need entropy solutions for different initial data. Hence, we briefly mention the notion of weak solution and entropy solution.

Definition 2.1 (Weak solutions) *A function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ is a weak solution of (1) with initial data $u_0 \in L^\infty(\mathbb{R})$ if the following identity holds for all test functions $\phi \in C^1(\mathbb{R}\times\mathbb{R}_+)$*

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}} u\phi_t + f(u)\phi_x\, dx\, dt + \int_{\mathbb{R}} u_0\phi(x,0)\, dx = 0. \tag{4}$$

Since weak solutions are not unique, therefore some extra conditions have to be imposed in order to pick a weak solution that describes the physical flow correctly. One such condition is entropy condition. Detailed characteristics of entropy solution can be found on [2].

Now we look at the explicit solutions for the Riemann problem. The Riemann problem is the initial value problem

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= \begin{cases} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \end{aligned} \tag{5}$$

Solution of this equation is constructed as

$$u(x,t) = \begin{cases} u_l & \text{for } x < h'(u_l)t \\ (h')^{-1}(\frac{x}{t}) & \text{for } h'(u_l)t < x < h'(u_r)t \\ u_r & \text{for } x > h'(u_r)t \end{cases} \tag{6}$$

where $h(u)$ is defined by

$$h(u) = \begin{cases} \sup\{g(u) \mid g \leq f \text{ and } g \text{ is convex on } [u_l, u_r]\} & \text{if } u_l < u_r, \\ \inf\{g(u) \mid g \geq f \text{ and } g \text{ is concave on } [u_r, u_l]\} & \text{if } u_r < u_l. \end{cases} \tag{7}$$

3 Numerical method

3.1 Finite volume scheme for general f

When designing the Finite difference method, it consists of four steps:

1. Discretizing the domain,
2. Fulfilling the equation at discrete points,
3. Replacing derivatives by finite differences,
4. Solve the discretized problem.

If we take a look on step 3, it requires that solutions should be smooth. However, solutions of conservations laws may not be differentiable or even not continuous. For this and other reasons, we must adapt the steps above to design a reasonable numerical method.

4 Numerical experiments

5 Convergence of the numerical method to entropy solution

References

[1] Ulrik Skre Fjordholm Siddhartha Mishra and Rémi Abgrall. "Numerical methods for conservation laws and related equations". In: - - (-), p. 21.

[2] Ulrik Skre Fjordholm Siddhartha Mishra and Rémi Abgrall. "Numerical methods for conservation laws and related equations". In: - - (-), p. 29.