Mandatory assignment 1 i MAT4270

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1 Problem 1.2.3

We have the wave equation:

$$u_{tt} = c^2 \nabla^2 u \tag{1}$$

And we want to show if

$$u(t.x, y) = e^{i(k_x x + k_y y - \omega t)}$$

solves eq(1)

$$\begin{aligned} u_t &= -i\omega u \\ u_{tt} &= -\omega^2 u \\ u_x &= ik_x u \\ u_{xx} &= -k_x^2 u \\ u_y &= ik_y u \\ u_{yy} &= -k_y^2 u \end{aligned}$$

If we now insert u_{tt}, u_{xx}, u_{yy} in equation (1) above, we get as follows:

$$\begin{aligned} u_{tt} &= c^2 \nabla^2 \\ -\omega^2 u &= -c^2 u(k_x^2 + k_y^2) \\ \omega^2 &= c^2(k_x^2 + k_y^2) \\ \omega &= c\sqrt{k_x^2 + k_y^2} \end{aligned}$$

As we have already showed that ω equals $c\sqrt{k_x^2 + k_y^2}$, it follows from that $u(t, x, y)$ solves our wave equation.

2 Problem 1.2.3

Assume that $m_x = m_y$ such that $k_x = k_y = k$, and $C = \frac{1}{\sqrt{2}}$ then ω equals $c\sqrt{2}k$. In this case, let:

$$u^n_{i,j} = e^{i(kh(i+j)-\tilde{\omega}n\Delta t)} \tag{2}$$

and insert Eq(2) into the following equation Eq(3)

$$u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j} = C^2(u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j} + u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}) \tag{3}$$

to deduce $\tilde{w} = c\sqrt{2}k = \omega$. As C is equal to $\frac{1}{\sqrt{2}}$ and $u^n_{i+1,j} = u^n_{i,j+1}$, $u^n_{i-1,j} = u^n_{i,j-1}$, Eq(3) simplifies to:

$$\begin{aligned}
u^{n+1}_{i,j} - 2u^n_{i,j} + u^{n-1}_{i,j} = u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j} \\
u^{n+1}_{i,j} + u^{n-1}_{i,j} = u^n_{i+1,j} + u^n_{i-1,j} \\
e^{i(kh(i+j)-\tilde{\omega}(n+1)\Delta t)} + e^{i(kh(i+j)-\tilde{\omega}(n-1)\Delta t)} = e^{i(kh(i+j+1)-\tilde{\omega}n\Delta t)} + e^{i(kh(i+j-1)-\tilde{\omega}n\Delta t)} \\
e^{i(kh(i+j)-\tilde{\omega}n\Delta t)}(e^{-i\tilde{\omega}\Delta t} + e^{i\tilde{\omega}\Delta t}) = e^{i(kh(i+j)-\tilde{\omega}n\Delta t}(e^{ikh} + e^{-ikh}) \\
e^{-i\tilde{\omega}\Delta t} + e^{i\tilde{\omega}\Delta t} = e^{ikh} + e^{-ikh} \\
2\cos\tilde{\omega}\Delta t = 2\cos kh \\
\cos\tilde{\omega}\Delta t = \cos kh \qquad \textit{take the cos}^{-1} \textit{ on both sides to get;} \\
\tilde{\omega}\Delta t = kh \\
\tilde{\omega} = \frac{kh}{\Delta t}
\end{aligned}$$

As $h = c\sqrt{2}\Delta t$, hence we get $\tilde{w} = c\sqrt{2}k = \omega$