Lecture 1: Some Fun Integrals

MATH E-156: Mathematical Foundations of Statistical Software

Theodore Hatch Whitfield

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The Gamma Function $\Gamma(x)$

The *gamma function*, denoted $\Gamma(x)$, is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \cdot dt$$

The Gamma Function $\Gamma(x)$

The gamma function is computed in an unusual manner:

- First, we select a value for x.
- The value of x defines a specific functional form for the integrand, which is now a function just of t.
- Then we integrate the integrand over t, from 0 to ∞.

In other words, the gamma function is the area under the curve that was defined by the value of x that we selected.

Here's how we calculate $\Gamma(1)$:

$$\begin{aligned}
\Gamma(1) &= \int_0^{\infty} t^{1-1} e^{-t} \cdot dt \\
&= \int_0^{\infty} e^{-t} \cdot dt \\
&= 1
\end{aligned}$$

The Gamma Function $\Gamma(x)$

Here's how we calculate $\Gamma(2)$:

$$\begin{aligned}\Gamma(2) &= \int_0^\infty t^{2-1} e^{-t} \cdot dt \\ &= \int_0^\infty t e^{-t} \cdot dt \\ &= 1\end{aligned}$$

For the moment, don't worry about actually evaluating this integral. (We'll discuss that soon!)

Instead, make sure that you understand the process by which $\Gamma(2)$ is defined.

The Gamma Function $\Gamma(x)$

Here's how we calculate $\Gamma(3)$:

$$\begin{aligned}\Gamma(3) &= \int_0^\infty t^{3-1}e^{-t}\cdot dt \\ &= \int_0^\infty t^2 e^{-t}\cdot dt \\ &= 2\end{aligned}$$

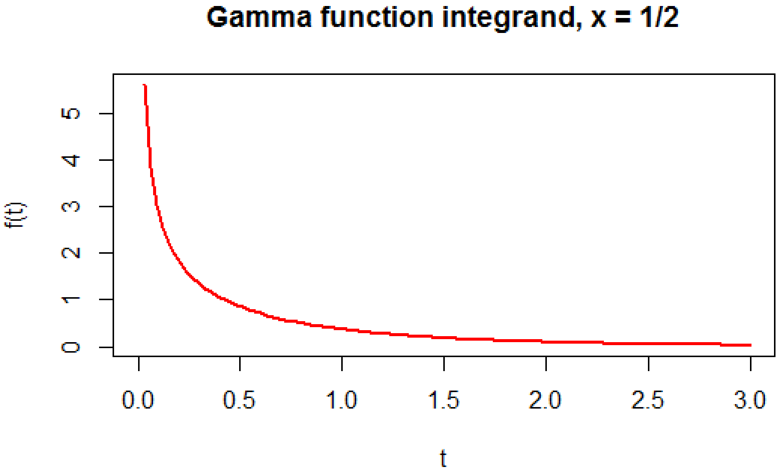
Again, for right now, don't worry about the mechanics of doing the calculation, but do try to understand what's going on.

What can we say about the shape of the integrand for different values of x?

For reference, here's the definition of the gamma function once again:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \cdot dt$$

When $x < 1$, the integrand has a vertical asymptote at $t = 0$, and then the function is always decreasing for all values of $t > 0$.

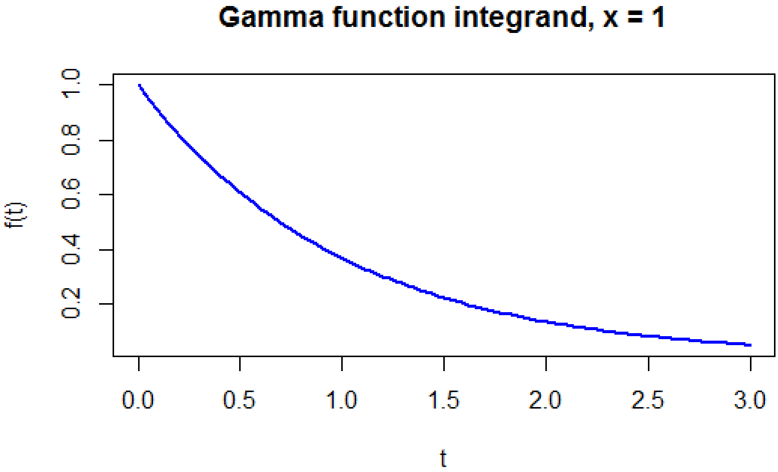

Gamma function integrand, x = 1/2
f(t)
0
1
2
3
4
5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t

The Gamma Function $\Gamma(x)$

What happens when $x = 1$?:

$$\begin{aligned}\Gamma(1) &= \int_0^\infty t^{1-1} e^{-t} \cdot dt \\ &= \int_0^\infty e^{-t} \cdot dt\end{aligned}$$

Thus, when $x = 1$, the integrand is just an exponential function, taking the value 1 at $t = 0$, and then decreasing for all values of $t > 0$.


Gamma function integrand, x = 1
f(t)
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t

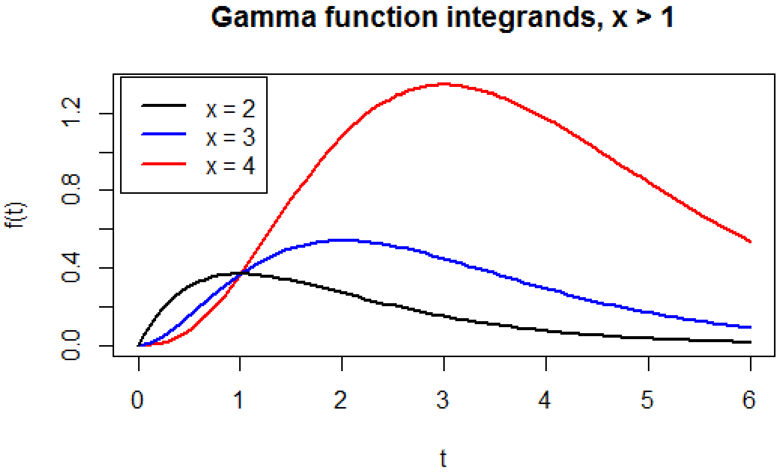
Once more, the definition of the gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \cdot dt$$

When $x > 1$, the integrand is 0 at $t = 0$, then is increasing for small values of t, but finally attains a maximum value at some value t_{max}, and decreases for all values $t > t_{\text{max}}$.

The larger the value of x, the bigger the hump, hence the greater the value of $\Gamma(x)$.

The Gamma Function $\Gamma(x)$



The Gamma Function $\Gamma(x)$

The gamma function has many, many interesting properties, and occurs throughout mathematics.

It's not something that is in any way specific to statistics or probability.

We'll be using this function a great deal in our course, so it's good to become comfortable with it.

The Gamma Function $\Gamma(x)$

The most important result on the gamma function is surely the *fundamental recurrence relation*, and so now we're going to investigate this.

The fundamental recurrence relation for the gamma function allows us to avoid doing a lot of work for certain integrals.

It also leads to an astonishing relation with another function that you might not suspect.

The Gamma Function $\Gamma(x)$

To start with, note that it's very easy to show that $\Gamma(1) = 1$:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{1-1} e^{-t} \cdot dt \\
&= \int_0^\infty e^{-t} \cdot dt \\
&= -e^{-t}\Big|_0^\infty \\
&= 1
\end{aligned}$$

So, to calculate $\Gamma(1)$, all we have to do is to evaluate a simple exponential integral.

How about $\Gamma(2)$?

Using our standard definition, $\Gamma(2)$ is:

$$\begin{aligned}\Gamma(2) &= \int_0^\infty t^{2-1} e^{-t} \cdot dt \\ &= \int_0^\infty t e^{-t} \cdot dt\end{aligned}$$

To evaluate this integral, we have to use integration by parts.

The formula for integration by parts is:

$$\int u dv = uv - \int v du$$

The trick to using integration by parts is knowing how to choose u and v.

As a general rule, when we have an integrand consisting of a power term times an exponential, we want to choose u to be the power term and dv to be the exponential.

For reference, here's $\Gamma(2)$:

$$\Gamma(2) = \int_0^\infty te^{-t} \cdot dt$$

When we choose u to be the power term t and dv to be the exponential term $e^{-t}dt$, we have:

$$\begin{aligned} u &= t \\ du &= dt \end{aligned}$$

$$\begin{aligned} dv &= e^{-t} \cdot dt \\ v &= -e^{-t} \end{aligned}$$

The Gamma Function $\Gamma(x)$

Now our integration by parts becomes:

$$\begin{aligned}
\Gamma(2) &= \int_0^\infty \underbrace{t}_{u}\underbrace{e^{-t}\cdot dt}_{dv} \\
&= \underbrace{t}_{u}\cdot\underbrace{-e^{-t}}_{v}\Big|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v}\cdot\underbrace{dt}_{du} \\
&= 0 + \int_0^\infty e^{-t}\cdot dt \\
&= \int_0^\infty e^{-t}\cdot dt
\end{aligned}$$

Now this is a simple exponential integral, and we know how to do this.

Incidentally, in this integration, you might be wondering why

$$t \cdot -e^{-t}\Big|_0^\infty = 0$$

The point is that at $t = 0$ the whole expression is just 0, while as t goes to infinity the exponential factor e^{-t} goes to 0 faster than the power term t increases.

This is true in general for all values of x, no matter how large:

$$t^x \cdot -e^{-t}\Big|_0^\infty = 0$$

So we've reduced the problem of evaluating $\Gamma(2)$ to the problem of evaluating a simple exponential integral, and we know how to do that:

$$\begin{aligned}\Gamma(2) &= \int_0^\infty e^{-t} \cdot dt \\ &= 1\end{aligned}$$

So $\Gamma(2) = 1$.

Note that in order to calculate $\Gamma(2)$, we had to do 1 integration by parts, and then a simple exponential integral.

How about $\Gamma(3)$?

Using the general definition, we have:

$$\begin{aligned}\Gamma(3) &= \int_0^{\infty} t^{3-2}e^{-t} \cdot dt \\ &= \int_0^{\infty} t^2 e^{-t} \cdot dt\end{aligned}$$

Once again, we have to use integration by parts to evaluate this integral.

We've seen that $\Gamma(3)$ is:

$$\Gamma(3) = \int_0^\infty t^2 e^{-t} \cdot dt$$

Using our standard integration by parts strategy, when we choose u to be the power term t and dv to be the exponential term $e^{-t}dt$, we have:

$$\begin{aligned} u &= t^2 \\ du &= 2t \cdot dt \end{aligned}$$

$$\begin{aligned} dv &= e^{-t} \cdot dt \\ v &= -e^{-t} \end{aligned}$$

Now when we do an integration by parts, we have:

$$\begin{aligned}
\Gamma(3) &= \int_0^\infty \underbrace{t^2}_{u} \underbrace{e^{-t} \cdot dt}_{dv} \\
&= \underbrace{t^2}_{u} \cdot \underbrace{-e^{-t}}_{v} \Big|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v} \cdot \underbrace{2t \cdot dt}_{du} \\
&= 0 + 2 \cdot \int_0^\infty te^{-t} \cdot dt \\
&= 2 \cdot \int_0^\infty te^{-t} \cdot dt
\end{aligned}$$

We've made some progress here, but we'll still have to do another integration by parts.

When we wanted to evaluate $\Gamma(2)$, we had to do 1 integration by parts, and one simple exponential integral.

When we wanted to evaluate $\Gamma(3)$, we had to do 2 integration by parts, and one simple exponential integral.

I'll spare you the gory details, but I hope you can see that if we were to try evaluate $\Gamma(4)$, we would have to perform 3 integrations by parts, and one simple exponential integral.

As a general rule, if we want to evaluate $\Gamma(n)$, we will have to do $n - 1$ integrations by parts, and one simple exponential integral.

Thus, if we have to calculate something like $\Gamma(50)$, we will have to do 49 integrations by parts, which is incredibly laborious.

Is there a better way?

Fortunately, there **is** a better way.

Let's look back to $\Gamma(2)$, and recall the end result of the integration by parts:

$$\begin{aligned} \Gamma(2) &= \int_0^\infty te^{-t} \cdot dt \\ &= 1 \cdot \int_0^\infty e^{-t} \cdot dt \end{aligned}$$

Notice that this last integral is $\Gamma(1)$.

So we can write:

$$\Gamma(2) = 1 \cdot \Gamma(1)$$

How about $\Gamma(3)$?

After the first integration by parts, we had:

$$\begin{aligned} \Gamma(3) &= \int_0^\infty t^2 e^{-t} \cdot dt \\ &= 2 \cdot \int_0^\infty t e^{-t} \cdot dt \end{aligned}$$

Now the final integral is just $\Gamma(2)$.

So we have:

$$\Gamma(3) = 2 \cdot \Gamma(2)$$

The Gamma Function $\Gamma(x)$

If we had worked out $\Gamma(4)$, we would have found that after our first integration by parts we would indeed have arrived at the relation:

$$\Gamma(4) = 3 \cdot \Gamma(3)$$

This suggests a general rule:

$$\Gamma(x) = (x-1) \cdot \Gamma(x-1)$$

Equivalently, we have:

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

Can we formally prove this?

The Gamma Function $\Gamma(x)$

The general formula for $\Gamma(x+1)$ is:

$$\Gamma(x) \quad = \quad \int_0^\infty t^{x-1} e^{-t} \cdot dt$$

Using our standard strategy, we have:

$$\begin{aligned} u &= t^{x-1} \\ du &= (x-1) \cdot t^{x-2} \\ \\ dv &= e^{-t} \\ v &= -e^{-t} \end{aligned}$$

Now our integration by parts becomes:

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty \underbrace{t^{x-1}}_{u} \underbrace{e^{-t} \cdot dt}_{dv} \\
&= \underbrace{t^x}_{u} \cdot \underbrace{-e^{-t}}_{v} \Bigg|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v} \cdot \underbrace{(x-1) \cdot t^{x-2} \cdot dt}_{du} \\
&= 0 \; + \; (x-1) \cdot \int_0^\infty t^{x-2} e^{-t} \cdot dt \\
&= (x-1) \cdot \Gamma(x-1)
\end{aligned}$$

And so we've established the fundamental recurrence relation!

The fundamental recurrence relation is powerful because it enables us to replace integrations by parts with multiplications.

To see how this works, imagine how we could calculate $\Gamma(6)$.

Recall that if we worked directly from the defintion, calculating $\Gamma(6)$ would require 5 integrations by parts and a simple exponential integral.

Instead, by repeatedly using the fundamental recurrence relation, we have:

$$\begin{aligned}
\Gamma(6) &= 5 \times \Gamma(5) \\
&= 5 \times 4 \times \Gamma(4) \\
&= 5 \times 4 \times 3 \times \Gamma(3) \\
&= 5 \times 4 \times 3 \times 2 \times \Gamma(2) \\
&= 5 \times 4 \times 3 \times 2 \times 1 \\
&= 120
\end{aligned}$$

So $\Gamma(6) = 120$.

We saved a lot of work by using the fundamental recurrence relation!

The Gamma Function $\Gamma(x)$

All of the concrete examples that we've looked at have involved gamma functions of positive whole numbers e.g. $\Gamma(1)$, $\Gamma(2)$, $\Gamma(3)$, etc.

However, if you look back in our proof of the fundamental recurrence relation, you'll notice that we never actually required that x be an integer e.g. a whole number.

In fact, x does **not** have to be an integer, and the fundamental recurrence relation is true for all real numbers $x > 0$.

Remember, if $x = 0$, then the integral won't converge, so $\Gamma(0)$ is undefined.

I said that the fundamental recurrence relation was very useful because it helped to simplify the evaluation of many integrals, and I hope you will now agree with that statement.

But I also said that the fundamental recurrence relation "...leads to an astonishing relation with another function that you might not suspect."

The Gamma Function $\Gamma(x)$

For your consideration, here's a table of the first few values of the gamma function:

n	$\Gamma(n)$
1	1
2	1
3	2
4	6
5	24
6	120

Do you notice anything remarkable about these values of $\Gamma(n)$?

Do you see a pattern?

Just for reference, let's update the table of values of $\Gamma(n)$ with the values of $(n-1)!$:

n	$\Gamma(n)$	$(n-1)!$
1	1	1
2	1	1
3	2	2
4	6	6
5	24	24
6	120	120

The Gamma Function $\Gamma(x)$

An astonishing fact: for any positive integer n,

$$\Gamma(n) = (n-1)!$$

You can see this visually in our derivation of $\Gamma(6)$ using the fundamental recurrence relation:

$$\Gamma(6) = \underbrace{5 \times 4 \times 3 \times 2 \times 1}_{5!}$$

The Gamma Function $\Gamma(x)$

In general, we have:

$$\begin{aligned}
\Gamma(n) &= (n-1)\times\Gamma(n-1) \\
&= (n-1)\times(n-2)\times\Gamma(n-2) \\
&\vdots \\
&= (n-1)\times(n-2)\times\ldots 3\times 2\times 1 \\
&= (n-1)!
\end{aligned}$$

Again: the fundamental recurrence relation is true for **all** values of $x > 0$.

But the factorial function is only defined for positive integers, so that's why we have to restrict the astonishing fact to positive integers.

One way to think of the gamma function is that it extends the factorial function to non-integral arguments.

Sometimes we might encounter an integral that looks very similar to a gamma function, but is not quite a gamma function because it has a constant multiplier in the exponent of the exponential.

For instance, we could have something like this:

$$\int_0^\infty x^4 e^{-3x} \cdot dx$$

Is this a gamma function?

Well, kinda sorta.

It's very close, except for the fact that the exponent of the exponential term is $-3x$.

If you go back to the definition of the gamma function, you'll notice that the integral there just had $-x$ for the exponent.

So, this is very close, except for that annoying factor of 3.

We can resolve this difficulty with a simple substitution:

$$\begin{aligned} s &= 3x \\ x &= \frac{s}{3} \\ dx &= \frac{ds}{3} \end{aligned}$$

Whenever we perform a substition in a definite integral, we also have to consider the limits of integration.

When $x = 0$, then $s = 0$, and as x goes to infinity, so does s.

So the limits of integration remain unchanged.

Now we can evaluate the original integral:

$$\begin{aligned} \int_0^\infty x^4 e^{-3x} \cdot dx &= \int_0^\infty \left(\frac{s}{3}\right)^4 e^{-s} \cdot \frac{ds}{3} \\ &= \frac{1}{3^5} \int_0^\infty s^4 e^{-s} \cdot ds \\ &= \frac{\Gamma(5)}{3^5} \end{aligned}$$

In general, we have the formula:

$$\int_0^\infty t^x e^{-qt} \cdot dt = \frac{\Gamma(x+1)}{q^{x+1}}$$

Rather than have all the fun and prove this result, I'm going to let you do the derivation all by yourself on the first homework.

Because this is kind of sort of a gamma function, I call it the "kinda sorta" gamma function.

I must warn you – the term "kinda sorta gamma function" is unique to this course, and if you use it around people who haven't taken this course, they will not have any idea of what you are talking about.

The Gamma Function $\Gamma(x)$

Let's return to the idea that the gamma function can take non-integer inputs.

There is no theoretical problem with non-integer inputs, and an expression such as $\Gamma(3.7)$ is completely well-defined.

There are however some computational difficulties that come up when we try to calculate this.

To see what the problem is, let's first think about why our calculations with integer inputs worked out.

You might have noticed that each time we did an integration by parts, we reduced the exponent of the power term by 1.

Eventually, if we started with an integer input, we ended up reducing the exponent of the power term to 0, in which case the integral was just a simple exponential integral which was easy to evaluate.

For instance, with $\Gamma(4)$, we have:

$$\begin{aligned}
\Gamma(4) &= \int_0^\infty t^3 e^{-t} \cdot dt \\
&= 3 \times \int_0^\infty t^2 e^{-t} \cdot dt \\
&= 3 \times 2 \times \int_0^\infty t^1 e^{-t} \cdot dt \\
&= 3 \times 2 \times 1 \times \int_0^\infty t^0 e^{-t} \cdot dt \\
&= 6
\end{aligned}$$

The crucial point is that if we start with an integer value of x, each time we perform an integration by parts we will reduce the exponent of the power term in the integrand by 1.

Eventually we will hit 0, at which point we can finish the integration.

But with a non-integer value of x, we will miss 0.

For instance,

$$\begin{aligned}\Gamma(3.7) &= \int_0^\infty t^{2.7} e^{-t} \cdot dt \\ &= 2.7 \times \int_0^\infty t^{1.7} e^{-t} \cdot dt \\ &= 2.7 \times 1.7 \times \int_0^\infty t^{0.7} e^{-t} \cdot dt\end{aligned}$$

And now we're stuck, because we won't be able reduce the exponent 0.7 to 0 by an integration by parts.

In this case, we need to use a computer to perform the integral numerically.

To be clear: the gamma function is perfectly well-defined for all positive values of x.

The only problem is that we can't do the integration in a nice closed form.

Microsoft Excel has a built-in function for numerically evaluating the gamma function, so for instance using the `gamma()` function we find that $\Gamma(3.7) = 4.170652$.

R can evaluate gamma functions with non-integer inputs, and in fact the function is also named `gamma()`. Note: R is case-sensitive, so you have to spell the function name with lower-case letters.

We can still use the recurrence relation, though:

$$\begin{aligned}\Gamma(3.7) &= 2.7 \times \Gamma(2.7) \\ &= 2.7 \times 1.7 \times \Gamma(1.7) \\ &= 2.7 \times 1.7 \times \int_0^\infty t^{0.7}\ e^{-t} \cdot dt\end{aligned}$$

So, to calculate $\Gamma(3.7)$, we just need to numerically evalute the integral

$$\int_0^\infty t^{0.7} e^{-t} \cdot dt$$

Using numerical software (e.g. R or Excel) we have:

$$\begin{aligned}\Gamma(1.7) &= \int_0^\infty t^{0.7} e^{-t} \cdot dt \\ &= 0.908639\end{aligned}$$

Notice that:

$$2.7 \times 1.7 \times 0.908639 = 4.170652$$

This is the same value that our software calculated for $\Gamma(3.7)$.

Here's one last startling fact about the gamma function:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

This is so wonderful that it would be cruel of me to deny you the enjoyment of deriving it for yourself, so you'll get to do that on the first homework!

The Gaussian Integral

The Gaussian Integral

The *Gaussian integral* is the integral:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx$$

This looks rather scary, and if you try to evaluate it using standard integration techniques such as substitution or integration by parts you'll find that none of these work.

In fact, it can be proved that the integrand is not an anti-derivative of any finite combination of elementary functions.

That is, there is no way to construct a function $f(x)$ using things like powers of x, trigonometric function, exponentials, square roots etc. such that

$$\frac{df(x)}{dx} = e^{-x^2}$$

You might think at this point that all hope is lost if we can't find an anti-derivative, but amaingly it is possible to evaluate this integral.

It will however require some trickery!

Note: This derivation is one of the few places in the course where you have to know something about multiple integration.

If you find it difficult to follow all the details, don't freak out.

The first bit of trickery is that we won't evaluate the integral directly, but will instead evaluate its square. Let's denote the integral with the letter I:

$$I = \int_{-\infty}^{+\infty} e^{-x^2} \cdot dx$$

Note that the integral in no way depends on the letter that we use for the integration:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx = \int_{-\infty}^{+\infty} e^{-y^2} \cdot dy$$

So, both of these integrals are equal to I.

The Gaussian Integral

We will evaluate I^2, which we will express as a double integral. When we write our squared integral I^2, we'll use an x for one of the integrals, and y for the other:

$$\begin{aligned} I^2 &= I \cdot I \\ &= \left(\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx \right) \cdot \left(\int_{-\infty}^{+\infty} e^{-y^2} \cdot dy \right) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} \cdot dx\ dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cdot dx\ dy \end{aligned}$$

At this point, it seems that we've made the problem even more complicated than what we started with, but astonishingly this double itegral is actually very easy to evaluate.

The (extremely clever) trick is to convert the integral from rectangular Cartesian coordinates x and y to polar co-ordinates r and θ.

This is a standard technique that's covered in multivariate calculus, but you might be a little rusty with this so let me remind you: in polar coordinates, $r^2 = x^2 + y^2$, and the differential element of area is $dx\ dy = r\ dr\ d\theta$.

The Gaussian Integral

Also, our double integral is over the entire plane: the integration over x ranges from $-\infty$ to $+\infty$ and the integration over y also ranges from $-\infty$ to $+\infty$.

In polar coordinates, we integrate over the entire plane by allowing r to range from 0 to ∞ and θ to range from 0 to 2π.

So, after we convert everything, we have:

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cdot dx\ dy &= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2} \cdot r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^{+\infty} re^{-r^2} \cdot dr\ d\theta\end{aligned}$$

Now it's easy to evaluate the inner integral over r:

$$\begin{aligned}
\int_0^{+\infty} re^{-r^2} \cdot dr &= -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=\infty} \\
&= -0 - \left(-\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

So the double integral is now reduced to a single integral:

$$\int_0^{2\pi} \left[\int_0^{+\infty} re^{-r^2} \cdot dr \right] d\theta \;=\; \int_0^{2\pi} \frac{1}{2} \cdot d\theta$$

This single integral is very easy:

$$\begin{aligned} \int_0^{2\pi} \frac{1}{2} \cdot d\theta &= \left. \frac{\theta}{2} \right|_{\theta=0}^{\theta=2\pi} \\ &= \frac{2\pi}{2} - 0 \\ &= \pi \end{aligned}$$

Incredible!!

Let's summarize what just happened.

First, instead of calculating I, we instead calculated I^2, which gave us a double integral in Cartesian coordinates.

We then converted the double integral to polar coordinates, evaluated it using elementary techniques, and in the end it came out to be π.

Don't forget the last step though: we originally set out to evaluate I, and we've shown that $I^2 = \pi$, so it must be that $I = \sqrt{\pi}$:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx = \sqrt{\pi}$$

Appendix: The Beta Function

Appendix: The Beta Function

We can't finish a lecture on the gamma function without mentioning the *beta function*:

$$\beta(a,b) = \int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp$$

Unlike the gamma function, the beta function takes two inputs rather than one.

But it's very similar to the gamma function, because it uses the same strategy of using the inputs to specify a particular integrand, and then the value of the beta function is the integral of this integrand i.e. it's the area under the curve.

Appendix: The Beta Function

It's possible to evaluate this integral:

$$\int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp \;=\; \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}$$

So I know you'll all agree: we couldn't have a lecture on the gamma function without mentioning this!

How do you prove this amazing formula?

There are many derivations, but they are all somewhat complicated and tricky.

You will not be responsible for the derivation of this beta function integral on any homeworks or exams.

I'm just mentioning it here as a little extra special thing. Cool!