Gaussian Beams for Underwater Ocean Acoustics

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Abstract

Notes:

- dynamic ray model = rtx model + extra info gathered about ray via Gaussian beams, aka, amplitude, phase, beamwidth

Contents

1 Introduction

Citation notes:

- Jensen et al. [2011]-computational ocean acoustics book
- Dushaw [2022] - sspd data stuff
- Červenỳ et al. [1982] - Gaussian beam method details from seismology

- Collins and Siegmann [2019] - parabollic wave equation method stuff
- Porter and Bucker [1987] - porter Gaussian beam
- Porter [2019] - porter Gaussian beam 2,3d
- Cervenỳ [2005] - got rccs derivation from this. cerveny rtx book
- Babich and Buldyrev [1991] - got $\beta = 1/2$. BL along Ω. Book on high frequency diffraction theory.

2 The Wave Equation

We have the 3-dimensional wave equation with spacial coordinates x, y, and z

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r}, t) = 0, \tag{2.1}$$

where $u = u(\mathbf{r}, t)$, $\mathbf{r} = (x, y, z)$, $c = c(\mathbf{r})$, and ∇^2 is Laplacian operator in 2-dimensions. Physically, x is the variable we will associate with range and z depth, later we will have invariance of y and only consider solutions of the 2-dimensional problem. We also have the wave propagation velocity or sound speed given as $V = c = c(\mathbf{r})$; in oceans we expect the sound speed to vary greatly with depth and little with a suitably small range. However, in considering sound speed variance with range we observe that at shallower depths there is a larger variation in sound speed, and in deep ocean we see sound speed convergence over range. Thus, for shallow water applications it is critical to have range dependence in the sound speed.

From this point, we have two objectives: the first is deriving the Helmholtz equation and from there obtaining and solving (with numerical methods) the so-called Eikonal equation. This solution will be our ray trace. The second objective is to derive and solve (using Gaussian beams) the parabolic wave equation. Then the ray trace will become the central ray of a Gaussian beam. This will enable us to develop a dynamic ray model which more closely resembles reality by providing a smoother transition into the shadow zone and taking into account attenuation of the wave. The beamwidth and phase fronts will also be seen to vary along the ray.

2.1 Helmholtz & Eikonal Equations

We will use the method of separation of variables on the wave equation (2.1). Assuming $u(\mathbf{r}, t)$ is separable we have

$$u(\mathbf{r}, t) = p(\mathbf{r})T(t). \tag{2.2}$$

Now substitute into (2.1)

$$\nabla^2(p(\mathbf{r})T(t)) - c^{-2}\frac{\partial^2}{\partial t^2}(p(\mathbf{r})T(t)) = 0, \tag{2.3}$$

$$T(t)\nabla^2(P(\mathbf{r})) - c^{-2}p(\mathbf{r})\frac{\partial^2 T(t)}{\partial t^2} = 0, \tag{2.4}$$

$$\frac{\nabla^2 p(\mathbf{r})}{p(\mathbf{r})} = \frac{1}{c^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}. \tag{2.5}$$

Noticing that the left-hand side of (2.5) only depends on $\mathbf{r}$ and the right-hand side only t. Thus, this can only be true if the left-hand side and the right-hand side of (2.5) are equal to the same constant.

$$\frac{\nabla^2 p(\mathbf{r})}{p(\mathbf{r})} = -k^2, \tag{2.6}$$

$$\frac{1}{c^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -k^2, \tag{2.7}$$

where $k^2 = \omega^2 c^{-2}$ and ω is the angular frequency. As we desire to be in the frequency domain we consider only the time-independent form, hence from (2.6)

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = 0, \tag{2.8}$$

$$\nabla^2 p(\mathbf{r}) + \frac{\omega^2}{c^2(\mathbf{r})}p(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}_0). \tag{2.9}$$

Introduction of the Dirac delta function in (2.9) is to account for a point source location $\mathbf{r}_0$ of the wave.

Following from Jensen et al. [2011], we seek the solution of the Helmholtz equation in ray series form

$$p(\mathbf{r}) = e^{\mathrm{i}\omega\tau(\mathbf{r})}\sum_{j=0}^{\infty}\frac{A_j(\mathbf{r})}{(\mathrm{i}\omega)^j}. \tag{2.10}$$

Now find the first order derivatives of the ray series $p(\mathbf{r})$ with respect to x

$$p_{,x} = \mathrm{i}\omega\tau_{,x}\mathrm{e}^{\mathrm{i}\omega\tau}\sum_{j=0}^{\infty}\frac{A_j(\mathbf{r})}{(\mathrm{i}\omega)^j} + \mathrm{e}^{\mathrm{i}\omega\tau}\sum_{j=0}^{\infty}\frac{A_{j,x}(\mathbf{r})}{(\mathrm{i}\omega)^j}, \tag{2.11}$$

$$= \mathrm{e}^{\mathrm{i}\omega\tau}\left\{\mathrm{i}\omega\tau_{,x}\sum_{j=0}^{\infty}\frac{A_j(\mathbf{r})}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,x}(\mathbf{r})}{(\mathrm{i}\omega)^j}\right\}, \tag{2.12}$$

where for $\tau = \tau(\mathbf{r})$ the dependency has been dropped for convenience. Also, the notion $p_{,x} = \partial p/\partial x$ has been used and will be used frequently in the future. Continuing on, we find the second order derivatives of the ray series $p(\mathbf{r})$ with respect to x

$$p_{,xx} = \mathrm{i}\omega\tau_{,xx}\mathrm{e}^{\mathrm{i}\omega\tau}\sum_{j=0}^{\infty}\frac{A_j(\mathbf{r})}{(\mathrm{i}\omega)^j} + (\mathrm{i}\omega\tau_x)^2\mathrm{e}^{i\omega\tau}\sum_{j=0}^{\infty}\frac{A_j(\mathbf{r})}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\tau_x\mathrm{e}^{i\omega\tau}\sum_{j=0}^{\infty}\frac{A_{j,x}(\mathbf{r})}{(\mathrm{i}\omega)^j} + \mathrm{e}^{i\omega\tau}\sum_{j=0}^{\infty}\frac{A_{j,xx}(\mathbf{r})}{(\mathrm{i}\omega)^j}, \tag{2.13}$$

$$= \mathrm{e}^{\mathrm{i}\omega\tau}\left\{[-\omega^2(\tau_{,x})^2 + \mathrm{i}\omega\tau_{,xx}]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\tau_{,x}\sum_{j=0}^{\infty}\frac{A_{j,x}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,xx}}{(\mathrm{i}\omega)^j}\right\}. \tag{2.14}$$

The same procedure is followed to find the other second order derivatives of the ray series $p(\mathbf{r})$:

$$p_{,yy} = \mathrm{e}^{\mathrm{i}\omega\tau}\left\{[-\omega^2(\tau_{,y})^2 + \mathrm{i}\omega\tau_{,yy}]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\tau_{,y}\sum_{j=0}^{\infty}\frac{A_{j,y}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,yy}}{(\mathrm{i}\omega)^j}\right\}, \tag{2.15}$$

$$p_{,zz} = \mathrm{e}^{\mathrm{i}\omega\tau}\left\{[-\omega^2(\tau_{,z})^2 + \mathrm{i}\omega\tau_{,zz}]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\tau_{,z}\sum_{j=0}^{\infty}\frac{A_{j,z}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,zz}}{(\mathrm{i}\omega)^j}\right\}. \tag{2.16}$$

From $\nabla^2 p = p_{,xx} + p_{,yy} + p_{,zz}$, we have

$$\nabla^2 p = \mathrm{e}^{\mathrm{i}\omega\tau}\Bigg\{\left[-\omega^2\left((\tau_{,x})^2 + (\tau_{,y})^2 + (\tau_{,z})^2\right) + \mathrm{i}\omega\left(\tau_{,xx} + \tau_{,yy} + \tau_{,zz}\right)\right]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\left(\tau_{,x}\sum_{j=0}^{\infty}\frac{A_{j,x}}{(\mathrm{i}\omega)^j} + \tau_{,y}\sum_{j=0}^{\infty}\frac{A_{j,y}}{(\mathrm{i}\omega)^j} + \tau_{,z}\sum_{j=0}^{\infty}\frac{A_{j,z}}{(\mathrm{i}\omega)^j}\right) + \left(\sum_{j=0}^{\infty}\frac{A_{j,xx}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,yy}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{A_{j,zz}}{(\mathrm{i}\omega)^j}\right)\Bigg\}. \tag{2.17}$$

Since we are in $\mathbb{R}^3$ under the standard basis we have $\mathbf{e}_x = (1,0,0)$, $\mathbf{e}_y = (0,1,0)$, and $\mathbf{e}_z = (0,0,1)$. From this we rewrite the sum

$$\tau_{,x}\sum_{j=0}^{\infty}\frac{A_{j,x}}{(\mathrm{i}\omega)^j} + \tau_{,y}\sum_{j=0}^{\infty}\frac{A_{j,y}}{(\mathrm{i}\omega)^j} + \tau_{,z}\sum_{j=0}^{\infty}\frac{A_{j,z}}{(\mathrm{i}\omega)^j} = (\tau_{,x}\mathbf{e}_x + \tau_{,y}\mathbf{e}_y + \tau_{,z}\mathbf{e}_z)\cdot\left(\sum_{j=0}^{\infty}\frac{A_{j,x}}{(\mathrm{i}\omega)^j}\mathbf{e}_x + \sum_{j=0}^{\infty}\frac{A_{j,y}}{(\mathrm{i}\omega)^j}\mathbf{e}_y + \sum_{j=0}^{\infty}\frac{A_{j,z}}{(\mathrm{i}\omega)^j}\mathbf{e}_z\right) \tag{2.18}$$

$$= (\tau_{,x}\mathbf{e}_x + \tau_{,y}\mathbf{e}_y + \tau_{,z}\mathbf{e}_z)\cdot\sum_{j=0}^{\infty}\frac{(A_{j,x}\mathbf{e}_x + A_{j,y}\mathbf{e}_y + A_{j,z}\mathbf{e}_z)}{(\mathrm{i}\omega)^j} \tag{2.19}$$

$$= \nabla\tau\cdot\sum_{j=0}^{\infty}\frac{\nabla A_j}{(\mathrm{i}\omega)^j}. \tag{2.20}$$

Using:

$$|\nabla\tau|^2 = \left((\tau_{,x})^2 + (\tau_{,y})^2 + (\tau_{,z})^2\right), \tag{2.21}$$

$$\nabla^2 = \tau_{,xx} + \tau_{,yy} + \tau_{,zz}, \tag{2.22}$$

$$\sum_{j=0}^{\infty} \frac{\nabla^2 A_j}{(\mathrm{i}\omega)^j} = \sum_{j=0}^{\infty} \frac{A_{j,xx}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty} \frac{A_{j,yy}}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty} \frac{A_{j,zz}}{(\mathrm{i}\omega)^j}, \tag{2.23}$$

and (2.20), we obtain

$$\nabla^2 p(\mathbf{r}) = \mathrm{e}^{\mathrm{i}\omega\tau}\left\{\left[-\omega^2|\nabla\tau|^2 + \mathrm{i}\omega\nabla^2\tau\right]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\nabla\tau\cdot\sum_{j=0}^{\infty}\frac{\nabla A_j}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{\nabla^2 A_j}{(\mathrm{i}\omega)^j}\right\}. \tag{2.24}$$

Now substitute $\nabla^2 p(\mathbf{r})$ (2.24) and $p(\mathbf{r})$ (2.10) into the Helmholtz equation, we have

$$\mathrm{e}^{\mathrm{i}\omega\tau}\left\{\left[-\omega^2|\nabla\tau|^2 + \mathrm{i}\omega\nabla^2\tau\right]\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j} + 2\mathrm{i}\omega\nabla\tau\cdot\sum_{j=0}^{\infty}\frac{\nabla A_j}{(\mathrm{i}\omega)^j} + \sum_{j=0}^{\infty}\frac{\nabla^2 A_j}{(\mathrm{i}\omega)^j}\right\} + \frac{\omega^2}{c^2(\mathbf{x})}e^{i\omega}\tau(\mathbf{x})\sum_{j=0}^{\infty}\frac{A_j(\mathbf{x})}{(i\omega)^j} = -\delta(\mathbf{x}-\mathbf{x}_0). \tag{2.25}$$

Considering only terms of $\mathcal{O}(\omega^2)$:

$$-|\nabla\tau|^2 e^{\mathrm{i}\omega\tau}\cancel{\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j}} + \frac{1}{c^2}e^{\mathrm{i}\omega\tau}\cancel{\sum_{j=0}^{\infty}\frac{A_j}{(\mathrm{i}\omega)^j}} = 0, \tag{2.26}$$

$$-|\nabla\tau|^2 + \frac{1}{c^2} = 0, \tag{2.27}$$

$$|\nabla\tau|^2 = c^{-2}(\mathbf{x}). \tag{2.28}$$

(2.28) is known as the Eikonal equation. From here we use the method of characteristics to get a system of ordinary differential equations then solve this system numerically to get the ray paths. Terms of $\mathcal{O}(\omega)$ and $\mathcal{O}(\omega^{1-j})$ are the so-called transport equations, which associate pressure and amplitude to each ray. The transport equations will not be solved here as the Gaussian beam method will be used instead.

As previously stated we now use the method of characteristics to reduce the Eikonal equation to a system of ordinary differential equations. In short, this is a coordinate transformation, from Cartesian to ray coordinates which is done by introducing a family of curves which are themselves perpendicular to the level curves of $\tau(\mathbf{r})$. With $\nabla\tau$ perpendicular to the wavefronts, we define the ray trajectory $\mathbf{x}(s)$ (where s is the arclength of the ray) with the following differential equation,

$$\frac{d\mathbf{x}}{ds} = c\nabla\tau. \tag{2.29}$$

An extra factor of c has been included so that $d\mathbf{x}/ds$ is of unit length. We easily check this with

$$\left|\frac{d\mathbf{x}}{ds}\right|^2 = c^2|\nabla\tau|^2 = c^2c^{-2} = 1. \tag{2.30}$$

We now move to eliminate $\tau(\mathbf{r})$ and write the ray equations in a form involving only $c(\mathbf{r})$. Considering only the x-component 2.29 we perform differentiation with respect to s

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = \frac{d}{ds}\left(\frac{\partial\tau}{\partial x}\right), \tag{2.31}$$

$$= \frac{\partial^2\tau}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2\tau}{\partial x\partial y}\frac{\partial y}{\partial s}, \tag{2.32}$$

$$= \frac{\partial^2\tau}{\partial x^2}\left(c\frac{\partial\tau}{\partial x}\right) + \frac{\partial^2\tau}{\partial x\partial y}\left(c\frac{\partial\tau}{\partial y}\right), \tag{2.33}$$

where we have used the substitution $\partial x/\partial s = c\,\partial\tau/\partial x$ and $\partial y/\partial s = c\,\partial\tau/\partial y$ from (2.29). Continuing on

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = c\left(\frac{\partial^2\tau}{\partial x^2}\frac{\partial\tau}{\partial x} + \frac{\partial^2\tau}{\partial x\partial y}\frac{\partial\tau}{\partial y}\right), \tag{2.34}$$

$$= c\left\{\left[\frac{\partial}{\partial x}\left(\frac{\partial\tau}{\partial x}\right)\right]\cdot\frac{\partial\tau}{\partial x} + \left[\frac{\partial}{\partial x}\left(\frac{\partial\tau}{\partial y}\right)\right]\cdot\frac{\partial\tau}{\partial y}\right\}, \tag{2.35}$$

$$= \frac{c}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2\right], \tag{2.36}$$

$$= \frac{c}{2}\frac{\partial}{\partial x}\left(\frac{1}{c^2}\right) = -\frac{1}{c^2}\frac{\partial c}{\partial x}. \tag{2.37}$$

Similarly, we also have:

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dy}{ds}\right) = -\frac{1}{c^2}\frac{\partial c}{\partial y}, \tag{2.38}$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = -\frac{1}{c^2}\frac{\partial c}{\partial z}. \tag{2.39}$$

Hence

$$\frac{d}{ds}\left(\frac{1}{c}\frac{d\mathbf{x}}{ds}\right) = -\frac{1}{c^2}\nabla c. \tag{2.40}$$

We switch to cylindrical coordinates $\mathbf{x} = (r, z)$,

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dr}{ds}\right) = -\frac{1}{c^2}\frac{\partial c}{\partial r}, \tag{2.41}$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = -\frac{1}{c^2}\frac{\partial c}{\partial z}. \tag{2.42}$$

Let

$$\frac{d\xi}{ds} = -\frac{1}{c^2}\frac{\partial c}{\partial r}, \tag{2.43}$$

$$\frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{\partial c}{\partial z}. \tag{2.44}$$

Hence

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dr}{ds}\right) = \frac{d\xi}{ds}, \tag{2.45}$$

$$\frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = \frac{d\zeta}{ds}. \tag{2.46}$$

Integrate with respect to s to get

$$\frac{dr}{ds} = c\xi(s), \tag{2.47}$$

$$\frac{dz}{ds} = c\zeta(s). \tag{2.48}$$

Combining (2.43), (2.44), (2.47), and (2.48) yields the desired system

$$\frac{dr}{ds} = c\xi(s), \quad \frac{dz}{ds} = c\zeta(s), \quad \frac{d\xi}{ds} = -\frac{1}{c^2}\frac{\partial c}{\partial r}, \quad \frac{d\zeta}{ds} = -\frac{1}{c^2}\frac{\partial c}{\partial z}, \tag{2.49}$$

with $c = c(\mathbf{x})$ and the initial conditions[1]

$$r = r_0, \quad z = z_0, \quad \xi_0 = \frac{\cos\theta_0}{c(\mathbf{0})}, \quad \zeta_0 = \frac{\sin\theta_0}{c(\mathbf{0})}. \tag{2.50}$$

This system will be solved with numerical methods, specifically, the Euler method and a fourth order Runge-Kutta method; yielding a ray trace subject to variable sound speed, an initial source location (r_0, z_0), and an initial launch angle θ_0. This result will be the standard ray system; the additional information gathered about the ray from Gaussian beams in conjunction with the standard ray system is the so-called dynamic ray system.

2.2 Parabolic Method

We know that for high frequency waves (large ω) the wave field will mostly propagate along rays; hence, we will use the parabolic method to solve the wave equation in a region close to a ray Ω. We will only consider time harmonic solutions for a circular frequency ω.

We starting again from the wave equation (2.1), the first step is to perform a coordinate transformation to orthogonal ray-centred coordinates connected to an arbitrary ray Ω. doing this coordinate transformation is not actually mandatory, however it is the simplest way to perform dynamic ray tracing. From there we will certain substitutions to arrive at the parabolic wave equation which will be solved using Gaussian beams...

[1] Note that the tangent and normal to the ray are $\mathbf{t}_{\text{ray}} = c[\xi(s), \zeta(s)]$ and $\mathbf{n}_{\text{ray}} = c[-\zeta(s), \xi(s)]$ respectively. These initial conditions ξ_0 and ζ_0 result from the initial geometry of the ray.

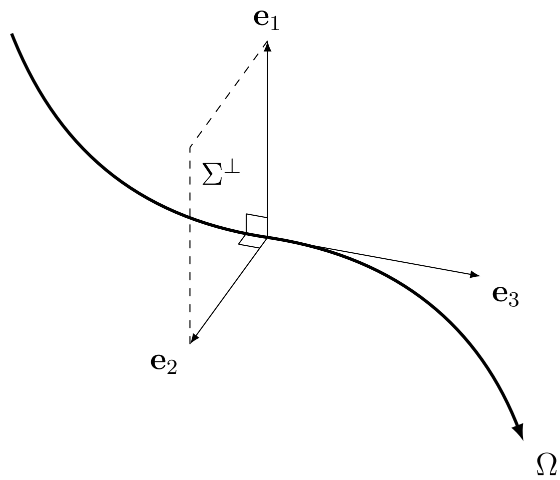

Figure 1: Ray-Centered Coordinate System

2.2.1 Ray-Centred Coordinates

We have a ray Ω which travels through the Cartesian space (x, y, z). We introduce the ray-centred coordinate system (q_1, q_2, q_3) which has the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, see (1). Let q_3 be a monotonic parameter along Ω, specifically we let $q_3 = s$ where s is arclength along the ray (we could choose a different monotonic parameter, for example travel-time T, $ds = V dt$). The basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ form a plane $\Sigma^\perp$ which is perpendicular to Ω. Then q_1 and q_2 the form a 2-dimensional Cartesian coordinate system in this plane perpendicular to Ω at $q_3 = s$ with origin Ω. Hence, Ω itself forms an axis of the coordinate system. The basis vectors of the ray-centred coordinate system connected with Ω form a right hand mutually perpendicular triplet of unit vectors.

Now find unit vectors $\mathbf{e}_i(s)$ along Ω by solving the first order vector differential equations

$$\frac{d}{ds}\mathbf{e}_i(s) = a_i(s)\mathbf{p}(s), \quad i = 1, 2, \tag{2.51}$$

where $a_i = a_i(s)$ is a continuous function of s which we are yet to find and $\mathbf{p}(s)$ is the slowness vector. We have scalar product of $\mathbf{p}(s)$ with itself

$$\mathbf{p}(s) \cdot \mathbf{p}(s) = \frac{1}{V^2}, \tag{2.52}$$

$$= |\nabla \tau|^2, \tag{2.53}$$

which is the Eikonal equation which we know from ray tracing. We will determine $a_i(s)$ to keep $\mathbf{e}_i$, $i = 1, 2$, perpendicular to Ω, hence, $\mathbf{e}_i \cdot \mathbf{p}(s) = 0$. We can take derivatives with respect to s of this scalar product to find $a_i(s)$

$$\frac{d}{ds}\left(\mathbf{e}_i \cdot \mathbf{p}(s)\right) = \frac{d}{ds}(\mathbf{e}_i) \cdot \mathbf{p}(s) + \mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p}(s)) = 0, \tag{2.54}$$

$$= (a_i(s)\mathbf{p}(s)) \cdot \mathbf{p}(s) + \mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p}(s)), \tag{2.55}$$

where

$$\frac{d}{ds}(\mathbf{e}_i) \cdot \mathbf{p}(s) = (a_i(s)\mathbf{p}(s)) \cdot \mathbf{p}(s). \tag{2.56}$$

Rearranging for $a_i(s)$

$$a_i = \frac{d}{ds}(\mathbf{e}_i) \cdot \frac{\mathbf{p}}{\mathbf{p} \cdot \mathbf{p}}, \tag{2.57}$$

where all s dependency has been omitted. We also have from (2.54)

$$\frac{d}{ds}(\mathbf{e}_i) \cdot \mathbf{p} = -\mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p}), \tag{2.58}$$

hence

$$\frac{d}{ds}(\mathbf{e}_i) \cdot \frac{\mathbf{p}}{\mathbf{p} \cdot \mathbf{p}} = -\mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p})\frac{1}{\mathbf{p} \cdot \mathbf{p}}. \tag{2.59}$$

Post multiply both sides by $\mathbf{p}$

$$\frac{d}{ds}(\mathbf{e}_i) = -\mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p})\frac{\mathbf{p}}{\mathbf{p} \cdot \mathbf{p}}. \tag{2.60}$$

Now using $\mathbf{p}(s) \cdot \mathbf{p}(s) = V^{-2}$ and

$$\mathbf{e}_i \cdot \frac{d}{ds}(\mathbf{p}) = \mathbf{e}_i \cdot \nabla \mathbf{p} = \mathbf{e}_i \cdot \nabla \left(\frac{1}{V}\right) = -\frac{1}{V^2}\mathbf{e}_i \cdot \nabla(V), \tag{2.61}$$

we have

$$\frac{d}{ds}(\mathbf{e}_i) = \frac{\cancel{V^2}}{\cancel{V^2}}(\mathbf{e}_i \cdot \nabla(V)) \cdot \mathbf{p}, \tag{2.62}$$

$$= (\mathbf{e}_i \cdot \nabla(V)) \cdot \mathbf{p}. \tag{2.63}$$

Now use this result to find $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$ along Ω. Assume $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$ satisfy the three conditions:

1. Both $\mathbf{e}_1(s)$ and $\mathbf{e}_2(s)$ are perpendicular to $\mathbf{p}_s$, i.e., perpendicular to the ray Ω

$$\mathbf{e}_i(s) \cdot \mathbf{p}(s) = 0$$

2. & 3. $\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$ and $\mathbf{e}_3(s) = \mathbf{t}(s)$ form a mutually perpendicular triplet of unit vectors at any point s.

$$\mathbf{e}_1(s) \cdot \mathbf{e}_2(s) = 0$$

$$\mathbf{e}_1(s) \cdot \mathbf{e}_1(s) = \mathbf{e}_2(s) \cdot \mathbf{e}_2(s) = 1$$

Now determine the ray-centred coordinates q_1, q_2, and $q_3 = s$ of any point $R' = R'[q_1, q_2, q_3]$ close to Ω. First construct plane $\Sigma^\perp$ perpendicular to Ω and passing through R'. Then find point of intersection of $\Sigma^\perp$ with Ω and denote by R. The plane $\Sigma^\perp$ is tangent to the wavefront at R. As R is on Ω $R = R[0, 0, s]$, i.e., $q_1 = q_2 = 0$ and $q_3 = s$, see (2).

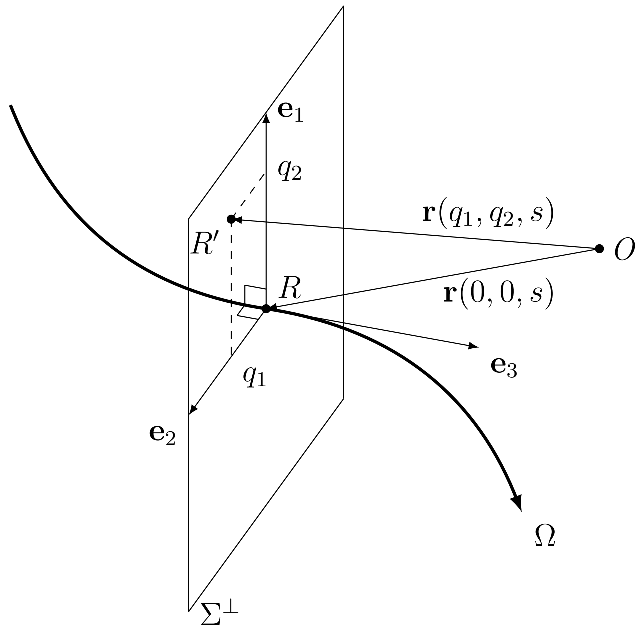

Figure 2: Geometry of the Ray-Centered Coordinate System

Note that q_3 coordinate of R and R' are the same, hence, $q_1(R')$ and $q_2(R')$ are found easily in the plane $\Sigma^\perp$ using the known basis vectors $\mathbf{e}_i$ $i = 1, 2$. The radius vector $\mathbf{r}$ of R' can be expressed in ray-centred coordinates.

$$\mathbf{r}(q_1, q_2, s) = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + \mathbf{r}(0, 0, s), \tag{2.64}$$

where $q_i = q_i(s)$, $\mathbf{e}_i = \mathbf{e}_i(s)$ with $i = 1, 2$, and $\mathbf{r}(0, 0, s)$ is the radius vector of R on Ω. (2.64) defines the ray-centred coordinates of R' assuming $\mathbf{e}_i = \mathbf{e}_i(s)$ and $\mathbf{r}(0, 0, s)$ are known.

The ray-centred coordinates q_1, q_2, and s of R' can be uniquely introduced if one and only one plane $\Sigma^\perp$ perpendicular to Ω and passing through R' can be constructed. For points far from Ω this condition may not be satisfied, therefore, only consider points R' close to Ω so that the ray-centred coordinates maybe introduced uniquely. This is the region of validity which itself depends on the curvature of the ray. For a slightly curving ray Ω we have a large region of validity and a high curvature ray Ω we have a smaller region of validity

We can show this coordinate system is in fact orthogonal by first finding the line element

$$d\ell^2 = d\mathbf{r} \cdot d\mathbf{r} = g_{ik} dq_i dq_k, \tag{2.65}$$

then showing the metric tensor g_{ik} is diagonal. Thus, take derivatives with respect to s of (2.64)

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}(0, 0, s)}{ds} + \frac{dq_1}{ds}\mathbf{e}_1 + q_1\frac{d\mathbf{e}_1}{ds} + \frac{dq_2}{ds}\mathbf{e}_2 + q_2\frac{d\mathbf{e}_2}{ds}, \tag{2.66}$$

$$d\mathbf{r} = \left[\frac{d\mathbf{r}(0, 0, s)}{ds} + q_1\frac{d\mathbf{e}_1}{ds} + q_2\frac{d\mathbf{e}_2}{ds}\right] ds + dq_1\mathbf{e}_1 + dq_2\mathbf{e}_2, \tag{2.67}$$

Now with (2.63) and by defining the tangent to Ω as $\mathbf{t} = \mathbf{e}_3$,we have

$$d\mathbf{r} = \left[1 + \frac{1}{V}(\mathbf{e}_1 \cdot \nabla v)q_1 + \frac{1}{V}(\mathbf{e}_2 \cdot \nabla v)q_2\right]\mathbf{t}ds + dq_1\mathbf{e}_1 + dq_2\mathbf{e}_2, \tag{2.68}$$

now assuming we are in the region of validity we write $V(q_1, q_2, s) = V(0, 0, s)$. Hence

$$d\mathbf{r} = \left[1 + \left(\frac{1}{V}\frac{\partial V}{\partial q_1}\bigg|_{q_1=q_2=0} q_1 + \frac{1}{V}\frac{\partial V}{\partial q_2}\bigg|_{q_1=q_2=0} q_2\right)\right]\mathbf{t}ds + dq_1\mathbf{e}_1 + dq_2\mathbf{e}_2, \tag{2.69}$$

$$= h\mathbf{t}ds + dq_1\mathbf{e}_1 + dq_2\mathbf{e}_1, \tag{2.70}$$

with the scale factor

$$h = 1 + \frac{1}{V}\frac{dV}{dq_i}\bigg|_{q_i=0} q_i, \quad i = 1, 2. \tag{2.71}$$

Hence we find the line element

$$d\ell = \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{h^2 ds^2 + dq_1^2 + dq_2^2}, \tag{2.72}$$

and thus we see $g_{11} = g_{22} = 1$, $g_{33} = h^2$, and $g_{ik} = 0$ for $i \neq k$. Hence the coordinate system is orthogonal as only the diagonal terms of the metric tensor are non-vanishing.

To transform the wave equation to ray-centred coordinates we need to find the Laplacian or the divergence of the gradient, we first will do this in general curvilinear coordinates then apply this to our coordinate system, as we now know what the metric tensor is. We have the general 3-dimensional line element $d\mathbf{r} = h_i dq_i$ for $i = 1, 2, 3$, hence

$$du = \nabla u \cdot d\mathbf{r}, \tag{2.73}$$

$$= \frac{\partial u}{\partial q_1}h_1 dq_1 + \frac{\partial u}{\partial q_2}h_2 dq_2 + \frac{\partial u}{\partial q_3}h_3 dq_3. \tag{2.74}$$

Notice $e_i \cdot d\mathbf{r} = \delta_{ij}(hdq)_j$ for $i = 1, 2, 3$ and $j = 1, 2, 3$, hence

$$\nabla u = \frac{1}{h_1}\frac{\partial u}{\partial q_1}e_1 + \frac{1}{h_2}\frac{\partial u}{\partial q_2}e_2 + \frac{1}{h_3}\frac{\partial u}{\partial q_3}e_3. \tag{2.75}$$

From the Voss-Weyl formula the divergence is given as

$$\mathrm{Div}(u) = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial q_i}\left(\sqrt{\det(g)}u^i\right), \tag{2.76}$$

where

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix} \tag{2.77}$$

is the metric tensor. Therefore,

$$\mathrm{Div}(u) =\frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{h_1h_2h_3}{h_i}u_i\right), \tag{2.78}$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(u_1h_2h_3)+\frac{\partial}{\partial q_2}(u_2h_1h_3)+\frac{\partial}{\partial q_3}(u_3h_1h_2)\right], \tag{2.79}$$

where $u^i = u_i/h_i$ is the relationship between covariant and contravariant components. Finally,

$$\nabla^2 u =\mathrm{Div}(\nabla u) = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial q_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial u}{\partial q_i}\right), \tag{2.80}$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial u}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial u}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial u}{\partial q_3}\right)\right] \tag{2.81}$$

Note that we can write

$$\nabla u = \frac{1}{h_i}\frac{\partial u}{\partial q_i}, \tag{2.82}$$

so alternatively,

$$\nabla^2 u = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial q_i}\left(\sqrt{\det(g)}g^{ij}\frac{\partial u}{\partial q_i}\right), \tag{2.83}$$

with $g^{ij} = h_i^{-1}h_j^{-1}\delta^{ij}$. For the final step of writing the ray centred coordinates it will be easiest to use (2.81) with $\partial/\partial q_3 = \partial/\partial s$, $h_1 = h_2 = 1$, and $h_3 = h$

$$\nabla^2 u =\frac{1}{h}\left[\frac{\partial}{\partial q_1}\left(h\frac{\partial u}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(h\frac{\partial u}{\partial q_2}\right)+\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial u}{\partial s}\right)\right], \tag{2.84}$$

$$=\frac{1}{h}\left[\frac{\partial h}{\partial q_1}\frac{\partial u}{\partial q_1}+h\frac{\partial^2 u}{\partial q_1^2}+\frac{\partial h}{\partial q_2}\frac{\partial u}{\partial q_2}+h\frac{\partial^2 u}{\partial q_2^2}+\frac{\partial}{\partial s}\left(\frac{1}{h}\right)\frac{\partial u}{\partial s}+\frac{1}{h}\frac{\partial^2 u}{\partial s^2}\right]. \tag{2.85}$$

As we are solving the wave equation in 2-dimensions we set $q_1 = n$ and $q_2 = 0$

$$\nabla^2 u = \frac{1}{h}\left[\frac{\partial h}{\partial n}\frac{\partial u}{\partial n}+h\frac{\partial^2 u}{\partial n^2}+\frac{\partial}{\partial s}\left(\frac{1}{h}\right)\frac{\partial u}{\partial s}+\frac{1}{h}\frac{\partial^2 u}{\partial s^2}\right]. \tag{2.86}$$

Finally we write the 2-dimensional wave equation in ray-centred coordinates

$$\frac{\partial h}{\partial n}\frac{\partial u}{\partial n}+h\frac{\partial^2 u}{\partial n^2}+\frac{\partial}{\partial s}\left(\frac{1}{h}\right)\frac{\partial u}{\partial s}+\frac{1}{h}\frac{\partial^2 u}{\partial s^2}=\frac{h}{V^2}\frac{\partial^2 u}{\partial t^2}, \tag{2.87}$$

which can be written as

$$h_{,n}u_{,n}+hu_{,nn}+(h^{-1})_{,s}u_{,s}+h^{-1}u_{,ss}=hV^{-2}u_{,tt}\,. \tag{2.88}$$

2.2.2 Parabolic Wave Equation

First we derive the Parabolic wave equation, to do this we make the substitution

$$u(s,n,t) = \mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}(s,n,\omega), \tag{2.89}$$

where

$$T = \int_{s_0}^{s} \frac{ds}{V} \tag{2.90}$$

is the travel time with s_0 initial arc length and s final. We take the travel time to be along along the ray Ω hence $V = V(s,0) = v(s)$. Now take derivatives of (2.89) and substitute them into (2.88). Starting with derivatives with respect to s

$$\begin{aligned}
\frac{\partial}{\partial s}(u(s,n,t)) =& \frac{\partial}{\partial s}(\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}(s,n,\omega)), && (2.91)\\
=& -\mathrm{i}\omega\frac{\partial}{\partial s}(t-T)\mathrm{e}^{-\mathrm{i}\omega(t-T)}\hat{u}(s,n,\omega) + \mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,s}(s,n,\omega), && (2.92)\\
=& \mathrm{e}^{-\mathrm{i}\omega[t-T]}\left(\frac{\mathrm{i}\omega}{v} + \hat{u}_{,s}(s,n,\omega)\right), && (2.93)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}(u(s,n,t)) = \mathrm{i}\omega\left(\frac{1}{v}\right)_{,s}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u} + \left(\frac{i\omega}{v}\right)^2\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u} + \frac{\mathrm{i}\omega}{v}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,s}&\\
+ \frac{\mathrm{i}\omega}{v}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,s} + \mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,ss}, && (2.94)\\
= \mathrm{e}^{-\mathrm{i}\omega[t-T]}\left(\mathrm{i}\omega\left(\frac{1}{v}\right)_{,s}\hat{u} - \frac{\omega^2}{v^2}\hat{u} + 2\frac{\mathrm{i}\omega}{v}\hat{u}_{,s} + \hat{u}_{,ss}\right). && (2.95)
\end{aligned}$$

Now for derivatives with respect to n

$$\begin{aligned}
u_{,n} =& \mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,n}, && (2.96)\\
u_{,nn} =& \mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,nn}, && (2.97)
\end{aligned}$$

and derivatives with respect to t

$$\begin{aligned}
u_{,t} =& \mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}, && (2.98)\\
u_{,tt} =& -\omega^2\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}. && (2.99)
\end{aligned}$$

Now substitute into (2.88)

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\omega[t-T]}\frac{1}{h}\left(\mathrm{i}\omega\left(\frac{1}{v}\right)_{,s}\hat{u} - \frac{\omega^2}{v^2}\hat{u} + 2\frac{\mathrm{i}\omega}{v}\hat{u}_{,s} + \hat{u}_{,ss}\right) + \mathrm{e}^{-\mathrm{i}\omega[t-T]}h\hat{u}_{,nn} + \frac{h}{V^2}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\omega^2\hat{u}&\\
+ \left(\frac{1}{h}\right)_{,s}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\left(\frac{\mathrm{i}\omega}{v}\hat{u} + \hat{u}_{,s}\right) + h_{,n}\mathrm{e}^{-\mathrm{i}\omega[t-T]}\hat{u}_{,n} = 0. && (2.100)
\end{aligned}$$

Simplify and rearrange

$$\frac{1}{h}\left(\mathrm{i}\omega\left(\frac{1}{v}\right)_{,s}\hat{u}-\frac{\omega^2}{v^2}\hat{u}+2\frac{\mathrm{i}\omega}{v}\hat{u}_{,s}+\hat{u}_{,ss}\right)+h\hat{u}_{,nn}+\frac{h}{V^2}\omega^2\hat{u} \\ +\left(\frac{1}{h}\right)_{,s}\left(\frac{\mathrm{i}\omega}{v}\hat{u}+\hat{u}_{,s}\right)+h_{,n}\hat{u}_{,n}=0, \quad (2.101)$$

$$\frac{1}{h}\left\{\left[\mathrm{i}\omega\left(\frac{1}{v}\right)_{,s}-\frac{\omega^2}{v^2}\right]\hat{u}+2\frac{\mathrm{i}\omega}{v}\hat{u}_{,s}+\hat{u}_{,ss}\right\}+h\hat{u}_{,nn}+\frac{h}{V^2}\omega^2\hat{u} \\ +\left(\frac{\mathrm{i}\omega}{v}\hat{u}+\hat{u}_{,s}\right)\left(\frac{1}{h}\right)_{,s}+h_{,n}\hat{u}_{,n}=0. \quad (2.102)$$

From Babich and Buldyrev [1991] the eigenfunctions of the bouncing ball type, that is..., are concentrated in a boundary layer of thickness $n=O(\omega^{-\beta})$. This follows from caustics which bounds the strip within which the corresponding eigenfunction is concentrated. Hence introduce a new coordinate ν to replace n ($\nu=O(1)$) with

$$\nu=\omega^{1/2}n. \quad (2.103)$$

Now substitute this into (2.102), we have

$$\hat{u}_{,n}=\omega^{1/2}\hat{u}_{,\nu} \quad , \quad (2.104)$$

$$\hat{u}_{,nn}=\omega\hat{u}_{,\nu\nu} \quad , \quad (2.105)$$

and note $(v^{-1})_{,s}=-v^{-2}v_{,s}$. Hence

$$\frac{\mathrm{i}\omega}{h}\left(\frac{1}{v}\right)_{,s}\hat{u}-\frac{\omega^2}{v^2h}\hat{u}+\frac{2\mathrm{i}\omega}{vh}\hat{u}_{,s}+\frac{1}{h}\hat{u}_{,ss} \\ +\omega h\hat{u}_{,\nu\nu}+\frac{h\omega^2}{V^2}\hat{u}+\frac{\mathrm{i}\omega}{v}\left(\frac{1}{h}\right)_{,s}\hat{u}+\left(\frac{1}{h}\right)_{,s}\hat{u}_{,s}\omega^{1/2}\hat{u}_{,\nu}h_{,n}=0, \quad (2.106)$$

$$\omega^2h\left(\frac{1}{V^2}-\frac{1}{h^2v^2}\right)\hat{u}+\omega\left[-\frac{\mathrm{i}}{hv^2}v_{,s}\hat{u}+\frac{\mathrm{i}}{v}\left(\frac{1}{h}\right)_{,s}\hat{u}+\frac{2\mathrm{i}}{vh}\hat{u}_{,s}+h\hat{u}_{,\nu\nu}\right] \\ +\omega^{1/2}\hat{u}_{,\nu}h_{,n}+\frac{1}{h}\hat{u}_{,ss}+\left(\frac{1}{h}\right)_{,s}\hat{u}_{,s}=0, \quad (2.107)$$

where $\hat{u}=\hat{u}(s,\nu,\omega)$. Note this equation is still equivalent to the wave equation we started with. Now solve (2.107) asymptotically for $\omega\to\infty$, hence neglect all terms of ω^γ for $\gamma<1$.

Terms with $\gamma < 1$ will be asymptotically negligible. Note that some coefficients are functions of ω, hence take care evaluating those terms. We see

$$\omega^2 h\left(\frac{1}{V^2} - \frac{1}{h^2 v^2}\right) \sim -\frac{\omega}{v^3}\nu v_{,nn}\ , \tag{2.108}$$

$$h \sim 1, \tag{2.109}$$

$$\left(\frac{1}{h}\right)_{,s} \sim 0. \tag{2.110}$$

Now substitute these into (2.107) and neglect all terms of ω^γ for $\gamma < 1$

$$-\frac{\omega}{v^3}\nu^2 v_{,nn}\hat{u} + \omega\left(-\frac{\mathrm{i}}{v^2}v_{,s}\hat{u} + \frac{2\mathrm{i}}{v}\hat{u}_{,s} + \hat{u}_{,\nu\nu}\right) = 0, \tag{2.111}$$

$$\frac{2\mathrm{i}}{v}\hat{u}_{,s} + \hat{u}_{,\nu\nu} - \left(\frac{1}{v^3}\nu^2 v_{,nn} + \frac{\mathrm{i}}{v^2}v_{,s}\right)\hat{u} = 0. \tag{2.112}$$

This is the parabolic wave equation we desired, where $\hat{u} = \hat{u}(s,\nu)$ is the leading term of the corresponding asymptotic series for $\hat{u}(s,\nu,\omega)$. Note that $\hat{u}(s,\nu)$ still depends on ω as $\nu^2 = \omega n^2$. This can be simplified further by making the substitution $\hat{u}(s,\nu) = v^{1/2}(s)W(s,\nu)$ along with

$$\hat{u}_{,s} = \frac{1}{2}v^{-1/2}v_{,s}W + v^{1/2}W_{,s}, \tag{2.113}$$

$$\hat{u}_{,\nu\nu} = v^{1/2}W_{,\nu\nu}, \tag{2.114}$$

hence

$$\frac{2\mathrm{i}}{v}\left(\frac{1}{2}v^{-1/2}v_{,s}W + v^{1/2}W_{,s}\right) + v^{1/2}W_{,\nu\nu} - \left(\frac{1}{v^3}\nu^2 v_{,nn} + \frac{\mathrm{i}}{v^2}v_{,s}\right)v^{1/2}(s)W(s,\nu) = 0, \tag{2.115}$$

$$\frac{\mathrm{i}}{v^{3/2}}v_{,s}W + \frac{2\mathrm{i}}{v}v^{1/2}W_{,s} + v^{1/2}W_{,\nu\nu} - \frac{1}{v^2}\nu^2 v_{,nn}v^{1/2}W - \frac{\mathrm{i}}{v^{3/2}}v_{,s}W = 0, \tag{2.116}$$

$$\frac{2\mathrm{i}}{v}W_{,s} + W_{,\nu\nu} - \frac{1}{v^2}\nu^2 v_{,nn}W = 0. \tag{2.117}$$

(2.117) is the final form of the parabolic wave equation, which we will solve using Gaussian beams in the following section. Note that after the substitution $\hat{u}(s,\nu) = v^{1/2}(s)W(s,\nu)$ we have

$$u(s,n,t) = v^{1/2}(s)\mathrm{e}^{-\mathrm{i}\omega(t-T)}W(s,\nu), \tag{2.118}$$

where $W(s,\nu)$ is the solution of (2.117) and that $W(s,\nu)$ still depends on ω as $\nu^2 = \omega n^2$.

2.2.3 Gaussian Beam Solution of the Parabolic Wave Equation

We seek solution of the parabolic wave equation (2.117) in the form of Gaussian beams centred on a ray Ω, hence

$$W(s,\nu) = A(s)\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2}, \tag{2.119}$$

where $\Gamma = \Gamma(s)$ is an unknown-complex valued function. We find the derivatives of (2.119)

$$
\begin{aligned}
W_{,s} &= A_{,s}\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2} + \frac{1}{2}A\mathrm{i}\nu^2\Gamma_{,s}\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2}, && (2.120)\\
W_{,\nu} &= \mathrm{i}\nu\Gamma A\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2}, && (2.121)\\
W_{,\nu\nu} &= \mathrm{i}\Gamma A\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2} - \nu^2\Gamma^2 A\mathrm{e}^{\mathrm{i}\nu^2\Gamma/2}, && (2.122)
\end{aligned}
$$

and substitute into (2.117)

$$
\begin{aligned}
\frac{2\mathrm{i}}{v}\left(A_{,s} + \frac{1}{2}\mathrm{i}\nu^2 A_{,s}\right) + \mathrm{i}A\Gamma - A\nu^2\Gamma^2 - \frac{1}{v^3}\nu^2 v_{,nn} &= 0, && (2.123)\\
\mathrm{i}\left(\frac{2}{v}A_{,s} + A\Gamma\right) - A\nu^2\left(\frac{1}{v}\Gamma_{,s} + \Gamma^2 + \frac{1}{v^3}v_{,nn}\right) &= 0. && (2.124)
\end{aligned}
$$

Set

$$
\begin{aligned}
\Gamma_{,s} + v\Gamma^2 + \frac{1}{v^2}v_{,nn} &= 0, && (2.125)\\
A_{,s} + \frac{1}{2}vA\Gamma &= 0. && (2.126)
\end{aligned}
$$

First consider (2.125); it is a first order non-linear (Riccati type) ordinary differential equation which cannot be solved analytically. We will aim to solve it numerically, to make this easier we shall rewrite it as a second order ordinary differential equation by introducing a new complex valued function $q = q(s)$ with

$$
\begin{aligned}
\Gamma(s) &= \frac{1}{vq}q_{,s}, && (2.127)\\
\Gamma_{,s} &= \frac{1}{vq}q_{,ss} - \frac{1}{v^2 q}v_{,s}q_{,s} - \frac{1}{vq^2}q_{,s}^2. && (2.128)
\end{aligned}
$$

Now substitute into (2.125)

$$
\begin{aligned}
\frac{1}{vq}q_{,ss} - \frac{1}{v^2 q}v_{,s}q_{,s} - \frac{1}{vq^2}q_{,s}^2 + \frac{1}{vq^2}q_{,s}^2 + \frac{1}{v^2}v_{,nn} &= 0, && (2.129)\\
\frac{1}{vq}q_{,ss} - \frac{1}{v^2 q}v_{,s}q_{,s} + \frac{1}{v^2}v_{,nn} &= 0, && (2.130)\\
vq_{,ss} - q_{,s}v_{,s} + qv_{,nn} &= 0. && (2.131)
\end{aligned}
$$

This can be again simplified to a system of two linear first-order ordinary differential equations by setting $q_{,s} = vp$ where $p = p(s)$. We see $q_{,ss} = v_{,s}p + vp_{,s}$, now substitute into (2.131)

$$
\begin{aligned}
vv_{,s}p + v^2 p_{,s} - vv_{,s}p + qv_{,nn} &= 0, && (2.132)\\
p_{,s} &= -\frac{1}{v^2}qv_{,nn}. && (2.133)
\end{aligned}
$$

We can also write

$$\Gamma = \frac{1}{vq} q_{,s} = \frac{p}{q}. \tag{2.134}$$

Hence the system of two linear first-order ordinary differential equations is

$$q_{,s} = vp, \quad p_{,s} = -\frac{1}{v^2} q v_{,nn}. \tag{2.135}$$

The system of differential equations (2.135) will be used to calculate geometric spreading of the Gaussian beam. Hence, we have the necessary information to construct a dynamic ray tracing system. We use the solution to the Eikonal equation to generate a central ray, then at each iteration using the above system we calculate the geometric spreading of the wave around this central ray. However, before doing this, there still remains more discussion on the Gaussian beam solution.

We can find the solution to (2.126) by first using (2.127) and seeing

$$\Gamma = \frac{1}{vq}\frac{dq}{ds}, \tag{2.136}$$

$$\int v\Gamma ds = \int \frac{1}{q} dq = \ln q, \tag{2.137}$$

$$v\Gamma = \frac{d\ln q}{ds}, \tag{2.138}$$

$$\Gamma = \frac{1}{v}\frac{d\ln q}{ds}. \tag{2.139}$$

Now substitute into (2.126)

$$A_{,s} + \frac{1}{2} A \frac{d\ln q}{ds} = 0, \tag{2.140}$$

$$A_{,s} = -\frac{1}{2} A \frac{d\ln q}{ds}, \tag{2.141}$$

$$= -\frac{1}{2}\frac{A}{q}\frac{dq}{ds}, \tag{2.142}$$

$$\int \frac{A_{,s}}{A} ds = -\frac{1}{2}\int \frac{dq}{q}, \tag{2.143}$$

$$\ln A = -\frac{1}{2}\ln q + \text{const.}, \tag{2.144}$$

$$A(s) = \Psi q^{-1/2}(s). \tag{2.145}$$

With Ψ a complex constant which remains constant along the entire ray but may be different on different rays, as such, we denote $\Psi = \Psi(\phi)$ with ϕ the ray parameter.

We can also find an expression for $u = u(s, n, t)$ by using inserting $\nu^2 = \omega n^2$, (2.119), (2.134), and (2.145) into (2.118).

$$u(s,n,t) = \Psi \left(\frac{v}{q}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\omega(t-T)} \mathrm{e}^{\mathrm{i}(1/2)\omega n^2 (p/q)}, \tag{2.146}$$

$$= \Psi \left(\frac{v}{q}\right)^{1/2} \exp\left\{-\mathrm{i}\omega\left(t - \int_{s_0}^{s} \frac{ds}{V}\right) + \frac{\mathrm{i}p\omega n^2}{2q}\right\}. \tag{2.147}$$

Where $p = p(s)$ and $q = q(s)$ are the solutions to the system (2.135) The function $q = q^{1/2}(s)$ at the initial point $s = s_0$ we chose the principle branch of the square root.

$$q^{1/2}(s_0) = |q(s_0)|^{1/2} \mathrm{e}^{\frac{1}{2}\mathrm{i}\arg q(s_0)}, \quad \text{for} \quad -\pi < \arg q(s_0) \leq \pi. \tag{2.148}$$

Using $q(s) \neq 0$ for $s \geq s_0$, the function $\arg q(s)$ is continuous, hence for arbitrary $s \geq s_0$ we see

$$q^{1/2}(s) = |q(s)|^{1/2} \mathrm{e}^{\frac{1}{2}\mathrm{i}\arg q(s)}, \quad \text{for} \quad -\pi < \arg q(s) \leq \pi. \tag{2.149}$$

In the solution far from a ray the functions $\Gamma(s)$, $p(s)$ and $q(s)$ are real and the function $q(s)$ may vanish, in particular at a caustic. However, for The solutions concentrated close to a ray we require the functions $\Gamma(s)$, $p(s)$ and $q(s)$ to be complex valued, with $\mathrm{im}(p/q) > 0$, this ensures the concentration of solutions close to the rays. Therefore, we require the initial conditions for the system (2.135) to also be complex valued.

In terms of uniqueness we note that (2.119) is not the only solution to the parabolic wave equation. It is possible to construct an infinite number of other possible solution concentrated close to the rays. However, for the purposes of our investigation we consider only a basic mode

$$W^0(s,\nu) = q^{-1/2} \exp\left\{\frac{\mathrm{i}\nu^2 p}{2q}\right\}. \tag{2.150}$$

We do not need such a general equation for all the solutions.

As said before our solution is in the form of a Gaussian beam centred on a ray Ω with the amplitude decreasing exponentially with increasing distance from the ray Ω. We can construct a matrix $\pi = \pi(s)$ of linearly independent real solutions of the system (2.135)

$$\pi(s) = \begin{pmatrix} q_1(s) & q_2(s) \\ p_1(s) & p_2(s) \end{pmatrix}, \tag{2.151}$$

where each column corresponds to one solution. We specify the two solutions q_1, p_1 and q_2, p_2 by the initial conditions

$$\pi(s_0) = \begin{pmatrix} 1 & 0 \\ 0 & v^{-1}(s_0) \end{pmatrix}. \tag{2.152}$$

Notice that

$$\det \pi(s) = q_1 p_2 - p_1 q_2 = v^{-1}(s_0), \tag{2.153}$$

Or that $\det \pi(s)$ does not change along the ray for any s. We can express any complex solution of the system (2.135) as two linearly independent real solutions

$$q(s) = z_1 q_1(s) + z_2 q_2(s), \tag{2.154}$$

$$p(s) = z_1 p_1(s) + z_2 p_2(s), \tag{2.155}$$

with z_1 and z_2 are (for now) unspecified complex constants. After some analysis, we see that selecting these constants in such a way ensures important properties of Gaussian beams.

In our solution (2.147) we observe a fraction of p/q; we can write this as

$$\frac{p}{q} = \frac{z_1 p_1(s) + z_2 p_2(s)}{z_1 q_1(s) + z_2 q_2(s)}. \tag{2.156}$$

Now by separating the real and imaginary parts of we can write (2.147) in a useful way

$$u(s,n,t) = \Psi \left[\frac{v(s)}{q(s)}\right]^{1/2} \exp\left\{-\mathrm{i}\omega\,(t-T) + \frac{\mathrm{i}\omega n^2}{2}\left[\mathrm{Re}\left(\frac{p(s)}{q(s)}\right) + \mathrm{i}\,\mathrm{Im}\left(\frac{p(s)}{q(s)}\right)\right]\right\} \tag{2.157}$$

Now set

$$K(s) = v\mathrm{Re}\left(\frac{p(s)}{q(s)}\right), \tag{2.158}$$

$$L(s) = \left[\frac{\omega}{2}\mathrm{Im}\left(\frac{p(s)}{q(s)}\right)\right]^{-1/2}, \tag{2.159}$$

hence

$$u(s,n,t) = \Psi \left[\frac{v(s)}{q(s)}\right]^{1/2} \exp\left\{-\mathrm{i}\omega\,(t-T) + \frac{\mathrm{i}\omega n^2}{2v}K(s) - \frac{n^2}{L^2(s)}\right\}. \tag{2.160}$$

Physically $K(s)$ represents the curvature of the phase front of the Gaussian beam, $L(s)$ is a frequency dependent effective half width.

It is possible to write this solution depending only on two complex-valued constants, Ψ and ϵ. We introduce $\epsilon = z_1/z_2$ with $z_2 \neq 0$, thus

$$q = z_2(\epsilon q_1 + q_2), \tag{2.161}$$

$$p = z_2(\epsilon p_1 + p_2), \tag{2.162}$$

hence

$$\frac{p}{q} = \frac{\epsilon p_1 + p_2}{\epsilon q_1 + q_2}. \tag{2.163}$$

Now $K(s)$ and $L(s)$ depend only on one complex-valued constant ϵ. Hence, $u(s,n,t)$ depends only on Ψ and ϵ.

It is useful to write ϵ in a particular way

$$\epsilon = S_0 = \mathrm{i}\frac{\omega}{2v(s_0)}L_M^2. \tag{2.164}$$

Hence, with (2.153) we can write

$$\mathrm{Im}\left(\frac{p(s)}{q(s)}\right) = \mathrm{Im}\left(\frac{\epsilon p_1 + p_2}{\epsilon q_1 + q_2}\right), \tag{2.165}$$

$$= \mathrm{Im}\left(\frac{S_0 p_1 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}p_1 + p_2}{S_0 q_1 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}q_1 + q_2}\right), \tag{2.166}$$

$$= \mathrm{Im}\left(\frac{\left[-(S_0 q_1 + q_2) + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}q_1\right]\left[(S_0 p_1 + p_2) + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}p_1\right]}{(S_0 q_1 + q_2)^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2}\right), \tag{2.167}$$

$$= \frac{(S_0 p_1 + p_2)\frac{\omega L_M^2}{2v(s_0)}q_1 - (S_0 q_1 + q_2)\frac{\omega L_M^2}{2v(s_0)}p_1}{(S_0 q_1 + q_2)^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2}, \tag{2.168}$$

$$= \frac{\frac{\omega L_M^2}{2v(s_0)}\left[S_0 p_1 q_1 + p_2 q_1 - S_0 q_1 p_1 - q_2 p_1\right]}{(S_0 q_1 + q_2)^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2}, \tag{2.169}$$

$$= \frac{\omega L_M^2}{2v^2(s_0)}\left[(S_0 q_1 + q_2)^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2\right]^{-1}. \tag{2.170}$$

Thus for $L(s)$ using (2.159), we have

$$L(s) = \left[\frac{\omega^2 L_M^2}{4v(s_0)}\left(\frac{1}{(S_0 q_1 + q_2)^2 + \frac{\omega^2 L_M^4}{4v^2(s_0)}q_1^2}\right)\right]^{-1/2}, \tag{2.171}$$

$$= \left[\frac{\omega^2 L_M^2}{4v^2(s_0)(S_0 q_1 + q_2)^2 + \omega^2 L_M^4 q_1^2}\right]^{-1/2}, \tag{2.172}$$

$$= \left[\frac{4v^2(s_0)(S_0 q_1 + q_2)^2 + \omega^2 L_M^4 q_1^2}{\omega^2 L_M^2}\right]^{1/2}, \tag{2.173}$$

$$= \left[L_M^2 q_1^2 + \left(\frac{2v(s_0)}{\omega L_M}\right)^2 (S_0 q_1 + q_2)^2\right]^{1/2}. \tag{2.174}$$

Now we can specify the conditions which must be satisfied along the ray

1. $q(s) \neq 0$. This guarantees the regularity of the Gaussian beam along the whole ray with finite amplitude at caustics.

2. $\mathrm{Im}(p/q) > 0$. This ensures the solution is concentrated close to the rays. We see this is in fact the case by observing in (2.170) the denominator will always be positive and we always have a positive frequency ω, so we only require that $L_M \neq 0$.

We can show the first condition is also true for $L_M \neq 0$ by calculating $q(s)q^*(s) \neq 0$ where $q^*(s)$ is the complex conjugate of $q(s)$. Using ϵ we have

$$q = S_0 q_1 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)} q_1 + q_2, \tag{2.175}$$

$$q^* = S_0 q_1 + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)} q_1 + q_2, \tag{2.176}$$

and hence,

$$\begin{aligned}
qq^* =& \left(\left(S_0 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right) q_1 + q_2\right)\left(\left(S_0 + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right) q_1 + q_2\right), && (2.177)\\
=& \left(S_0 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right)\left(S_0 + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right) q_1^2 && (2.178)\\
&+ \left(S_0 - \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right) q_1 q_2 + \left(S_0 + \mathrm{i}\frac{\omega L_M^2}{2v(s_0)}\right) q_1 q_2 + q_2^2, \\
=& S_0^2 q_1^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2 + 2S_0 q_1 q_2 + q_2^2, && (2.179)\\
=& (S_0 q_1 + q_2)^2 + \left(\frac{\omega L_M^2}{2v(s_0)}\right)^2 q_1^2 \neq 0. && (2.180)
\end{aligned}$$

We now aim to give a geometric interpretation of S_0 and L_M. To begin with, for simplicity, we assume the medium is homogeneous with constant velocity v_0. The solution to the system (2.135) with initial condition (2.152) is then

$$q_1(s) = 1, \quad q_2(s) = s - s_0, \quad p_1(s) = 0, \quad p_2(s) = v_0^{-1} \tag{2.181}$$

hence (2.174) reduces to

$$\begin{aligned}
L(s) =& \left[L_M^2 + \left(\frac{2v(s_0)}{\omega L_M}\right)^2 (S_0 + s - s_0)^2\right]^{1/2}, && (2.182)\\
L^2(s) =& L_M^2 + \left(\frac{2v(s_0)}{\omega L_M}\right)^2 (S_0 + s - s_0)^2, && (2.183)\\
\frac{L^2(s)}{L_M^2} =& 1 + \left(\frac{2v(s_0)}{\omega L_M^2}\right)^2 (S_0 + s - s_0)^2, && (2.184)\\
\frac{L^2(s)}{L_M^2} -& \left(\frac{2v(s_0)}{\omega L_M^2}\right)^2 (S_0 + s - s_0)^2 = 1. && (2.185)
\end{aligned}$$

This implies $L(s)$ is a hyperbola with a minimum at $s = s_M$, from $s_m - s_0 + S_0 = 0$. L_M is the effective half width of the beam at the point $s = s_M$; this means L_M is the minimum effective half width of the beam. S_0 is the distance of the minimum of the function $L(s)$ from the point $s = s_0$.

Consider now beams in a inhomogeneous medium with a minimum width at $s = s_0$. From this, We have

$$S_0 = 0, \quad \epsilon = -\mathrm{i}\frac{\omega L_M^2}{2v(s_0)}, \tag{2.186}$$

with (2.174) we see

$$L(s) = \left[L_M^2 q_1^2 + \left(\frac{2v(s_0)}{\omega L_M} \right)^2 q_2^2 \right]^{1/2}. \tag{2.187}$$

Assuming a homogeneous medium we have

$$L(s) = \left[L_M^2 + \left(\frac{2v(s_0)}{\omega L_M} \right)^2 (s - s_0)^2 \right]^{1/2}. \tag{2.188}$$

By choosing this ϵ with $L_M \neq 0$ we have seen conditions I and II ($q(s) \neq 0$ and $\mathrm{Im}(p/q) > 0$ respectfully) are automatically satisfied. However, we are still free to choose any non-trivial value for the quantity L_M. For our purposes of numerically solving the system (2.135) it is desirable to have beams which are as narrow as possible. In this case, we choose L_M which gives at a receiver a minimum value of the quantity $L(s)$. To achieve this simply find

$$\begin{aligned}
\frac{\partial L(s)}{\partial L_M} =& \frac{2L_M q_1^2 - \frac{8v^2(s_0)q_2^2}{\omega^2 L_M^3}}{2\left(L_M^2 q_1^2 + \frac{4v^2(s_0)q_2^2}{\omega^2 L_M^2} \right)} = 0, && (2.189)\\
2L_M q_1^2 =& \frac{8v^2(s_0)q_2^2}{\omega^2 L_M^3}, && (2.190)\\
2L_M^4 q_1^2 =& \frac{8v^2(s_0)q_2^2}{\omega^2}, && (2.191)\\
L_M^4 =& \frac{4v^2(s_0)}{\omega^2}\frac{q_2^2}{q_1^2}, && (2.192)\\
L_M^{\mathrm{opt}} =& \left(\frac{2v(s_0)}{\omega} \right)^{1/2} \left| \frac{q_2}{q_1} \right|^{1/2}. && (2.193)
\end{aligned}$$

Assuming a homogeneous medium with a constant velocity and $S_0 = 0$, the narrowest points of all hyperbole correspond to $s = s_0$. For $s \neq s_0$, the width of the hyperbole is controlled by L_M. Small values for the quantity L_M give narrow beams close to $s = s_0$, but quickly increasing width away from here. Large values of L_M give broad width close to $s = s_0$, but only increase slowly. L_M^{opt} gives a narrower beamwidth at any $s \neq s_0$ than any other L_M. By making the substitution of L_M^{opt} into (2.187) and (2.188) we find the formula for the envelope of all Gaussian beams, firstly for an inhomogeneous medium

$$L^{\text{env}}(s) = \left[\left(\frac{2v(s_0)}{\omega} \right) q_1 q_2 + \frac{4v^2(s_0)}{\omega^2} \frac{1}{\frac{2v(s_0)q_2}{\omega q_1}} q_2^2 \right]^{1/2}, \tag{2.194}$$

$$= \left[\left(\frac{2v(s_0)}{\omega} \right) q_1 q_2 + \frac{2v(s_0)}{\omega} q_1 q_2 \right]^{1/2}, \tag{2.195}$$

$$= 2 \left[\frac{v(s_0)}{\omega} q_1 q_2 \right]^{1/2}, \tag{2.196}$$

and for a homogeneous medium with constant velocity v_0

$$L^{\text{env}}(s) = 2 \left[\frac{v_0}{\omega} (s - s_0) \right]^{1/2}. \tag{2.197}$$

We now have everything needed to numerically perform dynamic ray tracing in a inhomogeneous medium. As we obtain additional information about the ray system from the use of Gaussian beams we can call this beam tracing.

3 Numerical Procedures

In this section, we will delve into the details of numerical procedures used in solving our ray tracing (2.49) and geometric spreading (2.135) systems. Combining these results yield dynamic ray tracing or beam tracing. We will focus on two specific techniques: the Euler method and the fourth-order Runge-Kutta (RK4) method. We will also introduce methods of performing appropriate boundary reflections, simulating the ocean surface and floor. In addition to this we will examine and account for range dependent sound speed profiles, this will improve shallow water and long range models where sound speed varies the most with range.

We can break down a numerical procedure into the following steps steps:

1.

3.1 Ray Tracing

3.1.1 Euler Method

3.1.2 RK4 Method

3.1.3 Boundary Conditions & Reflections

3.1.4 Range Dependent Sound Speed Profiles

In observation of real sound speed profiles we see the sound speed varies strongly with depth and little with suitably small range and large depth. However, for long range and shallow water applications, it is necessary to have range dependent sound speed.

3.2 Dynamic Ray Tracing

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