Master project: Numerical methods for nonlocal conservation laws

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1 Background and outline of the problem

Hyperbolic conservation laws are partial differential equations that can be written in the form

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where $\partial_t = \frac{\partial}{\partial t}$ is time differentiation, $\nabla\cdot$ is divergence, $u = u(x,t)$ is the unknown (either a scalar- or vector-valued function) and f is a given (usually nonlinear) function, the *flux function*. The initial data is u_0 and the PDE is required to hold for $t > 0$, $x \in \mathbb{R}^d$. A frequently used "toy example" is (1) with $d = 1$ and $f(u) = uv(u)$, where e.g. $v(u) = 1 - u$. This equation can be used for modelling traffic, and $u(x,t)$ represents the density of cars at position x at time t.

A closely related problem is the *nonlocal* equation

$$\begin{aligned} \partial_t u + \nabla \cdot (uV(u)) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{2}$$

where now V is a *nonlocal operator*, say, $V(u)(x,t) = \int_{\mathbb{R}} \omega(y) v(u(x+y,t))\,dy$ for some *nonlocal kernel* $\omega : \mathbb{R} \to \mathbb{R}$ satisfying $\omega \geq 0$ and $\int_{\mathbb{R}} \omega(y)\,dy = 1$. If we choose, say, $\omega = \omega_\varepsilon$, where $\omega_\varepsilon(y) = \varepsilon^{-1}\omega_1(y/\varepsilon)$ and ω_1 is some fixed kernel, then formally speaking, $V(u) \to v(u)$ as $\varepsilon \to 0$. However, it is not at all obvious whether the corresponding solutions u_ε of (2) converge to a solution of (1) (with $f(u) = uv(u)$) as $\varepsilon \to 0$.

The question of whether solutions of (1) can be "approximated" by solutions of (2) in the way described above, is mostly open, but the first positive result was published just recently [2]. They use specific choices of v and ω_ε, and the techniques they employ are also very specific to those choices and are somewhat esoteric.

A similar result was recently published by Coclite et al. [4], who used very different, but more standard, techniques.

The fact that $u^\varepsilon \to u$ as $\varepsilon \to 0$ was more fully resolved in the recent paper [5], where convergence is shown for a wide class of velocities v, kernels ω_ε, and initila data u_0.

The paper [6] designs a general class of numerical methods for (2). They prove that these numerical methods — under certain conditions that are stricter than those in [5] — are *asymptotic preserving*, namely that they converge to the entropy solution of (1) as both ε and the discretization parameter go to 0.

2 Main goals of the project

The main goals of the project are to read and understand the recent works on this problem; to design a numerical method for (2), and to prove convergence of this numerical method as the resolution increases.

3 Progress plan and milestones

The project can be divided into the following steps.

1. Read up on the latest papers on this subject, in particular [4, 5, 6]. It's not important to understand everything in these papers, but you need to get an idea of what is accomplished in each paper and roughly what techniques they use.

2. Write down a simple numerical method for (2), such as a Lax–Friedrichs-type scheme. Check if the scheme fits into the framework of [6].

3. Implement the numerical method in Python, Matlab, or some other programming language. Test the method in some numerical test cases that you find in the literature.

4. Try to determine, by numerical experiments, if the scheme seems to converge to the correct solutions as either $\varepsilon \to 0$, $h \to 0$ (where h is the discretization parameter), or both.

5. Try to prove that the method indeed converges to the correct solution (the entropy solution) in these limits.

References

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