7장 ARIMA 모델

Moving Average (MA) Processes

MA(1) processes

- moving average process of order [MA(1)] is $y_t - \mu = \varepsilon_t - \theta\varepsilon_{t-1}$
- where as before the ε_t 's are $WN(0, \sigma_\varepsilon^2)$
- can show that $E(y_t) = \mu$, $Var(y_t) = \sigma_\varepsilon^2(1+\theta^2)$ $\gamma(1) = -\theta\sigma_\varepsilon^2$, $\gamma(h) = 0$ *if* $|h| > 1$

$$\rho(1) = \frac{-\theta}{1+\theta^2}, \quad \rho(h) = 0 \;\; \textit{if} \;\; |h| > 1$$

General MA processes

- MA(q) process is $y_t - \mu = \varepsilon_t - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q}$
- can show that $\gamma(h) = 0$ and $\rho(h) = 0$ if $|h| > 1$
- formulas for $\gamma(h)$ and $\rho(h)$ when $|h| \le q$ are given in time series textbooks
- complicated but not be needed by us

Auto Regressive Moving Average (ARMA)

The backwards operator

- ARMA processes easily described with "backwards" operator, B
- backwards operator B is defined by $By_t = y_{t-1}$
- more generally, $B^k y_t = y_{t-k}$
- $B\,c = c$ for any constant c
- since a constant does not change with time

ARMA processes

- ARMA(p, q) process satisfies the equation

$$(1-\phi_1 B - \ldots - \phi_p B^p)(y_t - \mu) = (1-\theta_1 B - \ldots - \theta_q B^q)\varepsilon_t \qquad (1)$$

or $(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \ldots + \phi_p(y_{t-p} - \mu) + \varepsilon_t - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q}$

- white noise process is ARMA(0,0) since if $p = q = 0$, then (1) reduces to $(y_t - \mu) = \varepsilon_t$
- p and q stand for lag of AR and MA parts, respectively.
- *Note*: ARMA(p, 0) series is AR(p). ARMA(0,q) series is MA(q).

Auto Regressive Integrated Moving Average

- ARMA (autoregressive and moving average): stationary time series with complex autocorrelation behavior better modeled by mixed autoregressive and moving average processes
- ARIMA (autoregressive, integrated, moving average): based on stationary ARMA processes but are nonstationary
- ARIMA processes easily described with "differencing" operator, $\Delta = 1 - B$

The differencing operator

- differencing operator is $\Delta = 1 - B$ so that $\Delta y_t = y_t - By_t = y_t - y_{t-1}$
- differencing a time series produces a new time series consisting of the changes in the original series
- for example, if $p_t = \log(x_t)$ is the log price, then the log return is $r_t = \Delta p_t$
  differencing can be iterated
- for example, $\Delta^2 y_t = \Delta(\Delta y_t) = \Delta(y_t - y_{t-1}) = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$
  $= y_t - 2y_{t-1} + y_{t-2}$

From ARMA processes to ARIMA process

- often the first or second differences of nonstationary time series are stationary
- for example, the first differences of random walk (nonstationary) are white noise (stationary)
- a time series y_t is said to be ARIMA(p, d, q) if $\Delta^d y_t$ is ARMA(p, q)

$$(1-\phi_1 B - \ldots - \phi_p B^p)(1-B)^d (y_t - \mu) = (1-\theta_1 B - \ldots - \theta_q B^q)\varepsilon_t$$

or $$(1-\phi_1 B - \ldots - \phi_p B^p)\Delta^d (y_t - \mu) = (1-\theta_1 B - \ldots - \theta_q B^q)\varepsilon_t$$

- for example, if log returns (r_t) on an asset are
  ARMA (p, q), then the log prices (p_t) are ARIMA (p, 1, q)
- ARIMA procedures in SAS allow one to specify p, d and q
- an ARIMA(p, 0, q) model is the same as an ARMA(p, q) model
- ARIMA(p, 0, 0), ARMA(p, 0), and AR(p) models are the same
- Also, ARIMA(0, 0, q), ARMA(0, q), and MA(q) models are the same
- random walk is an ARIMA(0, 1, 0) model

Model Selection

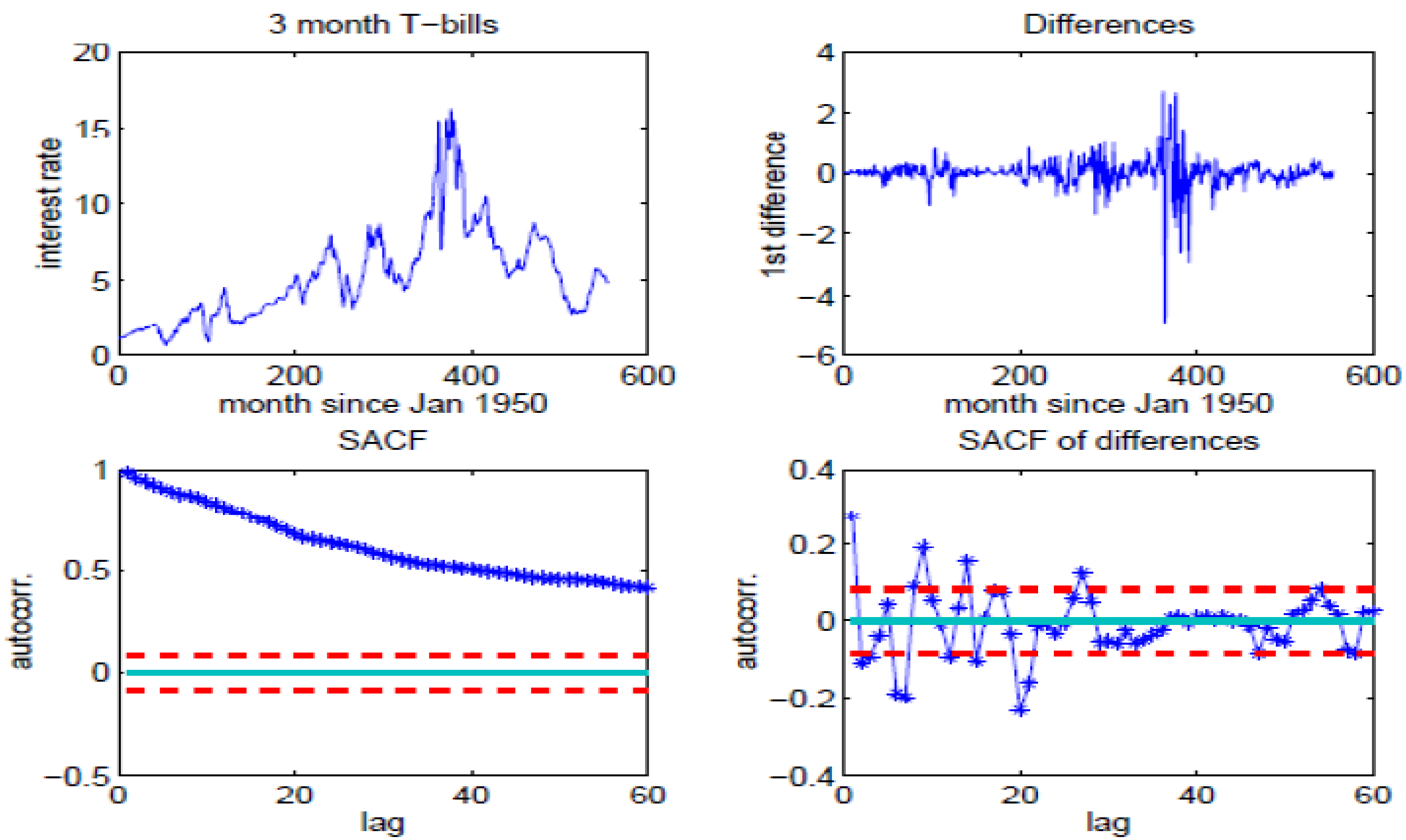
- once the parameters *p, d* and *q* selected, coefficients can be estimated by maximum likelihood
- but how do we choose *p, ;* and *q*?
- generally, *d* is either 0, 1, or 2

  chosen by looking at the SACF of y_t, Δy_t, and $\Delta^2 y_t$
- a sign that a process is nonstationary is that its SACF decays to zero very slowly
- if this is true of y_t then the original series is nonstationary
- should be differenced at least once
- if the SACF of Δy_t looks stationary, then we use $d = 1$
- otherwise, we look at the SACF of $\Delta^2 y_t$
- if this looks stationary we use $d = 2$.

Model Selection

- real time series where $\Delta^2 y_t$ did not look stationary are rare
- but if one were encountered then $d > 2$ would be used
- once d has been chosen we will fit ARMA(p, q) process to $\Delta^2 y_t$
- but still need p and q
- comparing various choices of p and q by some criterion that measures how well a model fits

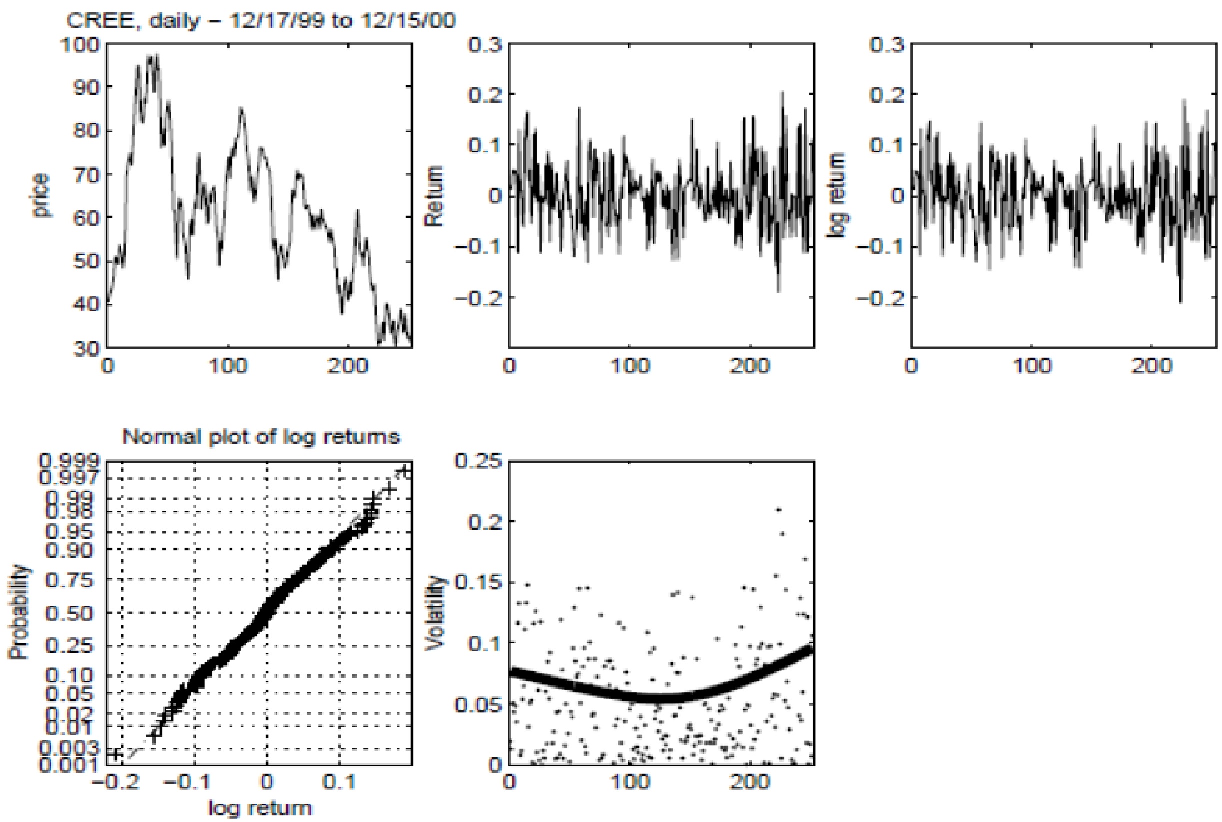
Using the SACF to choose *d*



SACF's show that one should use $d = 1$

Using ARIMA on SAS : Cree data

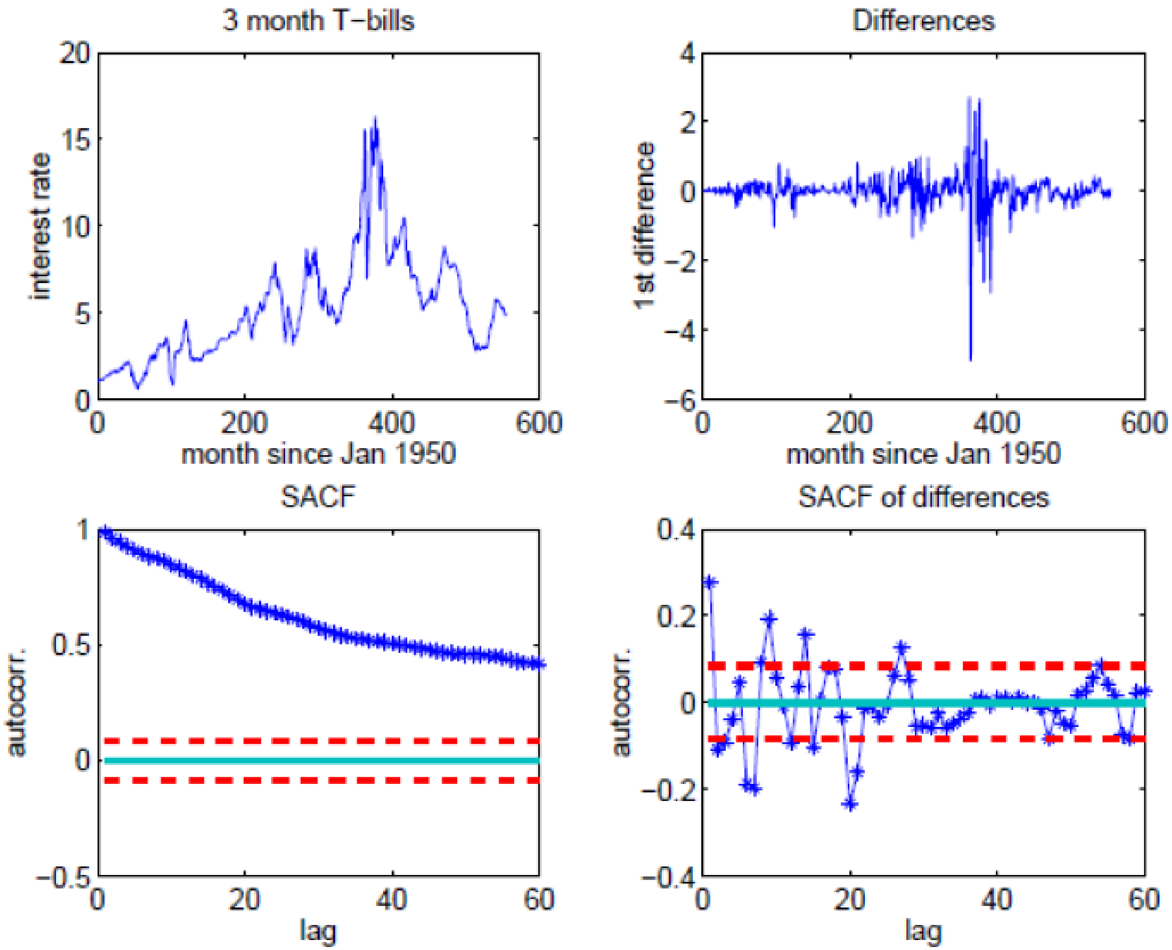
- in this example, we will illustrate fitting an ARMA model in SAS
- will use daily log returns on Cree from December 1999 to December 2000.



Example : Three-month Treasury bill rates

- our empirical results : log returns have little autocorrelation
  but not exactly white noise
- other financial time series do have substantial autocorrelation
- Example : monthly interest rates on three-month US
  Treasury bills from December 1950 until February 1996
- data come from Example 16.1 of Pindyck and Rubin (1998),
  Econometric Models and Economic Forecasts
- rates are plotted in next figure
- first differences look somewhat stationary
- we will fit ARMA models to the first differences

Example : Three-month Treasury bill rates



Forecasting using R

- Forecasting

```
y=arima.sim(list(order=c(2,1,3), ar=c(0.7, 0.2),ma=c(0.1,0.2,0.1)), n=300);  #AR(2) simulation
ts.plot(y);

a=arima(y,order=c(2,1,3));  # AR(2) estimation

predict(arima(y, order=c(2,1,3)), n.ahead=10)  # forecasting up to 10 time ahead
```

- Many packages *tseries*, *forecast*,… provide forecasting functions

- Best model fitting using *forecast* package

```
library(forecast);
y1=arima.sim(list(order=c(2,1,3), ar=c(0.7, 0.2),ma=c(0.1,0.2,0.1)), n=300);

auto.arima(y1);
```

Invertibility

Invertibility

- ARMA(p, q) process in (1) can be written as $\alpha(L)y_t = \beta(L)\varepsilon_t$

  With $\alpha(z) = \sum_{k=0}^{p} \alpha_k z^k$, $\beta(z) = \sum_{k=0}^{q} \beta_k z^k$, $\varepsilon_t \sim WN(0, \sigma^2)$

  - y_t is invertible if and only if y_t can be represented as $y_t = \gamma(L)\varepsilon_t$
  - ε_t is invertible if and only if ε_t can be written as $\varepsilon_t = \rho(L)y_t$

- If y_t is an AR(p) process, then it can be written as $\alpha(L)y_t = \varepsilon_t$
  - If y_t is invertible, then $y_t = \gamma(L)\varepsilon_t$
  - This implies that $y_t = \alpha(L)^{-1}\varepsilon_t = \gamma(L)\varepsilon_t$, where $\alpha(L) \neq 0$

    $\Rightarrow$ y_t is a weakly stationary
  - In particular, $y_t = \sum_{k=0}^{\infty} \varphi_k \varepsilon_{t-k}$, i.e., MA(∞).

Reading lists

[1] Tsay, R.S. (2005), Analysis of Financial Time Series, Wiley, New Jersey, USA.

[2] Cryer, J.D., Chan, K. (2008), Time Series Analysis with Applications in R, Springer, New York, USA.

[3] Gallant, A. R. and Goebel, J. J. (1976) Nonlinear Regression with Autoregressive Errors, *Journal of the American Statistical Association*, 71, 961–967.

[4] 경제시계열분석 (2002), 박준용, 장유순, 한상범, 경문사