Master project:
Analysis of nonlocal predator-prey models

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1 Background and outline of the problem

The *Lotka–Volterra model*, or *predator-prey model*, is the system of ordinary differential equations

$$\begin{cases} \dot{u} = \alpha u - \beta uv \\ \dot{v} = -\gamma u + \delta uv. \end{cases} \tag{1}$$

Here, $u(t)$ is the size of a population of prey animals, $v(t)$ the population of predators, and $\alpha, \beta, \gamma, \delta \geq 0$ are given constants which prescribe, respectively, the rate at which prey are born, the rate at which predators kill prey, the rate at which predators die off naturally, and the rate at which predators benefit from eating prey.

The simple model (1) does not take into account geographical mobility, i.e., the movement of individuals through space. The purpose of the present project is to derive a partial differential equation (PDE) modelling the interaction between the species as well as their movement through space, analyse the properties of this equation (existence, uniqueness, stability, qualitative properties), and/or design a numerical method for the PDE and analyse this method.

One such model is (see Colombo and Rossi [1])

$$\begin{cases} \partial_t u - \mu \Delta u = \alpha u - \beta uv \\ \partial_t v + \nabla \cdot (vW) = -\gamma u + \delta uv \end{cases} \qquad \text{for } x \in \Omega, t > 0 \tag{2}$$

for some $\Omega \subset \mathbb{R}^d$. The unknowns u and v are now functions both of space $x \in \Omega$ and time $t \in (0, \infty)$. The right-hand sides are the same as in the Lotka–Volterra models, so locally, at each x, we have the same Lotka–Volterra dynamics. But in addition, the prey population is *diffused in space* as in the diffusion equation $\partial_t u - \mu \Delta u = 0$ (where $\mu > 0$ is a constant), and the predator population is *transported in space* along the velocity field $W = W(x, t)$. One reasonable choice of W is to set

$$W(x, t) = \kappa \nabla u(x, t)$$

for some $\kappa > 0$, i.e., the predators move in the direction in which the number of prey increases the most. Another reasonable choice is

$$W(x,t) = \kappa \nabla \big(\omega * u\big)(x,t)$$

where $\omega\colon \mathbb{R}^d \to \mathbb{R}$ is some nonnegative function, and $*$ denotes convolution,

$$(\omega * u)(x,t) = \int_{\mathbb{R}^d} \omega(y) u(x-y,t)\, dy.$$

The latter choice leads to a *non-local PDE*, in which the velocity of predators at position x is determined by counting the number of prey at nearby points $x-y$, weigh that number by a weight $\omega(y)$, and then find which direction this quantity increases the most.

2 Main goals and milestones of the project

The purpose of the project is to study one such non-local predator-prey model (or a few different models). Some possible milestones of the project are as follows.

1. Read up on the existing literature on non-local predator-prey models, starting with [1].
2. In the student's own words, formulate a proof of existence of a solution of the initial-value problem for the non-local PDE on $\Omega = \mathbb{R}^d$. The most straightforward way of accomplishing this is to use a fixed point iteration.
3. Generalize the proof to work in an arbitrary (smooth enough) domain $\Omega \subset \mathbb{R}^d$.
4. Investigate the possibility of restricting the movement of predators to a subset $\Omega_0 \subset \Omega$.
5. Investigate non-linear models in which the velocity W of predators depends on v itself. For instance, the choice

   $$W = \kappa(1-v)\nabla u \tag{3}$$

   would have the effect of putting an upper limit ($v = 1$ in this example) on the density of predators per square meter.
6. Design a numerical method for the non-local PDE. Prove that the method converges as the resolution increases.

The first three milestones is the basis of the thesis. The last three milestones are projects which (to the best of my knowledge) do not exist in the literature, and would constitute research projects in their own right. Depending on the student's progress and findings, the final thesis could consist of one or more of these projects.

References

[1] R. M. Colombo and E. Rossi. Hyperbolic Predators vs. Parabolic Preys. Preprint available from https://arxiv.org/pdf/1402.2099 (2014).