Fundamentals of acoustics

What is a wave, how do we arrive at the wave equation and how to use it.

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What is a wave?

- Movement of disturbance (information) from one point to another
- Movement takes place at finite speed
- Shape of the disturbance is arbitrary
- Longitudinal and transverse waves
- Medium might be required

Derivation of the wave equation

As always we start with some reasonable assumptions

1. The medium is homogeneous
2. The medium is not moving
3. The medium can be considered as an ideal gas
4. Its change of state can be considered as adiabatic
5. Pressure and density fluctuations are small compared to their static values

Derivation of the wave equation

Involved physical quantities

- Time t — $[t] = \mathrm{s}$
- Position $\mathbf{x}$ — $[\mathbf{x}] = \mathrm{m}$
- Particle velocity $\mathbf{v}(\mathbf{x}, t)$ — $[\mathbf{v}] = \frac{\mathrm{m}}{\mathrm{s}}$
- Density of the medium $\varrho(\mathbf{x}, t)$ — $[\varrho] = \frac{\mathrm{kg}}{\mathrm{m}^3}$
- Sound pressure $p(\mathbf{x}, t)$ — $[p] = \mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{kg}}{\mathrm{ms}^2}$

Derivation of the wave equation

Part I

Conservation of mass $\Rightarrow$ continuity equation

$$\partial_t \varrho(\mathbf{x}, t) + \nabla(\varrho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)) = 0$$

Derivation of the wave equation

Part II

Conservation of momentum $\Rightarrow$ Euler equation

$$\varrho(\mathbf{x}, t)\partial_t \mathbf{v}(\mathbf{x}, t) + \varrho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)\nabla\mathbf{v}(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) = 0$$

Derivation of the wave equation

Part III

Behavior of ideal gas and conservation of energy $\Rightarrow$ connection between pressure and density

$$\partial_t p(\mathbf{x}, t) = c^2 \partial_t \varrho(\mathbf{x}, t)$$

with $c^2 = \gamma R T_0$ the propagation speed.

Derivation of the wave equation

Part IV

Small changes in pressure, density and small particle velocity compared to propagation speed $\Rightarrow$ homogeneous wave equation

$$\nabla^2 p(\mathbf{x}, t) - \frac{1}{c^2} \partial_t^2 p(\mathbf{x}, t) = 0$$

Helmholtz equation

With the help of a Fourier transform we obtain the Helmholtz equation

$$\nabla^2 P(\mathbf{x}, \omega) + k^2 P(\mathbf{x}, \omega) = 0$$

with the acoustic wave number $k^2 = \frac{\omega^2}{c^2}$.

$\Rightarrow$ Formulation of the wave equation in the frequency domain

Solutions of the one-dimensional wave equation

Guess trigonometric functions as solution

$$p(x,t) = Ae^{\pm ia(x-ct)}$$

Insert into wave equation to check

$$\begin{aligned}\partial_x^2 p(x,t) - \frac{1}{c^2}\partial_t^2 p(x,t) &= (\pm ia)^2 Ae^{\pm ia(x-ct)} - \frac{1}{c^2}(\pm iac)^2 Ae^{\pm ia(x-ct)} \\ &= 0\end{aligned}$$

Solutions of the one-dimensional wave equation

Guess more generic function

$$p(x,t) = f(ct - x)$$

Insert into wave equation to check

$$\begin{aligned}\partial_x^2 p(x,t) - \frac{1}{c^2}\partial_t^2 p(x,t) &= f''(ct-x) - \frac{1}{c^2}c^2 f''(ct-x)\\ &= 0\end{aligned}$$

Solutions of the three-dimensional wave equation

D'Alembert solution of wave equation in Cartesian coordinates

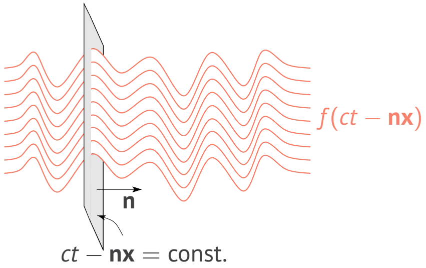
$$p(\mathbf{x}, t) = f(ct - \mathbf{n}\mathbf{x})$$

- f arbitrary, two-times differentiable function
- $\mathbf{n}$ unit vector with arbitrary direction
- $ct - \mathbf{n}\mathbf{x} = \text{const.}$ describes a plane in space at time t

Representation in frequency domain:

$$\begin{aligned} P(\mathbf{x}, \omega) &= \hat{P}(\omega)\mathrm{e}^{-i\frac{\omega}{c}\mathbf{n}\mathbf{x}} \\ &= \hat{P}(\omega)\mathrm{e}^{-i\mathbf{k}\mathbf{x}} \end{aligned}$$

with $\mathbf{k} = \frac{\omega}{c}\mathbf{n}$.



Wave field decomposition

An arbitrary wave field can be decomposed into orthogonal Eigen-solutions of the wave equation.

Eigen-solutions of the homogeneous wave equation

- Cartesian coordinates: plane waves
- Spherical coordinates: spherical harmonics
- Cylindrical coordinates: cylindrical harmonics

This can be useful for recording and reproducing sound fields.

The inhomogeneous wave equation

$$\nabla^2 p(\mathbf{x}, t) - \frac{1}{c^2}\partial_t^2 p(\mathbf{x}, t) = -q(\mathbf{x}, t)$$

- Homogeneous wave equation: $q(\mathbf{x}, t) = 0$
- Specific inhomogeneities can be interpreted as sources
- Point-type, line-type and planar inhomogeneities

Acoustic monopole sources

With $Q(\omega, t) = A(\omega)\,\delta\,(\mathbf{x} - \mathbf{x}_\mathrm{s})$ we obtain the sound field of a

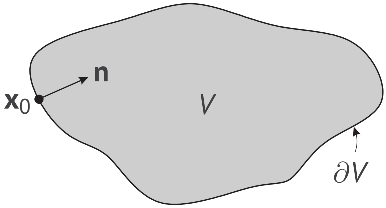
point source $$P(\mathbf{x}, \omega) = A(\omega)\frac{1}{4\pi}\frac{\mathrm{e}^{-\mathrm{i}\frac{\omega}{c}|\mathbf{x}-\mathbf{x}_\mathrm{s}|}}{|\mathbf{x} - \mathbf{x}_\mathrm{s}|}$$

With $Q(\omega, t) = A(\omega)\,\delta\,(x - x_\mathrm{s})\,\delta\,(y - y_\mathrm{s})$ we obtain the sound field of a

line source $$P(\mathbf{x}, \omega) = -A(\omega)\frac{\mathrm{i}}{4}H_0^{(2)}\left(\frac{\omega}{c}r\right),$$

with $r^2 = (x - x_\mathrm{s})^2 + (y - y_\mathrm{s})^2$, $H_0^{(2)}$ Hankel function of 2nd type, 0th order.

Boundary conditions



For a complete solution of the wave equation, boundary conditions are required:

- Dirichlet

$$P(\mathbf{x}_0, \omega) = f(\mathbf{x}_0, \omega)$$

- Neumann

$$\partial_{\mathbf{n}} P(\mathbf{x}_0, \omega) = f(\mathbf{x}_0, \omega)$$