6장 AR 모델

Stationary Processes

- often a time series has same type of random behavior from one time period to the next
  - outside temperature: each summer is similar to the past summers
  - interest rates and returns on equities
- *stationary stochastic processes* are probability models for such series
- process stationary if behavior unchanged by shifts in time
- a process is *weakly stationary* if its mean, variance, and covariance are unchanged by time shifts
- thus $X_1, X_2, \dots$ is a weakly stationary process if

$$\boxed{\begin{array}{l} E(X_i) = \mu \quad \text{(a constant) for all } i \\ Var(X_i) = \sigma^2 \quad \text{(a constant) for all } i \\ Corr(X_i, X_j) = \rho(|i-j|) \text{ for all } i \text{ and } j \text{ for some function } \rho \end{array}}$$

- the correlation between two observations depends only on the time distance between them (called the lag)

Stationary Processes

- example: correlation between X_2 and X_5 = correlation between X_7 and X_{10}
- ρ is the correlation function

  Note that $\rho(h) = \rho(-h)$
- covariance between X_t and X_{t+h} is denoted by $\gamma(h)$
- $\gamma(\cdot)$ *is* called the autocovariance function
- Note that $\gamma(h) = \sigma^2 \rho(h)$ and that $\gamma(0) = \sigma^2$ since $\rho(0) = 1$

  $\gamma(h) = \gamma(-h)$
- many financial time series not stationary
- but the *changes* in these time series may be stationary: $z_t = y_t - y_{t-1}$ or $z_t = (1-B)y_t$
- Lag operator B.

Expected Value and Variance

- Continuous variable

$$\text{Mean:}\quad E(X) = \mu = \int_{-\infty}^{\infty} x f(x)dx$$

$$\text{Variance:}\quad V(X) = E[(X-\mu)^2] = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Correlation

- Let X and Y be jointly distributed random variables. The correlation between X and Y is:

$$\boxed{\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \qquad (-1 \le \rho \le 1)}$$

$$Cov(X,Y) = E(X - E(X))(Y - E(Y))$$

$\rho = 1$: X and Y have a perfect <u>positive linear</u> relationship.

$\rho = -1$: X and Y have a perfect <u>negative linear</u> relationship.

$\rho = 0$: X and Y have <u>no linear</u> relationship.

- $Corr(X,X) = \dfrac{Cov(X,X)}{\sqrt{V(X) \cdot V(X)}} = 1$

- For time series data $X_1, X_2, \ldots,$

$r(h) = Cov(X_i, X_{i+h})$ for $h \neq 0$ is **auto-covariance** of X

$\rho(h) = Corr(X_i, X_{i+h})$ for $h \neq 0$ is **autocorrelation** of X (or ACF)

$h : time - lag$

Correlation

- estimate autocovariance and autocorrelation with

$$\gamma(h) = n^{-1}\sum_{j=1}^{n-h}(x_{j+h} - \bar{x})(x_j - \bar{x}) \quad \text{and} \quad \hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \quad h = 1,\ 2, \ldots:$$

Note: $\rho(k) = \dfrac{r(k)}{r(0)}$ and $\hat{\rho}(h)$ is *SACF*.

Independence Condition

- X and Y are independent random variables if, for every x and y:

Discrete case:	$P(x, y) = P(x)P(y)$ or $P(x \mid y) = P(x)$
Continuous case:	$f(x, y) = f(x)f(y)$ or $f(x \mid y) = f(x)$

- X and Y are independent, they are always uncorrelated ($Corr(X, Y) = 0$)

- But, the other way is not held all the time.

- Uncorrelated two random variables are always independent if they follow a normal distribution.

Independent and Identical Distribution (IID)

- $X_1, X_2, \ldots$ are *i.i.d.* if they are independent and identical.
- $X_1, X_2, \ldots$ are identical if they are from same distribution and same parameter values.
- Time series data are usually dependent. ⇨ necessary for time series modeling.

Conditional Expectation

- If $g(Y)$ is a function of Y, the conditional expectation of $g(Y)$ given that $X = x$ is denoted by $E(g(Y) \mid x)$:

$$\boxed{\text{Continuous case: } E(g(Y) \mid x) = \int_{-\infty}^{\infty} g(y) f(y \mid x) dy}$$

- If X and Y are any two random variables, then $E[X] = E[E(X \mid Y)]$.

Proof) $E[E(X \mid Y)] = \int [\int xf(x \mid y)dx] f(y)dy \;\; = \int \int xf(x, y)dxdy \;\; = \int xf(x)dx$

- For two random variables X and Y, $V(X) = E[V(X \mid Y)] + V[E(X \mid Y)]$.

Partial Correlation

- Used to measure the correlation between X and Y deleting (adjusting) the effect of Z

$$\rho_{XY,Z} = \frac{E\{[X - E(X \mid Z)][Y - E(Y \mid Z)]\}}{\sqrt{E[X - E(X \mid Z)]^2 \cdot E[Y - E(Y \mid Z)]^2}}$$

- Compute the correlation using the error terms of regression models

  error term of a regression model: $X - E(X \mid Z) \;\Rightarrow\; X^*$

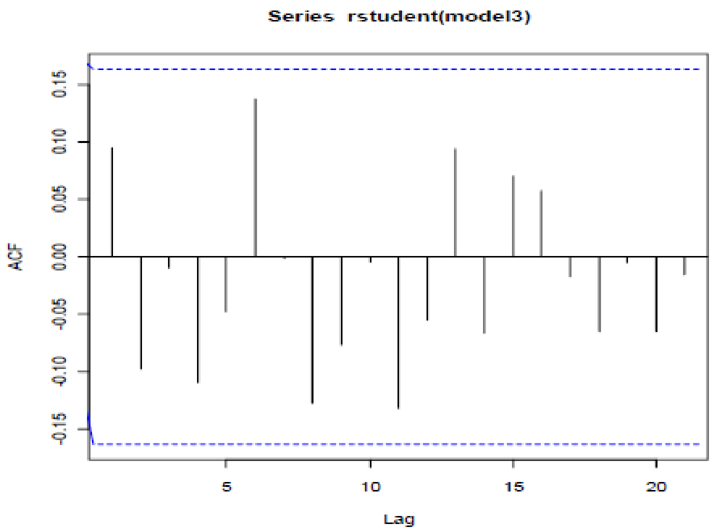
  error term of a regression model: $Y - E(Y \mid Z) \;\Rightarrow\; Y^*$

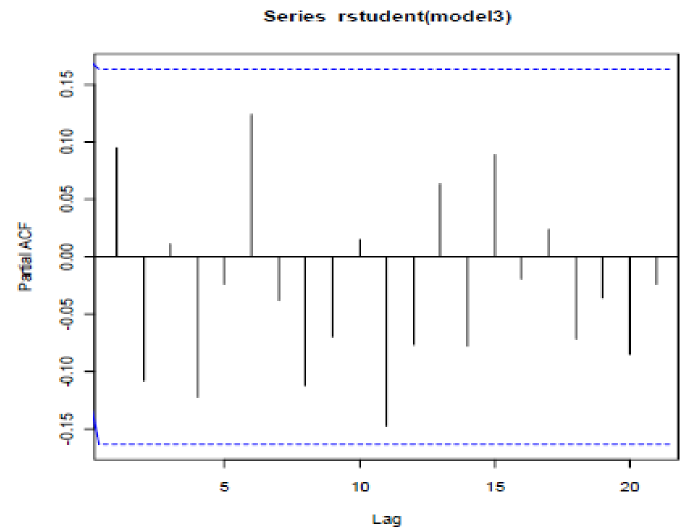
- For time series data $X_1, X_2, \ldots,$ *partial correlation function (PACF)* ϕ_{kk} is correlation between X_t and X_{t+k} deleting the effect of $X_{t+1}, X_{t+2}, \ldots, X_{t+k-1}$

$$\phi_{kk} = Corr(X_t^*, X_{t+k}^*),$$

where
$$X_t^* = X_t - E(X_t \mid X_{t+1}, X_{t+2}, \ldots, X_{t+k-1}) = X_t - (a_1 X_{t+1} + a_2 X_{t+2} + \cdots + a_{k-1} X_{t+k-1})$$
$$X_{t+k}^* = X_{t+k} - E(X_{t+k} \mid X_{t+1}, X_{t+2}, \ldots, X_{t+k-1}) = X_{t+k} - (b_1 X_{t+1} + b_2 X_{t+2} + \cdots + b_{k-1} X_{t+k-1})$$

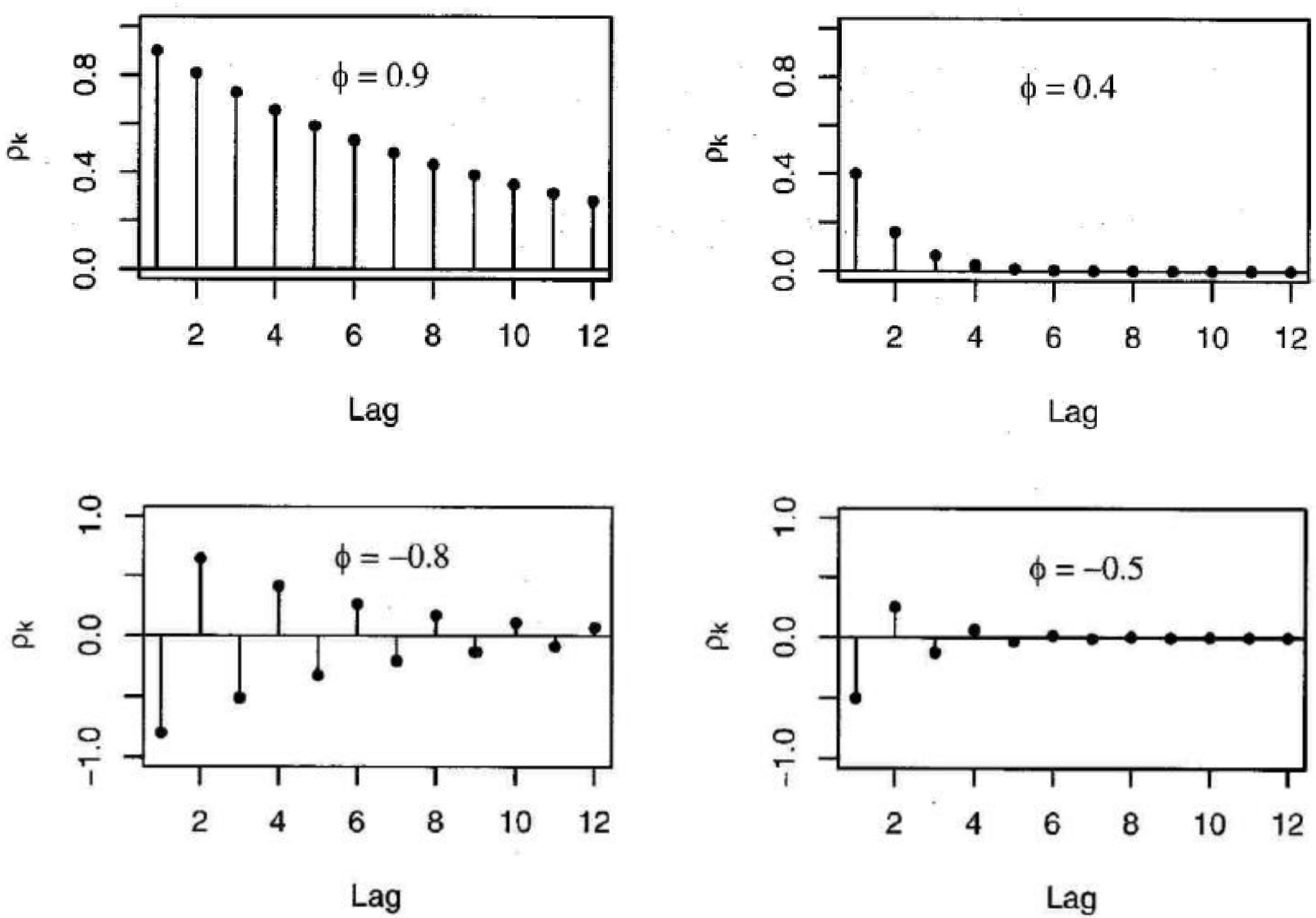
SACF and SPACF





- Sample PACF (SPACF) is similar.
- SACF and SPACF are applied to lag variable selection in ARMA models.
- ACF for several AR(1) model: $Y_t = \phi Y_{t-1} + e_t$

SACF and SPACF



White Noise

- simplest example of stationary process:
  No Correlation Case

- $X_1, X_2, \ldots$ is White noise or $WN(\mu, \sigma^2)$ if

  Note: Distribution specification is not required.
  It does not have to be a normal distribution

$$
\boxed{
\begin{array}{l}
E(X_i) = \mu \quad \text{for all } i \\
Var(X_i) = \sigma^2 \quad \text{(a constant) for all } i \\
Corr(X_i, X_j) = 0 \quad \text{for all} \quad i \neq j
\end{array}
}
$$

- If $X_1, X_2, \ldots$ IID normal then process is *Gaussian white noise process*

- white noise process is weakly stationary with

$$
\boxed{
\begin{array}{ll}
 & \rho(0) = 1 \;\; and \;\; \rho(t) = 0 \text{ if } t \neq 0 \\
\text{so that} & \gamma(0) = \sigma^2 \; and \;\; \gamma(t) = 0 \;\; \text{if } t \neq 0
\end{array}
}
$$

- WN is uninteresting in itself
  – but is the building block of important models ⇨ usually used as error terms.

- It is interesting to know if a financial time series, e.g., of net returns, is WN.

Gauss-Markov Theorem

- The OLS estimates β_0, β_1 for $y = \beta_0 + \beta_1 x + \varepsilon$ is BLUE under the three conditions:

$$E(\varepsilon) = 0$$

$$V(\varepsilon) = \sigma^2$$

$$Corr(\varepsilon_i, \varepsilon_j) = 0$$

- BLUE is Best Linear Unbiased Estimator.

Auto Regressive (AR) processes

AR(1) processes

- time series models with correlation built from WN
- in AR processes y_t is modeled as a weighted average of past observations plus a white noise "error"
- AR(1) is simplest AR process
- $\varepsilon_1, \varepsilon_2, \ldots$ are $WN(0, \sigma_\varepsilon^2)$
- $y_1, y_2, \ldots$ is an AR(1) process if

  $y_t - \mu = \phi(y_{t-1} - \mu) + \varepsilon_t$ for all t

> <u>three parameters</u>:
> μ : mean, σ_ε^2 : variance of one-step ahead prediction errors
> ϕ : a correlation parameter

- $y_t = (1-\phi)\mu + \phi y_{t-1} + \varepsilon_t$
- compare with linear regression model, $y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$

Auto Regressive (AR) processes

- In general, $\beta_1 = corr(x, y) \times \frac{\sigma_y}{\sigma_x}$, where $y = \beta_0 + \beta_1 x + \varepsilon$

- If $y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$, then $\alpha_1 = corr(y_{t-1}, y_t) \times \frac{\sigma_{y_t}}{\sigma_{y_{t-1}}}$

- Under the stationary condition, $\alpha_1 = ?$

- Range of correlation?

Auto Regressive (AR) processes

- $\beta_0 = (1-\phi)\mu$ is called the "constant" in computer output
- μ is called the "mean" in the output
- When $|\phi| < 1$ then

$$\boxed{y_t = \mu + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots = \mu + \sum_{h=0}^{\infty}\phi^h\varepsilon_{t-h}}$$

- infinite moving average [MA(∞)] representation
- If $|\phi| < 1$, then $y_1, \ldots$ is a weakly stationary process

  since $|\phi| < 1$, $\phi^h \to 0$ as the lag $h \to \infty$
- *Note*: if $\{\varepsilon_t\}$ is a strictly *<u>independent</u>* zero-mean random variable, then a stationary time series $\{y_t\}$ is ***linear***. Otherwise the series is ***nonlinear***.

Auto Regressive (AR) processes

Properties of a stationary AR(1) process

- When $|\phi| < 1$ (stationarity), then

$$E(y_t) = \mu \quad \forall t$$

$$\gamma(0) = Var(y_t) = \frac{\sigma_\varepsilon^2}{1-\phi^2} \quad \forall t$$

$$\gamma(h) = Cov(y_t, y_{t+h}) = \frac{\sigma_\varepsilon^2 \phi^{|h|}}{1-\phi^2} \quad \forall t$$

$$\rho(h) = Corr(y_t, y_{t+h}) = \phi^{|h|} \quad \forall t$$

Only if $|\phi| < 1$ and only for AR(1) processes

- if $|\phi| \geq 1$, then the AR(1) process is nonstationary, and the mean, variance, and correlation are not constant

Auto Regressive (AR) processes

- Formulas 1–4 can be proved using

$$y_t = \mu + \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots + = \mu + \sum_{h=0}^{\infty}\phi^h\varepsilon_{t-h}$$

For example

$$Var(y_t) = Var(\sum_{h=0}^{\infty}\phi^h\varepsilon_{t-h}) = \sigma_\varepsilon^2\sum_{h=0}^{\infty}\phi^{2h} = \frac{\sigma_\varepsilon^2}{1-\phi^2}$$

- Also, for $h > 0$

$$\gamma(h) = Cov(\sum_{i=0}^{\infty}\varepsilon_{t-i}\phi^i, \sum_{j=0}^{\infty}\varepsilon_{t+h-j}\phi^j) = \frac{\sigma_\varepsilon^2\phi^{|h|}}{1-\phi^2}$$

- distinguish between σ_ε^2 = variance of $\varepsilon_1, \varepsilon_{2,\ldots}$ and $\gamma(0)$ = variance of $y_1, y_2, \ldots$

Non-Stationary AR(1) Processes

Random Walk

- if $\phi = 1$ (*unit root case*) then $y_t = y_{t-1} + \varepsilon_t$
- not stationary
- random walk process
- $y_t = y_{t-1} + \varepsilon_t = (y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots = y_0 + \varepsilon_1 + \ldots + \varepsilon_t = y_0 + \sum_{i=1}^{t} \varepsilon_i$
- start at the process at an arbitrary point y_0 then $E(y_t | y_0) = y_0$ for all t
- and $Var(y_t | y_0) = t\sigma_\varepsilon^2$
- A shock on ε_t on time $t = 0$ is transient in the stationary process, whereas it is permanent in the unit root series.
- Stationary processes tend to have mean reversion, whereas unit root processes tend to move irregularly off the mean (there is even no stationary level of mean to revert).

No mean reversion

- Unit root processes include the unpredictable stochastic trend.

Non-Stationary AR(1) Processes

When $|\phi| > 1$, an AR(1) process has explosive behavior

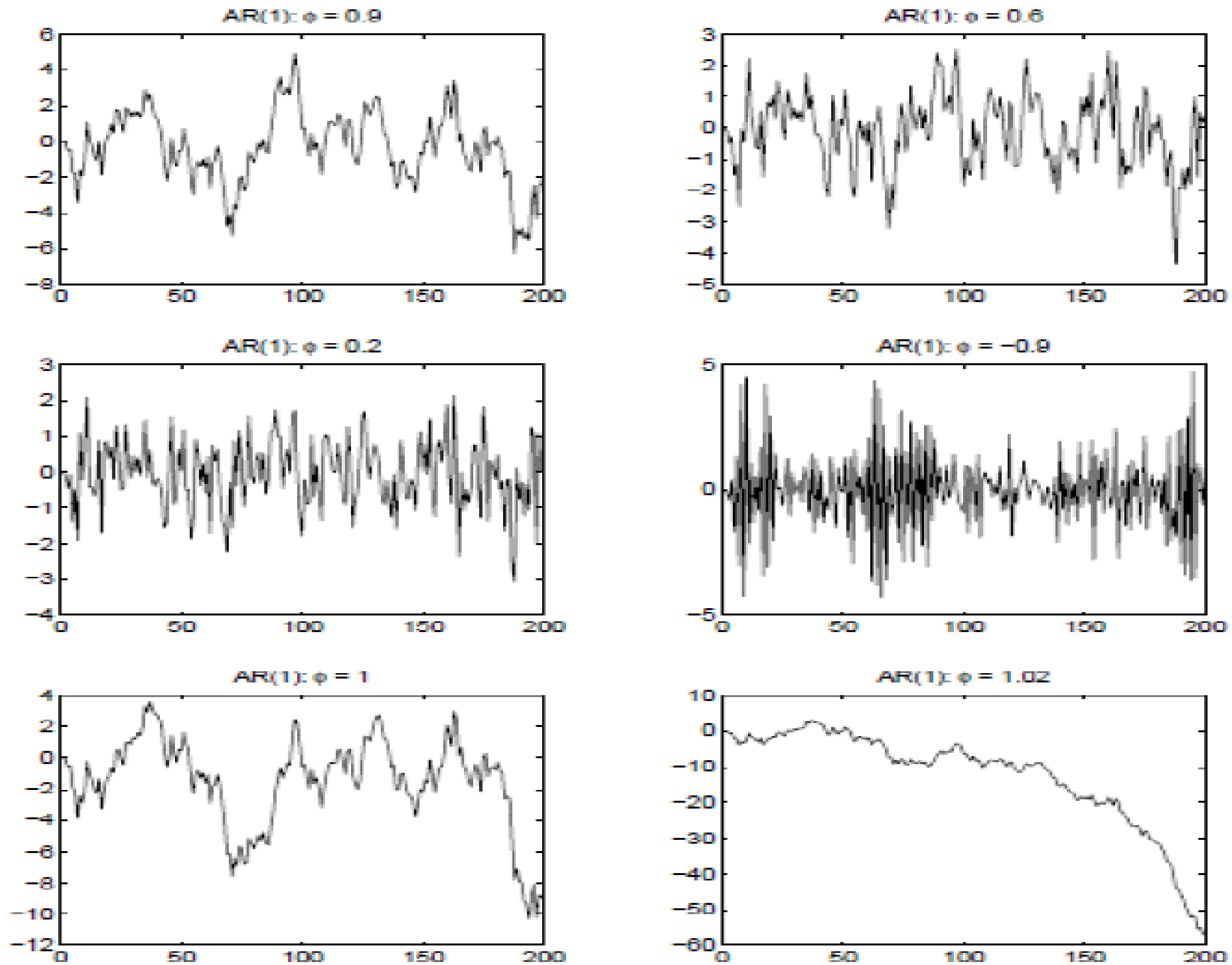
- Suppose an explosive AR(1) process starts at $y_0 = 0$ and has $\mu = 0$ Then

$$\boxed{\begin{aligned} y_t &= \phi y_{t-1} + \varepsilon_t = \phi(\phi y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \quad = \phi^2 y_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &= \ldots \quad = \varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \ldots + \phi^{t-1}\varepsilon_1 + \phi^t y_0 \end{aligned}}$$

- Therefore, $E(y_t) = \phi^t y_0$ and
- $Var(y_t) = \sigma_\varepsilon{}^2(1 + \phi^2 + \phi^4 + \ldots + \phi^{2(t-1)}) = \sigma_\varepsilon{}^2 \dfrac{\phi^{2t} - 1}{\phi^2 - 1}$

  Since $|\phi| > 1$, variance increases geometrically fast at $t \to \infty$
- increasing variance makes the random walk "wander" AR(1) processes when $|\phi| > 1$
- Explosive AR processes not widely used in econometrics since economic growth usually is not explosive.

Non-Stationary AR(1) Processes



(n = 200)

Unit Root Tests

- Test for $H_0: \phi = 1$ v.s. $H_1: |\phi| < 1$

  $H_0: \phi = 1 \Rightarrow$ nonstationary $\qquad H_1: |\phi| < 1 \Rightarrow$ stationary

- Dickey-Fuller Test $\qquad y_t = \phi y_{t-1} + \varepsilon_t$, where $\varepsilon_t \sim WN(0, \sigma^2)$

- If $\phi = 1$, y_t are referred to as integrated with order one or I(1).

  $\Rightarrow \; y_t - y_{t-1} = \Delta y_t = \varepsilon_t$

  Note: ε_t is not integrated, i.e., I(0).

- Order of integration: I(d).

  It reports the minimum number of differences required to obtain a stationary series.

  A time series is integrated of order d if $(1-B)^d X_t$ yields a stationary process.

- Three types
  - Zero mean: $y_t = \phi y_{t-1} + \varepsilon_t$
  - Single mean: $y_t = \mu + \phi y_{t-1} + \varepsilon_t$
  - Trend: $y_t = \mu + \delta \cdot t + \phi y_{t-1} + \varepsilon_t$

Unit Root Tests

- Augmented Dickey-Fuller test

$$\Delta y_t = \phi y_{t-1} + \sum_{i=1}^{p} \alpha_i \Delta y_{t-i} + \varepsilon_t \qquad (1)$$

H_0: ϕ=0 (unit root exists) # coefficient of lag 1 of y: (z.lag.1)

- Why? Let us use AR(3) example

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} = \varepsilon_t$$

$$y_t - \rho_1 y_{t-1} - \rho_2 \Delta y_{t-2} - \rho_3 \Delta y_{t-3} = \varepsilon_t \qquad (2)$$

H_0: ρ_1=1 (unit root exists) Here $\rho_1 = \phi_1 + \phi_2 + \phi_3$

Subtract y_{t-1} on both sides of (2), then (1) is obtained, where $\phi = \rho_1 - 1$

Unit Root Tests

- The following SAS code performs augmented Dickey-Fuller tests with autoregressive orders 2 and 5.

```
proc arima data=test;
identify var=x stationarity=(adf=(2,5));
run;
```

- Phillips-Perron test is using nonparametric estimation skill
- Unlike the null hypothesis of the Dickey-Fuller and Phillips-Perron tests, the null hypothesis of the KPSS states that the time series is stationary.
- R-code for unit root tests

```
library(tseries);
######
x=rnorm(1000); y=diffinv(x); # x has no unit-root, but y contains a unit-root.
adf.text(x); pp.test(x); kpss.test(x);
```

Estimation

- AR(1) model is a linear regression model
- one creates a lagged variable in y and uses this as the "x-variable"
- The least squares estimation: minimize $$\sum_{t=2}^{n}\left[\{y_t-\mu\}-\{\phi(y_{t-1}-\mu)\}\right]^2$$
- If the errors are *Gaussian* white noise then OLS =MLE
- In SAS, use the "AUTOREG" or the "ARIMA" procedure
- SAS provides maximum likelihood estimates (ML), unconditional least squares estimates (ULS), Yule-Walker estimates (YW: default), iterative Yule-Walker estimates (ITYW)

```
proc autoreg data=b;
    model y = time / nlag=2 method=ml     backstep;
    output out=p p=yhat pm=ytrend
    lcl=lcl ucl=ucl;
    run;
```

Estimation

- The output data set includes all the variables in the input data set, the forecast values (YHAT), the predicted trend (YTREND), and the upper (UCL) and lower (LCL) 95% confidence limits.
- Backstep: stepwise variable selection method
- For more about Yule-Walker estimates, see Gallant and Goebel (1976)

Residuals

$$\hat{\varepsilon}_t = y_t - \hat{\mu} - \hat{\phi}(y_{t-1} - \hat{\mu})$$

- estimate $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ since $\varepsilon_t = y_t - \mu - \phi(y_{t-1} - \mu)$
- used to check that $y_1, y_2, \ldots, y_n$ is an AR(1) process
- autocorrelation in residuals evidence against AR(1) assumption
- to test for residual autocorrelation use SAS's autocorrelation plots
- can also use the Ljung-Box test
  null hypothesis is that autocorrelations up to a specified lag are zero

Autocorrelation Tests

- **Durbin-Watson Test**

  Suppose $e_t = \rho e_{t-1} + \varepsilon_t$, where $|\rho| < 1$ with ε_t is iid normal distributed.

  Test $H_0 : \rho = 0$ is testing no first order autocorrelation in e_t .

R-code:

- Autocorrelation test for OLS residual

```
library(lmtest)
dwtest(Revenue~Assets, data=bankdat)
```

- The number of lags can be specified using the max.lag argument

```
library(car)
results =lm(Y ~ x1 + x2)
durbin.watson(results,max.lag=2)
```

SAS code:

- Autocorrelation test for OLS residual

```
proc autoreg data=a;
	model y = time / dw=4 dwprob;
run;
```

Autocorrelation Tests

- Runs test, Portmanteau test: Box test (Box-Pierce, Ljung-Box)

```
### Runs test
library(tseries); x=rnorm(50); x1=factor(ifelse(x>=median(x), 1, 0));
runs.test(x1, a='less');

data(milk); x2=factor(ifelse(milk>=median(milk), 1, 0));
runs.test(x2, a='less');


### Portmanteau test: Box test (Box-Pierce, Ljung-Box)
Box.test(milk, t=c('B'));  # Box-Pierce test
Box.test(milk, t=c('L'))   # Ljung-Box test

y=arima.sim(list(order=c(1,0,0), ar=0.4), n=300);  #AR(1) simulation
ts.plot(y);
a=arima(y,order=c(1,0,0));  # AR(1) estimation
Box.test(a$residuals, lag=2);
Box.test(a$residuals, lag=2, type="Ljung-Box")
```

Example : GE daily returns

- The SAS output comes from running the following program (**proc autoreg**)

```
options linesize = 72 ;
data ge ;
infile 'c:\courses\or473\data\ge.dat' ;
input close ;
logP = log(close) ;
logR = dif(logP) ;
run ;
title 'GE - Daily prices, Dec 17, 1999 to Dec 15, 2000' ;
title2 'AR(1)' ;
proc autoreg ;
model logR =/nlag = 1 ;
run ;
```

AR(p) Models

- y_t is AR(p) process if $$\boxed{(y_t-\mu)=\phi_1(y_{t-1}-\mu)+\phi_2(y_{t-2}-\mu)+\ldots+\phi_p(y_{t-p}-\mu)+\varepsilon_t}$$
- here $\varepsilon_1,\ldots,\varepsilon_n$ is $WN(0,\sigma_\varepsilon^2)$
- multiple linear regression model with lagged values of the time series as the "x-variables"
- model can be re-expressed as
$$y_t=\beta_0+\phi_1 y_{t-1}+\ldots+\phi_p y_{t-p}+\varepsilon_t$$
$$\text{here} \quad \beta_0=\{1-(\phi_1+\ldots+\phi_p)\}\,\mu$$
- least-squares estimator minimizes $$\boxed{\sum_{t=p+1}^{n}\{y_t-(\beta_0+\phi_1 y_{t-1}+\ldots+\phi_p y_{t-p})\}^2}$$
- least-squares estimator can be calculated using a multiple linear regression program
- one must create "x-variables" by lagging the timeseries with lags 1 throught p
- easier to use the ARIMA command in SAS's AUTOREG procedures
- these do the lagging automatically

Forecasting

- AR models can forecast future values
- consider forecasting using an AR(1) process
- have data $y_1, \ldots, y_n$
- and estimates $\hat{\mu}$ and $\hat{\phi}$
- remember $y_{n+1} = \mu + \phi(y_n - \mu) + \varepsilon_{n+1}$ and $E(\varepsilon_{n+1} \,|\, y_1, \ldots, y_n) = 0$

  so we estimate y_{n+1} by $\hat{y}_{n+1} := \hat{\mu} + \hat{\phi}(y_n - \hat{\mu})$

  and y_{n+2} by $\hat{y}_{n+2} := \hat{\mu} + \hat{\phi}(\hat{y}_{n+1} - \hat{\mu}) = \hat{\mu} + \hat{\phi}\left\{\hat{\phi}(y_n - \hat{\mu})\right\}$ etc.
- in general, $\hat{y}_{n+k} = \hat{\mu} + \hat{\phi}^k (y_n - \hat{\mu})$
- if $\hat{\phi} < 1$ then as k increases forecasts decay exponentially fast to $\hat{\mu}$
- forecasting general AR(p) processes is similar

Forecasting

- Example: for an AR(2) process
  - $y_{n+1} = \mu + \phi_1(y_n - \mu) + \phi_2(y_{n-1} - \mu) + \varepsilon_{n+1}$
  - therefore $\hat{y}_{n+1} := \hat{\mu} + \hat{\phi}_1(y_n - \mu) + \hat{\phi}_2(y_{n-1} - \hat{\mu})$

    $\hat{y}_{n+2} := \hat{\mu} + \hat{\phi}_1(\hat{y}_{n+1} - \hat{\mu}) + \hat{\phi}_2(y_n - \hat{\mu})$, etc.
- the forecasts can be generated automatically by statistical software SAS

Forecasting using R

- Forecasting

```
y=arima.sim(list(order=c(2,0,0), ar=c(0.7, 0.2)), n=300);  #AR(2) simulation
ts.plot(y);

a=arima(y,order=c(2,0,0));  # AR(2) estimation

predict(arima(y, order=c(2,0,0)), n.ahead=3)  # forecasting up to 3 time ahead
```

- Many packages *tseries, forecast*,… provide forecasting functions

- Best model fitting using *forecast* package (…not very accurate…)

```
library(forecast);
y1=arima.sim(list(order=c(1,0,0), ar=0.4), n=300);

auto.arima(y1);
```

Co-integration Tests

Co-integration Tests

- If we find the response and predictor variables are integrated (non-stationary), then we might suspect the spurious regression problem (see Granger and Newbold, 1974) from the models.
- A regression model involving the non-stationary series can spuriously lead to a significant relationship between unrelated series. ⇨ a spurious regression problem
- However, Engle and Granger (1987) claimed that if the error term is stationary, in which case, the non-stationary time series are said to be co-integrated. Then the relationship between variables is interpreted to be in long-run equilibrium.
- Technically, Hamilton has shown that the OLS estimates for the coefficients of the regression model are consistent under the existence of the co-integration (Hamilton, 1994, pp. 590~591).
- Test for the non-stationairy of each response and predictor variable
  ⇒ test for a unit root of each variable
- Test for the co-integrating relationship between variables
  ⇒ test for a unit root in the residuals of the co-integration regression

Co-integration Tests

- Let y_t and x_t be integrated (non-stationary), and $y_t = \beta X_t + e_t$

  If y_t and x_t are co-integrated, then estimates of e_t would be I(0).

  If not, estimates of e_t would be also non-stationary for some β.

- Evidence of co-integration implies that a variable captures the dominant source of persistent innovations in the other variable over this period $\Rightarrow$ interested in long term relationship

- Phillips-Ouliaris Co-integration Test
  - Computes the Phillips-Ouliaris test for the null hypothesis that x is not co-integrated.
  - The unit root is estimated from a regression of the first variable (column) of x on the remaining variables of x without a constant and a linear trend.
  - R-code

```
### Phillips-Ouliars Co-integration test
x=diffinv(rnorm(1000)); y=2-3*x+rnorm(x, sd=5);
z=ts(cbind(x,y));          # x and y are co-integrated
x11(); plot(z);
po.test(z);  # null: no co-integration
```

Co-integration Tests

- Johansen Co-integration test: useful for Vector Auto Regression (VAR)

```
### Johansen Co-integration test: useful for vector AR
data(denmark) ;
sjd <- denmark[, c("LRM", "LRY", "IBO", "IDE")] ;
head(sjd);

sjd.vecm <- ca.jo(sjd, ecdet = "const", type="eigen", K=2, spec="longrun", season=4) ;
summary(sjd.vecm);
?ca.jo
#https://rpubs.com/sdkshihsoj/ATSA#:~:text=r%20is%20the%20rank%20of,at%20least%20two%20time%20series.
```

```
ca.jo(x, type = c("eigen", "trace"), ecdet = c("none", "const", "trend"), K = 2, spec=c("longrun", "transitory"), season = NULL, dumvar = NULL)
```

x	Data matrix to be investigated for cointegration.
type	The test to be conducted, either ‘eigen’ or ‘trace’.
ecdet	Character, ‘none’ for no intercept in cointegration, ‘const’ for constant term in cointegration and ‘trend’ for trend variable in cointegration.
K	The lag order of the series (levels) in the VAR.
spec	Determines the specification of the VECM, see details below.
season	If seasonal dummies should be included, the data frequency must be set accordingly,*i.e* ‘4’ for quarterly data.
dumvar	If dummy variables should be included, a matrix with row dimension equal to x can be provided.

Co-integration Tests

- If the co-integration relationship is detected, the error correction model (ECM) is usually applied to model the dynamic relationship among the co-integrated variables.

- ECM employs the differenced variables to transform the original series into a stationary process.

- Example

$$\boxed{\Delta Y_t = \alpha + \sum_{i=1}^{k} \Delta Y_{t-i} + \sum_{i=1}^{k} \Delta RE_{t-i} + \sum_{i=1}^{k} \Delta I_{t-i} + u_t}$$

Granger Causality

- For some $k > 0$, if $E(y_{t+k} - E(y_{t+k} \mid F_t))^2 < E(y_{t+k} - E(y_{t+k} \mid \Im_t))^2$,

  then we say that x Granger-causes y

  Note: F_t denotes the information set of x and y available at time t

  $\Im_t$ denotes the information set of y available at time t

- Null hypothesis: x does not Granger-cause y

```
### Granger Causality test
library(lmtest); data(ChickEgg);
grangertest(chicken~egg, order=3, data=ChickEgg);  # egg granger-caused chicken

grangertest(egg~chicken, order=3, data=ChickEgg);
# alternative way to give same result
grangertest(ChickEgg, order=3);
grangertest(ChickEgg[,1], ChickEgg[,2], order=3);
```

Reading lists

[1] Cryer, J.D., Chan, K. (2008), Time Series Analysis with Applications in R, Springer, New York, USA.

[2] Enger RF and Granger CWJ (1987). Conintegration and error correction: representation, estimation, and testing. *Econometrica* **55**: 251-276.

[3] Gallant, A. R. and Goebel, J. J. (1976) Nonlinear Regression with Autoregressive Errors, *Journal of the American Statistical Association*, 71, 961–967.

[4] Granger CWJ and Newbold P (1974). Spurious regressions in econometrics. *J Econometrics* **2**: 111-120.

[5] Hamilton J (1994). *Time Series Analysis*. Princeton, New Jersey.

[6] Tsay, R.S. (2005), Analysis of Financial Time Series, Wiley, New Jersey, USA.

[7] 경제시계열분석 (2002), 박준용, 장유순, 한상범, 경문사