1 Composite Rotations

Two frames A and B are initially coincident. Frame B then undergoes the following sequence of transformations:

$$ {}^{A}_{B}R = [R_x(\frac{\pi}{3})[[R_x(\frac{\pi}{2})R_y(\frac{\pi}{4})]R_z(\frac{\pi}{6})]]R_y(\frac{\pi}{3}) $$

2 Transformations Matrices

Two frames A and B are initially coincident. Frame B then undergoes the following transformations:
Write the transformation matrices ${}^{A}_{B}T$ and ${}^{B}_{A}T$

$$ {}^{A}_{B}T = T(z,\frac{\pi}{2}) * T(0,3,0) * T(x,\frac{\pi}{2}) $$

$$ {}^{A}_{B}T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}_{B}T = \begin{bmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

To get ${}^{B}_{A}T$

$$ {}^{B}_{A}T = \begin{bmatrix} 0 & 1 & 0 & -(0+0+0) \\ 0 & 0 & 1 & -(0+0+0) \\ 1 & 0 & 0 & -(-3+0+0) \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{A}T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

3 Quaternions to Rotations

Let $q = a + bi + cj + dk$ be a unit quaternion. In the lecture notes it is stated that its associated rotation matrix is

$$R = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}$$

Claim. Show that **R** is a rotation matrix.

Proof.
(a) each of its column has length 1;

This is the first column

$$q_1 = \sqrt{(2(a^2+b^2)-1)^2 + (2(bc+ad))^2 + (2(bd-ac))^2}$$

using symbolab, I got the result

$$\begin{aligned} \mid q_1 \mid &= \sqrt{4a^4 + 8a^2b^2 + 4a^2c^2 + 4a^2d^2 - 4a^2 + 4b^4 - 4b^2 + 4b^2c^2 + 4b^2d^2 + 1}, simplify \\ &= \sqrt{4a^2(a^2 + 2b^2 + d^2 - 1) + 4b^2(b^2 - 1 + c^2 + d^2) + 1} \\ &= \sqrt{4a^2(-b^2) + 4b^2(a^2) + 1} \\ &= \sqrt{1} = 1 \end{aligned}$$

check the other columns

$$\begin{aligned} \mid q_2 \mid &= \sqrt{(2(bc-ad))^2 + (2(a^2+c^2)-1)^2 + (2(cd+ab))^2} \\ &= \sqrt{4c^4 + 4b^2c^2 + 8c^2a^2 + 4c^2d^2 - 4c^2 + 4a^4 - 4a^2 + 4b^2a^2 + 4a^2d^2 + 1}, simplify \\ &= \sqrt{4c^2(c^2 + b^2 + 2a^2 + d^2 - 1) + 4a^2(a^2 - 1 + b^2 + d^2) + 1} \\ &= \sqrt{4c^2(-a^2) + 4a^2(c^2) + 1} \\ &= \sqrt{1} = 1 \end{aligned}$$

$$\begin{aligned} \mid q_3 \mid &= \sqrt{(2(bd+ac))^2 + (2(cd-ab))^2 + (2(a^2+d^2)-1)^2} \\ &= \sqrt{4d^4 + 4b^2d^2 + 8d^2a^2 + 4d^2c^2 - 4d^2 + 4a^4 - 4a^2 + 4b^2a^2 + 4a^2c^2 + 1}, simplify \\ &= \sqrt{4d^2(d^2 + b^2 + 2a^2 + c^2 - 1) + 4a^2(a^2 - 1 + b^2 + c^2) + 1} \\ &= \sqrt{4d^2(-a^2) + 4a^2(d^2) + 1} \\ &= \sqrt{1} = 1 \end{aligned}$$

(b) its columns are mutually orthogonal;

where $q_0 * q_n = 0$ means that the column are orthogonal where $\mathbf{q}$ = the column

$$\begin{aligned} q_1 * q_2 &= (2(a^2+b^2)-1)(2(bc-ad)) + (2(bc+ad))(2(a^2+c^2)-1) + (2(bd-ac))(2(cd+ab)) \\ &= 4b^3c - 4bc + 4bc^3 + 4a^2bc + 4bcd^2 \\ &= 4b(b^2c - c + c^3 + a^2c + cd^2) \\ &= 4bc(b^2 - 1 + c^2 + a^2 + d^2) = 4bc(1-1) \\ &= 4bc(0) = 0 \end{aligned}$$

$$\begin{aligned} q_1 * q_3 &= (2(a^2+b^2)-1)(2(bd+ac)) + (2(bc+ad))(2(cd-ab)) + (2(bd-ac))(2(a^2+d^2)-1) \\ &= 4b^3d - 4bd + 4bd^3 + 4a^2bd + 4bdc^2 \\ &= 4b(b^2d - d + d^3 + a^2d + dc^2) \\ &= 4bd(b^2 - 1 + d^2 + a^2 + c^2) = 4bd(1-1) \\ &= 4bd(0) = 0 \end{aligned}$$

$$\begin{aligned} q_2 * q_3 &= (2(bc-ad))(2(bd+ac)) + (2(a^2+c^2)-1)(2(cd-ab)) + (2(cd+ab))(2(a^2+d^2)-1) \\ &= 4b^3a - 4ba + 4ba^3 + 4d^2ba + 4bac^2 \\ &= 4b(b^2a - a + a^3 + d^2a + ac^2) \\ &= 4ba(b^2 - 1 + a^2 + d^2 + c^2) = 4ba(1-1) \\ &= 4ba(0) = 0 \end{aligned}$$

(c) its determinant is 1.

$$\begin{aligned} det(R) = [2(a^2+b^2)-1] \cdot det &\begin{bmatrix} 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix} \\ -[2(bc-ad)] \cdot det &\begin{bmatrix} 2(bc+ad) & 2(cd-ab) \\ 2(bd-ac) & 2(a^2+d^2)-1 \end{bmatrix} \\ +[2(bd+ac)] \cdot det &\begin{bmatrix} 2(bc+ad) & 2(a^2+c^2)-1 \\ 2(bd-ac) & 2(cd+ab) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= 8a^6 + 16a^4b^2 + 16a^4c^2 + 16a^4d^2 - 12a^4 + 8a^2b^4 \\ &- 12a^2b^2 + 8a^2c^4 + 16a^2b^2c^2 - 12a^2c^2 + 8a^2d^4 + 16a^2b^2d^2 \\ &- 12a^2d^2 + 16a^2c^2d^2 + 6a^2 + 2b^2 + 2c^2 + 2d^2 - 1 \\ &= 16a^4(1-a^2) - 12a^2(1-a^2) + 8a^6 + 2(1-a^2) + 8a^2(1-a^4) - 16a^2 + 12a^4 + 6a^2 - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

4 Change of Coordinates

For each of the required points, if the answer is positive, show how it can be computed, and if the answer is negative explain why it cannot be computed.

$$
\begin{aligned}
{}^{B}p &= {}^{B}_{A}T({}^{A}p) \\
{}^{C}p &= ({}^{C}_{W}T)({}^{W}_{B}T)({}^{B}_{A}T)({}^{A}p) \\
{}^{W}p &= ({}^{W}_{B}T)({}^{B}_{A}T)({}^{A}p)
\end{aligned}
$$

5 Quarternions

$$p = 1 + 2i - 3k$$

$$q = 5 + 4j + 2k$$

1. the product of **pq**

$$
\begin{aligned}
pq &= (1 + 2i - 3k)(5 + 4j + 2k) \\
&= 5 + 4j - 13k + 10i + 8ij + 4ik - 12jk - 6k^2 \\
&= 5 + 4j - 13k + 10i + 8k - 4j - 12i + 6 \\
&= 11 - 2i - 5k
\end{aligned}
$$

2. the norm of the product **pq**

$$
\begin{aligned}
&pq = 11 - 2i - 5k \\
&norm = \sqrt{11^2 + (-2)^2 + (-5)^2} = \sqrt{150}
\end{aligned}
$$