Time-Harmonic Line-Force Acting on the Surface of a Semi-Infinite Elastic Half-Space.

Finley Boulton

University of Liverpool
Thesis Presentation for MATH499

12/5/2022

Background.

Dispersion of waves in (and on) solids was famously considered in [Lamb(1904)] where he obtains surface displacements generated by surface point and line sources acting on an elastic half spaces.

Background.

Much work has subsequently been done by many on the broad class of problems involving waves in half spaces from surface and buried sources. Such as

- [Heelan(1953)]
- [Miller and Pursey(1954)]
- [Graff(1991)]

Introduction - The problem.

Consider a linearly elastic half space where

- x, z-plane is the surface,
- x, y-plane is vertical plane with $y > 0$ pointing down into the solid.

We apply a time-harmonic line load normal to the surface from $-a < x < a$, and with invariance with respect to z. And thus, we have plane strain $u_z = \partial/\partial z = 0$.

Introduction - The plan.

We wish to solve for the displacement equations of motion for u_x, u_y.

- Consider the governing equations of the system,
- Obtain exact solutions in terms of definite integrals using Fourier transforms,
- Asymptotically approximate said integrals in the far field for case of interior waves,
- The special case of surface waves will be considered separately,
- Compare approximated results to results obtained using finite element analysis.

Governing Equations - Displacement.

We start with (from the Linear Momentum Balance Law)

$$\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla\left(\nabla \cdot \mathbf{u}\right) + \rho\mathbf{F} = \rho\ddot{\mathbf{u}}, \tag{1}$$

where λ and μ are the Lamé elastic constants, ∇^2 is Laplace's operator, and ρ is density.

We note the dilatation Δ and rigid rotation W under conditions of plane strain

$$\Delta = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}, \tag{2}$$

$$W\mathbf{k} = \nabla \times \mathbf{u} = \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_x}{\partial y}\right)\mathbf{k}. \tag{3}$$

Governing Equation - Wave Equations

Using the dilatation Δ and rigid rotation W We derive the two equations

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial x} + \mu \frac{\partial W}{\partial y} + \rho\omega^2 u_x = 0, \tag{4}$$

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial y} - \mu \frac{\partial W}{\partial x} + \rho\omega^2 u_y = 0. \tag{5}$$

From here we can isolate the wave equations

$$\nabla^2 \Delta + k_1^2 \Delta = 0, \tag{6}$$

$$\nabla^2 W + k_2^2 W = 0. \tag{7}$$

Governing Equations - Stresses.

Using a similar procedure and with Cauchy's stress tensor we see

$$\sigma_{yy} = \frac{\mu^2}{\rho\omega^2}\left[2\frac{\partial^2 W}{\partial x \partial y} - k^4\frac{\partial^2 \Delta}{\partial y^2} - k^2\left(k^2 - 2\right)\frac{\partial^2 \Delta}{\partial x^2}\right], \tag{8}$$

$$\sigma_{xy} = \frac{\mu^2}{\rho\omega^2}\left(\frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial y^2} - 2k^2\frac{\partial^2 \Delta}{\partial x \partial y}\right), \tag{9}$$

$$\sigma_{yz} = 0. \tag{10}$$

Where $k^2 = (\lambda + 2\mu)/\mu = k_2^2/k_1^2$.

Governing Equations - Boundary Conditions.

The boundary conditions for the problem result from force being applied normal to the surface and being of unit magnitude in the region $|x| < a$. They are (omitting time variation),

$$\sigma_{yy}(x,0) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}, \quad \sigma_{xy}(x,0) = 0. \tag{11}$$

If the forcing were to be non-normal to the surface then we would see $\sigma_{xy}(x,0) \neq 0$.

Fourier Transform.

We use the standard Fourier transform on the spatial variable x.

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \quad f(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(x) e^{i\xi x} d\xi. \tag{12}$$

Fourier Transform - Displacements & Wave equations.

On the displacements (2),(3)

$$\bar{u}_x = -\frac{1}{k_2^2}\left[\frac{d\bar{W}}{dy} + i\xi k^2\bar{\Delta}\right], \tag{13}$$

$$\bar{u}_y = -\frac{1}{k_2^2}\left[k^2\frac{d\bar{\Delta}}{dy} - i\xi\bar{W}\right]. \tag{14}$$

On the wave equations (4),(5)

$$\frac{d^2\bar{\Delta}}{dy^2} - (\xi^2 + k_1^2)\bar{\Delta} = 0, \tag{15}$$

$$\frac{d^2\bar{W}}{dy^2} - (\xi^2 + k_2^2)\bar{W} = 0. \tag{16}$$

Fourier Transform - Stresses & Boundary Conditions.

On the stresses (8),(9)

$$\bar{\sigma}_{yy} = \frac{\mu^2}{\rho\omega^2}\left[2i\xi\frac{d\bar{W}}{dy} - k^4\frac{d\bar{\Delta}}{dy^2} + k^2\left(k^2-2\right)\xi^2\bar{\Delta}\right], \tag{17}$$

$$\bar{\sigma}_{xy} = -\frac{\mu^2}{\rho\omega^2}\left(\xi^2\bar{W} + \frac{d^2\bar{W}}{dy^2} + 2i\xi k^2\frac{d\bar{\Delta}}{dy}\right). \tag{18}$$

On the boundary conditions (11)

$$\bar{\sigma}_{yy}(\xi,0) = \frac{2\sin\xi a}{\xi}, \quad \bar{\sigma}_{xy}(\xi,0) = 0. \tag{19}$$

Fourier Transform - Intermediate Steps.

Solve (15),(16)

$$\bar{\Delta} = Ae^{-(\xi^2 - k_1^2)^{\frac{1}{2}} y}, \tag{20}$$

$$\bar{W} = Be^{-(\xi^2 - k_2^2)^{\frac{1}{2}} y}. \tag{21}$$

Substitute into the transformed stresses (17), (18) and evaluate on the transformed boundary (19) to obtain A, B, and hence

$$\bar{\Delta} = \frac{2\rho\omega^2(k_2^2 - 2\xi^2)}{\mu^2 k^2 \xi F(\xi)} \sin(\xi a) e^{-(\xi^2 - k_1^2)^{1/2} y}, \tag{22}$$

$$\bar{W} = -\frac{4i(\xi^2 - k_1^2)^{1/2} \rho\omega^2}{\mu^2 F(\xi)} \sin(\xi a) e^{-(\xi^2 - k_2^2)^{1/2} y}. \tag{23}$$

With $F(\xi) = \left(2\xi^2 - k_2^2\right)^2 - 4\xi^2(\xi^2 - k_1^2)^{1/2}(\xi^2 - k_2^2)^{1/2}$.

Fourier Transform - Exact Solution.

Substitute (22), (23) into our transformed displacements (13), (14), then simply apply the inverse Fourier transform

$$u_x(x,y) = \frac{1}{i\mu\pi}\int_{-\infty}^{\infty} \frac{\sin\xi a}{F_0(\xi)} \left[2(\xi^2-1)^{1/2}(\xi^2-k^2)^{1/2}e^{-(\xi^2-k^2)^{1/2}y} \right. \\ \left. +(k^2-2\xi^2)e^{-(\xi^2-1)^{1/2}y}\right] e^{i\xi x}d\xi, \quad (24)$$

$$u_y(x,y) = \frac{1}{\mu\pi}\int_{-\infty}^{\infty} \frac{(\xi^2-1)^{1/2}\sin\xi a}{\xi F_0(\xi)} \left[(k^2-2\xi^2)e^{-(\xi^2-1)^{1/2}y} \right. \\ \left. +2\xi^2 e^{-(\xi^2-k^2)^{1/2}y}\right] e^{i\xi x}d\xi. \quad (25)$$

Where $F_0(\xi) = \left(2\xi^2-k^2\right)^2 - 4\xi^2(\xi^2-1)^{1/2}(\xi^2-k^2)^{1/2}$.

Integrands - Things to Note.

- k_1 is used as a normalising factor for all length parameters, $k_1^2 = 1 \implies k^2 = k_2^2$. Hence, $x \sim k_1 x$, $y \sim k_1 y$, $\xi \sim \xi / k_1$.
- Later we will let $a \to 0$ but let load per unit length $2a$ be const. This implies $\sin \xi a$ is replaced by ξa and $\sin \xi a / \xi$ is replaced by a.

Integrands - Poles.

Determined by $F_0(\xi) = 0$ which is given by $\xi = \pm\xi_R \in \mathbb{R}$. This depends on our choice of k, and thus our choice of material parameters.

$$
\begin{aligned}
\nu = 1/4, &\quad \xi_R = \pm 1.883889091\ldots\ , \\
\nu = 1/3, &\quad \xi_R = \pm 2.144712535\ldots\ .
\end{aligned}
\tag{26}
$$

Integrands - Branch Points and Cuts.

Determined by $(\xi^2 - 1)^{1/2}$, $(\xi^2 - k^2)^{1/2}$, and hence they are located at

$$\xi = \pm 1, \quad \pm k. \tag{27}$$

The branch cuts will lie along hyperbolas. These hyperbolas will degenerate to cuts along the real and imaginary axis.

The Contour of Integration.

- Assume $x > 0$ so that the contour is closed in the upper half plane,
- Poles at $\xi = \pm\xi_R$ are, respectively, excluded and included,
- Strategic geometry around branch points remains to be determined using the method of steepest descent.

The full contour is given by

$$\oint = \int_{\Gamma_1} + \int_{\Gamma_\alpha} + \int_{\Gamma_\beta} + \int_{\Gamma_2} + \int_{-R}^{R} = -2\pi i \sum \text{Res}, \tag{28}$$

The Contour of Integration.

As stated in [Graff(1991)]

- Contribution from the poles (where we will neglect contribution from the branch cuts using Cauchy's residue theorem) will lead to surface waves,
- Contribution from the branch cuts lead to interior bulk waves (where we will neglect the contribution from the poles using the method of steepest descent),
- Contribution from Γ_1, Γ_2 will vanish as $R \to \infty$,
- Contribution along the real axis is what we are interested in.

Evaluation of Branch Cut Integrals.

We see the displacements (24) and (25) have the form

$$I_1 = \int_{-\infty}^{\infty} \chi(\xi) e^{i\xi x - (\xi^2 - m)^{1/2} y} d\xi, \tag{29}$$

where m is 1 or k. Switching to polar coordinates and writing

$$f(\xi) = f_1 + if_2 = -(\xi^2 - m)^{1/2} \cos\theta + i\xi \sin\theta, \tag{30}$$

we then have

$$I_1 = \int_{-\infty}^{\infty} \chi(\xi) e^{R(f_1 + if_2)} d\xi = \int_{-\infty}^{\infty} \chi(\xi) e^{Rf_1} e^{iRf_2} d\xi. \tag{31}$$

Method of Steepest Descent.

We need to approximate I_1. The idea is to deform our contour such that f_2 is const.

To achieve this we perform a change of variables such that our contour has been deformed through saddle points of $f(\xi)$.

In this transformed form our integral I_1 can be approximated using asymptotic methods, in this case an alternate form of Watson's lemma.

Method of Steepest Descent.

The coordinate transformation we will use is

$$f(\xi) = f(\xi_0) - u^2. \tag{32}$$

This results in

$$I_1 = \pm \frac{\sqrt{2} e^{i\theta} e^{Rf(\xi_0)}}{|f''(\xi_0)|^{1/2}} \int_{-\infty}^{\infty} \chi(\xi(u)) e^{-Ru^2} du. \tag{33}$$

Now using Watson's lemma

$$I_1 \sim \pm \frac{\sqrt{2} e^{i\theta} e^{Rf(\xi_0)}}{|f''(\xi_0)|^{1/2}} \sum_{n=0}^{\infty} \chi^{(2n)}(u) \frac{(n - \frac{1}{2})!}{R^{n+1/2}}, \tag{34}$$

or

$$I_1 \sim e^{Rf(\xi_0)} \frac{d\xi}{du} \left(\chi(0) \frac{\sqrt{\pi}}{R^{1/2}} + \chi''(0) \frac{\sqrt{\pi}}{2R^{3/2}} + \chi^{(4)}(0) \frac{3\sqrt{\pi}}{4R^{5/2}} + \dots \right). \tag{35}$$

Method of Steepest Descent.

Before continuing on there are many things that remain to be determined

- Value ξ_0 where saddle points occur,
- $f(\xi_0)$, $f''(\xi_0)$, $\xi(u)$,
- The shape of the deformed contour
- Behaviour of the path away from the saddle point,
- Behaviour of the path around the vicinity of the pole,

Method of Steepest Descent.

Only after the saddle points have been properly established and the steepest descent path properly determined can we give the first order asymptotic approximation for I_1,

$$I_1 \sim \sqrt{\frac{2\pi m}{R}} e^{i(\pi/4 - Rm)} \cos\theta \chi(-m \sin\theta). \tag{36}$$

Where m can be 1 or k.

Method of Steepest Descent.

From here, by comparing back to our original displacement integrals, we can

- Obtain our displacements,
- Evaluate χ functions,
- Find $F_0(\xi_0)$
- Let $a \to 0$,
- Switch coordinate systems,
- Plot the real parts of them, etc.

Displacements.

They can be expressed in either polar or Cartesian. The polar forms are much more elegant so they will be given here, but the Cartesian forms are used for plotting.

$$u_R \sim \sqrt{\frac{2}{R\pi}} \frac{ae^{i(3\pi/4-R)} \cos\theta(k^2 - 2\sin^2\theta)}{\mu F_0(\sin\theta)}, \tag{37}$$

$$u_\theta \sim \sqrt{\frac{2k^5}{R\pi}} \frac{ae^{i(5\pi/4-kR)} \sin 2\theta(k^2\sin^2\theta - 1)^{1/2}}{\mu F_0(k\sin\theta)}. \tag{38}$$

Where

$$F_0(-m\sin\theta) = (2m^2\sin^2\theta - k^2)^2 - \\ 4m^2\sin^2\theta(m^2\sin^2\theta - 1)^{1/2}(m^2\sin^2\theta - k^2)^{1/2}. \tag{39}$$

Surface Waves.

Arise when $\theta = \pi/2$ or $y = 0$. Considering our earlier displacements (24) and (25), we see in the limit $a \to 0$

$$u_x(x,0) = \frac{a}{i\mu\pi}\int_{-\infty}^{\infty} \frac{\xi}{F_0(\xi)}\left[2(\xi^2-1)^{1/2}(\xi^2-k^2)^{1/2} + (k^2-2\xi^2)\right] e^{i\xi x}\,d\xi, \tag{40}$$

$$u_y(x,0) = \frac{ak^2}{\mu\pi}\int_{-\infty}^{\infty} \frac{(\xi^2-1)^{1/2}}{F_0(\xi)} e^{i\xi x}\,d\xi. \tag{41}$$

Surface Waves.

We have by Cauchy's residue theorem

$$\oint_\gamma f(z)dz = 2\pi i \sum_k \mathrm{Res}(f, c_k). \tag{42}$$

Thus

$$u_x(x,0) = -\frac{2a\xi_R e^{-i\xi_R x}}{\mu F_0'(-\xi_R)}\left[2(\xi_R^2-1)^{1/2}(\xi_R^2-k^2)^{1/2}+(k^2-2\xi_R^2)\right], \tag{43}$$

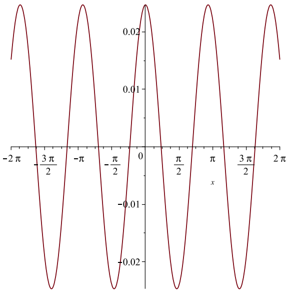
$$u_y(x,0) = \frac{2iak^2(\xi_R^2-1)^{1/2}}{\mu F_0'(-\xi_R)}e^{-i\xi_R x}. \tag{44}$$

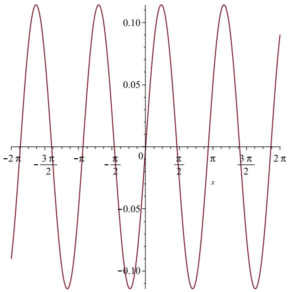
Finite Element Numerical Analysis.

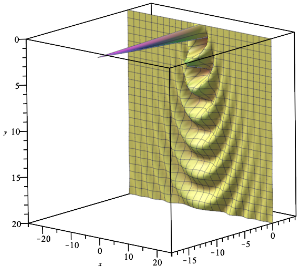
For this analysis we will use COMSOL multiphysics. We begin by setting up the problem;

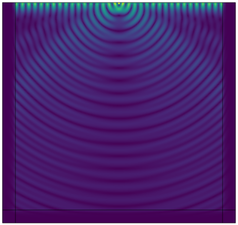
- We create a finite 2d rectangular domain with a unit magnitude time harmonic point source on the surface,
- The boundary on the upper surface is left free,
- On the bottom and sides of the domain we expand the domain to add perfectly matched layers,
- Specify zero initial values for the internal displacement fields,
- Specify material parameters in accordance with what we used for the asymptotic model,
- Set an extremely fine mesh.

We then solve the resulting problem in the frequency domain (choosing a desired frequency) to obtain solutions


0.02
0.01
0
−0.01
−0.02
−2 π
−3π/2
−π
−π/2
π/2
π
3π/2
2 π
x


0.10
0.05
0
−0.05
−0.10
−2 π
−3π/2
−π
−π/2
π/2
π
3π/2
2 π
x


0
5
y 10
15
20
-20
-10
0
10
20
x
-15
-10
-5
0



References.

Karl F Graff.
Wave motion in elastic solids.
Dover, 1991.

Patrick Aidan Heelan.
Radiation from a cylindrical source of finite length.
Geophysics, 18(3):685–696, 1953.

Horace Lamb.
I. on the propagation of tremors over the surface of an elastic solid.
Philosophical Transactions of the Royal Society of London. Series A, Containing papers of a mathematical or physical character, 203(359-371):1–42, 1904.

GF Miller and H Pursey.
The field and radiation impedance of mechanical radiators on the free surface of a semi-infinite isotropic solid.
Proceedings of the Royal Society of London. Series A.