Lecture 1: Some Fun Integrals

MATH E-158: Mathematical Foundations of Statistical Software

Spring 2018

1 Introduction

Now it's fun time!!! We'll start off by learning about a new special function, known as the *gamma function*, denoted $\Gamma(x)$, that we'll encounter throughout the course. There's nothing specifically "statistical" about this function, and in fact it occurs in many areas of mathematics outside of probability, so it's useful to have an acquaintance with it. Next, we'll see how to evaluate an apparently intractable integral using some very clever trickery. Finally, I'll present another integral evaluation, but you won't be responsible for this, and I'm including it just as some extra bonus material.

2 The Gamma Function

2.1 Definition

The first step is just to define the gamma function. So: the *gamma function*, denoted by $\Gamma(x)$, is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \cdot dt,\ x > 0$$

This might look a little strange to you if you haven't seen it before, and indeed it is an uncommon way of defining a function. It's helpful to think about this in 3 steps:

1. First we select a value of x, with the condition that x be a positive number.
2. Once we've selected x, this determines the specific form of the integrand, which now is just a function of t. (The *integrand* is the function that is being integrated i.e. it's the function inside the integral.)
3. We then integrate t from 0 to infinity, and this gives us the output of the gamma function for the value of x we selected as the input.

In other words, the value of x picks out a particular form for the integrand, and then the gamma function is just the area under the curve for this integrand.

It helps to look at some specific examples. First, let's consider very simple cases where x is a positive integer:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\cdot dt \\
&= \int_0^\infty e^{-t}\cdot dt \\
\Gamma(2) &= \int_0^\infty t^{2-1}e^{-t}\cdot dt \\
&= \int_0^\infty t^{1}e^{-t}\cdot dt \\
\Gamma(3) &= \int_0^\infty t^{3-1}e^{-t}\cdot dt \\
&= \int_0^\infty t^{2}e^{-t}\cdot dt
\end{aligned}$$

Note that there was nothing in our definition that required x to be an integer, so the gamma function is also defined for any real value, just as long as it is positive:

$$\begin{aligned}
\Gamma(3.5) &= \int_0^\infty t^{2.5}e^{-t}\cdot dt \\
\Gamma(\pi/2) &= \int_0^\infty t^{\pi/2-1}e^{-t}\cdot dt \\
\Gamma(1/2) &= \int_0^\infty t^{-1/2}e^{-t}\cdot dt
\end{aligned}$$

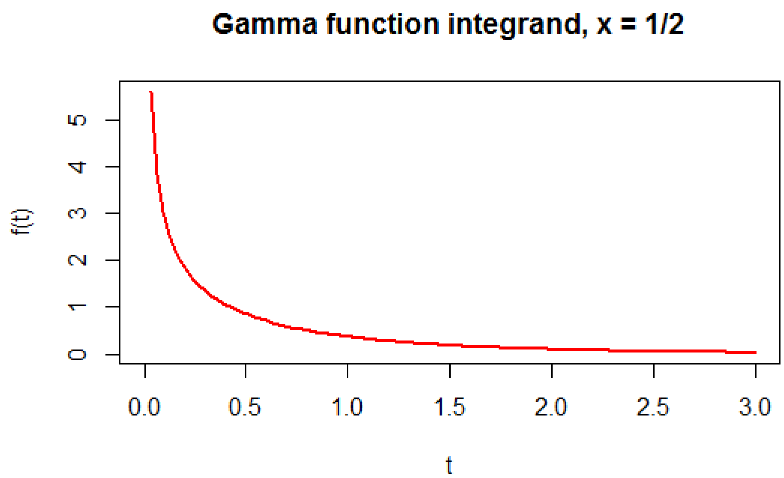
Notice in the last example that, by the definition of the gamma function, x is required to be positive, but because the exponent in the integrand is $x-1$, the actual exponent itself is negative. In fact, this will be true for all values of x less than 1.

2.2 General Properties of the Integrand

Let's think a little more generally about the integrand $f(x,t)$:

$$f(x,t) = t^{x-1}\cdot e^{-t}$$

This function is comprised of two factors: a "power" term t^{x-1}, and an exponential term e^{-t}. As before, it helps to look at some specific cases.



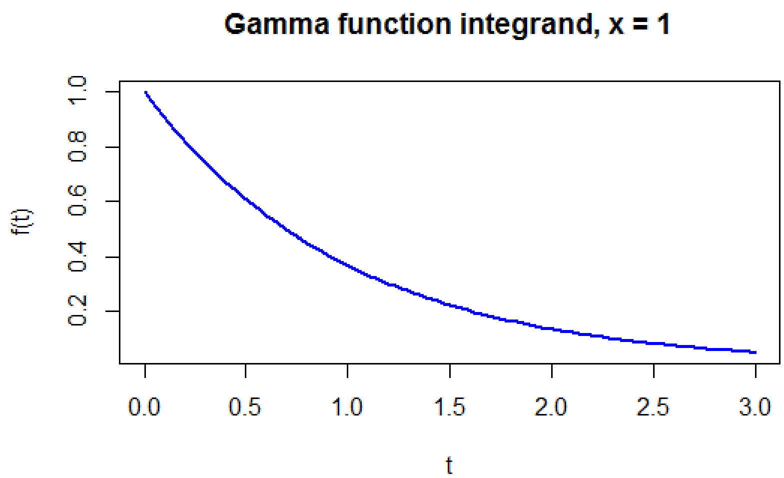
When $0 < x < 1$, then the factor t^{x-1} will have a negative exponent, so that the factor will be undefined at 0, and thus will have a vertical asymptote at $t = 0$. The factor t^{x-1} will always be decreasing for $t > 0$, and so will the factor e^{-t}, and since the integrand is the product of these two factors it too will always be decreasing. To summarize: for the case where $0 < x < 1$, the integrand will have a vertical asymptote at 0 and then will always be decreasing. The first figure is a graph of the integrand when $x = 1/2$:

Next, when $x = 1$, then this is just an exponential function:

$$f(x = 1, t) = e^{-t}$$

At $t = 0$, this has the value 0, and then the function is always decreasing for all values of $t > 0$. The second figure is a graph of this function.

For the case where $x > 1$, things get a little more complicated. In this case, now the factor t^{x-1} will have a positive exponent, and so will be an *increasing* function for all t. On the other hand, e^{-t} will always be a decreasing function. What happens to the integrand as a whole? Well, t^{x-1} has polynomial growth, while e^{-t} has exponential growth, and for sufficiently large values of t exponential growth dominates polynomial growth, so eventually the exponential factor will become dominant and the integrand will go to 0. However, for small values of t, the polynomial growth rate is in fact stronger than the exponential decay, and thus the integrand will be an increasing function for small values of t. Eventually however the exponential decay will predominate, and then the integrand will start to decrease, and will decrease from then on. Thus, the integrand will have the shape of a hump, and for larger values of x the hump gets bigger and taller, and the declining point occurs for larger values of t. This means that



as x gets larger, the value of the integral (and thus the value of $\Gamma(x)$) will get larger. Our third figure is a graph of three of these integrands, for x values of 2, 3, and 4:

2.3 Evaluating the Gamma Function

OK, all of this is very nice, but how do we calculate specific values of this function? Well, for $x = 1$, this is just a simple exponential integral:

$$\begin{aligned} \int_0^\infty e^{-t} \cdot dt &= -e^{-t}\big|_0^\infty \\ &= -0 - (-e^0) \\ &= 1 \end{aligned}$$

For the case $x = 2$, we need to do a little work. The integral has the form

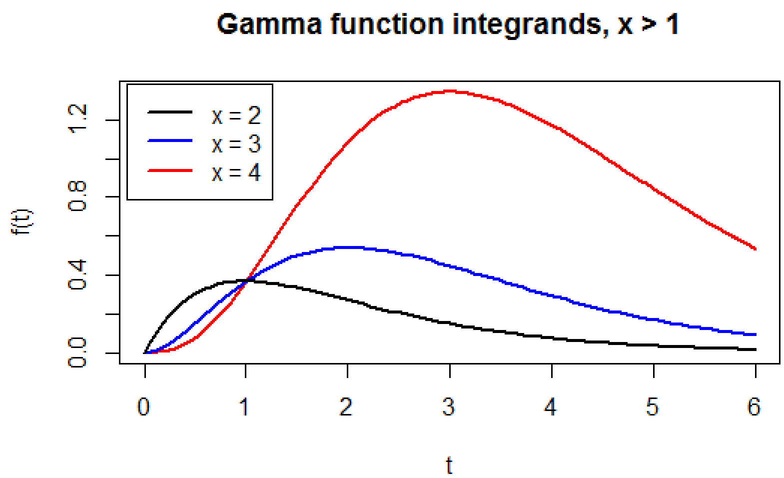
$$\Gamma(2) = \int_0^\infty t^{2-1} e^{-t} \cdot dt \tag{1}$$

$$\tag{2}$$

$$= \int_0^\infty t^1 e^{-t} \cdot dt \tag{3}$$

For this, we need to use integration by parts:

$$\int_0^\infty u \cdot dv = u \cdot v|_0^\infty - \int_0^\infty v \cdot du$$



As a general rule, when we have an integrand consisting of a power term times an exponential, we want to choose u to be the power term and dv to be the exponential. So we'll choose these values for u and v:

$$\begin{aligned} u &= t \\ du &= dt \\ v &= -e^{-t} \\ dv &= e^{-t} \end{aligned}$$

Then the integration formula becomes:

$$\begin{aligned} \Gamma(2) &= \int_0^\infty \underbrace{t}_{u} \underbrace{e^{-t} \cdot dt}_{dv} \\ &= \underbrace{t}_{u} \cdot \underbrace{-e^{-t}}_{v} \Big|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v} \cdot \underbrace{dt}_{du} \\ &= 0 + \int_0^\infty e^{-t} \cdot dt \\ &= \int_0^\infty e^{-t} \cdot dt \\ &= 1 \end{aligned}$$

Notice that in the last line the integral is in fact just $\Gamma(1)$, and we can use our previous calculation to evaluate this. Note that in order to calculate $\Gamma(2)$, we had to do 1 integration by parts, and then a simple exponential integral.

Incidentally, in this integration, you might be wondering why

$$t \cdot -e^{-t}\Big|_0^\infty = 0$$

The point is that at $t = 0$ the whole expression is just 0, while as t goes to infinity the exponential factor e^{-t} goes to 0 faster than the power term t increases. This is true in general for all values of x, no matter how large:

$$t^x \cdot -e^{-t}\Big|_0^\infty = 0$$

For $\Gamma(3)$, the integral is

$$\begin{aligned}\Gamma(3) &= \int_0^\infty t^{3-1}e^{-t} \cdot dt \\ &= \int_0^\infty t^2 e^{-t} \cdot dt\end{aligned}$$

We'll have to use integration by parts again. Using our standard integration by parts strategy, we choose u to be the power term t and dv to be the exponential term $e^{-t}dt$:

$$\begin{aligned}u &= t^2 \\ du &= 2t \cdot dt \\ \\ dv &= e^{-t} \cdot dt \\ v &= -e^{-t}\end{aligned}$$

Then the integration by parts gives us

$$\begin{aligned}\Gamma(3) &= \int_0^\infty \underbrace{t^2}_{u}\underbrace{e^{-t} \cdot dt}_{dv} \\ &= \underbrace{t^2}_{u} \cdot \underbrace{-e^{-t}}_{v}\Bigg|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v} \cdot \underbrace{2t \cdot dt}_{du} \\ &= 0 \ + \ 2 \cdot \int_0^\infty te^{-t} \cdot dt \\ &= 2 \cdot \int_0^\infty te^{-t} \cdot dt\end{aligned}$$

We've made some progress here, but we'll still have to do another integration by parts, at which point we'll be left with a simple exponential integral, which is easy to solve. So to evaluate $\Gamma(3)$, we have to do 2 integrations by parts, and a simple exponential integral.

We can keep on doing this process for ever larger values of x. The strategy is to use integration by parts, and each time we do it we reduce the value of the exponent of the power term in the integrand, until eventually it vanishes, and then all we have to evaluate is a simple exponential integral. But as x gets larger, we have to do more integrations by parts:

- When we wanted to evaluate $\Gamma(2)$, we had to do 1 integration by parts, and one simple exponential integral.
- When we wanted to evaluate $\Gamma(3)$, we had to do 2 integration by parts, and one simple exponential integral.
- I'll spare you the gory details, but I hope you can see that if we were to try evaluate $\Gamma(4)$, we would have to perform 3 integrations by parts, and one simple exponential integral.

As a general rule, if we want to evaluate $\Gamma(n)$, we will have to do $n-1$ integrations by parts, and one simple exponential integral. Thus, if we have to calculate something like $\Gamma(50)$, we will have to do 49 integrations by parts, which is incredibly laborious. Is there a better way?

2.4 The Fundamental Recurrence Relation for the Gamma Function

Fortunately, there **is** a better way. Let's look back to $\Gamma(2)$, and recall the end result of the integration by parts:

$$\begin{aligned}\Gamma(2) &= \int_0^\infty te^{-t}\cdot dt \\ &= 1\cdot\int_0^\infty e^{-t}\cdot dt\end{aligned}$$

Notice that this last integral is $\Gamma(1)$. So we can write:

$$\Gamma(2) = 1\cdot\Gamma(1)$$

What happens when we evaluate $\Gamma(3)$? After the first integration by parts, we had:

$$\begin{aligned}\Gamma(3) &= \int_0^\infty t^2e^{-t}\cdot dt \\ &= 2\cdot\int_0^\infty te^{-t}\cdot dt\end{aligned}$$

Now the final integral is just $\Gamma(2)$. So we have:

$$\Gamma(3) = 2\cdot\Gamma(2)$$

If we had worked out $\Gamma(4)$, we would have found that after our first integration by parts we would indeed have arrived at the relation:

$$\Gamma(4) = 3 \cdot \Gamma(3)$$

This suggests a general rule:

$$\Gamma(x) = (x-1) \cdot \Gamma(x-1)$$

Equivalently, we have:

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

Let's formally prove this identity. The general formula for $\Gamma(x+1)$ is:

$$\Gamma(x) \quad = \quad \int_0^\infty t^{x-1} e^{-t} \cdot dt$$

Using our standard strategy, we have:

$$\begin{aligned} u &= t^{x-1} \\ du &= (x-1) \cdot t^{x-2} \\ \\ dv &= e^{-t} \\ v &= -e^{-t} \end{aligned}$$

Now our integration by parts becomes:

$$\begin{aligned} \Gamma(x) &= \int_0^\infty \underbrace{t^{x-1}}_{u} \underbrace{e^{-t} \cdot dt}_{dv} \\ &= \underbrace{t^x}_{u} \cdot \underbrace{-e^{-t}}_{v} \Big|_0^\infty - \int_0^\infty \underbrace{-e^{-t}}_{v} \cdot \underbrace{(x-1) \cdot t^{x-2} \cdot dt}_{du} \\ &= 0 \;+\; (x-1) \cdot \int_0^\infty t^{x-2} e^{-t} \cdot dt \\ &= (x-1) \cdot \Gamma(x-1) \end{aligned}$$

And so we've established the fundamental recurrence relation!

The fundamental recurrence relation is powerful because it enables us to replace integrations by parts with multiplications. To see how this works, imagine how to calculate $\Gamma(6)$. Recall that if we worked directly from the defintion, calculating $\Gamma(6)$ would require 5 integrations by parts and a simple exponential integral. Instead, by repeatedly using the fundamental recurrence relation, we have:

$$
\begin{aligned}
\Gamma(6) &= 5 \times \Gamma(5) \\
&= 5 \times 4 \times \Gamma(4) \\
&= 5 \times 4 \times 3 \times \Gamma(3) \\
&= 5 \times 4 \times 3 \times 2 \times \Gamma(2) \\
&= 5 \times 4 \times 3 \times 2 \times 1 \\
&= 120
\end{aligned}
$$

So $\Gamma(6) = 120$. We saved a lot of work by using the fundamental recurrence relation!

All of the concrete examples that we've looked at have involved gamma functions of positive whole numbers e.g. $\Gamma(1)$, $\Gamma(2)$, $\Gamma(3)$, etc. However, If you look back in our proof of the fundamental recurrence relation, you'll notice that we never actually required that x be an integer e.g. a whole number. In fact, x does **not** have to be an integer, and the fundamental recurrence relation is true for all real numbers $x > 0$. Remember, if $x = 0$, then the integral won't converge, so $\Gamma(0)$ is undefined.

2.5 An Astonishing Relationship

The fundamental recurrence relation is very useful, because it enables us to simplify the calculation of many otherwise difficult integrals. But it also leads to an astonishing and unexpected relationship between the gamma function and another mathematical function.

For your consideration, here's a table of the first few values of the gamma function:

n	$\Gamma(n)$
1	1
2	1
3	2
4	6
5	24
6	120

Do you notice anything remarkable about these values of $\Gamma(n)$? Do you see a pattern?

Just for reference, let's update the table of values of $\Gamma(n)$ with the values of

$(n-1)!$:

n	$\Gamma(n)$	$(n-1)!$
1	1	1
2	1	1
3	2	2
4	6	6
5	24	24
6	120	120

We are lead to an astonishing fact: for any positive integer n,

$$\Gamma(n) = (n-1)!$$

You can see this visually in our derivation of $\Gamma(6)$ using the fundamental recurrence relation:

$$\Gamma(6) = \underbrace{5 \times 4 \times 3 \times 2 \times 1}_{5!}$$

In general, we have:

$$\begin{aligned}
\Gamma(n) &= (n-1) \times \Gamma(n-1) \\
&= (n-1) \times (n-2) \times \Gamma(n-2) \\
&\vdots \\
&= (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1 \\
&= (n-1)!
\end{aligned}$$

So we started with a very strange looking thing, a function defined by an integral, and we have arrived at something very familiar, the factorial function.

In the last paragraph, you might have noticed that I required n to be a positive *integer*. Why was that? After all, we've seen that the gamma function is defined for all real $x > 0$, and the recurrence relation doesn't require x to be an integer. The answer is that this doesn't have to do with the gamma function, but with the factorial function, because by convention this is restricted to integers. Typically we reserve the "!" notation for integers, so it's OK to say 2! but it's weird to write 2.7!. However, if we *were* prepared to write the unorthodox expression 2.7!, then we would also say that

$$2.7! = \Gamma(3.7) = 4.170652$$

But really you shouldn't write this; instead, use the gamma function notation for non-integral x, and write $\Gamma(x)$. If nothing else, you will impress your family and friends with your knowledge of this mysterious, obscure, and quite terrifying mathematical symbol.

Warning!! It's important to remember in all of this that you have to remember to subtract 1. It's wrong to say that $\Gamma(n) = n!$, but instead we have

$\Gamma(n) = (n-1)!$. The recursion relation is also subtly different: for the factorial function we have

$$n! = n \cdot (n-1)!$$

while for the gamma function we have

$$\Gamma(n) = (n-1) \cdot \Gamma(n-1)$$

As an aside, one thing that many people find puzzling is why $0! = 1$. The real answer is because that's how we choose to define the symbol $0!$, and a lot of very nice formulas are true only when we do choose to define $0!$ in this way. But if we think about the factorial function in terms of the gamma function, then this identity is automatically true without any "choosing", because

$$0! = \Gamma(1) = 1$$

2.6 The "Kinda Sorta" Gamma Function

Sometimes we might encounter an integral that looks very similar to a gamma function, but is not quite a gamma function because it has a constant multiplier in the exponent of the exponential. For instance, we could have something like this:

$$\int_0^\infty x^4 e^{-3x} \cdot dx$$

Is this a gamma function? Well, kinda sorta. It's very close, except for the fact that the exponent of the exponential term is $-3x$. If you go back to the definition of the gamma function, you'll notice that the integral there just had $-x$ for the exponent. So, this is very close, except for that annoying factor of 3. But we can remove that with a simple substitution:

$$\begin{aligned} s &= 3x \\ x &= \frac{s}{3} \\ dx &= \frac{ds}{3} \end{aligned}$$

Whenever we perform a substition in a definite integral, we also have to consider the limits of integration. When $x = 0$, then $s = 0$, and as x goes to infinity, so does s. So the limits of integration remain unchanged. Now we can evaluate the original integral:

$$\begin{aligned} \int_0^\infty x^4 e^{-3x} \cdot dx &= \int_0^\infty \left(\frac{s}{3}\right)^4 e^{-s} \cdot \frac{ds}{3} \\ &= \frac{1}{3^5} \int_0^\infty s^4 e^{-s} \cdot ds \\ &= \frac{\Gamma(5)}{3^5} \end{aligned}$$

In general, we have the formula:

$$\int_0^\infty t^x e^{-qt} \cdot dt = \frac{\Gamma(x+1)}{q^{x+1}}$$

Rather than have all the fun and prove this result, I'm going to let you do the derivation all by yourself on the first homework.

I must warn you – the term "kinda sorta gamma function" is unique to this course, and if you use it around people who haven't taken this course, they will not have any idea of what you are talking about.

2.7 Non-Integral Values of x

Full disclosure – I've been cheating (but only just a little bit!). Recall that the gamma function is defined for *all* real values of x greater than 0. But in much of the discussion above, I only considered *integer* values of x. There was a reason for that: each integration by part reduces the exponent of the power term of the integrand by 1, so eventually the exponent will become 0, and then it's easy to do the integral. For instance, with $\Gamma(4)$, we have:

$$\begin{aligned}
\Gamma(4) &= \int_0^\infty t^3 e^{-t} \cdot dt \\
&= 3 \times \int_0^\infty t^2 e^{-t} \cdot dt \\
&= 3 \times 2 \times \int_0^\infty t^1 e^{-t} \cdot dt \\
&= 3 \times 2 \times 1 \times \int_0^\infty t^0 e^{-t} \cdot dt \\
&= 6
\end{aligned}$$

The crucial point is that if we start with an integer value of x, each time we perform an integration by parts we will reduce the exponent of the power term in the integrand by 1. Eventually we will hit 0, at which point we can finish the integration.

If x *isn't* an integer, then this strategy doesn't work, because we'll miss the value 0 in this process, and so the polynomial term will never go away, and we'll never have the nice integral with just the exponential factor. For instance,

$$\begin{aligned}
\Gamma(3.7) &= \int_0^\infty t^{2.7} e^{-t} \cdot dt \\
&= 2.7 \times \int_0^\infty t^{1.7} e^{-t} \cdot dt
\end{aligned}$$

$$= \quad 2.7 \times 1.7 \times \int_0^\infty t^{0.7} e^{-t} \cdot dt$$

And now we're stuck, because we won't be able reduce the exponent 0.7 to 0 by an integration by parts.

Is there a computational trick for when x is not an integer? Alas, this time the answer is: no. If x is not an integer, then we can't compute the value of $\Gamma(x)$ with some simple closed form expression. Instead, this has to be evaluated numerically i.e. a computer has to do a lot of arithmetic to generate a numerical approximation to the true value of $\Gamma(x)$. However, the fundamental recursion relation still comes into play here, and if you go back to our derivation of the formula you'll notice that nothing depended on x being an integer. Thus, we can still do simplifications like this:

$$\begin{aligned}
\Gamma(3.7) &= 2.7 \times \Gamma(2.7) \\
&= 2.7 \times 1.7 \times \Gamma(1.7) \\
&= 2.7 \times 1.7 \times \int_0^\infty t^{0.7}\ e^{-t} \cdot dt
\end{aligned}$$

But now we have to use a computer to calculate the numerical value of this integral. Microsoft Excel can do this using the built-in function `gamma()`, and using this we have $\Gamma(1.7) = 0.908639$. Finally, plugging this into our expression, we end up with $\Gamma(3.7) = 4.170652$. Of course, since Excel has the gamma function already built-in, we could have just used that to begin with, and in practice that is exactly what people do. R also has a built-in gamma function, which is also called `gamma`. Warning: R is case-sensitive, so you have to write the function with all lower-case letters.

Even if there isn't a nice algebraic formula for the gamma function for non-integer values of x, it's still conceptually very appealing, because it allows us to *extend* the value of the factorial function. That is, the gamma function still gives us the values of the factorial function for integer values of x, but then it's also defined for non-integral values so that the recursion relation still holds.

2.8 One More Result

Before we finish, there's one last point to consider. Although there's no general method for obtaining a closed-form solution for the gamma function for an arbitrary real value of x, it turns out that there is one special value of x which can be computed:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

I hope you are charmed by this perhaps unexpected result, because it will be one of your homework problems. Of course, all the recursion relations still hold, so that

$$\begin{aligned}\Gamma\left(\frac{7}{2}\right) &= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\left(\frac{1}{2}\right)\\ &= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\\ &= 3.323351\end{aligned}$$

3 The Gaussian Integral

The *Gaussian integral* is the integral:

$$\int_{-\infty}^{+\infty} e^{-x^2}\cdot dx$$

This looks rather scary, and if you try to evaluate it using standard integration techniques such as substitution or integration by parts you'll find that none of these work. In fact, it can be proved that the integrand is not an anti-derivative of any finite combination of elementary functions. That is, there is no way to construct a function $f(x)$ using things like powers of x, trigonometric function, exponentials, square roots etc. such that

$$\frac{df(x)}{dx} = e^{-x^2}$$

You might think at this point that all hope is lost if we can't find an anti-derivative, but amaingly it is possible to evaluate this integral. It will however require some trickery!

The first bit of trickery is that we won't evaluate the integral directly, but will instead evaluate its square. Let's denote the integral with the letter I:

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\cdot dx$$

Note that the integral in no way depends on the letter that we use for the integration:

$$\int_{-\infty}^{+\infty} e^{-x^2}\cdot dx = \int_{-\infty}^{+\infty} e^{-y^2}\cdot dy$$

So, both of these integrals are equal to I. We will evaluate I^2, which we will express as a double integral. When we write our squared integral I^2, we'll use an x for one of the integrals, and y for the other:

$$\begin{aligned}
I^2 &= I \cdot I \\
&= \left(\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx\right) \cdot \left(\int_{-\infty}^{+\infty} e^{-y^2} \cdot dy\right) \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} \cdot dx\ dy \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cdot dx\ dy
\end{aligned}$$

At this point, it seems that we've made the problem even more complicated than what we started with, but astonishingly this double itegral is actually very easy to evaluate. The (extremely clever) trick is to convert the integral from rectangular Cartesian coordinates x and y to polar co-ordinates r and θ. This is a standard technique that's covered in multivariate calculus, but you might be a little rusty with this so let me remind you: in polar coordinates, $r^2 = x^2 + y^2$, and the differential element of area is $dx\ dy = r\ dr\ d\theta$. Also, our double integral is over the entire plane: the integration over x ranges from $-\infty$ to $+\infty$ and the integration over y also ranges from $-\infty$ to $+\infty$. In polar coordinates, we integrate over the entire plane by allowing r to range from 0 to ∞ and θ to range from 0 to 2π. So, after we convert everything, we have:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} \cdot dx\ dy &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} \cdot r\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^{+\infty} re^{-r^2} \cdot dr\ d\theta
\end{aligned}$$

Now it's easy to evaluate the inner integral over r:

$$\begin{aligned}
\int_0^{+\infty} re^{-r^2} \cdot dr &= -\frac{1}{2} e^{-r^2} \Big|_{r=0}^{r=\infty} \\
&= -0 - \left(-\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}$$

So the double integral is now reduced to a single integral:

$$\int_0^{2\pi} \left[\int_0^{+\infty} re^{-r^2} \cdot dr\right] d\theta = \int_0^{2\pi} \frac{1}{2} \cdot d\theta$$

This single integral is very easy:

$$\int_0^{2\pi} \frac{1}{2} \cdot d\theta = \frac{\theta}{2} \Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{2\pi}{2} - 0$$

$$= \pi$$

Incredible! Once we performed the trick of converting to polar coordinates, we were able to evaluate the big double integral using elementary techniques, and in the end it came out to be π. Don't forget the last step though: we originally set out to evaluate I, and we've shown that $I^2 = \pi$, so it must be that $I = \sqrt{\pi}$:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cdot dx = \sqrt{\pi}$$

Personally, I've always found this result mysterious, almost mystical: π is something that has to do with circles, and yet there is nothing about the integrand e^{-x^2} that seems particularly "circular".

4 Appendix: The Beta Function

Warning! I'm including this material as an added bonus. You're not responsible for it, and it won't be part of any homework or exam. It's just a little extra in case you're interested. Don't invest time in this if you haven't mastered the previous material.

We can't finish a lecture on the gamma function without mentioning the *beta function*:

$$\beta(a, b) = \int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp$$

Unlike the gamma function, the beta function takes two inputs rather than one.

But it's very similar to the gamma function, because it uses the same strategy of using the inputs to specify a particular integrand, and then the value of the beta function is the integral of this integrand i.e. it's the area under the curve. It's possible to evaluate this integral:

$$\int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp \;=\; \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}$$

So I know you'll all agree: we couldn't have a lecture on the gamma function without mentioning this!

In this appendix, I will show the derivation of this remarkable formula:

$$\int_0^1 t^{a-1}(1-t)^{b-1} \cdot dt = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)}$$

You are not responsible for this derivation in our course, and I'm including it just for people who might be curious.

First, let's multiply through by $\Gamma(a+b)$:

$$\Gamma(a+b)\cdot\int_0^1 t^{a-1}(1-t)^{b-1}\cdot dt = \Gamma(a)\cdot\Gamma(b)$$

Our strategy will be to combine the two integrals on the right-hand side of this identity into one double integral, so as with the Gaussian integral we'll need to use different dummy variables for the integration:

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\cdot dx$$

$$\Gamma(b) = \int_0^\infty y^{b-1}e^{-y}\cdot dy$$

Now let's multiply these two integrals together:

$$\begin{aligned}\Gamma(a)\cdot\Gamma(b) &= \left(\int_0^\infty x^{a-1}e^{-x}\cdot dx\right)\cdot\left(\int_0^\infty y^{b-1}e^{-y}\cdot dy\right)\\ &= \int_0^\infty\int_0^\infty x^{a-1}e^{-x}\cdot y^{b-1}e^{-y}\cdot dx\ dy\\ &= \int_0^\infty\int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)}\cdot dx\ dy\end{aligned}$$

We now embark on a series of three substitutions to evaluate this integral. To start with, let's think a little about the inner integral, which I'll delineate with brackets:

$$\int_0^\infty\int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)}\cdot dx\ dy = \int_0^\infty\left[\int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)}\cdot\cdot dx\right]dy$$

In this inner integral, we are integrating over x, and thus y is effective a constant. We'll start our series of substitutions with a new variable u:

$$\begin{aligned}u &= x/y\\ x &= yu\\ dx &= y\ du\end{aligned}$$

Now the double integral becomes:

$$\begin{aligned}
\int_0^\infty \left[\int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)} \cdot \cdot dx\right] dy &= \int_0^\infty \left[\int_0^\infty (yu)^{a-1}y^{b-1}e^{-(yu+y)} \cdot \cdot du\right] dy \\
&= \int_0^\infty \left[\int_0^\infty (yu)^{a-1}y^{b-1}e^{-(u+1)y} \cdot \cdot ydu\right] dy \\
&= \int_0^\infty \left[\int_0^\infty u^{a-1}y^{a+b-1}e^{-(u+1)y} \cdot \cdot du\right] dy \\
&= \int_0^\infty \left[\int_0^\infty u^{a-1}y^{a+b-1}e^{-(u+1)y} \cdot \cdot dy\right] du
\end{aligned}$$

Notice that in the last line we swapped the order of integration, so that now the inner integral is over y, and inside the brackets u is effectively a constant. We eventually want to extract $\Gamma(a+b)$ from this mess, and we know that gamma functions have exponential terms of the form e^{-s}, not $e^-(u+1)y$. So let's make this substitution:

$$\begin{aligned}
s &= (u+1)y \\
y &= \frac{s}{u+1} \\
dy &= \frac{ds}{u+1}
\end{aligned}$$

Substituting this into the double integral, we have:

$$\begin{aligned}
&\int_0^\infty \left[\int_0^\infty u^{a-1}y^{a+b-1}e^{-(u+1)y} \cdot \cdot dy\right] du \\
&= \int_0^\infty \left[\int_0^\infty u^{a-1}\left(\frac{s}{u+1}\right)^{a+b-1} e^{-s} \cdot \cdot \frac{ds}{u+1}\right] du \\
&= \left(\int_0^\infty s^{a+b-1}e^{-s} \cdot ds\right) \cdot \left(\int_0^\infty \left(\frac{u}{u+1}\right)^{a-1} \left(\frac{1}{u+1}\right)^{b+1} \cdot du\right) \\
&= \Gamma(a+b) \cdot \int_0^\infty \left(\frac{u}{u+1}\right)^{a-1} \left(\frac{1}{u+1}\right)^{b+1} \cdot du
\end{aligned}$$

And magically the $\Gamma(a+b)$ has appeared! By the way, that $b+1$ is **not** a typo, and you should make sure that you understand all the algebra that produced it.

We're still not done, because the integral is still not in the right form. Now we'll make our third substitution:

$$\begin{aligned}
p &= \frac{u}{u+1} \\
1-p &= \frac{1}{u+1} \\
u &= \frac{p}{1-p} \\
du &= \frac{dp}{1-p} + \frac{pdp}{(1-p)^2} \\
&= \frac{dp}{(1-p)^2}
\end{aligned}$$

We also have to be careful about the limits of integration with this substitution. When $u = 0$, then $p = 0$ as well, but as u goes to infinity, p goes to 1. So we have:

$$\begin{aligned}
\int_0^\infty \left(\frac{u}{u+1}\right)^{a-1} \left(\frac{1}{u+1}\right)^{b-1} \cdot du &= \int_0^1 p^{a-1}(1-p)^{b+1} \cdot \frac{dp}{(1-p)^2} \\
&= \int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp
\end{aligned}$$

And now the integral is in the right form.

Let's put all of this together:

$$\begin{aligned}
\Gamma(a) \cdot \Gamma(b) &= \left(\int_0^\infty x^{a-1}e^{-x} \cdot dx\right) \cdot \left(\int_0^\infty y^{b-1}e^{-y} \cdot dy\right) \\
&= \int_0^\infty \int_0^\infty x^{a-1}y^{b-1}e^{-(x+y)} \cdot dx\ dy \\
&= \int_0^\infty \int_0^\infty u^{a-1}y^{a+b-1}e^{-(u+1)y} \cdot \cdot dy\ du \\
&= \Gamma(a+b) \cdot \int_0^\infty \left(\frac{u}{u+1}\right)^{a-1} \left(\frac{1}{u+1}\right)^{b+1} \cdot du \\
&= \Gamma(a+b) \cdot \int_0^1 p^{a-1}(1-p)^{b-1} \cdot dp
\end{aligned}$$