Exercise 10.2:22. Psychological studies of stimulus and response often attempt to treat these as numerical variables, s and r related by an equation $r = f(s)$. It is sometimes hypothesized that f satisfies a differential equation of the form

$$\frac{dr}{ds} = k\frac{r^n}{s} \quad \text{with } k > 0$$

Which of the two hypotheses on the exponent n, $n = 0$ or $n = 1$ is consistent with the following values of (r, s):

$$\{(0.5, 1), (1, 2), (3, 6)\}$$

Solution. By seperation of variables we have

$$\int \frac{1}{r^n} dr = \int \frac{k}{s} ds$$

so if $n = 1$ we have $\log(r) = \log(s^k) + c$. Therefore

$$r = K \cdot s^k$$

Suppose $\{(0.5, 1), (1, 2), (3, 6)\}$ are all on the graph of some solution, then since $(r, s) = (0.5, 1)$ is on the graph $K = 0.5$. Since $(1, 2)$ is on the graph then $2 = 2^k$ so $k = 1$. Since $6 \cdot 0.5 = 3$ then it follows that $(3, 6)$ is also on the graph and thus $n = 1$ is consistent with the data.

If instead $n = 0$ then $r = \log(s^k) + c$. Suppose $\{(0.5, 1), (1, 2), (3, 6)\}$ are all on the graph of some solution. Since $(r, s) = (0.5, 1)$ is on the graph then $c = 0.5$. Since $(r, s) = (1, 2)$ is on the graph then $k = \frac{1}{2\log(2)}$. Now

$$\frac{\log(6)}{2\log(2)} + \frac{1}{2} = \frac{\log(3) + 1}{2} \neq 3$$

So $n = 0$ is not consistent with the data. $\square$

Exercise 10.2:26. Show that the differential equation

$$\frac{dy}{dx} = y + x$$

cannot be written in the form

$$g(y)\frac{dy}{dx} = f(x)$$

and therefore cannot be solved by separation of variables

Solution. Suppose $\frac{dy}{dx} = x + y = \frac{f(x)}{g(y)}$ and assume without loss of generality that $x, y > 0$ then

$$\frac{\partial}{\partial y}\left(\frac{\partial \log(f(x)) - \log(g(y))}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{f'(x)}{f(x)}\right) = 0$$

However

$$\frac{\partial}{\partial y}\left(\frac{\partial \log(x + y)}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x + y}\right) = -\frac{1}{(x + y)^2} \neq 0$$

$\square$

Exercise 10.2:28. (a) Let $F(x,y)$ be a homogeneous function of degree zero, then the substitution $y = xu$ transforms the differential equation

$$\frac{dy}{dx} = F(x,y)$$

into

$$\frac{du}{dx} = \frac{F(1,u) - u}{x}$$

(b) Show that $F(x,y) = \frac{(x^2+y^2)}{2xy}$ is homogeneous and use the subsitution of part (a) to change the equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

into an equation of the form

$$\frac{du}{dx} = G(x,u)$$

(c) Solve the last differential equation of part (b), and substitute u with $\frac{y}{x}$ in the resulting solution.

Solution. (a) let $y = x \cdot u(x)$ then

$$F(x,y) = \frac{dy}{dx} = x\frac{du}{dx} + u(x)$$

since F is homegeneous of degree zero then $F(x,y) = F(x,xu) = F(1,u)$ so

$$\frac{du}{dx} = \frac{F(1,u) - u}{x}$$

(b) Computing

$$F(tx,ty) = \frac{(tx)^2 + (ty)^2}{2txty} = \frac{t^2(x^2+y^2)}{2t^2xy} == \frac{x^2+y^2}{2xy}$$

so F is homogeneous of degree zero. So by part (a)

$$\frac{du}{dx} = \frac{\frac{1^2+u^2}{2u} - u}{x} = \frac{1-u^2}{2ux}$$

so by separation of variables

$$\int \frac{2u}{1-u^2}du = \int \frac{1}{x}dx$$

let $v = 1 - u^2$ then $dv = -2udu$ so

$$\int \frac{1}{v}dv = -\int \frac{1}{x}dx$$

hence

$$\log(1-u^2) = \log(v) = \log\left(\frac{1}{x}\right) + c$$

so applying the exponential function on both sides we have

$$1 - u^2 = K \cdot \frac{1}{x}$$

hence

$$u(x) = \sqrt{1 - \frac{K}{x}}$$

Therefore

$$y(x) = x\sqrt{1 - \frac{K}{x}}$$

We verify first note that since $y = x\sqrt{1 - \frac{K}{x}}$ then $\frac{y^2}{x^2} = 1 - \frac{K}{x}$ so

$$K = x - \frac{y^2}{x} = \frac{x^2 - y^2}{x^2}$$

therefore we can verify as follows

$$\begin{aligned}
\frac{dy}{dx} &= \sqrt{1 - \frac{K}{x}} + \frac{x}{2\sqrt{1 - \frac{K}{x}}} \cdot \frac{K}{x^2} \\
&= \frac{y}{x} + \frac{K}{2y} \\
&= \frac{y}{x} + \frac{x^2 - y^2}{2xy} \\
&= \frac{2y^2}{2xy} + \frac{x^2 - y^2}{2xy} \\
&= \frac{y^2 + x^2}{2xy}
\end{aligned}$$

□

Exercise 10.3:6. In Exercises 6 and 7 find the general solution to the differential equation, then find a paticular solution satisfying the initial condition.

$$\frac{dy}{dx} = y + 1, \quad y(0) = 1$$

Solution. By seperation of variables

$$\int \frac{1}{1+y} dy = \int 1 dx$$

so $\log(1 + y) = x + c$ hence $1 + y = e^{x+c}$ so the general solution is given by

$$y(x) = Ke^x - 1$$

If $y(0) = 1$ then $K - 1 = 1$ so $K = 2$ hence the particular solution is

$$y(x) = 2e^x - 1$$

□

Exercise 10.3:7.

$$2\frac{dy}{dx} = xy \qquad y(1) = 0$$

Solution. By separation of variables

$$\int \frac{1}{y} dy = \int \frac{1}{2} x dx$$

so $\log(y) = \frac{x^2}{4} + c$ for some constant c, therefore

$$y(x) = e^{\frac{x^2}{4}+c} = Ke^{\frac{x^2}{4}}$$

with $K = e^c$. If $y(1) = 0$ then $K = 0$, thus the specific solution becomes $y(x) \equiv 0$. □

Exercise 10.3:9. Salt solution enters a 100-gallon tank of initially pure water from two different sources. One source provides water containing 1 pound of salt per gallon at a rate of 2 gallons per minute. A second source provides 3 gallons of salt solution per minute at a varying concentration $C(t) = 2e^{-2t}$, measured in pounds of salt per gallon. Assume that the contents of the tank are kept thoroughly mixed at all times and that solution is drawn off at a rate of 5 gallons per minute. Find the amount of salt in the tank at an arbitrary time $t > 0$.

Solution. First we set up the differential equation

$$\frac{ds}{dt} = 2 + 3 \cdot C(t) - \frac{s}{20} = 2 + 6e^{-2t} - \frac{s}{20}$$

Rewriting this into the form $s' + \frac{1}{20}s = 2 + 6e^{-2t}$ and using the panzer formula

$$\begin{aligned} s(t) =& e^{\frac{-t}{20}} \left(\int e^{\frac{t}{20}} \left(2 + 6e^{-2t}\right) dt \right) \\ =& e^{\frac{-t}{20}} \left(\int 2e^{\frac{t}{20}} dt + \int 6e^{\frac{-39t}{20}} dt \right) \\ =& e^{\frac{-t}{20}} \left(\frac{2}{20} e^{\frac{t}{20}} - \frac{49 \cdot 6}{25} e^{\frac{-39t}{20}} + c \right) \\ =& \frac{2}{20} - \frac{39 \cdot 6}{20} e^{-2t} + c \end{aligned}$$

Since $s(0) = 0$ then

$$c = \frac{39 \cdot 6}{20} - \frac{2}{20}$$

so

$$s(t) = \frac{234}{20} \left(1 - e^{-2t}\right)$$

where is is the number amount of salt in the water measured in pounds. □

Exercise 10.3:16. Suppose that a metal bar initially at 300 F is immersed in a water bath at 100 F for 30 minutes and then is transferred to another water bath at 50 F. Assume the validity of Newton's law described in Example 5 of the text. (a) What will the temperature of the bar be after an additional 30 minutes, assuming the cooling coefficientfor the iron in water is $k = 0.1$? (b) Suppose that initially the bar is cooled for 30 minutes in air at 100, for which the cooling coefficient is only $k = 0.07$ and is then immersed in water for 30 minutes. What will the temperature of the bar be at the end of the hour?

Solution. (a) By Newtons law of cooling

$$\frac{du}{dt} = k(100 - u)$$

By separation of variables

$$\int \frac{1}{u - 100} du = \int -k dt$$

As long as $u > 100$ the right hand side is positive so $\log(u - 100) = -kt + c$ hence

$$u(t) = 100 + Ce^{-kt}$$

Since the initial temperature is 300 F, then $u(0) = 300$ hence $C = 200$. Letting $k = 0.1$ we have

$$u(30) = 100 + 200 \cdot e^{-3}$$

So after 30 minutes the temperature is $100 + 200 \cdot e^{-3}$ which is greater than 50. Let $\tilde{u}$ be a solution to $\frac{d\tilde{u}}{dt} = k(50 - u)$ with $\tilde{u}(0) = 100 + 200 \cdot e^{-3}$. Solving using separation of variables we find

$$\tilde{u}(t) = 50 + Ce^{-kt}$$

Since $\tilde{u}(0) = 100 + 200 \cdot e^{-3}$ then $C = 50 + 200 \cdot e^{-3}$ so after 1 hour the temperature would be

$$\tilde{u}(30) = 50 + \left(50 + 200 \cdot e^{-3}\right) \cdot e^{-3} = 50 + 50 \cdot e^{-3} + 200 \cdot e^{-6} \approx 53$$

(b) After 30 minutes we have $u(30) = 100 + 200 \cdot e^{-0.07 \cdot 30}$ so after 1 hour the temperature would be

$$\tilde{u}(30) = 50 + \left(50 + 200 \cdot e^{-0.07 \cdot 30}\right) \cdot e^{-3} \approx 54$$

□

Exercise 3.1:2. Exercises 2 and 4 give information about a linear function f. In each case find the matrix A that represents f in the form $f(x) = Ax$ and determine whether the functions one-to-one.

$$f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

Solution.

$$A = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$$

Since $(2, 1)$ and $(1, 1)$ is obviously linearly independent, then A is injective. □

Exercise 3.1:4.

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \qquad f\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

Solution.

$$A = \begin{pmatrix}-1 & 1 & 2\\1 & -1 & 1\end{pmatrix}$$

Since $-(1, -1) = (-1, 1)$ then $(-1, 1)$ and $(1, -1)$ are linearly dependent, so A is not injective. $(f(1, -1, 0) = 0 = f(0, 0, 0))$ □

Exercise 3.1:6. Exercises 6 and 8 give information about linear functions f. In each case find $f(\vec{e}_k)$.

$$f\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix} \qquad f\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$$

Solution.

$$f\begin{pmatrix}1\\0\end{pmatrix}=f\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\1\end{pmatrix}$$
$$f\begin{pmatrix}0\\1\end{pmatrix}=f\left(\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix}+\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}\frac{3}{2}\\0\end{pmatrix}$$

□

Exercise 3.1:8.

$$f\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad f\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix} \qquad f\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

Solution.

$$f\begin{pmatrix}1\\0\\0\end{pmatrix}=f\left(\begin{pmatrix}2\\1\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\1\end{pmatrix}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix}$$
$$f\begin{pmatrix}0\\1\\0\end{pmatrix}=f\left(\begin{pmatrix}2\\1\\0\end{pmatrix}-2\begin{pmatrix}1\\0\\0\end{pmatrix}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix}-2\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-5\\0\\-1\end{pmatrix}$$
$$f\begin{pmatrix}0\\0\\1\end{pmatrix}=f\left(\begin{pmatrix}0\\1\\1\end{pmatrix}-\begin{pmatrix}0\\1\\0\end{pmatrix}\right)=\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}-5\\0\\-1\end{pmatrix}=\begin{pmatrix}7\\0\\1\end{pmatrix}$$

□

Exercise 3.1:12. Find the matrix that represents the composition $g\circ f$. Also, say what the domain and range of $g\circ f$ are.

$$f(\vec{x})=\begin{pmatrix}1&2&0\\2&2&3\end{pmatrix}\vec{x} \qquad \text{and} \qquad g(\vec{x})=\begin{pmatrix}2&0\\3&-1\end{pmatrix}\vec{x}$$

Solution.

$$g\circ f(\vec{x})=\begin{pmatrix}2&4&0\\1&4&-3\end{pmatrix}\vec{x}$$

The domain is $\mathbb{R}^3$ and the image is $\mathbb{R}^2$. □

Exercise 3.1:17. (a) Show that

$$U=\begin{pmatrix}1&0&0\\0&0&-1\\0&1&0\end{pmatrix} \qquad \text{and} \qquad V=\begin{pmatrix}0&0&1\\0&1&0\\-1&0&0\end{pmatrix}$$

represents 90 degrees rotation of $\mathbb{R}^3$ about the x_1-axis and x_2-axis respectively. Find the matrix W that represents a 90 degree rotation about the x_3-axis. Also find U^{-1} and V^{-1} which represent rotations in the opposite direction. (b) Compute UVU^{-1} and VUV^{-1} and interpret the results geometrically by checking out what they do to basis vectors.

Solution. (a) First we compute

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_3 \\ x_2 \end{pmatrix}$$

Since U fixes the first coordinate, this is some linear transformation around the x_1-axis. Let θ denote the angle between $\vec{x}$ and $U\vec{x}$ measured in a plane parallel to x_2, x_3 plane. Then

$$\cos(\theta) \cdot (x_2^2 + x_3^2)^2 = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \bullet \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix} = 0$$

so U is a 90 degree rotation around the x_1 axis. For V we likewise compute

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_2 \\ -x_1 \end{pmatrix}$$

Since V fixes the second coordinate, this is a linear transformation around the x_2-axis. Let θ denote the angle between $\vec{x}$ and $V\vec{x}$ measured in a plane parallel to x_3, x_1 plane. Then

$$\cos(\theta) \cdot (x_3^2 + x_1^2)^2 = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} \bullet \begin{pmatrix} x_3 \\ -x_1 \end{pmatrix} = 0$$

so V is a 90 degree rotation around the x_2 axis. Let

$$W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ x_3 \end{pmatrix}$$

Since W fixes the third coordinate, this is a linear transformation around the x_3-axis. Let θ denote the angle between $\vec{x}$ and $W\vec{x}$ measured in a plane parallel to x_1, x_2 plane. Then

$$\cos(\theta) \cdot (x_1^2 + x_2^2)^2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = 0$$

so W represents a 90 degree rotation around the x_3-axis.

$$U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{and} \qquad V^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(b)

$$UVU^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = W \qquad \text{and} \qquad VUV^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = W^{-1}$$

The geometric meaning is that first rotating 90 degrees clockwise around the x_1 axis, then rotating 90 degrees counter clockwise around the x_2 axis and then rotating 90 degrees counterclockwise around the x_1 axis is the same rotating 90 degrees around the x_3 axis. $\square$

Exercise 3.1:19. Let $\vec{n}$ be the unit vector $\left(\frac{3}{7}, \frac{6}{7}, \frac{2}{6}\right)$, and let $P_{\vec{n}} : \mathbb{R}^3 \to \mathbb{R}^3$ be the associated projection function as in Example 6. Find the matrix of $P_{\vec{n}}$ by finding the image of each of the standard basis vectors under it.

Solution. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ denote the standard basis vectors, then

$$P_{\vec{n}}(\vec{e}_1) = (\vec{e}_1 \bullet \vec{n}) \cdot \vec{n} = \frac{3}{7} \cdot \vec{n} = \begin{pmatrix} \frac{9}{49} \\ \frac{18}{49} \\ \frac{6}{49} \end{pmatrix}$$

$$P_{\vec{n}}(\vec{e}_2) = (\vec{e}_2 \bullet \vec{n}) \cdot \vec{n} = \frac{6}{7} \cdot \vec{n} = \begin{pmatrix} \frac{18}{49} \\ \frac{36}{49} \\ \frac{12}{49} \end{pmatrix}$$

$$P_{\vec{n}}(\vec{e}_3) = (\vec{e}_3 \bullet \vec{n}) \cdot \vec{n} = \frac{2}{7} \cdot \vec{n} = \begin{pmatrix} \frac{6}{49} \\ \frac{12}{49} \\ \frac{4}{49} \end{pmatrix}$$

So the matrix of $P_{\vec{n}}$ is given by

$$A = \begin{pmatrix} | & | & | \\ P_{\vec{n}}(\vec{e}_1) & P_{\vec{n}}(\vec{e}_2) & P_{\vec{n}}(\vec{e}_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} \frac{9}{49} & \frac{18}{49} & \frac{6}{49} \\ \frac{18}{49} & \frac{36}{49} & \frac{12}{49} \\ \frac{6}{49} & \frac{12}{49} & \frac{4}{49} \end{pmatrix}$$

$\square$