# Problem Set 2: Review of Probability Theory SOLUTIONS

MATH E-156: Mathematical Foundations of Statistical Software

Due February 5, 2018

# Problem 1

## Problem Statement

The random variable $X$ has this categorical distribution:

| $k$ | $\Pr(X = k)$ |
|---|---|
| 25 | 0.10 |
| 40 | 0.25 |
| 50 | 0.20 |
| 55 | 0.30 |
| 60 | 0.15 |

**Part (a)** Calculate $\mathrm{E}[X]$, the expected value of $X$.

**Part (b)** Calculate $\mathrm{E}[X^2]$, the second moment of $X$.

**Part (c)** Calculate $\mathrm{Var}[X]$, the variance of $X$.

The problem solution starts on the next page.

## Problem Solution

**Part (a)** Calculate $\mathrm{E}[X]$, the expected value of $X$.

**Solution** For a categorical random variable, we can use a tabular form to calculate the expected value:

| $k$ | $\Pr(X=k)$ | $k \cdot \Pr(X=k)$ |
|---|---|---|
| 25 | 0.10 | 2.5 |
| 40 | 0.25 | 10.0 |
| 50 | 0.20 | 10.0 |
| 55 | 0.30 | 16.5 |
| 60 | 0.15 | 9.0 |
| | Total | 48.0 |

So the expected value of $X$ is $\mathrm{E}[X] = 48$.

**Part (b)** Calculate $\mathrm{E}[X^2]$, the second moment of $X$.

**Solution** Again, we can use a tabular form for this calculation:

| $k$ | $\Pr(X=k)$ | $k^2 \cdot \Pr(X=k)$ |
|---|---|---|
| 25 | 0.10 | 62.5 |
| 40 | 0.25 | 400.0 |
| 50 | 0.20 | 500.0 |
| 55 | 0.30 | 907.5 |
| 60 | 0.15 | 540.0 |
| | Total | 2,410.0 |

So the second moment of $X$ is $\mathrm{E}[X^2] = 2,410$.

**Part (c)** Calculate $\mathrm{Var}[X]$, the variance of $X$.

**Solution** We can use the standard formula for calculating the variance:

$$\begin{aligned} \mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\ &= 2,410 - (48)^2 \\ &= 106 \end{aligned}$$

So the variance of $X$ is $\mathrm{Var}[X] = 106$.

If you wanted to, you could also calculate this using the definition of variance:

$$\mathrm{Var}[X] = \sum_{k \in \Omega_X} (k - \mathrm{E}[X])^2 \cdot \Pr(X = k)$$

Again, a tabular form works well here:

| $k$ | $\Pr(X = k)$ | $k - \mathrm{E}[X]$ | $(k - \mathrm{E}[X])^2 \cdot \Pr(X = k)$ |
|---|---|---|---|
| 25 | 0.10 | -23.0 | 52.9 |
| 40 | 0.25 | -8.0 | 16.0 |
| 50 | 0.20 | 2.0 | 0.8 |
| 55 | 0.30 | 7.0 | 14.7 |
| 60 | 0.15 | 12 | 21.6 |
| | | Total | 106.0 |

Again, we arrive at the same conclusion that the variance of $X$ is $\mathrm{Var}[X] = 106$.

# Problem 2

In this problem we will explore a new discrete probability distribution called the *geometric* distribution.

## Problem Statement

A discrete random variable $X$ has a *geometric* distribution with parameter $p$ if for all non-negative integer values of $k$ it has the probability mass function:

$$\Pr(X = k) = p^k \cdot (1 - p), \quad \text{integer } k \geq 0$$

Notice here that the support of $X$ is infinite, unlike the categorical distribution that we examined in lecture. However, our theory for discrete random variables works the exact same way for random variables with infinite support as for those with finite support..

**Part (a)** Show that the probabilities defined by this mass function do indeed sum to one when summing over all non-negative integers.

**Part (b)** Calculate $\mathrm{E}[X]$, the expected value of a geometric random variable.

**Part (c)** Calculate $\mathrm{Var}[X]$, the variance of a geometric random variable.

**Hint** You might find these power series from elementary calculus to be useful:

$$\begin{aligned}
\sum_{k=0}^{\infty} p^k &= \frac{1}{1-p} \\
\sum_{k=0}^{\infty} k p^k &= \frac{p}{(1-p)^2} \\
\sum_{k=0}^{\infty} k^2 p^k &= \frac{p+p^2}{(1-p)^3}
\end{aligned}$$

## Problem Solution

**Part (a)** Show that the probabilities defined by this mass function do indeed sum to one when summing over all non-negative integers.

**Solution** Let's sum up the probabilities over the full support of $X$, which in this case is the set of all non-negative integers:

$$\begin{aligned}\sum_{k=0}^{\infty} p^k \cdot (1-p) &= (1-p) \cdot \sum_{k=0}^{\infty} p^k \\ &= (1-p) \cdot \left(\frac{1}{1-p}\right) \\ &= 1\end{aligned}$$

So the probabilities really do sum to 1.

**Part (b)** Calculate $\mathrm{E}[X]$, the expected value of a geometric random variable.

**Solution** Now we use the standard formula for the expected value for a discrete random variable:

$$\begin{aligned}\mathrm{E}[X] &= \sum_{k \in \Omega_X} k \cdot \Pr(X = k) \\ &= \sum_{k=0}^{\infty} k \cdot p^k (1-p) \\ &= (1-p) \cdot \sum_{k=0}^{\infty} k p^k \\ &= (1-p) \cdot \frac{p}{(1-p)^2} \\ &= \frac{p}{1-p}\end{aligned}$$

So the expected value of $X$ is $\dfrac{p}{1-p}$.

**Part (c)** Calculate Var$[X]$, the variance of a geometric random variable.

**Solution** First, we need the second moment:

$$\begin{aligned}
\mathrm{E}[X^2] &= \sum_{k \in \Omega_X} k^2 \cdot \Pr(X = k) \\
&= \sum_{k=0}^{\infty} k^2 \cdot p^k (1-p) \\
&= (1-p) \cdot \sum_{k=0}^{\infty} k^2 p^k \\
&= (1-p) \cdot \frac{p + p^2}{(1-p)^3} \\
&= \frac{p + p^2}{(1-p)^2}
\end{aligned}$$

Now we can calculate the variance:

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= \frac{p + p^2}{(1-p)^2} - \left(\frac{p}{1-p}\right)^2 \\
&= \frac{p + p^2}{(1-p)^2} - \frac{p^2}{(1-p)^2} \\
&= \frac{p}{(1-p)^2}
\end{aligned}$$

So the variance of $X$ is $\frac{p}{(1-p)^2}$.

# Problem 3

## Problem Statement

Let $X$ be an exponential random variable with parameter $\lambda$, so that

$$f_X(x) = \lambda e^{-\lambda x}$$

**Part (a)** Derive an algebraic expression for the $k$th moment of $X$, denoted $\mathrm{E}[X^k]$.

**Part (b)** Using the formula from Part (a), $\mathrm{E}[X]$, the expected value of $X$.

**Part (c)** Calculate $\mathrm{Var}[X]$, the variance of $X$.

**Part (d)** Show that $F_X(x)$, the cumulative probability function for $X$, is:

$$F_X(x) = 1 - \exp(-\lambda x)$$

## Problem Solution

**Part (a)** Derive an algebraic expression for the $k$th moment of $X$, denoted $\mathrm{E}[X^k]$.

**Solution** By the general definition of the $k$th moment for a continuous random variable, we have:

$$\begin{aligned}
\mathrm{E}[X^k] &= \int_{\Omega_X} x^k \cdot f_X(x) \cdot dx \\
&= \int_0^\infty x^k \cdot \lambda e^{-\lambda x} \cdot dx \\
&= \lambda \cdot \int_0 x^k e^{-\lambda x} \cdot dx
\end{aligned}$$

Now what is this integral? It's a "kinda sorta" gamma function, and from Lecture 1 we have a formula for this:

$$\int_0 x^k e^{-\lambda x} \cdot dx = \frac{\Gamma(k+1)}{\lambda^{k+1}}$$

Now we have:

$$\begin{aligned}
\lambda \cdot \int_0 x^k e^{-\lambda x} \cdot dx &= \lambda \cdot \left( \frac{\Gamma(k+1)}{\lambda^{k+1}} \right) \\
&= \frac{k!}{\lambda^k}
\end{aligned}$$

**Part (b)** Using the formula from Part (a), $\mathrm{E}[X]$, the expected value of $X$.

**Solution** The expected value of $X$ is just the first moment of $X$, so we set $k = 1$ in our formula for $\mathrm{E}[X^k]$:

$$\begin{aligned}
\mathrm{E}[X^1] &= \frac{1!}{\lambda^1} \\
&= \frac{1}{\lambda}
\end{aligned}$$

So the expected value of $X$ is $\mathrm{E}[X] = \dfrac{1}{\lambda}$.

**Part (c)** Calculate $\mathrm{Var}[X]$, the variance of $X$.

**Solution** First, we need the second moment of $X$, so we set $k = 2$ in the general formula:

$$\begin{aligned}
\mathrm{E}[X^2] &= \frac{2!}{\lambda^2} \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

Then we use the standard formula for the variance:

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Let's summarize the results of this problem:

$$\begin{aligned}
\mathrm{E}[X^k] &= \frac{k!}{\lambda^k} \\
\mathrm{E}[X] &= \frac{1}{\lambda} \\
\mathrm{Var}[X] &= \frac{1}{\lambda^2}
\end{aligned}$$

# Problem 4

General moments and variance of 1-parameter Pareto

## Problem Statement

A random variable $X$ has the *1-parameter Pareto* distribution with parameter $\alpha > 0$ and lower limit of support $\theta > 0$ if it has the density function

$$f_X(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta$$

I admit that it seems to weird to call something a "1-parameter" distribution when it seems to have two parameters, but the variable $\theta$ only serves to define the support of the random variable, so conventionally it is not considered to be a real parameter.

**Part (a)** Show that the function $f_X(x)$ is a valid density function. Hint: be careful about the support.

**Part (b)** Derive an expression for the $k$th moment $\mathrm{E}[X^k]$. Do you need to place any restrictions on any values?

**Part (c)** Derive $\mathrm{Var}[X]$, the variance of $X$. Do we need to make any assumptions about the values of $\alpha$ or $\theta$?

## Problem Solution

**Part (a)** Show that the function $f_X(x)$ is a valid density function.

**Solution** In order to show that the function is a valid density function, we must integrate over the full support of the distribution. Here's where the hint comes in: the support of the random variable only starts at $\theta$, so the lower limit of integration must be $\theta$, not 0:

$$\begin{aligned}
\int_{\Omega_X} f_X(x) \cdot dx &= \int_\theta^\infty \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \cdot dx \\
&= -\frac{\theta^\alpha}{x^\alpha}\bigg|_\theta^\infty \\
&= 0 - \left(-\frac{\theta^\alpha}{\theta^\alpha}\right) \\
&= 1
\end{aligned}$$

**Part (b)** Derive an expression for the $k$th moment $\mathrm{E}[X^k]$. Do you need to place any restrictions on any values?

**Solution** Let's first set up the integral for the $k$th moment:

$$\begin{aligned}
\mathrm{E}[X^k] &= \int_{\Omega_X} x^k \cdot f_X(x) \cdot dx \\
&= \int_\theta^\infty x^k \cdot \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \cdot dx \\
&= \int_\theta^\infty \alpha\theta^\alpha x^{k-\alpha-1} \cdot dx
\end{aligned}$$

We have to be careful here. Notice the exponent on the $x$: if this is greater thanor equal to -1, then the integral will not converge i.e. as we integrate out to infinity the integral will blow up. So we must have:

$$k - \alpha - 1 < -1$$

From this it follows that $k < \alpha$. So we have an interesting phenomenon here: not all the moments are defined, and this depends on the value of $\alpha$.

Let's get back to our integral:

$$\begin{aligned}
\int_\theta^\infty \alpha\theta^\alpha x^{k-\alpha-1} \cdot dx &= \left.\frac{\alpha\theta^\alpha \cdot x^{k-\alpha}}{k-\alpha}\right|_\theta^\infty \\
&= 0 - \frac{\theta^\alpha \cdot \theta^{k-\alpha}}{k-\alpha} \\
&= \frac{\theta^k}{\alpha - k}
\end{aligned}$$

Notice that because of the restriction that $k < \alpha$ the term at infinity becomes 0; if $k \geq \alpha$ then this would have gone to infinity and the integral would not have converged. Also, since $k < \alpha$, the final expression is positive.

**Part (c)** Derive Var$[X]$, the variance of $X$.

**Solution** For the variance, we need to calculate the expected value and the second moment, and this means that $alpha > 2$. So we will assume this in what follows, and note that if $\alpha \leq 2$ then the variance does not exist.

Let's calculate the first moment using our general formula for the $k$th moment, with $k = 1$:

$$\begin{aligned}
\mathrm{E}[X^1] &= \frac{\theta^1}{\alpha - 1} \\
&= \frac{\theta}{\alpha - 1}
\end{aligned}$$

For the second moment, we use the formula with $k = 2$:

$$\mathrm{E}[X^2] = \frac{\theta^2}{\alpha - 2} \tag{1}$$

Now we can calculate the variance:

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= \frac{\theta^2}{\alpha - 2} - \left(\frac{\theta}{\alpha - 1}\right)^2 \\
&= \frac{\theta^2}{\alpha - 2} - \frac{\theta^2}{(\alpha - 1)^2} \\
&= \frac{\theta^2(\alpha - 1)^2 - \theta^2(\alpha - 2)}{(\alpha - 2)(\alpha - 1)^2} \\
&= \frac{\alpha^2\theta^2 - 2\alpha\theta^2 + \theta^2 - \alpha\theta^2 + \theta^2}{(\alpha - 2)(\alpha - 1)^2} \\
&= \frac{\alpha^2\theta^2 - 3\alpha\theta^2 + 2\theta^2}{(\alpha - 2)(\alpha - 1)^2} \\
&= \frac{\theta^2(\alpha - 1)(\alpha - 2)}{(\alpha - 2)(\alpha - 1)^2} \\
&= \frac{\theta^2}{\alpha - 1}
\end{aligned}$$

So the variance of $X$ is $\dfrac{\theta^2}{\alpha - 1}$ if $\alpha > 2$, and undefined otherwise.

# Problem 5

For the next 3 problems, we will develop the properties of a probability distribution known as a *2-parameter Pareto* distribution. First, we want to make sure that we have a valid probability density function.

## Problem Statement

Let $X$ be a continuous random variable with support on the non-negative real numbers. Suppose for some value of $c$ that $X$ has the density function:

$$f_X(x) = \frac{c}{(x+\theta)^{\alpha+1}}$$

**Problem** Determine the value of $c$.

**Hint** The value of $c$ has to be such that the density function integrates to 1.

## Problem Solution

We know that any valid probability density function must integrate to 1 over its support:

$$\int_{\Omega_X} f_X(x) \cdot dx = 1$$

In our case, this integral is:

$$\begin{aligned} \int_{\Omega_X} f_X(x) \cdot dx &= \int_0^\infty \frac{c}{(x+\theta)^{\alpha+1}} \cdot dx \\ &= c \cdot \int_0^\infty \frac{1}{(x+\theta)^{\alpha+1}} \cdot dx \end{aligned}$$

So we need to solve the equation:

$$c \cdot \int_0^\infty \frac{1}{(x+\theta)^{\alpha+1}} \cdot dx = 1$$

Let's calculate the integral:

$$\begin{aligned}
\int_0^\infty \frac{1}{(x+\theta)^{\alpha+1}} \cdot dx &= -\frac{1}{\alpha} \cdot \frac{1}{(x+\theta)^{\alpha}} \bigg|_{x=0}^{x=\infty} \\
&= \left(-\frac{1}{\alpha} \cdot 0\right) - \left(-\frac{1}{\alpha} \cdot \frac{1}{(0+\theta)^{\alpha}}\right) \\
&= \frac{1}{\alpha \cdot \theta^{\alpha}}
\end{aligned}$$

So now our equation is:

$$c \cdot \left(\frac{1}{\alpha \cdot \theta^{\alpha}}\right) = 1$$

Thus,

$$c = \alpha \cdot \theta^{\alpha}$$

Then the full density function for this distribution is:

$$f_X(x) = \frac{\alpha \cdot \theta^{\alpha}}{(x+\theta)^{\alpha+1}}$$

# Problem 6

Now that we have a valid density function for the 2-parameter Pareto distribution, let's derive its cumulative probability function.

## Problem Statement

Let $X$ be a 2-parameter Pareto distribution with parameters $\alpha$ and $\theta$, so that the density function for $X$ is:

$$f_X(x) = \frac{\alpha \cdot \theta^\alpha}{(x+\theta)^{\alpha+1}}$$

**Problem** Derive an expression for $F_X(x) = \Pr(X \leq x)$, the cumulative probability function of $X$.

## Problem Solution

The general formula for the cumulative probability function for a continuous random variable is:

$$F_X(x) = \Pr(X \leq x) = \int_{s \leq x} f_X(s) \cdot ds$$

In the current problem, this becomes:

$$\begin{aligned}
F_X(x) &= \int_0^x \frac{\alpha \cdot \theta^\alpha}{(s+\theta)^{\alpha+1}} \cdot ds \\
&= -\frac{\theta^\alpha}{(s+\theta)^\alpha}\bigg|_0^x \\
&= \left(-\frac{\theta^\alpha}{(x+\theta)^\alpha}\right) - \left(-\frac{\theta^\alpha}{(0+\theta)^\alpha}\right) \\
&= 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha
\end{aligned}$$

# Problem 7

## Problem Statement

Let $X$ be a 2-parameter Pareto distribution, with $\alpha > 1$.

**Problem** Using calculus, derive $\mathrm{E}[X]$, the expected value of $X$.

## Problem Solution

The density for a 2-parameter Pareto distribution is

$$f(x) = \frac{\alpha \cdot \theta^\alpha}{(x+\theta)^{\alpha+1}}$$

To calculate the expected value, we have to evaluate the integral:

$$\begin{aligned} \mathrm{E}[X] &= \int_0^\infty x \cdot f(x) \cdot dx \\ &= \int_0^\infty x \cdot \frac{\alpha \cdot \theta^\alpha}{(x+\theta)^{\alpha+1}} \cdot dx \end{aligned}$$

That denominator is annoying, so let's do a substitution to simplify it:

$$\begin{aligned} s &= x + \theta \\ x &= s - \theta \\ dx &= ds \end{aligned}$$

Also, we have to keep in mind the limits of integration: when $x = 0$, then $s = \theta$, and when $x = \infty$, then $s = \infty$. Now we have everything we need to perform the substitution:

$$
\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty x \cdot \frac{\alpha \cdot \theta^\alpha}{(x+\theta)^{\alpha+1}} \cdot dx \\
&= \alpha \cdot \theta^\alpha \int_0^\infty \frac{x}{(x+\theta)^{\alpha+1}} \cdot dx \\
&= \alpha \cdot \theta^\alpha \int_\theta^\infty \frac{s-\theta}{s^{\alpha+1}} \cdot ds \\
&= \alpha \cdot \theta^\alpha \int_\theta^\infty \frac{1}{s^\alpha} - \frac{\theta}{s^{\alpha+1}} \cdot ds \\
&= \alpha \cdot \theta^\alpha \cdot \left[ -\frac{1}{\alpha-1} s^{-(\alpha-1)} + \frac{\theta}{\alpha} s^{-\alpha} \right]_\theta^\infty \\
&= \alpha \cdot \theta^\alpha \cdot \left[ -0 + 0 - \left( -\frac{1}{\alpha-1} \theta^{-(\alpha-1)} + \frac{\theta}{\alpha} \theta^{-\alpha} \right) \right] \\
&= \alpha \cdot \theta^\alpha \cdot \left[ \frac{\theta^{-\alpha+1}}{\alpha-1} - \frac{\theta^{-\alpha+1}}{\alpha} \right] \\
&= \alpha \cdot \theta^\alpha \cdot \left[ \frac{\theta^{-\alpha+1}}{\alpha \cdot (\alpha-1)} \right] \\
&= \frac{\theta}{\alpha-1}
\end{aligned}
$$

Thus, the expected value of the 2-parameter Pareto distribution is $\theta/(\alpha-1)$.

# Problem 8

Sometimes we can use information about a quantile to calculate a parameter value.

## Problem Statement

**Part (a)** Let $X$ be a random variable that has a 2-parameter Pareto distribution with known $alpha = 2$. Suppose the 43.75 percentile is 200. What is $\theta$?

**Part (b)** Let $Y$ be a random variable with an exponential distribution, and suppose that the 63.21 percentile is 4. What is the probability that $X > 5$?

**Part (c)** Let $W$ be a random variable with an exponential distribution with parameter $\lambda$. Let $m$ be the median of the distribution i.e. $m$ is the 50th percentile. Derive an algebraic expression for $\lambda$ in terms of $m$.

## Problem Solution

**Part (a)** Let $X$ be a random variable that has a 2-parameter Pareto distribution with known $alpha = 2$. Suppose the 43.75 percentile is 200. What is $\theta$?

**Solution** If the 43.75 percentile is 200, then that means that $F_X(200) = 43.75$. From Problem 6, the formula for the cumulative probability function with known parameter $\alpha = 2$ is:

$$F_X(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^2$$

Substituting in the value $x = 200$ and $F_X(x) = 0.4375$, we have:

$$0.4375 = 1 - \left(\frac{\theta}{200 + \theta}\right)^2$$

Rearranging, we have:

$$\left(\frac{\theta}{200 + \theta}\right)^2 = 0.5625$$

Now we can take square roots:

$$\frac{\theta}{200 + \theta} = 0.75$$

Solving for $\theta$ gives $\theta = 600$.

**Part (b)** Let $Y$ be a random variable with an exponential distribution, and suppose that the 63.2121 percentile is 4. What is the probability that $X > 5$?

**Solution** In order to calculate the probability that $X > 5$, we need to know the value of the parameter $\lambda$. From Problem 3 Part (d) we know that the cumulative probability function for an exponential distribution with parameter $\lambda$ is:

$$F_X(x) = 1 - \exp(-\lambda x)$$

Substituting in $x = 4$ and $F_X(4) = 0.632121$, we have

$$0.632121 = 1 - \exp(-4\lambda)$$

Rearranging, we have

$$\exp(-4\lambda) = 0.367879$$

Taking the natural logarithm of both sides, we obtain

$$4\lambda = -1$$

Thus $\lambda = 0.25$.

Now that we have the value of the parameter $\lambda$, we can calculate the probability that $X > 5$. First, note that this is a survival probability: $\Pr(X > 5)$ is just $S_X(5)$, and we can obtain a formula for this using the formula for the cumulative probability function:

$$\begin{aligned} \Pr(X > x) &= S_X(x) \\ &= 1 - F_X(x) \\ &= 1 - (1 - \exp(-\lambda x)) \\ &= \exp(-\lambda x) \end{aligned}$$

Now we have:

$$\begin{aligned}
\Pr(X > 5) &= S_X(5) \\
&= \exp(-0.25 \times 5) \\
&= \exp(-1.25) \\
&= 0.26850
\end{aligned}$$

So, the probability that $X > 5$ is 0.26850.

**Part (c)** Let $W$ be a random variable with an exponential distribution with parameter $\lambda$. Let $m$ be the median of the distribution i.e. $m$ is the 50th percentile. Derive an algebraic expression for $\lambda$ in terms of $m$.

**Solution** The median $m$ of a distribution is the 50th percentile, which means that $F_X(m) = 0.5$. Using the formula for the cumulative probability function for an exponential distribution, we have:

$$0.5 = 1 - \exp(-\lambda m)$$

Rearranging, we have:

$$\exp(-\lambda m) = 0.5$$

Taking the natural logarithm of both sides, we have:

$$-\lambda m = \ln(0.5) = -\ln 2$$

Finally:

$$\lambda = \frac{\ln 2}{m}$$

Incidentally, we know from Problem 3, Part (b), that

$$\mathrm{E}[X] = \frac{1}{\lambda}$$

Therefore,

$$\lambda = \frac{1}{\mathrm{E}[X]}$$

Equating the two expressions for $\lambda$, we obtain:

$$\frac{1}{\mathrm{E}[X]} = \frac{\ln 2}{m}$$

Clearing denominators, we arrive at:

$$m = \ln 2 \cdot \mathrm{E}[X]$$

This is always true for all exponential distributions, and since $\ln 2 < 1$ it implies that the median is always less than the expected value. This is because the distribution has a long tail, so the values far away pull the expected value up but leave the median unaltered.