# 5장 Decomposition 모델1

서강대 김명석

# Time Series Decomposition

- Time series ***decomposition*** seeks to separate a time series $Y$ into four components
  - Trend ($T$): deterministic trend
  - Cycle ($C$): long term period
  - Seasonal ($S$): short term period
  - Irregular ($I$): stochastic disturbance

- These components are assumed to follow either an additive or a multiplicative model.

- *Additive versus Multiplicative Models*

| ***Model*** | ***Components*** | ***Used For*** |
|---|---|---|
| Additive | $Y = T + C + S + I$ | Data of similar magnitude (short run or trend-free data) with constant *absolute* growth or decline. |
| Multiplicative | $Y = T \times C \times S \times I$ | Data of increasing or decreasing magnitude (long run or trended data) with constant *percent* growth or decline. |

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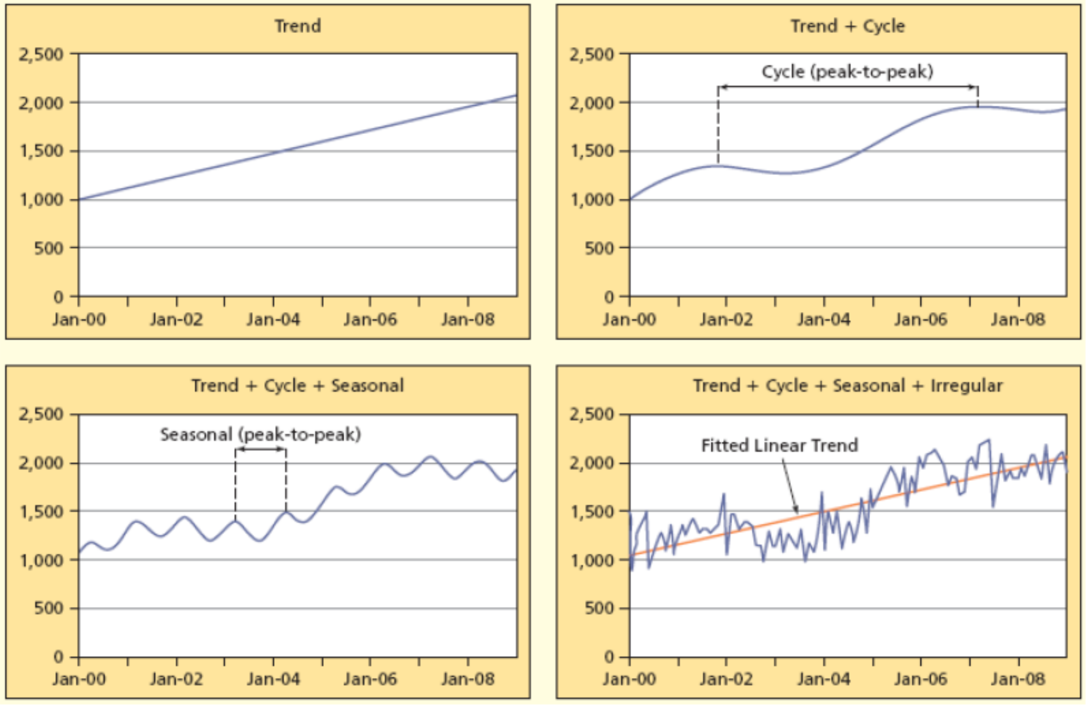
# Time Series Decomposition

- The multiplicative model becomes additive if logarithms are taken (for nonnegative data)

$$\boxed{\log(Y) = \log(T \times C \times S \times I) = \log(T) + \log(C) + \log(S) + \log(I)}$$

- Here is a graphical view of the 4 components of a hypothetical time series.

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# Additive vs. Multiplicative

DGP 1: $y_t = 10 + 3\cos\left(2\pi t \frac{10}{300} + 8\right) + \varepsilon_t$ additive seasonal

DGP 2: $y_t = (20 + 0.1t) + \left(10 + 3\cos\left(2\pi t \frac{10}{300} + 8\right)\right) + \varepsilon_t$ additive seasonal linear trend

DGP 3: $y_t = (20 + 0.1t)\left(10 + 3\cos\left(2\pi t \frac{10}{300} + 8\right)\right) + \varepsilon_t$ multiplicative seasonal linear trend

DGP 4: $y_t = (200 + 0.001t^2) + \left(10 + 3\cos\left(2\pi t \frac{10}{300} + 8\right)\right) + \varepsilon_t$ add season nonlinear trend

DGP 5: $y_t = (20 + 0.001t^2)\left(10 + 3\cos\left(2\pi t \frac{10}{300} + 8\right)\right) + \varepsilon_t$ mult season nonlinear trend

where $\varepsilon_t \sim N(0, \sigma^2)$. $\sigma$ is set to be 0.01, 0.1, 0.5, 1 and 2 for DGPs 1, 2 and 4, 1, 5, 10, 20 and 30 for DGP 3, and 1, 10, 30, 40 and 50 for DGP 5.

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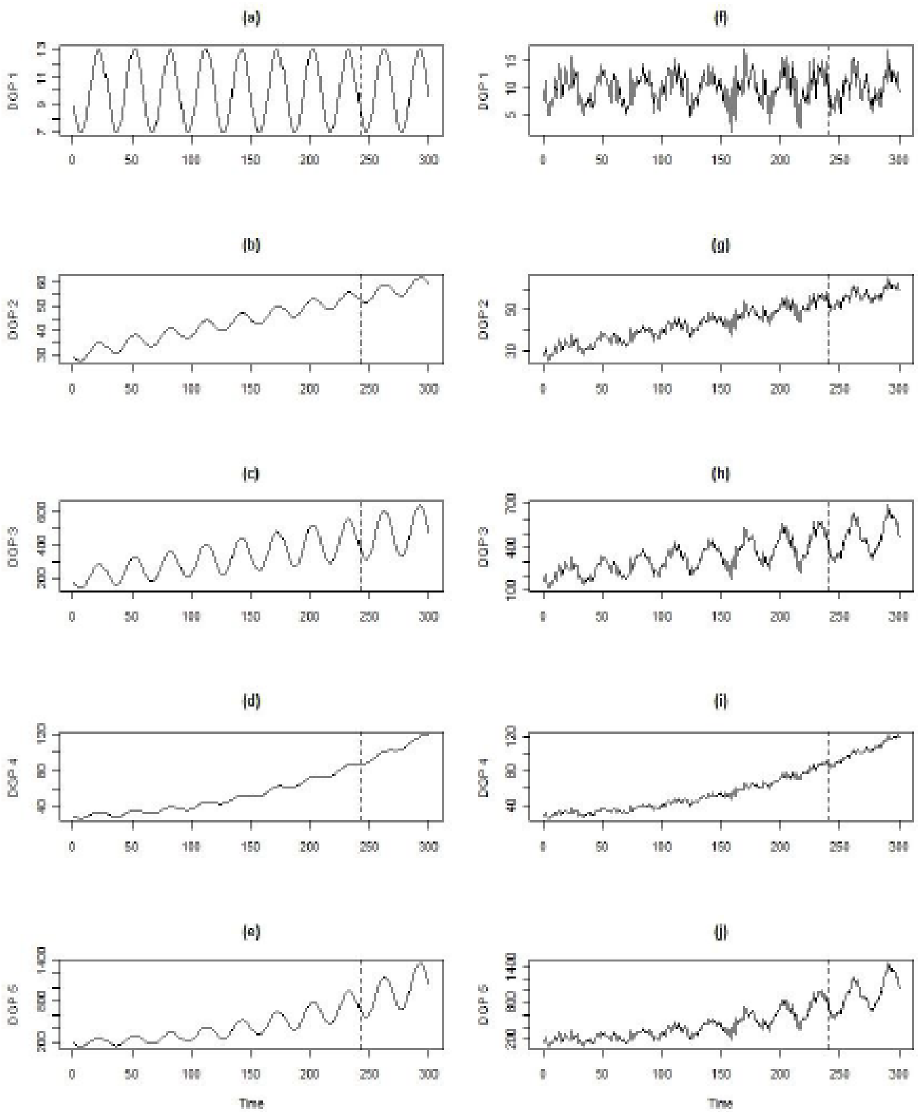
# Additive vs. Multiplicative



Panels (a) to (e) indicate GDPs 1 to 5 with the smallest levels of noises and panels (f) to (j) indicate DGPs 1 to 5 with the largest levels of noise.

서강대 김명석

# Time Series Decomposition

- *Trend* ($T$) is the general movement over all years ($t$ = 1, 2, ..., $n$).
- Trends may be steady and predictable, increasing, decreasing, or staying the same.
- A mathematical trend can be fitted to any data but may or may not be useful for predictions.
- *Cycle* ($C$) is a repetitive up-and-down movement *about a trend* that covers several years.
- *Seasonal* ($S$) is a repetitive cyclical pattern *within a year* (or within a week, day, or other time period).
- By definition, annual data have no seasonality.
- *Irregular* ($I$) is a random disturbance that follows no pattern.
- It is also called the *error* component or *random noise* reflecting all factors other than trend, cycle and seasonality.
- Short run forecasts are best if data are irregular.

# Trend Forecasting

- The following three trend models are especially useful in business applications for $t = 1, 2, \ldots n$

$$\begin{aligned} T_t &= a + bt && \text{(linear trend)} \\ T_t &= a \cdot e^{bt} && \text{(exponential trend)} \\ T_t &= a + bt + ct^2 && \text{(quadratic trend)} \end{aligned}$$

The ***linear* trend model** has the form $T_t = a + bt$

- It is the simplest model and may suffice for short-run forecasting or as a baseline model.
- Linear trend is fitted by using ordinary least squares formulas.
- Note: instead of using the actual time values (e.g., years), use an index t = $x_t$ = 1, 2, 3,

$$\hat{b} = \frac{\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})}{\sum_{t=1}^{n}(x_t - \bar{x})^2}, \quad \hat{a} = \bar{y} - \hat{b}\bar{x}$$

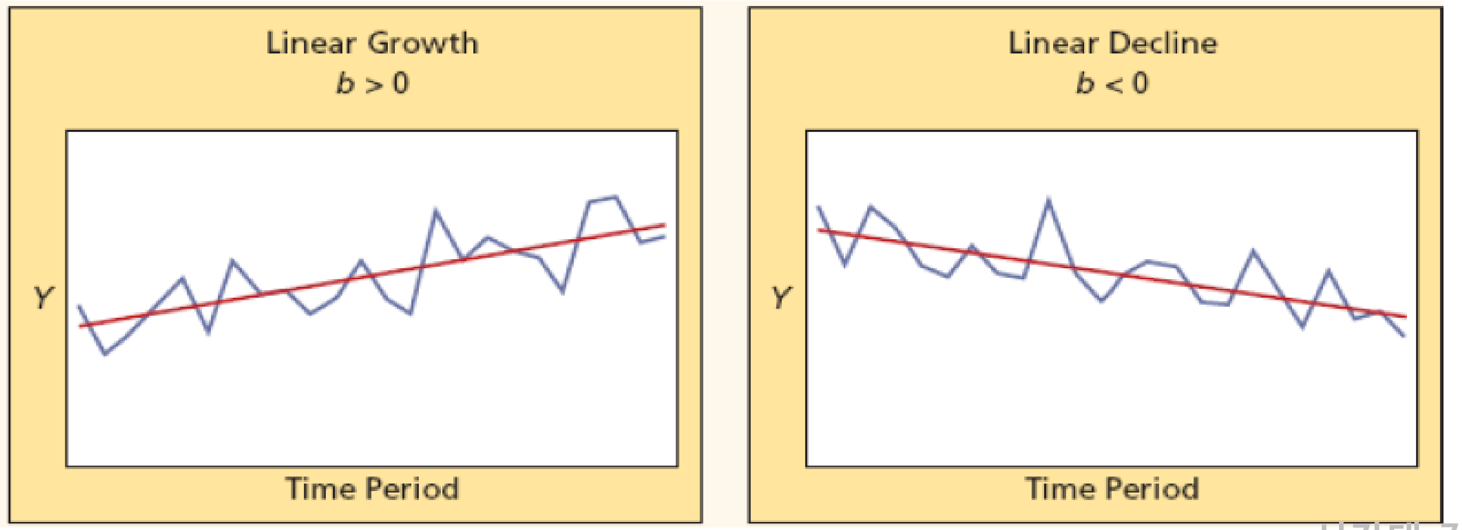
서강대 김명석

# Trend Forecasting

- Once the slope and intercept have been calculated, a forecast can be made for any future time period (e.g., year) by using the fitted model.

- For example,

$$\begin{array}{l}
\text{For 2003 } (t=6)\text{: } y_6 = 22{,}860 - 235(6) = 21{,}450 \\
\text{For 2004 } (t=7)\text{: } y_7 = 22{,}860 - 235(7) = 21{,}215 \\
\text{For 2005 } (t=8)\text{: } y_8 = 22{,}860 - 235(8) = 20{,}980
\end{array}$$



# Trend Forecasting

- $R^2$ can be calculated as

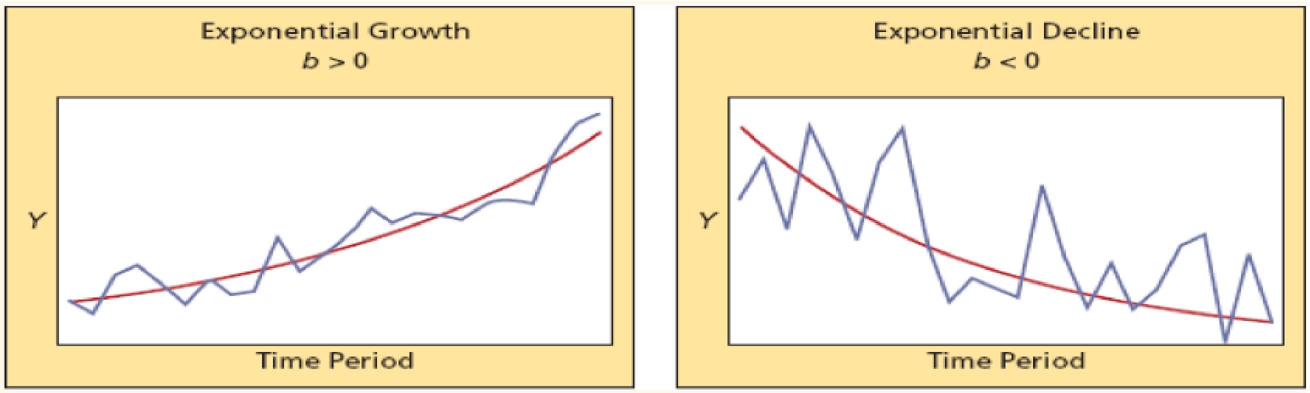
$$\textit{Coefficient of determination:} \quad R^2 = 1 - \frac{\sum_{t=1}^{n}(y_t - \hat{y}_t)^2}{\sum_{t=1}^{n}(y_t - \bar{y})^2}$$

- An $R^2$ close to 1.0 would indicate a good fit to the *past* data.
- However, more information is needed since the forecast is simply a projection of current trend assuming that nothing changes.

The ***exponential trend* model** has the form $T_t = ae^{bt}$

- Useful for a time series that grows or declines at the same *rate* ($b$) in each time period.
- This model is often preferred for financial data or data that covers a longer period of time.
- You can compare two growth rates in two time series variables with dissimilar data units (i.e., a percent growth rate is *unit-free*)

# Trend Forecasting




- There may not be much difference between a linear and exponential model when the growth rate is small and the data set covers only a few time periods.
- Calculations of the exponential trend are done by using a transformed variable $z_t = \ln(y_t)$ to produce a linear equation so that the least squares formulas can be used.

$$\hat{b} = \frac{\sum_{t=1}^{n}(t-\bar{t})(z_t-\bar{z})}{\sum_{t=1}^{n}(t-\bar{t})^2} = \frac{32.12329}{82.5} = 0.38937$$

$$\ln \hat{a} = \bar{z} - \hat{b}\bar{t} = 3.481731 - (0.3893732)(5.5) = 1.340178$$

# Trend Forecasting

- Once the least squares calculations are completed, transform the intercept back to the original units by exponentiation to get the correct intercept.
- For example, if $\ln \hat{a} = 1.340178$ and $\hat{b} = .3893732$, $\hat{a} = e^{1.340178} = 3.8197$
- In the final form, the fitted trend line would be $\hat{y}_t = \hat{a} \cdot e^{\hat{b}t} = 3.8197e^{0.38937t}$
- A forecast can be made for any future time period (e.g., year) by using the fitted model.
- For example,

$$y_{11} = 3.8197e^{.38937(11)} = 276.8$$
$$y_{12} = 3.8197e^{.38937(12)} = 408.5$$
$$y_{13} = 3.8197e^{.38937(13)} = 603.0$$

- All calculations of $R^2$ are done in terms of $z_t = \ln(y_t)$.
- An $R^2$ close to 1.0 would indicate a good fit to the *past* data.

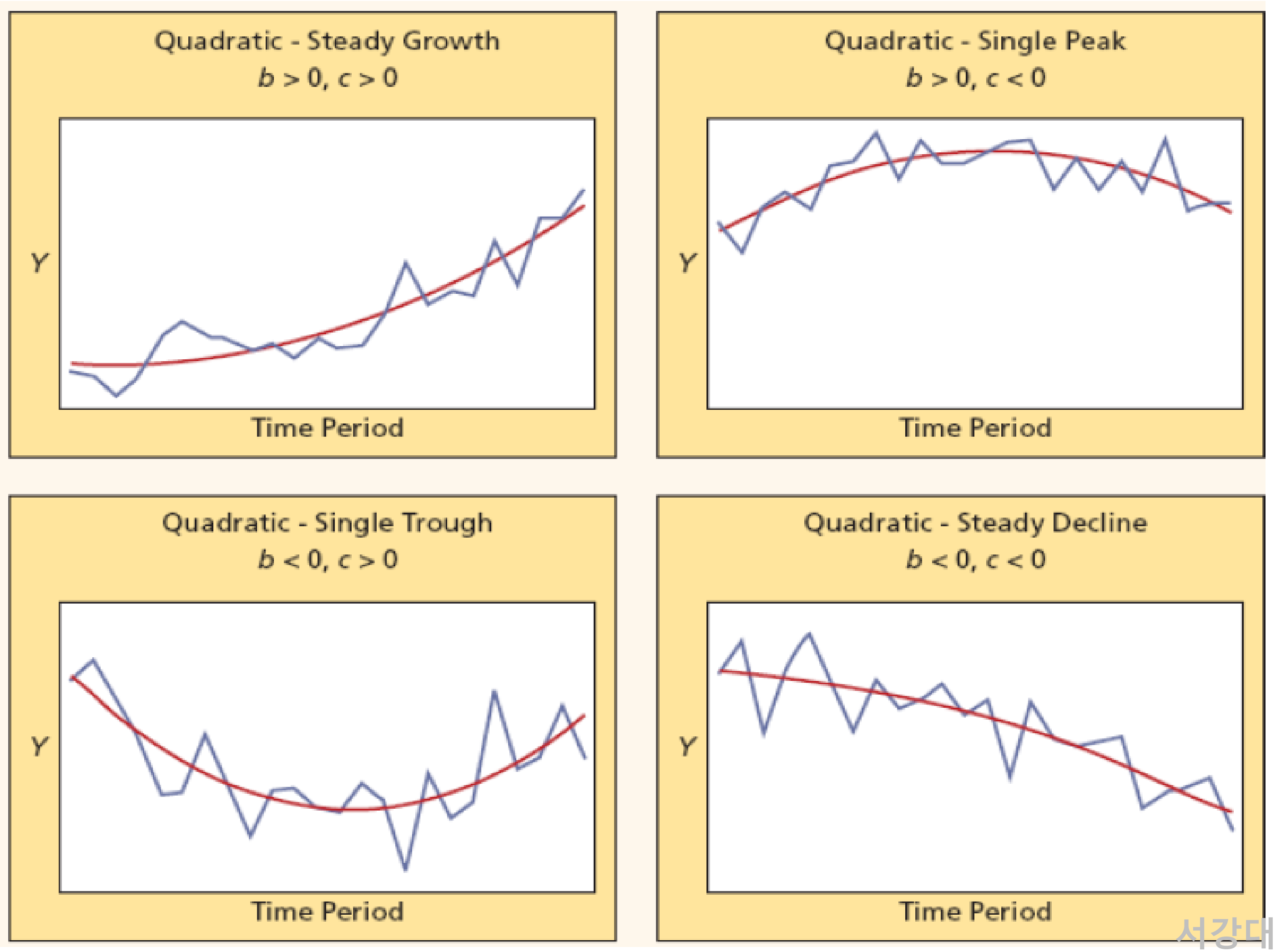
$$R^2 = 1 - \frac{\sum_{t=1}^{n}(z_t - \hat{z}_t)^2}{\sum_{t=1}^{n}(z_t - \bar{z})^2}$$

# Trend Forecasting

A ***quadratic trend*** **model** has the form $T_t = a + bt + ct^2$

- If $c$ = 0, then the quadratic model becomes a linear model (i.e., the linear model is a special case of the quadratic model).
- Fitting a quadratic model is a way of checking for nonlinearity.
- If $c$ does not differ significantly from zero, then the linear model would suffice.
- Depending on the values of $b$ and $c$, the quadratic model can assume any of four shapes
- Because the quadratic trend model $T_t = a + bt + ct^2$ is a multiple regression with two predictors ($t$ and $t^2$), the least squares calculations can be obtained from computer software.

# Trend Forecasting



# Trend Forecasting

- Criteria for selecting a trend forecasting model

| *Criterion* | *Ask Yourself* |
|---|---|
| Occam's Razor | Would a simpler model suffice? |
| Overall fit | How does the trend fit the past data? |
| Believability | Does the extrapolated trend "look right"? |
| Fit to recent data | Does the fitted trend match the last few data points? |

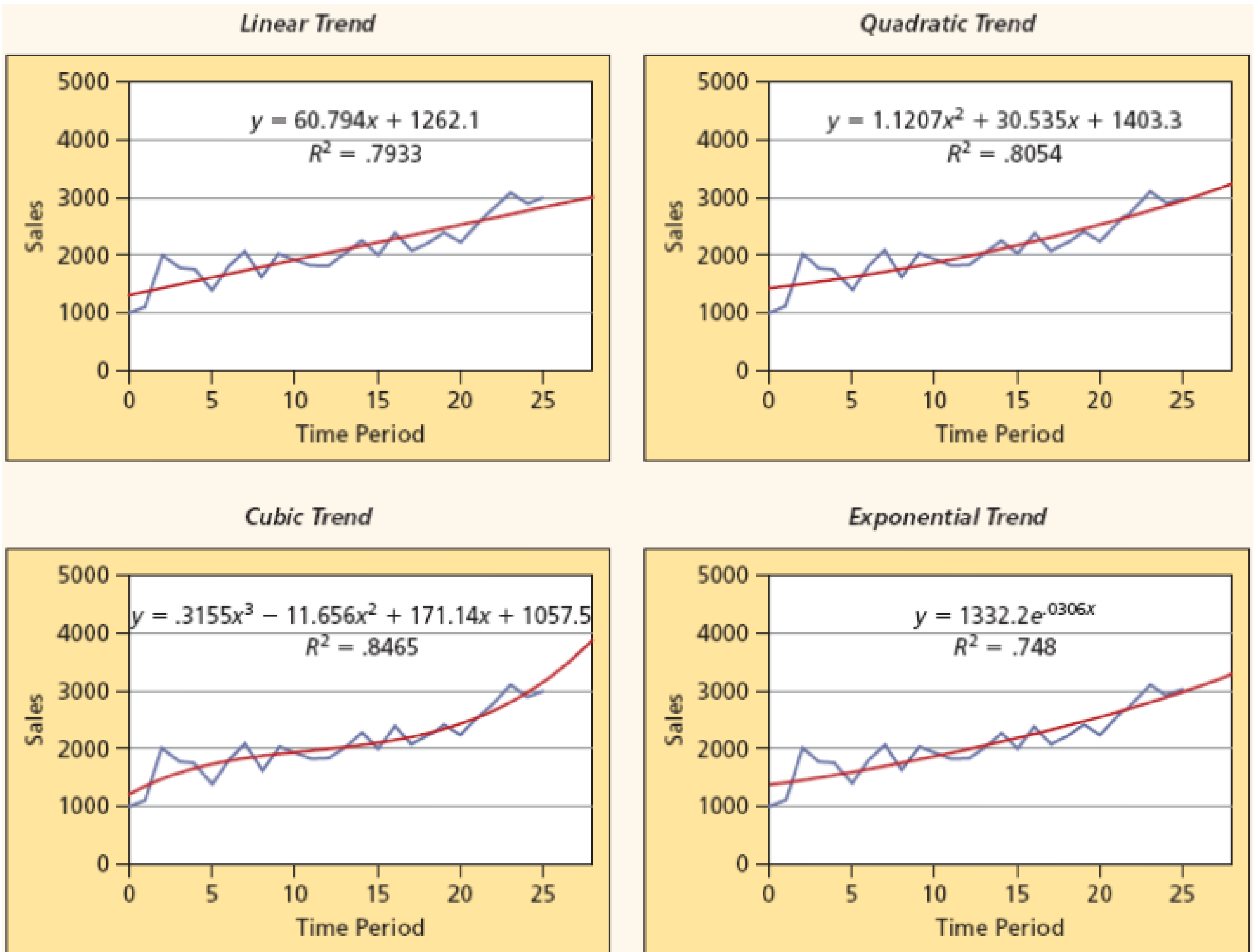
- $R^2$ can usually be increased by choosing a more complex model.
- But $R^2$ measures fit to the *past* data.
- Look at **forecasts** (i.e., extrapolated trends) to see which of four fitted trends using the same data give the best fit.
- Any trend model's forecasts become less reliable as they are extrapolated farther into the future.

# Trend Forecasting

- Consider the following three trend models

| *Model* | *Pro* | *Con* |
|---|---|---|
| Linear | 1. Simple, familiar to everyone.<br>2. May suffice for short-run data. | 1. Assumes constant slope.<br>2. Can't capture nonlinear change. |
| Exponential | 1. Familiar to financial analysts.<br>2. Shows compound percent growth rate. | 1. Some managers are unfamiliar with $e^x$.<br>2. Data values must be positive. |
| Quadratic | 1. Useful for data with a turning point.<br>2. Useful test for nonlinearity. | 1. Complex and no intuitive interpretation.<br>2. Gives weird forecasts if extrapolated too far. |

# Trend Forecasting



# Assessing Fit

- "Fit" refers to how well the estimated trend model matches the observed historical past data.
- These fit statistics are most useful in comparing different trend models for the same data.
- All the statistics (especially the *MSD*) are affected by unusual residuals.

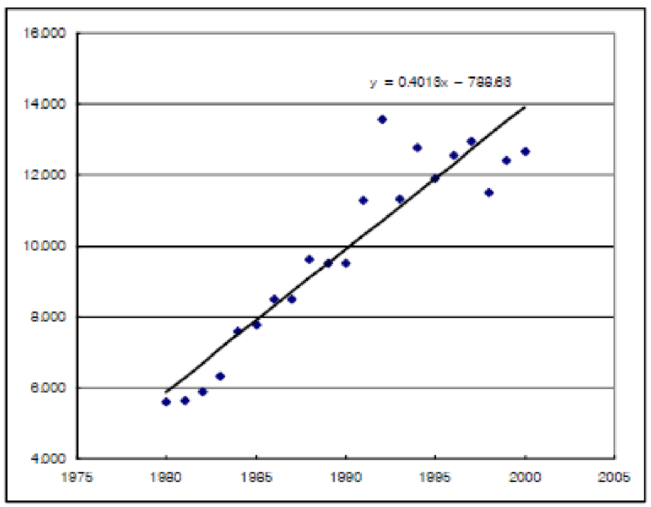
| ***Statistic*** | ***Description*** | ***Pro*** | ***Con*** |
|---|---|---|---|
| $R^2 = 1 - \dfrac{\sum_{t=1}^{n}(y_t - \hat{y}_t)^2}{\sum_{t=1}^{n}(y_t - \bar{y}_t)^2}$ | Coefficient of determination | 1. Unit-free measure.<br>2. Very common. | 1. Often interpreted incorrectly (e.g., "percent of correct predictions"). |
| $MAPE = \dfrac{100}{n}\sum_{t=1}^{n}\dfrac{\lvert y_t - \hat{y}_t\rvert}{y_t}$ | Mean Absolute Percent Error (MAPE) | 1. Unit-free measure (%).<br>2. Intuitive meaning. | 1. Requires $y_t > 0$.<br>2. Lacks nice math properties. |
| $MAD = \dfrac{1}{n}\sum_{t=1}^{n}\lvert y_t - \hat{y}_t\rvert$ | Mean Absolute Deviation (MAD) | 1. Intuitive meaning.<br>2. Same units as $y_t$. | 1. Not unit-free.<br>2. Lacks nice math properties. |
| $MSD = \dfrac{1}{n}\sum_{t=1}^{n}(y_t - \hat{y}_t)^2$ | Mean Squared Deviation (MSD) | 1. Nice math properties.<br>2. Penalizes big errors more. | 1. Nonintuitive meaning.<br>2. Rarely reported. |
| $SE = \sqrt{\sum_{t=1}^{n}\dfrac{(y_t - \hat{y}_t)^2}{n-2}}$ | Standard error | 1. Same units as $y_t$.<br>2. For confidence intervals. | 1. Nonintuitive meaning. |

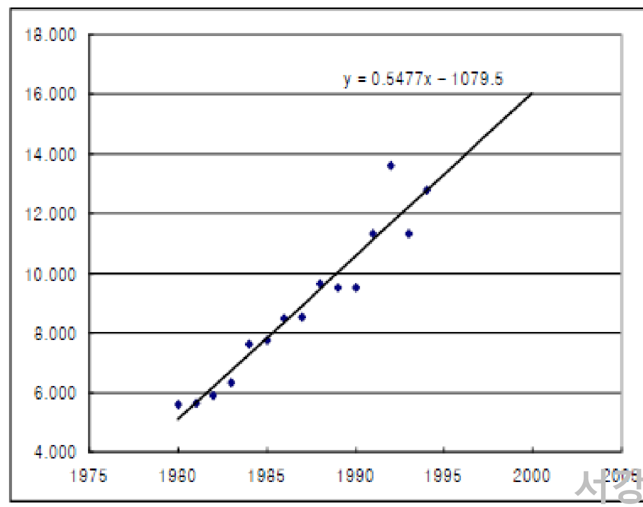
# Assessing Fit

- The standard error ($SE$) is useful if we want to make a prediction interval for a forecast.

$$\hat{y}_t \pm t_{n-2}\mathrm{SE}\sqrt{1 + \frac{1}{n} + \frac{(x_t - \bar{x})^2}{\sum_{t=1}^{n}(x_t - \bar{x})^2}}$$

- In sample accuracy & out of sample accuracy
- One time ahead forecast & $k$ time ahead forecast



# Smoothing Techniques

***Moving Averages*** ⇨ delete irregularity

- In cases where the time series $y_1, y_2, \ldots, y_n$ is erratic or has no consistent trend, there may be little point in fitting a trend line.
- A conservative approach is to calculate either a *trailing* or *centered moving average*.

***Trailing Moving Average (TMA)***

$$\boxed{\hat{y}_t = \frac{y_t + y_{t-1} + \cdots + y_{t-m+1}}{m}}$$

- The *TMA* simply averages over the last $m$ periods.
- The *TMA* smoothes the past fluctuations in the time series in order to see the pattern more clearly.
- Choosing a larger $m$ yields a "smoother" *TMA* but requires more data.
- The value of $\hat{y}_t$ may also be used as a forecast for period $t+1$.
- There is no way to update the moving average beyond the observed data range.
- This is a *one-period-ahead forecast*.

# Smoothing Techniques

***Centered Moving Average (CMA)***

- The *CMA* smoothing method looks forward *and* backward in time to express the current "forecast" as a mean of the current observation *and* observations on either side of the current data.

$$\boxed{\hat{y}_t = \frac{y_{t-1} + y_t + y_{t+1}}{3}}$$

- When $m$ is odd ($m = 3, 5$, etc.), the *CMA* is easy to calculate.
- When $m$ is even, the mean of an even number of data points would lie between two data points and would not be correctly centered.
- For example, $m = 4$, we would average $y_{t-1}$ through $y_{t+1}$, then average $y_t$ through $y_{t+2}$ and finally average the two averages.
- In this case, we would take a double moving average to get the resulting *CMA* centered properly. (taking an average for two averaged CMAs)

# Smoothing Techniques

## ***Exponential Smoothing***

- The *exponential smoothing* model is a special kind of moving average.
- Its one-period-ahead forecasting technique is utilized for data that has up-and-down movements but no *consistent* trend.
- The updating formula is $\boxed{F_{t+1} = \alpha y_t + (1-\alpha)F_t}$ where

$$\boxed{\begin{aligned} F_{t+1} &= \text{the forecast for the next period} \\ \alpha &= \text{the "smoothing constant" } (0 \le \alpha \le 1) \\ y_t &= \text{the actual data value in period } t \\ F_t &= \text{the previous forecast for period } t \end{aligned}}$$

- The next forecast $F_{t+1}$ is a weighted average of $y_t$ (the current data) and $F_t$ (the previous forecast).
- The value of $\alpha$ (the *smoothing constant*) is the weight given to the latest data.
- A small value of $\alpha$ would give low weight to the most recent observation.

# Smoothing Techniques

- A small value of $\alpha$ would give heavy weight to the previous forecast.
- The larger the value of $\alpha$, the more quickly the forecasts adapt to recent data.
- If $\alpha$ = 1, there is no smoothing at all and the forecast for the next period is the same as the latest data point.
- The effect of our choice of $\alpha$ on the forecast diminishes as time increases.
- To see this, replace $F_t$ with $F_{t-1}$ and repeat this type of substitution indefinitely to obtain

$$\boxed{F_{t+1} = \alpha y_t + \alpha(1-\alpha) y_{t-1} + \alpha(1-\alpha)^2 y_{t-2} + \alpha(1-\alpha)^3 y_{t-3} + \cdots}$$

- The next forecast depends on all the prior data.
- Optimal $\alpha$ is the one providing minimum prediction errors.
- Note that $F_{t+1}$ depends on $F_t$, which in turn depends on $F_{t-1}$, and so on all the way back to $F_1$.

# Smoothing Techniques

- Where do we get the initial forecast $F_1$ (i.e., how do we initialize the process)?

- *Method A*

  Use the first data value. Set $F_1 = y_1$

  Although simple, if $y_1$ is unusual, it could take a few iterations for the forecasts to stabilize.

- *Method B*

  Average the first 6 data values. Set $F_1 = (y_1 + y_2 + + y_3 + y_4 + y_5 + y_6) / 6$.

  This method consumes more data and is still vulnerable to unusual $y$-values.

- *Method C*

  Backward extrapolation. Set $F_1$ = prediction from *backcasting*

- *Backcasting* fits a trend to the data *in reverse order* and extrapolates the trend to predict the initial value.

# Smoothing Techniques

- Single exponential smoothing is for *trendless* data.
- For data with a trend, use *Holt's method* with *two* smoothing constants (one for *trend* and one for *level*).
- For data with both trend and seasonality, use *Winters's method* with *three* smoothing constants (for *trend*, *level*, and *seasonality*.)
- Exponential smoothing has proven through the years to be very useful in many forecasting situations. It was first suggested by C.C. Holt in 1957 and was meant to be used for non-seasonal time series showing no trend. He later offered a procedure (1958) that does handle trends. Winters(1965) generalized the method to include seasonality, hence the name "Holt-Winters Method".

# Seasonality (Multiplicative Model)

- When the data periodicity is monthly or quarterly, calculate a seasonal index and use it to *deseasonalize* it.

- For the multiplicative model, a seasonal index is a *ratio*.

- The seasonal indexes must sum to 12 for monthly data or to 4 for quarterly data.

  Step 1: Calculate a centered moving average (*CMA*) for each month (quarter).

  Step 2: Divide each observed $y_t$ value by the *MA* to obtain seasonal ratios.

  Step 3: Average the seasonal ratios by the month (quarter) to get raw seasonal indexes.

  Step 4: Adjust the raw seasonal indexes so they sum to 12 (monthly) or 4 (quarterly).

  Step 5: Divide each $y_t$ by its seasonal index to get deseasonalized data.

- Estimate a regression model using *seasonal binaries* as predictors in order to address seasonality.

# Trend & Seasonality (Multiplicative Model)

- Steps of Multiplicative Time Series Model

  1. Calculate the centered moving averages (CMAs).
  2. Center the CMAs on integer-valued periods.
  3. Determine the seasonal and irregular factors ( $S_t I_t$ ).
  4. Determine the average seasonal factors.
  5. Scale the seasonal factors ( $S_t$ ).
  6. Determine the deseasonalized data ( $Y_t / S_t$ ).
  7. Determine a trend line of the deseasonalized data.
  8. Determine the deseasonalized predictions.
  9. Take into account the seasonality.

# Trend & Seasonality (Additive Model)

- For example, for quarterly data, the fourth quarter binary *Qtr4* (arbitrarily chosen), would be excluded in order to prevent multicollinearity.

$$Sales = 161 + 14.4\,Time + 89.8\,Qtr1 + 12.9\,Qtr2 - 83.6\,Qtr3$$

$$Y = T + S + I$$

- Periodic extreme event can be modeled using dummy variable, which can be applied to forecasting (example: holiday)

| Year | Quarter | Sales | Time | Qtr1 | Qtr2 | Qtr3 |
|---|---|---|---|---|---|---|
| 2002 | 1 | 259 | 1 | 1 | 0 | 0 |
| | 2 | 236 | 2 | 0 | 1 | 0 |
| | 3 | 164 | 3 | 0 | 0 | 1 |
| | 4 | 222 | 4 | 0 | 0 | 0 |
| 2003 | 1 | 306 | 5 | 1 | 0 | 0 |
| | 2 | 300 | 6 | 0 | 1 | 0 |
| | 3 | 189 | 7 | 0 | 0 | 1 |
| | 4 | 275 | 8 | 0 | 0 | 0 |
| 2004 | 1 | 379 | 9 | 1 | 0 | 0 |
| | 2 | 262 | 10 | 0 | 1 | 0 |
| | 3 | 242 | 11 | 0 | 0 | 1 |
| | 4 | 296 | 12 | 0 | 0 | 0 |
| 2005 | 1 | 369 | 13 | 1 | 0 | 0 |
| | 2 | 373 | 14 | 0 | 1 | 0 |
| | 3 | 255 | 15 | 0 | 0 | 1 |
| | 4 | 374 | 16 | 0 | 0 | 0 |
| 2006 | 1 | 515 | 17 | 1 | 0 | 0 |
| | 2 | 373 | 18 | 0 | 1 | 0 |
| | 3 | 339 | 19 | 0 | 0 | 1 |
| | 4 | 519 | 20 | 0 | 0 | 0 |
| 2007 | 1 | 626 | 21 | 1 | 0 | 0 |
| | 2 | 535 | 22 | 0 | 1 | 0 |
| | 3 | 397 | 23 | 0 | 0 | 1 |
| | 4 | 488 | 24 | 0 | 0 | 0 |

# Trend & Seasonality (Multiplicative Model)

- Example: Terry's Tie shop

Business at Terry's Tie Shop can be viewed as falling into three distinct seasons: (1) Christmas (November and December); (2) Father's Day (late May to mid June); and (3) all other times. Average weekly sales ($) during each of the three seasons during the past four years are shown on the next slide.

| | Season | | |
|---|---|---|---|
| Year | 1 | 2 | 3 |
| 1 | 1856 | 2012 | 985 |
| 2 | 1995 | 2168 | 1072 |
| 3 | 2241 | 2306 | 1105 |
| 4 | 2280 | 2408 | 1120 |

Determine a forecast for the average weekly sales in year 5 for each of the three seasons.

# Trend & Seasonality (Multiplicative Model)

- Step 1: There are three distinct seasons in each year. Hence, take a three-season moving average to eliminate seasonal and irregular factors. For example:

1st CMA = (1856 + 2012 + 985)/3 = 1617.67
2nd CMA = (2012 + 985 + 1995)/3 = 1664.00

- Step 2: The first centered moving average computed in step 1 (1617.67) will be centered on season 2 of year 1. Note that the moving averages from step 1 center themselves on integer-valued periods because $n$ is an odd number.

| | Season | | |
|---|---|---|---|
| Year | 1 | 2 | 3 |
| 1 | 1856 | 2012 | 985 |
| 2 | 1995 | 2168 | 1072 |
| 3 | 2241 | 2306 | 1105 |
| 4 | 2280 | 2408 | 1120 |

# Trend & Seasonality (Multiplicative Model)

- Step 3: Isolate the trend and cyclical components. For each period $t$, this is given by:

  $S_t = Y_t$ / (Moving average for period t)

- Step 4: Averaging all $S_t$ values corresponding to that season:

Season 1: (1.163 + 1.196 + 1.181) /3 = 1.180
Season 2: (1.244 + 1.242 + 1.224 + 1.244) /4 = 1.236
Season 3: (.592 + .587 + .582) /3 = 0.587

| Year | Season | Dollar Sales ($Y_t$) | Moving Average | $S_t$ |
|---|---|---|---|---|
| 1 | 1 | 1856 | | |
| | 2 | 2012 | *1617.67 | **1.244 |
| | 3 | 985 | 1664.00 | 0.592 |
| 2 | 1 | 1995 | 1716.00 | 1.163 |
| | 2 | 2168 | 1745.00 | 1.242 |
| | 3 | 1072 | 1827.00 | 0.587 |
| 3 | 1 | 2241 | 1873.00 | 1.196 |

# Trend & Seasonality (Multiplicative Model)

- Step 5: Average the seasonal factors = (1.180 + 1.238 + .587)/3 = 1.002. Then, divide each seasonal factor by the average of the seasonal factors.

Season 1: 1.180/1.002 = 1.178
Season 2: 1.238/1.002 = 1.236
Season 3: 0.587/1.002 = 0.586 → sum: 3.000

| Year | Season | Dollar Sales ($Y_t$) | Moving Average | $S_t$ | Scaled $S_t$ | $Y_t / S_t$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1856 | | | 1.178 | 1576 |
| | 2 | 2012 | *1617.67 | **1.244 | 1.236 | 1628 |
| | 3 | 985 | 1664.00 | 0.592 | 0.586 | 1681 |
| 2 | 1 | 1995 | 1716.00 | 1.163 | 1.178 | 1694 |
| | 2 | 2168 | 1745.00 | 1.242 | 1.236 | 1754 |
| | 3 | 1072 | 1827.00 | 0.587 | 0.587 | 1829 |
| 3 | 1 | 2241 | 1873.00 | 1.196 | 1.178 | 1902 |

# Trend & Seasonality (Multiplicative Model)

- Step 6: Divide the data point values, $Y_t$ , by $S_t$
- Step 7: Using the least squares method for $t$ = 1, 2, ..., 12, gives:

$$T_t = 1580.11 + 33.96t$$

- Step 8: Substitute $t$ = 13, 14, and 15 into the least squares equation:

$$\begin{aligned} T_{13} &= 1580.11 + (33.96)(13) = 2022 \\ T_{14} &= 1580.11 + (33.96)(14) = 2056 \\ T_{15} &= 1580.11 + (33.96)(15) = 2090 \end{aligned}$$

- Step 9: Multiply each deseasonalized prediction by its seasonal factor to give the following forecasts for year 5:

| | |
|---|---|
| Season 1: | (1.178)(2022) = 2382 |
| Season 2: | (1.236)(2056) = 2541 |
| Season 3: | (0.586)(2090) = 1225 |

# Reading lists


[1] Doane, D.P., Seward, L.E. (2007) Applied Statistics in Business and Economics (Ch. 14), McGraw Hill, Boston, USA.

[2] Cryer, J.D., Chan, K. (2008) Time Series Analysis with Applications in R, Springer, New York, USA.

[3] M. Kendall and A. Stuart (1983) *The Advanced Theory of Statistics*, 3, Griffin. pp. 410–414.