Problem Set 1 SOLUTIONS

MATH E-158: Introduction to Bayesian Inference

Due September 11, 2017

Problem 1

Problem Statement

Part (a) What is the value of $\Gamma(9)$?

Part (b) What is the value of $\Gamma(5.5)$?

Part (c) What is the value of $\dfrac{\Gamma(8)}{\Gamma(5)} \cdot \dfrac{\Gamma(7.1)}{\Gamma(10.1)}$?

Part (d) Obtain an expression in terms of a gamma function divided by a constant for the integral

$$\int_0^\infty t^{3.6}\ e^{-2.7t} \cdot dt$$

Once you've expressed this in terms of a gamma function divided by a constant, use software (e.g. Excel or R) to obtain the numerical value.

Problem Solution

Part (a) Since we want to determine the value of the gamma function for an integer argument, we can just use the factorial function:

$$\begin{aligned}\Gamma(9) &= 8! \\ &= 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 40,320\end{aligned}$$

I sincerely hope that you did not attempt to do 7 integrations by parts.

Part (b) Here we need to use the fundamental recurrence relation, and the fact that $\Gamma(0.5) = \sqrt{\pi}$:

$$\begin{aligned}\Gamma(5.5) &= 4.5 \times \Gamma(4.5) \\ &= 4.5 \times 3.5 \times \Gamma(3.5) \\ &= 4.5 \times 3.5 \times 2.5 \times \Gamma(2.5) \\ &= 4.5 \times 3.5 \times 2.5 \times 1.5 \times \Gamma(1.5) \\ &= 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \Gamma(0.5) \\ &= 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \sqrt{\pi} \\ &= 52.34278\end{aligned}$$

Part (c) Let's first reduce each of the two terms individually. For the first term, we have:

$$\begin{aligned}\frac{\Gamma(8)}{\Gamma(5)} &= \frac{7 \times 6 \times 5 \times \Gamma(5)}{\Gamma(5)} \\ &= 7 \times 6 \times 5 \\ &= 210\end{aligned}$$

For the second term, we have:

$$\begin{aligned}\frac{\Gamma(7.1)}{\Gamma(10.1)} &= \frac{\Gamma(7.1)}{9.1 \times 8.1 \times 7.1 \times \Gamma(7.1)} \\ &= \frac{1}{9.1 \times 8.1 \times 7.1}\end{aligned}$$

Putting these two results together, we have

$$\begin{aligned}\frac{\Gamma(8)}{\Gamma(5)} \cdot \frac{\Gamma(7.1)}{\Gamma(10.1)} &= \frac{210}{9.1 \times 8.1 \times 7.1} \\ &= 0.40127\end{aligned}$$

Part (d) Obtain an expression in terms of a gamma function divided by a constant for the integral

$$\int_0^\infty t^3.6\ e^{-2.7t} \cdot dt$$

Once you've expressed this in terms of a gamma function divided by a constant, use software (e.g. Excel or R) to obtain the numerical value.

Solution This is a "kinda sorta" gamma function, and we can use the formula from lecture:

$$\int_0^\infty t^3.6\ e^{-2.7t} \cdot dt \ = \ \frac{\Gamma(4.6)}{2.7^{4.6}}$$

This is the solution to the first part of the problem, because the right-hand side is a gamma function divided by a constant. When we evaluate this expression numerically, we obtain

$$\frac{\Gamma(4.6)}{2.7^{4.6}} \ = \ 0.13875$$

Thus,

$$\int_0^\infty t^3.6\ e^{-2.7t} \cdot dt \ = \ 0.13875$$

Problem 2

Problem Statement

Using the recurrence relation, we can actually define the gamma function for negative numbers, as long as they are not integers (i.e. whole numbers). Calculate the value of $\Gamma(-5/2)$.

Problem Solution

The fundamental recursion relation for the gamma function is:

$$\Gamma(x) = (x-1) \cdot \Gamma(x-1)$$

This enables us to express the gamma function for x in terms of the gamma function for $x-1$. Now we just use the recurrence relation repeatedly:

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \left(-\frac{1}{2}\right) \cdot \Gamma\left(-\frac{1}{2}\right) \\
&= \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \Gamma\left(-\frac{3}{2}\right) \\
&= \left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \cdot \Gamma\left(-\frac{5}{2}\right) \\
&= -\frac{15}{8} \cdot \Gamma\left(-\frac{5}{2}\right)
\end{aligned}$$

Solving for $\Gamma(-5/2)$, we have

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15} \cdot \Gamma\left(\frac{1}{2}\right)$$

Remember from lecture the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

We can substitute this in to obtain

$$\begin{aligned}\Gamma\left(-\frac{5}{2}\right) &= -\frac{8}{15}\cdot\Gamma\left(\frac{1}{2}\right)\\ &= -\frac{8}{15}\cdot\sqrt{\pi}\\ &= -0.94531\end{aligned}$$

Problem 3

Problem Statement

Part (a) The binomial coefficient is defined for non-negative integers r and k with $r \geq k$ as

$$\binom{r}{k} = \frac{r!}{k! \cdot (r-k)!}$$

Use the gamma function to define a generalized form of the binomial coefficient, where r is still non-negative but is not necessarily an integer. Your generalized form should return the same answer as before when r is an integer. Note that k is still restricted to integer values.

Part (b) What is the value of $\binom{8.5}{4}$?

Problem Solution

Part (a) When r is an integer, we want our "generalized" binomial coefficient to have the same value as the standard binomial coefficient. We can achieve this by replacing the $r!$ term in the denominator with $\Gamma(r+1)$, which will equal $r!$ when r is an integer, and likewise replacing $(r-k)!$ with $\Gamma(r-k+1)$:

$$\binom{r}{k} = \frac{r!}{k! \cdot (r-k)!} = \frac{\Gamma(r+1)}{k! \cdot \Gamma(r-k+1)}$$

If we wanted to, we could replace $k!$ in the denominator with $\Gamma(k+1)$, but since k is still restricted to integer values, this doesn't gain us anything.

Part (b) When

$$\begin{aligned}
\binom{8.5}{4} &= \frac{\Gamma(8.5+1)}{4!\cdot\Gamma(8.5-4+1)} \\
&= \frac{\Gamma(9.5)}{4!\cdot\Gamma(5.5)} \\
&= \frac{8.5\times 7.5\times 6.5\times 5.5\times\Gamma(5.5)}{4!\cdot\Gamma(5.5)} \\
&= \frac{8.5\times 7.5\times 6.5\times 5.5}{4!} \\
&= 94.96094
\end{aligned}$$

Incidentally, note that

$$\binom{8}{4} = 70$$

Also, note that

$$\binom{9}{4} = 126$$

So, if you tried to linearly interpolate the value of $\binom{8.5}{4}$, you'd get

$$\begin{aligned}
\frac{\binom{8}{4}+\binom{9}{4}}{2} &= \frac{70+126}{2} \\
&= \frac{196}{2} \\
&= 98
\end{aligned}$$

This is the wrong way to do this problem, but it's not totally crazy, and the interpolated value is approximately correct, so it shows that our definition of the generalized binomial coefficient is a reasonable one.

Problem 4

Problem Statement

Calculate the value of $\Gamma(4)$ directly from the definition i.e. evaluate the integral

$$\Gamma(4) \;=\; \int_0^\infty t^3\; e^{-t} \cdot dt$$

You'll have to use integration by parts for this. You can use the fact that

$$\int_0^\infty t^2 e^{-t} \cdot dt = 2$$

Problem Solution

Recall that integration by parts has the form

$$\int_a^b u \cdot dv = u \cdot v|_a^b - \int_a^b v \cdot du$$

The integrand we want to integrate has the form of a power term t^3 and an exponential term e^{-t}. The basic rule of thumb for this type of integrand is to set u equal to the power term and dv equal to the exponential term:

$$\begin{aligned} u &= t^3 \\ du &= 3t^2 \cdot dt \\ dv &= e^{-t} \cdot dt \\ v &= -e^{-t} \end{aligned}$$

Now the integration by parts formula becomes:

$$\begin{aligned} \int_0^\infty t^3\; e^{-t} \cdot dv &= \left[t^3 \cdot -e^{-t}\right]_0^\infty - \int_0^{-\infty} -e^{-t} \cdot 3t^2\; dt \\ &= \left[t^3 \cdot -e^{-t}\right]_0^\infty \;+\; 3\int_0^{-\infty} t^2 e^{-t} \cdot dt \end{aligned}$$

Now consider the first term on the right-hand side, the expression in brackets:

$$\left[t^3 \cdot -e^{-t}\right]_0^\infty$$

When t goes to infinity, so does t^3, but e^{-t} goes to 0, and the exponential term drops off more rapidly than the power term increases, so we have

$$\lim_{t \to \infty} t^3 \cdot e^{-t} = 0$$

For the lower limit, when $t = 0$, then $t^3 = 0$, and

$$t^3 \cdot e^{-t}\Big|_0 = 0$$

So the bracketed term vanishes:

$$\left[t^3 \cdot -e^{-t}\right]_0^\infty = 0$$

Thus we have

$$\begin{aligned}
\int_0^\infty t^3\ e^{-t} \cdot dv &= \left[t^3 \cdot -e^{-t}\right]_0^\infty + 3\int_0^{-\infty} t^2 e^{-t} \cdot dt \\
&= 0 + 3\int_0^{-\infty} t^2 e^{-t} \cdot dt \\
&= 3\int_0^{-\infty} t^2 e^{-t} \cdot dt
\end{aligned}$$

Now we can use the integral given in the problem statement:

$$\int_0^\infty t^2 e^{-t} \cdot ds = 2$$

Putting this all together, we have

$$\begin{aligned}
\int_0^\infty t^3\ e^{-t} \cdot dv &= 3\int_0^{-\infty} t^2 e^{-t} \cdot dt \\
&= 3 \cdot 2 \\
&= 6
\end{aligned}$$

Problem 5

Problem Statement

Evaluate the integral

$$\int_0^\infty t^5 \; e^{-3t} \cdot dt$$

In this problem, you should explicitly evaluate this integral. You can use any and all results from sections 2.1 to 2.4 of the reading, but don't use any formulas after that! If you think there is a formula that is applicable to this problem, you are welcome to use it to check your work, but for this problem I want to see you do the integral and work with gamma functions.

Problem Solution

This looks like it's very close to a gamma function, but the problem is the $3t$ in the exponential factor – a gamma function is defined in terms of e^{-t}, not e^{-3t}. So we'll do a substitution to fix this, and a good substitution for this is $s = 3t$. Then

$$\begin{aligned} s &= 3t \\ t &= \frac{s}{3} \\ dt &= \frac{ds}{3} \end{aligned}$$

Note that when t goes to infinity, then so does s, and when t equals 0, then so does s as well, so the limits of integration remain unchanged. Then, using our substitution, we have:

$$\begin{aligned}
\int_0^\infty t^5 \; e^{-3t} \cdot dt &= \int_0^\infty \left(\frac{s}{3}\right)^5 \; e^{-s} \cdot \frac{ds}{3} \\
&= \frac{1}{3^6} \int_0^\infty t^5 \; e^{-t} \cdot dt \\
&= \frac{1}{3^6} \cdot \Gamma(6) \\
&= \frac{120}{729} \\
&= 0.16461
\end{aligned}$$

Perhaps you noticed that this is a "kinda sorta" gamma function, so you could have checked your work using the formula:

$$\begin{aligned}
\int_0^\infty t^5 \; e^{-3t} \cdot dt &= \frac{\Gamma(5+1)}{3^{5+1}} \\
&= \frac{\Gamma(6)}{3^6} \\
&= \frac{120}{729} \\
&= 0.16461
\end{aligned}$$

So our answer is the same when we used the formula from lecture.

Problem 6

Problem Statement

Derive an algebraic expression for the "kinda sorta" gamma function:

$$\int_0^\infty t^k e^{-\beta t} \cdot ds, \quad \beta > 0$$

You must explicitly evaluate the integral, although you are welcome to check your answer with anything that you might find in lecture.

Problem Solution

This is basically the same problem as before, except that now we consider the general case. We need to make the substitution $s = \beta t$:

$$\begin{aligned} s &= \beta t \\ t &= \frac{s}{\beta} \\ dt &= \frac{ds}{\beta} \end{aligned}$$

As in the previous problem, the limits of integration are unchanged, because as t goes to infinity, so does s, and when t is 0, so also is s. Now we can perform the substitution:

$$\begin{aligned} \int_0^\infty t^k e^{-\beta t} \cdot dt &= \int_0^\infty \left(\frac{s}{\beta}\right)^k e^{-s} \cdot \frac{ds}{\beta} \\ &= \int_0^\infty \frac{s^k}{\beta^{k+1}} e^{-s} \cdot ds \\ &= \frac{1}{\beta^{k+1}} \int_0^\infty s^{(k+1)-1} e^{-s} \cdot ds \\ &= \frac{\Gamma(k+1)}{\beta^{k+1}} \end{aligned}$$

Compare this with the previous problem, where $k = 5$ and $\beta = 3$, and you'll find that this general formula gives the same answer as before (which it should!).

Problem 7 (Graduate)

Problem Statement

Evaluate the integral

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2} \cdot \frac{x^2}{\sigma^2}\right\} \cdot dx$$

To do this, you should first note that the integrand here is symmetric about 0, so we have:

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2} \cdot \frac{x^2}{\sigma^2}\right\} \cdot dx = 2 \cdot \int_{0}^{\infty} \exp\left\{-\frac{1}{2} \cdot \frac{x^2}{\sigma^2}\right\} \cdot dx$$

Similarly, the Gaussian integral has the same symmetry property:

$$\int_{-\infty}^{\infty} \exp\left\{-x^2\right\} \cdot dx = 2 \cdot \int_{0}^{\infty} \exp\left\{-x^2\right\} \cdot dx$$

By making a clever substitution, you should be able to convert our original integral into a problem involving a Gaussian integral, and then you can evaluate this using the result from lecture.

WARNING!! DO NOT attempt to solve this by constructing a double integral and converting to polar coordinates.

Problem Solution

We'll use the substitution

$$s^2 = \frac{1}{2} \cdot \frac{x^2}{\sigma^2}$$

Then

$$\begin{aligned} s &= \frac{x}{\sqrt{2}\sigma} \\ x &= \sqrt{2}\sigma s \\ dx &= \sqrt{2}\sigma ds \end{aligned}$$

What are the limits of integration with this substitution? At the upper end, as x goes to infinity, so will s:

$$
\begin{aligned}
\lim_{x\to\infty} s &= \lim_{x\to\infty} \frac{x}{\sqrt{2}\sigma} \\
&= \infty
\end{aligned}
$$

On the lower end, when x is 0, so is s. So the limits of integration are unchanged. When we substitute all of this into the integral, we have:

$$
\begin{aligned}
\int_0^\infty \exp\left\{-\frac{1}{2}\cdot\frac{x^2}{\sigma^2}\right\}\cdot dx &= \int_0^\infty \exp\left\{-s^2\right\}\cdot\sqrt{2}\sigma ds \\
&= \sqrt{2}\sigma\cdot\int_0^\infty \exp\left\{-s^2\right\}\cdot ds \\
&= \sqrt{2}\sigma\cdot\left(\frac{1}{2}\int_\infty^\infty \exp\left\{-s^2\right\}\cdot ds\right) \\
&= \sqrt{2}\sigma\cdot\left(\frac{1}{2}\sqrt{\pi}\right) \\
&= \frac{\sqrt{2\pi}\sigma}{2}
\end{aligned}
$$

Finally, for the original integral, we have:

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\cdot\frac{x^2}{\sigma^2}\right\}\cdot dx &= 2\cdot\int_0^\infty \exp\left\{-\frac{1}{2}\cdot\frac{x^2}{\sigma^2}\right\}\cdot dx \\
&= 2\cdot\frac{\sqrt{2\pi}\sigma}{2} \\
&= \sqrt{2\pi}\sigma
\end{aligned}
$$

You might be wondering why we went through all contortions of working with the half-integrals. The reason is that our substitution involved s^2, and when we take square roots we have to consider both positive and negative values. By just working over the positive numbers, we avoid this problem.

Problem 8 (Graduate)

Problem Statement

In lecture, we saw the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-s^2} \cdot ds = \sqrt{\pi}$$

Use this to derive the result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hint: the integral goes from $-\infty$ to $+\infty$, but the Gamma function is defined by an integral from 0 to $+\infty$. So the first step is to obtain a new integral that has the appropriate range of integration. You actually can do this with a conceptual argument; no algebra manipulation required. Then it's just a matter of doing the right substitution to put the integral into the form of a gamma function.

Problem Solution

We first need to deal with issue of the range of integration: the gamma function is defined by an integral that ranges from 0 to $+\infty$, and so we need to find some sort of integral formula on that range, given our initial result:

$$\int_{-\infty}^{\infty} e^{-s^2} \cdot ds = \sqrt{\pi}$$

The trick here is to think about the symmetry of the integrand:

$$f(s) = e^{-s^2}$$

Notice that this function is symmetric with respect to 0: that is, the value of the integrand for $-s$ is the same as the value for s, because the integrand only depends on s^2. For instance, $f(-2)$ is the same as $f(2)$:

$$\begin{aligned} f(-2) &= e^{-(-2)^2} \\ &= e^{-4} \\ &= e^{-(2)^2} \\ &= f(2) \end{aligned}$$

This is true for *every* value of s: $f(-s)$ will always be equal to $f(s)$. Thus, the integrand is symmetric with respect to 0, in that the image of the function to the right of 0 is the mirror image of the function to the left. That means that the areas under the curve for the left- and right-hand sides are equal, so that the integral from 0 to $+\infty$ must be one half of the total:

$$\begin{aligned} \int_0^{+\infty} e^{-s^2} \cdot ds &= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-s^2} \cdot ds \\ &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

We can generalize this result: if $f(-s)$ equals $f(s)$ for all s, then we say that $f(s)$ is an *even* function, and for all values of a we have:

$$\int_0^{+a} f(s) \cdot ds = \frac{1}{2} \int_{-a}^{+a} f(s) \cdot ds$$

Of course, this is true only if $f(s)$ is an even function.

Now for the second step of the problem. At this point, we've derived this result:

$$\int_0^{+\infty} e^{-s^2} \cdot ds = \frac{\sqrt{\pi}}{2}$$

Now we have to deal with another problem: what about the s^2 in the exponent? The gamma function is defined with an exponential factor of the form e^{-s}, but here we have an exponential factor of the form e^{-s^2}, so we can't use this integral directly as a gamma function. The trick here is to make the substitution $t = s^2$:

$$\begin{aligned} t &= s^2 \\ s &= t^{1/2} \\ ds &= \frac{1}{2}\, t^{-1/2} \cdot dt \end{aligned}$$

When $s = 0$, then $t = 0$ as well, and similarly when $s = \infty$, then $t = \infty$ as well, so the limits of integration remain unchanged. Thus

$$\begin{aligned} \int_0^\infty e^{-s^2} \cdot ds &= \int_0^\infty e^{-t} \cdot \frac{1}{2}\, t^{-1/2} \cdot dt \\ &= \frac{1}{2} \int_0^{+\infty} t^{-1/2}\, e^{-t} \cdot dt \\ &= \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) \end{aligned}$$

Notice what happened in the last step: once we finished with all the algebra, we ended up with an integral that was *exactly* in the form required for a gamma function, and since the exponent on the power term was $-1/2$, that means that x was $1/2$ (remember in the definition of the gamma function that the exponent of the power term is $x - 1$). So, we now have

$$\int_0^\infty e^{-s^2} \cdot ds = \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right)$$

But earlier we said that

$$\int_0^\infty e^{-s^2} \cdot ds = \frac{\sqrt{\pi}}{2}$$

So therefore

$$\frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

And finally we conclude

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$