Fourier transformation

Explains the mathematical concept of space, norm, and basis. Introduces Fourier transform and convolution.

WS 16/17
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Mathematical methods

Concepts of space and Fourier transform are connected and very important.

- Investigation of vibrating string, planet orbits
- Fourier transformation (1759-1805, Lagrange, Gauss, Fourier, Euler)
- Hilbert space (1900, Hilbert, Lebesgue, Quantum mechanics)

Euclidean space $\mathbb{R}^N$

$\mathbf{x}$ and $\mathbf{y}$ are vectors $(x_1, x_2, \ldots, x_N)$ and $(y_1, y_2, \ldots, y_N)$.

Scalar product:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{n=1}^{N} x_n \, y_n$$

Norm:

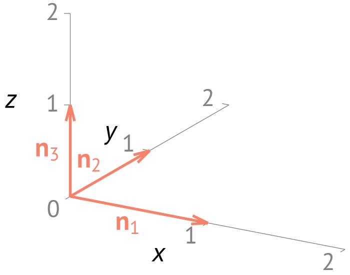
$$\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}} = \left(\sum_{n=1}^{N} x_n^2 \right)^{\frac{1}{2}}$$

Basis:

$$\mathbf{n}_1 = (1, 0, \ldots, 0), \mathbf{n}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{n}_n = (0, 0, \ldots, 1)$$

Euclidean space $\mathbb{R}^N$

Example: $\mathbb{R}^3$



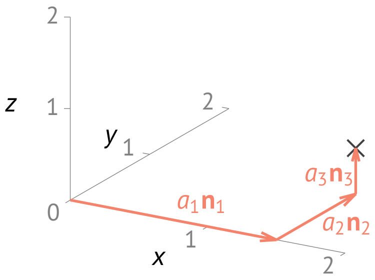
A space is spanned by its basis, here $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$.

$$\langle \mathbf{n}_i, \mathbf{n}_j \rangle = \delta_{ij}$$

$$\|\mathbf{n}_i\| = \|\mathbf{n}_j\| = 1$$

Euclidean space $\mathbb{R}^N$

Example: $\mathbb{R}^3$



Any point can be constructed by a linear combination of the basis.

$$\mathbf{x} = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3 = \sum_{n=1}^{3} a_n\mathbf{n}_n$$

$$a_n = \langle \mathbf{x}, \mathbf{n}_n \rangle$$

Complex space $\mathbb{C}$

$z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$ are complex numbers.

Scalar product:

$$\langle z_1, z_2 \rangle := z_1 z_2^* = (x_1 + \mathrm{i}y_1)(x_2 - \mathrm{i}y_2)$$

Norm:

$$\|z_1\| := \langle z_1, z_1 \rangle^{\frac{1}{2}} = (z_1 z_1^*)^{\frac{1}{2}}$$

Basis:

$$1, \mathrm{i}$$

Hilbert space L^2

$f(x)$ and $g(x)$ are square-integrable functions.

Scalar product:

$$\langle f, g \rangle := \int_a^b f(x)\, g(x)^*\, \mathrm{d}x$$

Norm:

$$\|f\|^2 := \langle f, f \rangle = \int_a^b |f(x)|^2\, \mathrm{d}x$$

Basis:

$$\mathrm{e}^{\mathrm{i}\omega x} = \cos(\omega x) + \mathrm{i}\sin(\omega x)$$

trigonometric functions

Fourier series

Every periodic signal can be approximated by a finite summation of basis functions.

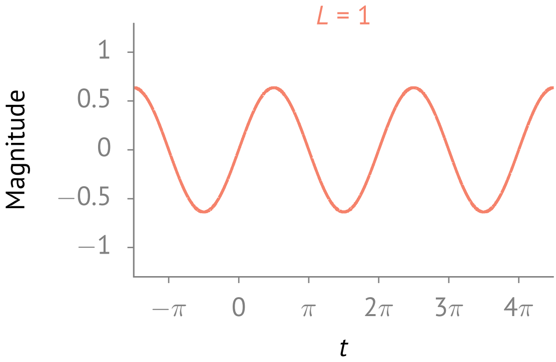
$$f(x) = \sum_{n=1}^{N} a_n \mathrm{e}^{\mathrm{i}n\omega x}$$

$$a_n(\omega) = \left\langle f(x), \mathrm{e}^{-\mathrm{i}n\omega x} \right\rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \mathrm{e}^{-\mathrm{i}n\omega x} \, \mathrm{d}x$$

Fourier series

Example 1: sawtooth wave

$$\text{sawtooth}(t) = \sum_{n=1}^{N} \frac{2\sin(nt)}{n\pi}$$



Fourier series

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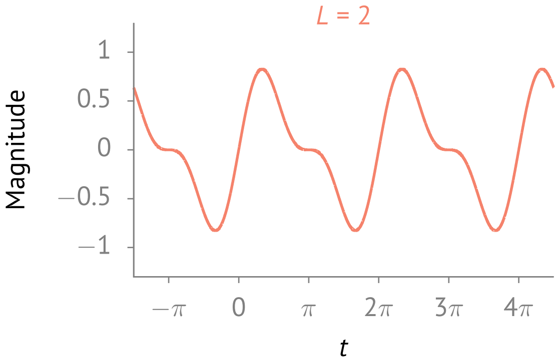

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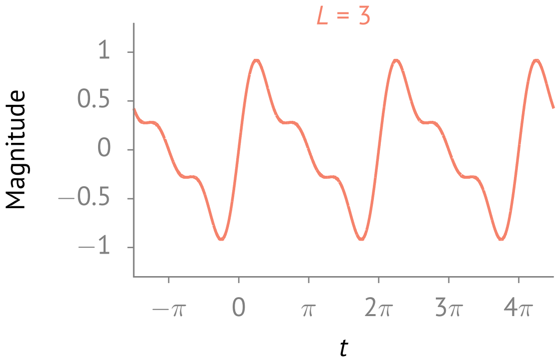

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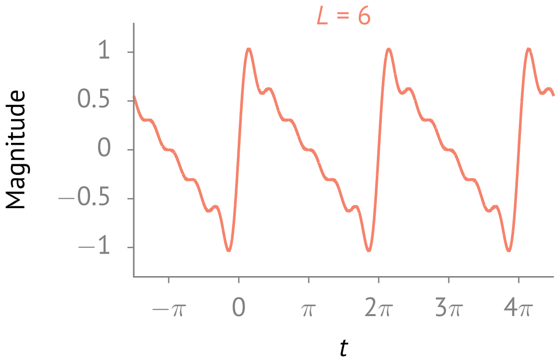

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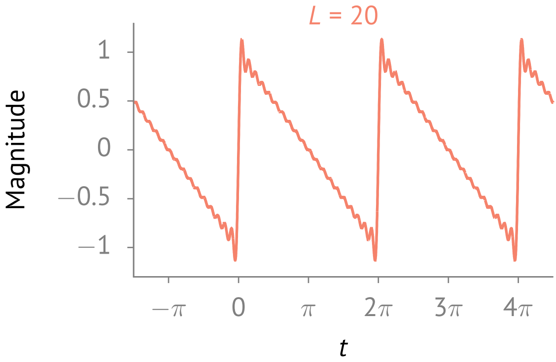

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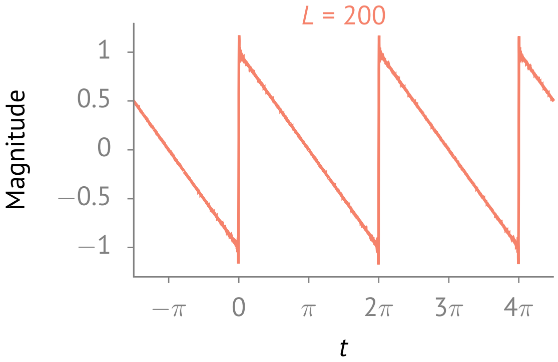
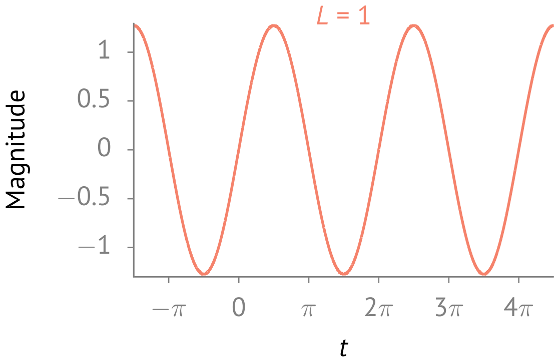


Fourier series

Example 2: square wave

$$\text{square}(t) = \sum_{n=1}^{N} \frac{4\sin((2n-1)t)}{(2n-1)\pi}$$



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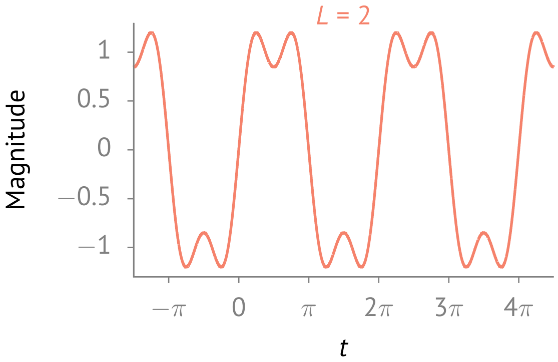

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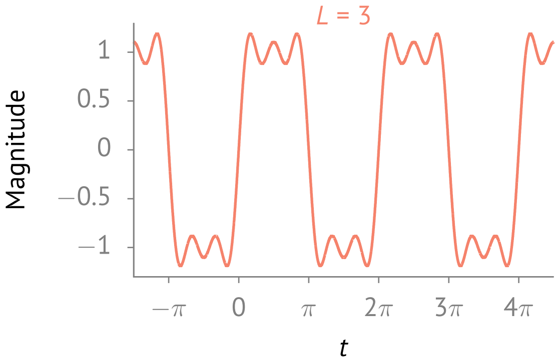

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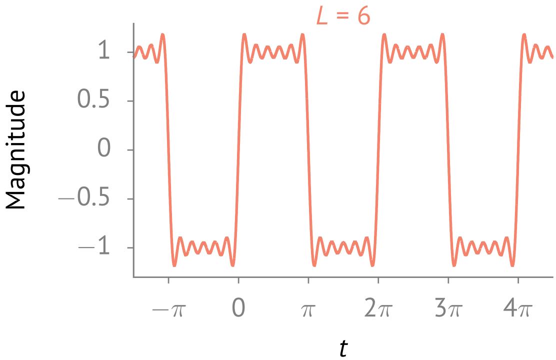

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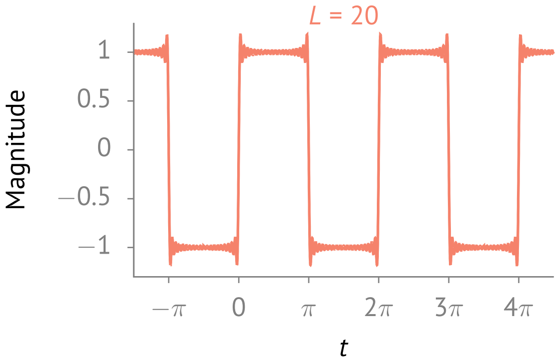

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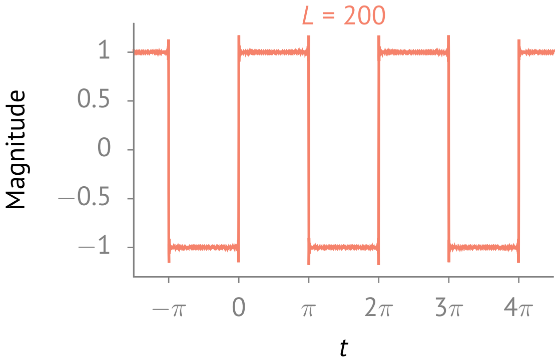
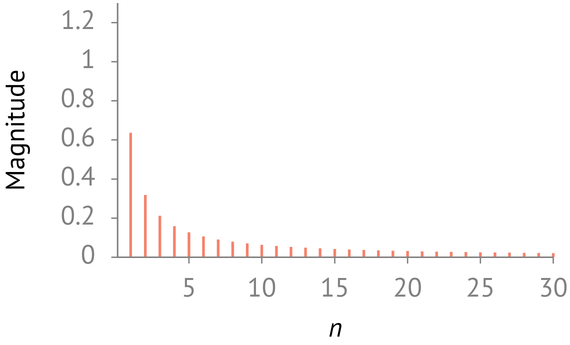


Fourier series

Example 1: sawtooth wave

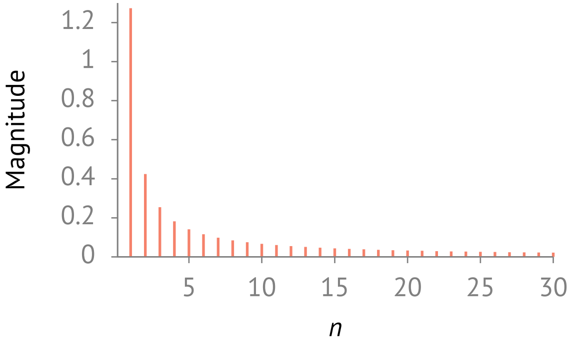
$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \mathrm{sawtooth}(t)\, \mathrm{e}^{-\mathrm{i}nt}\, \mathrm{d}t$$



Fourier series

Example 2: square wave

$$a_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \mathrm{square}(t)\, e^{-int}\, dt$$



Fourier transformation

Temporal Fourier transformation:

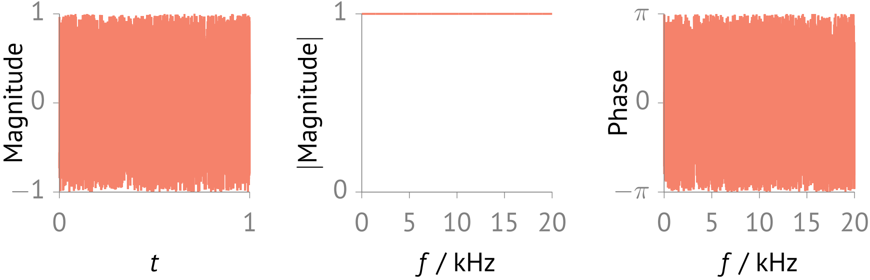
$$S(\omega) = \mathcal{F}\left\{s(t)\right\} := \int_{-\infty}^{\infty} s(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$$

Inverse temporal Fourier transformation:

$$s(t) = \mathcal{F}^{-1}\left\{S(\omega)\right\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega$$

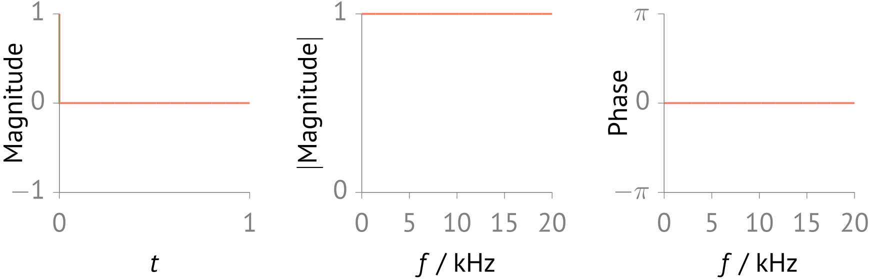
Fourier transformation

White noise



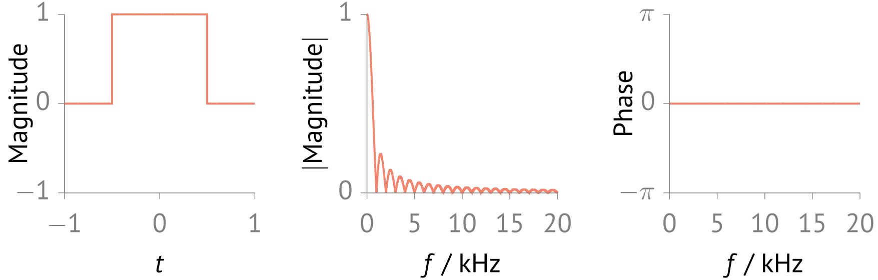
Fourier transformation

Dirac function



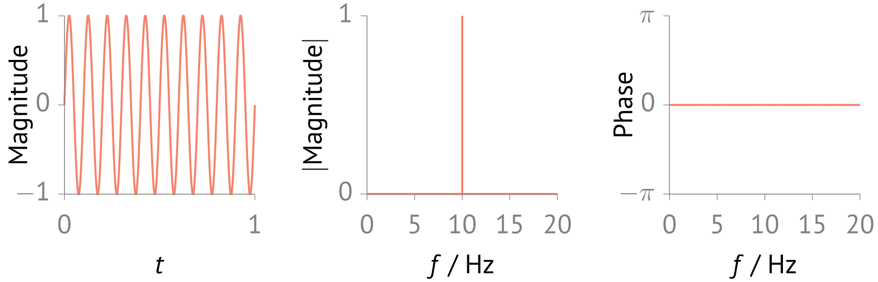
Fourier transformation

Rectangular function



Fourier transformation

Sinusoid



Convolution theorem

Convolution:

$$(f * g)(t) := f(t) * g(t) := \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau$$

Convolution theorem:

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\} \cdot \mathcal{F}\{g\}$$

Take home

- Every acoustical signal is a combination of sinusoid tones with different frequency, amplitude and phase
- Fourier transformation is a way to characterize time signals