OG-ITA: Documentation for the Large-scale Dynamic General Equilibrium Overlapping Generations Model for Policy Analysis of Italy

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December 2017

(version 2017.12.a)

Preface

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Part I

Introduction

Chapter 1

Introduction

The overlapping generations model is a workhorse of dynamic fiscal analysis. `OG-ITA` is dynamic in that households in the model make consumption, savings, and labor supply decisions based on their expectations over their entire lifetime, not just the current period. Because `OG-ITA` is a general equilibrium model, behavioral changes by households and firms can cause macroeconomic variables and prices to adjust.

But the main characteristic that differentiates the overlapping generations model from other dynamic general equilibrium models is its realistic modeling of the finite lifetimes of individuals and the cross-sectional age heterogeneity that exists in the economy. One can make a strong case that age heterogeneity and income heterogeneity are two of the main sources of diversity that explain much of the behavior in which we are interested for policy analysis.

`OG-ITA` can be summarized as having the following characteristics.

- Households
  - overlapping generations of finitely lived households
  - households are forward looking and see to maximize their expected lifetime utility, which is a function of consumption, labor supply, and bequests
  - households choose consumption, savings, and labor supply every period.
  - The only uncertainty households face is with respect to their mortality risk
  - realistic demographics: mortality rates, fertility rates, immigration rates, population growth, and population distribution dynamics
  - heterogeneous lifetime income groups within each age cohort, calibrated from Italian data
  - incorporation of detailed household tax data from [TODO]
  - calibrated intentional and unintentional bequests by households to surviving generations
- Firms
  - representative perfectly competitive firm maximizes static profits with general CES production function by choosing capital and labor demand

  - exogenous productivity growth is labor augmenting technological change
  - firms face a corporate income tax as well as various depreciation deductions and tax treatments

- Government
  - government collects tax revenue from households and firms
  - government distributes transfers to households
  - government spends resources on public goods
  - government can run deficits and surpluses
  - a stabilization rule (budget closure rule) must be implemented at some point in the time path if government debt is growing at a rate permanently different from GDP.

- Aggregate, market clearing, and international
  - Aggregate model is deterministic (no aggregate shocks)
  - Three markets must clear: capital, labor, and goods markets

We will update this document as more detail is added to the model.

Chapter 2

Exogenous Inputs and Endogenous Output

In this chapter, list the exogenous inputs to the model, options, and where the values come from (weak calibration vs. strong calibration). Point to the respective chapters for some of the inputs.

Also go through the output of the model, endogenous variables, and potential tables and pictures to produce.

2.1 Exogenous Parameters

List all the exogenous parameters that are outputs of the model here.

Table 2.1: List of exogenous parameters and baseline calibration values

Symbol	Description	Value
S	Maximum periods in economically active household life	80
E	Number of periods of youth economically outside the model	$\text{round}\left(\frac{S}{4}\right) = 20$
R	Retirement age (period)	$E + \text{round}\left(\frac{9}{16}S\right) = 65$
T_1	Number of periods to steady state for initial time path guesses	160
T_2	Maximum number of periods to steady state for nonsteady-state equilibrium	160
ν	Dampening parameter for TPI	0.4
$\{\{\omega_{s,0}\}_{s=1}^{E+S}\}_{t=0}^{T_2+S-1}$	Initial population distribution by age	(see Ch. 3)
$\{f_s\}_{s=1}^{E+S}$	Fertility rates by age	(see Sec. 3.1)
$\{i_s\}_{s=1}^{E+S}$	Immigration rates by age	(see Sec. 3.2)
$\{\rho_s\}_{s=0}^{E+S}$	Mortality rates by age	(see Sec. 3.3)

2.2 Endogenous Variables

List all the endogenous variables that are outputs of the model here.

Part II

Households Theory

Chapter 3

Demographics

We start the `OG-ITA` section on modeling the household with a description of the demographics of the model. Nishiyama (2015) and DeBacker et al. (2017) have recently shown that demographic dynamics are likely the biggest influence on macroeconomic time series, exhibiting more influence than fiscal variables or household preference parameters.

In this chapter, we characterize the equations and parameters that govern the transition dynamics of the population distribution by age. In `OG-ITA`, we take the approach of taking mortality rates and fertility rates from outside estimates. But we estimate our immigration rates as residuals using the mortality rates, fertility rates, and at least two consecutive periods of population distribution data. This approach makes sense if one modeling a country in which in one is not confident in the immigration rate data. If the country has good immigration data, then the immigration residual approach we describe below can be skipped.

We define $\omega_{s,t}$ as the number of households of age s alive at time t. A measure $\omega_{1,t}$ of households is born in each period t and live for up to $E+S$ periods, with $S \geq 4$.[1] Households are termed "youth", and do not participate in market activity during ages $1 \leq s \leq E$. The households enter the workforce and economy in period $E+1$ and remain in the workforce until they unexpectedly die or live until age $s = E+S$. We model the population with households age $s \leq E$ outside of the workforce and economy in order most closely match the empirical population dynamics.

The population of agents of each age in each period $\omega_{s,t}$ evolves according to the following function,

$$\begin{aligned} \omega_{1,t+1} &= (1-\rho_0)\sum_{s=1}^{E+S} f_s\omega_{s,t} + i_1\omega_{1,t} \quad \forall t \\ \omega_{s+1,t+1} &= (1-\rho_s)\omega_{s,t} + i_{s+1}\omega_{s+1,t} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{3.1}$$

where $f_s \geq 0$ is an age-specific fertility rate, i_s is an age-specific net immigration rate, ρ_s is an age-specific mortality hazard rate, and ρ_0 is an infant mortality rate.[2] The total population in the economy N_t at any period is simply the sum of households in the economy,

[1] Theoretically, the model works without loss of generality for $S \geq 3$. However, because we are calibrating the ages outside of the economy to be one-fourth of S (e.g., ages 21 to 100 in the economy, and ages 1 to 20 outside of the economy), it is convenient for S to be at least 4.

[2] The parameter ρ_s is the probability that a household of age s dies before age $s+1$.

the population growth rate in any period t from the previous period $t-1$ is $g_{n,t}$, $\tilde{N}_t$ is the working age population, and $\tilde{g}_{n,t}$ is the working age population growth rate in any period t from the previous period $t-1$.

$$N_t \equiv \sum_{s=1}^{E+S} \omega_{s,t} \quad \forall t \tag{3.2}$$

$$g_{n,t+1} \equiv \frac{N_{t+1}}{N_t} - 1 \quad \forall t \tag{3.3}$$

$$\tilde{N}_t \equiv \sum_{s=E+1}^{E+S} \omega_{s,t} \quad \forall t \tag{3.4}$$

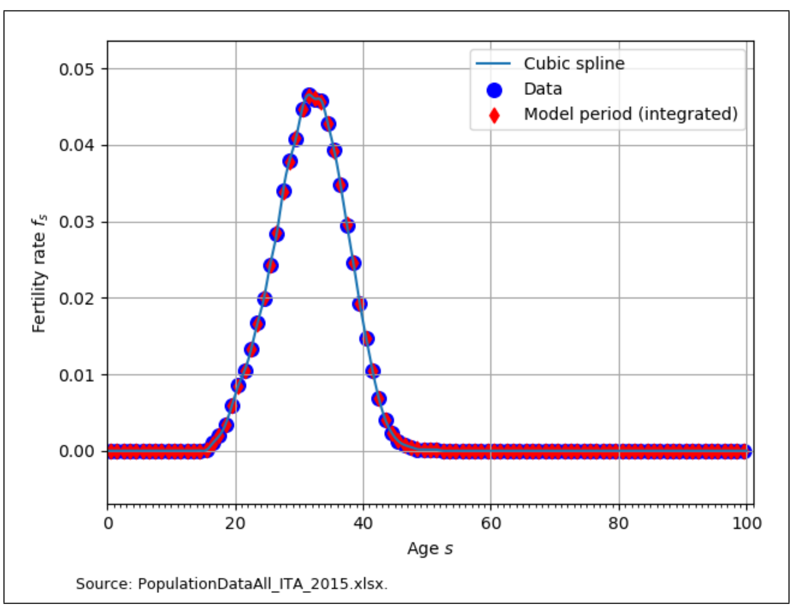
$$\tilde{g}_{n,t+1} \equiv \frac{\tilde{N}_{t+1}}{\tilde{N}_t} - 1 \quad \forall t \tag{3.5}$$

We discuss the approach to estimating fertility rates f_s, mortality rates ρ_s, and immigration rates i_s in Sections 3.1, 3.2, and 3.3.

3.1 Fertility rates

In `OG-ITA`, we assume that the fertility rates for each age cohort f_s are constant across time. However, this assumption is conceptually straightforward to relax. Our data for Italian fertility rates by age come from [TODO: add citation]. Figure 3.1 shows the fertility-rate data and the estimated average fertility rates for $E+S=100$.

Figure 3.1: Fertility rates by age (f_s) for $E+S=$ 100



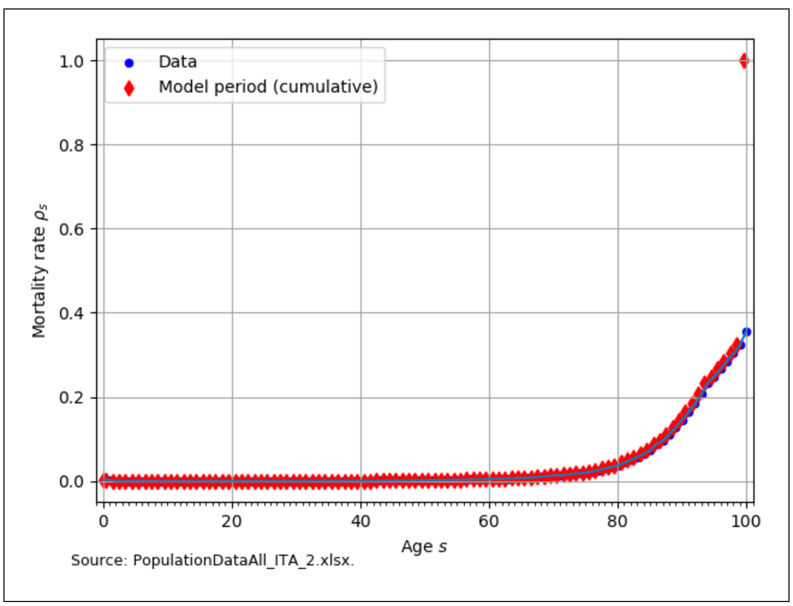
The large blue circles are the 2015 Italian fertility rate data from [TODO: add citation]. In order to get our cubic spline interpolating function to fit better at the endpoints we added to fertility rates of zero to ages 9 and 10, and we added two fertility rates of zero to ages 55 and 56. The blue line in Figure 3.1 shows the cubic spline interpolated function of the data.

The red diamonds in Figure 3.1 are the average fertility rate in age bins spanning households born at the beginning of period 1 (time = 0) and dying at the end of their 100th year. Let the total number of model years that a household lives be $E + S \leq 100$. Then the span from the beginning of period 1 (the beginning of year 0) to the end of period 100 (the end of year 99) is divided up into $E + S$ bins of equal length. We calculate the average fertility rate in each of the $E + S$ model-period bins as the average population-weighted fertility rate in that span. The red diamonds in Figure 3.1 are the average fertility rates displayed at the midpoint in each of the $E + S$ model-period bins.

3.2 Mortality rates

The mortality rates in our model ρ_s are a one-period hazard rate and represent the probability of dying within one year, given that an household is alive at the beginning of period s. We assume that the mortality rates for each age cohort ρ_s are constant across time. The infant mortality rate of $\rho_0 = 0.0033$ comes from the 2016 estimate of Index Mundi.[3] The data for Italian mortality rates by age come from [TODO: get citation]. Figure 3.2 shows the mortality rate data and the corresponding model-period mortality rates for $E + S = 100$. We constrain the mortality rate to be 1.0 or 100 percent at the maximum age of 100.

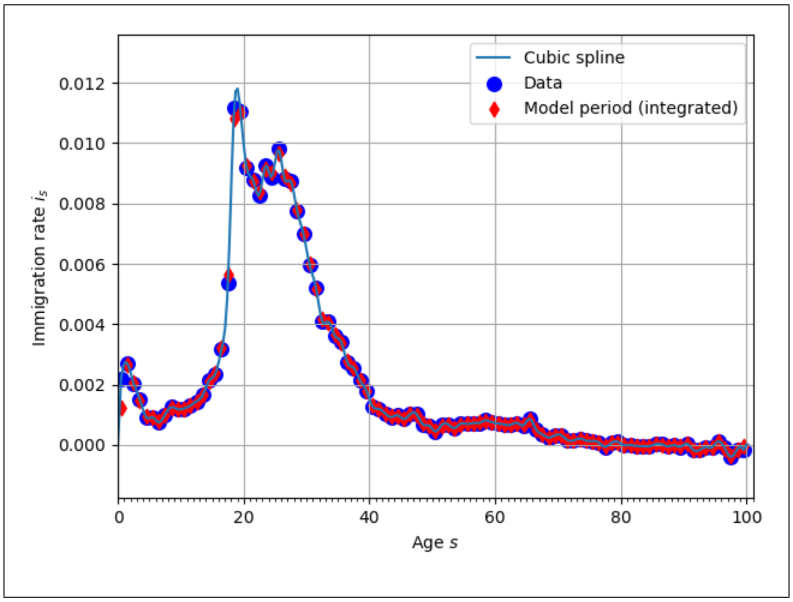
Figure 3.2: Mortality rates by age (ρ_s) for $E+S =$ 100



[3]Index Mundi estimates 3.3 infant deaths per 1,000 live births in 2016. See https://www.indexmundi.com/italy/infant_mortality_rate.html.

3.3 Immigration rates

Figure 3.3: Immigration rates by age (ρ_s) for $E+S=100$



At the end of Section 3.4, we describe a small adjustment that we make to the immigration rates after a certain number of periods in order to make computation of the transition path equilibrium of the model compute more robustly.

3.4 Population steady-state and transition path

This model requires information about mortality rates ρ_s in order to solve for the household's problem each period. It also requires the steady-state stationary population distribution $\bar{\omega}_s$ and population growth rate $\bar{g}_n$ as well as the full transition path of the stationary population distribution $\hat{\omega}_{s,t}$ and population grow rate $\tilde{g}_{n,t}$ from the current state to the steady-state. To solve for the steady-state and the transition path of the stationary population distribution, we write the stationary population dynamic equations (3.6) and their matrix representation (3.7).

$$\begin{aligned} \hat{\omega}_{1,t+1} &= \frac{(1-\rho_0)\sum_{s=1}^{E+S} f_s\hat{\omega}_{s,t} + i_1\hat{\omega}_{1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \\ \hat{\omega}_{s+1,t+1} &= \frac{(1-\rho_s)\hat{\omega}_{s,t} + i_{s+1}\hat{\omega}_{s+1,t}}{1+\tilde{g}_{n,t+1}} \quad \forall t \quad \text{and} \quad 1 \leq s \leq E+S-1 \end{aligned} \tag{3.6}$$

$$\begin{bmatrix} \hat{\omega}_{1,t+1} \\ \hat{\omega}_{2,t+1} \\ \hat{\omega}_{2,t+1} \\ \vdots \\ \hat{\omega}_{E+S-1,t+1} \\ \hat{\omega}_{E+S,t+1} \end{bmatrix} = \frac{1}{1+g_{n,t+1}} \times ...$$

$$\begin{bmatrix} (1-\rho_0)f_1 + i_1 & (1-\rho_0)f_2 & (1-\rho_0)f_3 & \dots & (1-\rho_0)f_{E+S-1} & (1-\rho_0)f_{E+S} \\ 1-\rho_1 & i_2 & 0 & \dots & 0 & 0 \\ 0 & 1-\rho_2 & i_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & i_{E+S-1} & 0 \\ 0 & 0 & 0 & \dots & 1-\rho_{E+S-1} & i_{E+S} \end{bmatrix} \begin{bmatrix} \hat{\omega}_{1,t} \\ \hat{\omega}_{2,t} \\ \hat{\omega}_{2,t} \\ \vdots \\ \hat{\omega}_{E+S-1,t} \\ \hat{\omega}_{E+S,t} \end{bmatrix} \tag{3.7}$$

We can write system (3.7) more simply in the following way.

$$\hat{\boldsymbol{\omega}}_{t+1} = \frac{1}{1+g_{n,t+1}} \boldsymbol{\Omega} \hat{\boldsymbol{\omega}}_t \quad \forall t \tag{3.8}$$

The stationary steady-state population distribution $\bar{\boldsymbol{\omega}}$ is the eigenvector $\boldsymbol{\omega}$ with eigenvalue $(1+\bar{g}_n)$ of the matrix $\boldsymbol{\Omega}$ that satisfies the following version of (3.8).

$$(1+\bar{g}_n)\bar{\boldsymbol{\omega}} = \boldsymbol{\Omega}\bar{\boldsymbol{\omega}} \tag{3.9}$$

Proposition 3.1. If the age $s = 1$ immigration rate is $i_1 > -(1-\rho_0)f_1$ and the other immigration rates are strictly positive $i_s > 0$ for all $s \geq 2$ such that all elements of $\boldsymbol{\Omega}$ are nonnegative, then there exists a unique positive real eigenvector $\bar{\boldsymbol{\omega}}$ of the matrix $\boldsymbol{\Omega}$, and it is a stable equilibrium.

Proof. First, note that the matrix $\boldsymbol{\Omega}$ is square and non-negative. This is enough for a general version of the Perron-Frobenius Theorem to state that a positive real eigenvector exists with a positive real eigenvalue. This is not yet enough for uniqueness. For it to be unique by a version of the Perron-Fobenius Theorem, we need to know that the matrix is irreducible. This can be easily shown. The matrix is of the form

$$\boldsymbol{\Omega} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & 0 & \dots & 0 & 0 & 0 \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

Where each * is strictly positive. It is clear to see that taking powers of the matrix causes the sub-diagonal positive elements to be moved down a row and another row of positive entries is added at the top. None of these go to zero since the elements were all non-negative to begin with.

$$\mathbf{\Omega}^2 = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}; \quad \mathbf{\Omega}^{S+E-1} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \end{bmatrix}$$

$$\mathbf{\Omega}^{S+E} = \begin{bmatrix} * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{bmatrix}$$

Existence of an $m \in \mathbb{N}$ such that $\left(\mathbf{\Omega}^{\mathbf{m}}\right)_{ij} \neq 0 \;\; (> 0)$ is one of the definitions of an irreducible (primitive) matrix. It is equivalent to saying that the directed graph associated with the matrix is strongly connected. Now the Perron-Frobenius Theorem for irreducible matrices gives us that the equilibrium vector is unique.

We also know from that theorem that the eigenvalue associated with the positive real eigenvector will be real and positive. This eigenvalue, p, is the Perron eigenvalue and it is the steady state population growth rate of the model. By the PF Theorem for irreducible matrices, $|\lambda_i| \leq p$ for all eigenvalues λ_i and there will be exactly h eigenvalues that are equal, where h is the period of the matrix. Since our matrix $\mathbf{\Omega}$ is aperiodic, the steady state growth rate is the unique largest eigenvalue in magnitude. This implies that almost all initial vectors will converge to this eigenvector under iteration. □

For a full treatment and proof of the Perron-Frobenius Theorem, see Suzumura (1983). Because the population growth process is exogenous to the model, we calibrate it to annual age data for age years $s = 1$ to $s = 100$.

Figure 3.4 shows the steady-state population distribution $\bar{\boldsymbol{\omega}}$ and the population distribution after 240 periods $\hat{\boldsymbol{\omega}}_{240}$. Although the two distributions look very close to each other, they are not exactly the same.

Further, we find that the maximum absolute difference between the population levels $\hat{\omega}_{s,t}$ and $\hat{\omega}_{s,t+1}$ was 4.85×10^{-5} after 240 periods. That is to say, that after 240 periods, given the estimated mortality, fertility, and immigration rates, the population has not achieved its steady state. For convergence in our solution method over a reasonable time horizon, we want the population to reach a stationary distribution after T_1 periods. To do this, we artificially impose that the population distribution in period $t = T_1 = 240$ $(3S)$ is the population steady-state. As can be seen from Figure 3.4, this assumption is not very restrictive. Figure 3.5 shows the change in immigration rates that would make the period $t = T_1 = 240$ population distribution equal be the steady-state. The maximum absolute difference between any two corresponding immigration rates in Figure 3.5 is 0.00096.

The most recent year of population data come from [TODO: Get citation] 2015 population estimates. We use those data and the population transition matrix (3.8) to age it to

Figure 3.4: Theoretical steady-state population distribution vs. population distribution at period $t = 120$

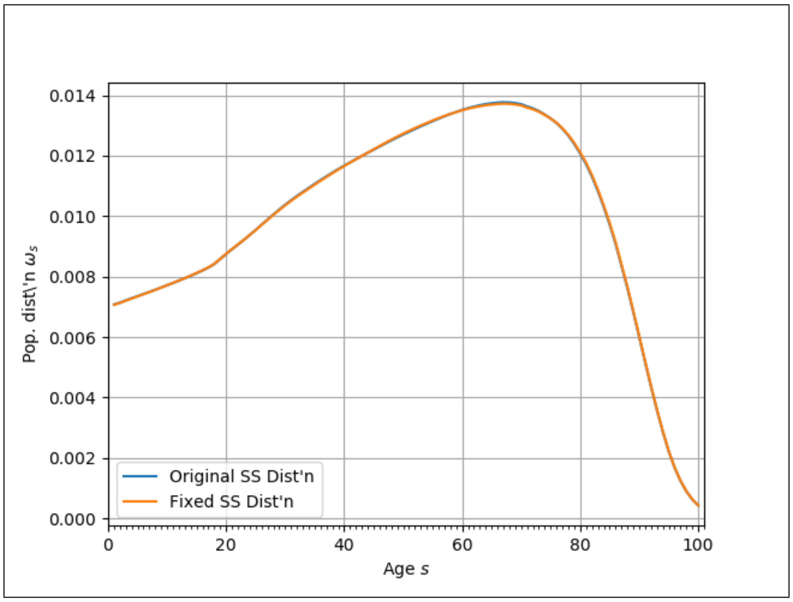
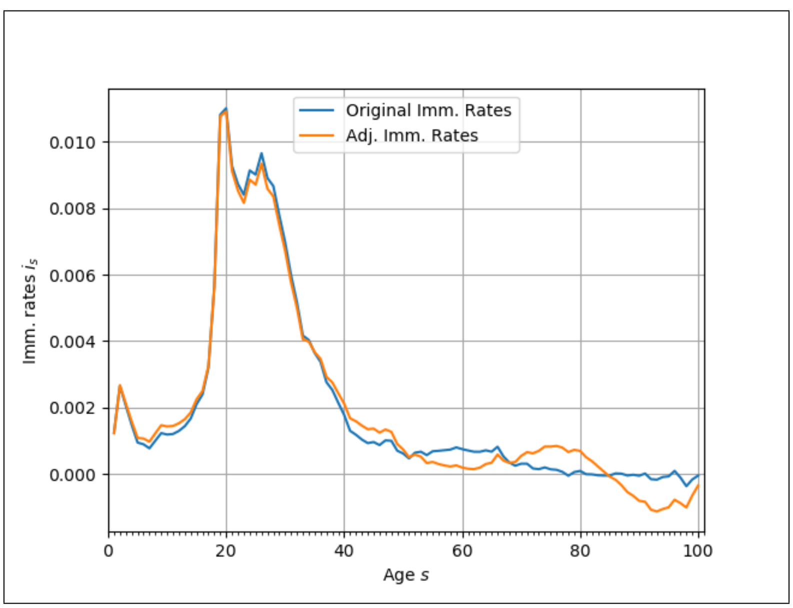


Figure 3.5: Original immigration rates vs. adjusted immigration rates to make fixed steady-state population distribution



the current model year of 2018. We then use (3.8) to generate the transition path of the population distribution over the time period of the model. Figure 3.6 shows the progression from the 2015 population data to the fixed steady-state at period $t = 240$. The time path of the growth rate of the economically active population $\tilde{g}_{n,t}$ is shown in Figure 3.7.

Figure 3.6: Stationary population distribution at periods along transition path

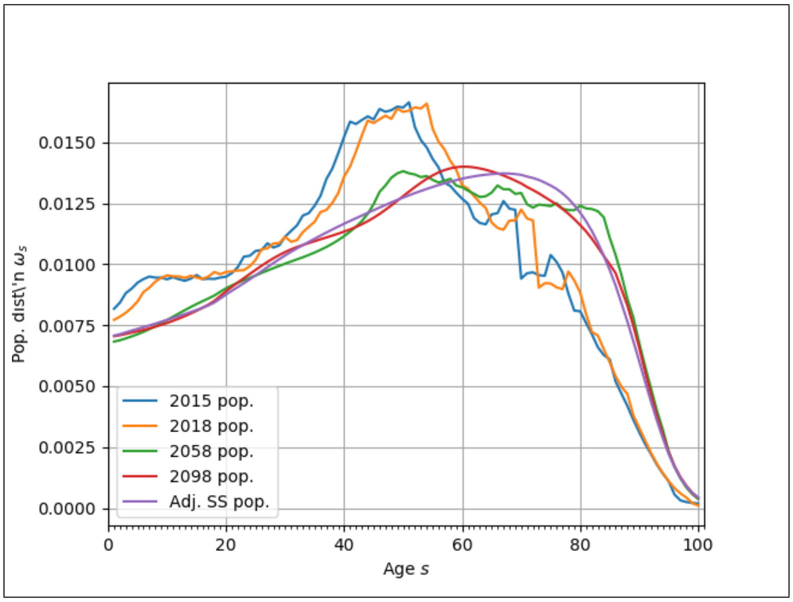
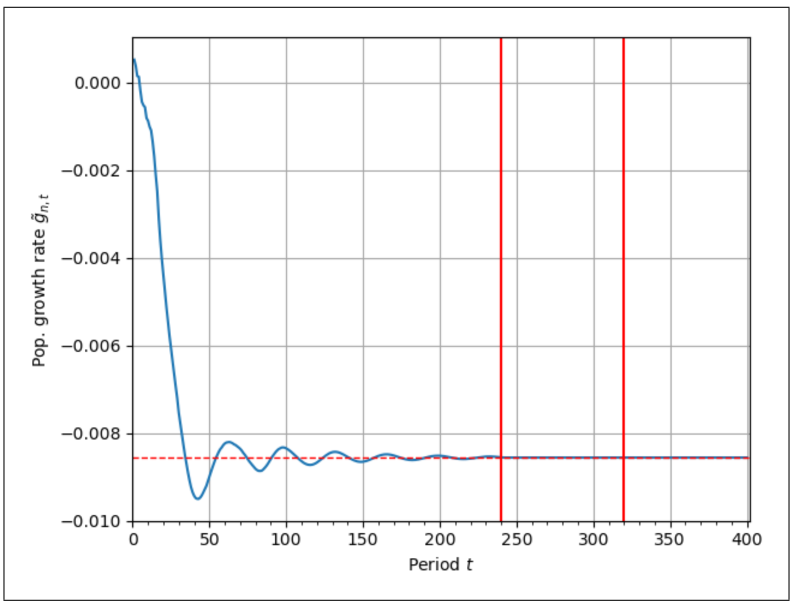


Figure 3.7: Time path of the population growth rate $\tilde{g}_{n,t}$



Chapter 4

Lifetime Earnings Profiles

Among households in `OG-ITA`, we model both age heterogeneity and within-age ability heterogeneity. We use this ability or productivity heterogeneity to generate the income heterogeneity that we see in the data.

Differences among workers' productivity in terms of ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this chapter, we characterize this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned. In `OG-ITA`, households' labor income comes from the equilibrium wage and the agent's endogenous quantity of labor supply. In this section, we augment the labor income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with that ability path.

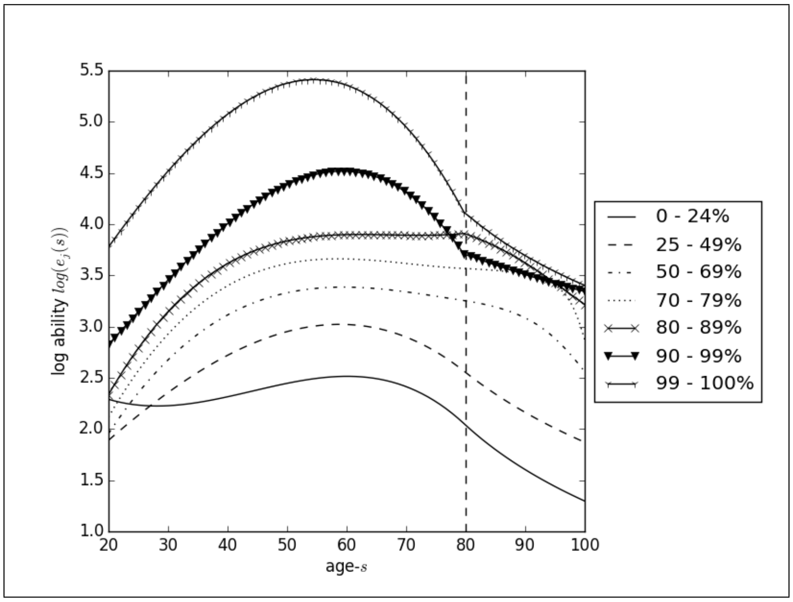
$$\text{labor income:} \quad x_{j,s,t} \equiv w_t e_{j,s} n_{j,s,t} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S \tag{??}$$

In this specification, w_t is an equilibrium wage representing a portion of labor income that is common to all workers. Individual quantity of labor supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labor productivity factor that augments or diminishes the productivity of a worker's labor supply relative to average productivity.

We calibrate deterministic ability paths such that each lifetime income group has a different life-cycle profile of earnings. The distribution on income and wealth are often focal components of macroeconomic models. As such, we use a calibration of deterministic lifetime ability paths from DeBacker et al. (2017b) that can represent U.S. earners in the top 1% of the distribution of lifetime income. Piketty and Saez (2003) show that income and wealth attributable to these households has shown the greatest growth in recent decades. The data come from the U.S. Internal Revenue Services's (IRS) Statistics of Income program (SOI) Continuous Work History Sample (CWHS). DeBacker et al. (2017b) match the SOI data with Social Security Administration (SSA) data on age and Current Population Survey (CPS) data on hours in order to generate a non-top-coded measure of hourly wage.

Figure 4.1 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to labor income percentiles $\boldsymbol{\lambda} = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Because there are few individuals above age 80 in the data, DeBacker et al. (2017b) extrapolate these estimates for model ages 80-100 using an arctan function.

Figure 4.1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$



We calibrate the model such that each lifetime income group has a different life-cycle profile of earnings. Since the distribution on income and wealth are key aspects of our model, we calibrate these processes so that we can represent earners in the top 1 percent of the distribution of lifetime income. It is income and wealth attributable to these households that has shown the greatest growth in recent decades (see, for example, Piketty and Saez (2003)). In order to have observations on the earnings of those at very top of the distribution that are not subject to top-coding we use data from the Internal Revenue Services's (IRS) Statistics of Income program (SOI).

4.1 Continuous Work History Sample

The SOI data we draw from are the Continuous Work History Sample (CWHS). From this CWHS, we use a panel that is a 1-in-5000 random sample of tax filers from 1991 to 2009. For each filer-year observation we are able to observe detailed information reported on Form 1040 and the associated forms and schedules. We are also able to merge these tax data with Social Security Administration (SSA) records to get information on the age and gender of the primary and secondary filers. Our model variable of effective labor units maps into wage rates, because the market wage rate in the model, w_t, is constant across households. Earnings per hour thus depend upon effective labor units and equal $e_{j,s,t} \times w_t$ for household in lifetime income group j, with age s, in year t. Income tax data, however, do not contain information on hourly earnings or hours works. Rather, we only observe total earned income (wage and salaries plus self-employment income) over the tax year. In order to find hourly earnings for tax filers, we use an imputation procedure. This is described in detail in DeBacker and Ramnath (2017). The methodology applies an imputation for hours worked for a filing unit based on a model of hours worked for a filing unit estimated from the Current Population

Survey (CPS) for the years 1992-2010.[1] We then use the imputed hours to calculate hourly earnings rates for tax filing units in the CWHS.

We exclude from our sample filer-year observations with earned income (wages and salaries plus business income) of less than $1,250. We further exclude those with positive annual wages, but with hourly wages below $5.00 (in 2005$). We also drop one observation where the hourly wage rate exceeds $25,000.[2] Economic life in the model runs from age 21 to 100. Our data have few observations on filers with ages exceeding 80 years old. Our sample is therefore restricted to those from ages 21 to 80. After these restrictions, our final sample size is 333,381 filer-year observations.

4.2 Lifetime Income

In our model, labor supply and savings, and thus lifetime income, are endogenous. We therefore define lifetime income as the present value of lifetime labor endowments and not the value of lifetime labor earnings. Note that our data are at the tax filing unit. We take this unit to be equivalent to a household. Because of differences in household structure (i.e., singles versus couples), our definition of lifetime labor income will be in per adult terms. In particular, for filing units with a primary and secondary filer, our imputed wage represents the average hourly earnings between the two. When calculating lifetime income we assign single and couple households the same labor endowment. This has the effect of making our lifetime income metric a per adult metric, there is therefore not an over-representation of couple households in the higher lifetime income groups simply because their time endowment is higher than for singles. We use the following approach to measure the lifetime income.

First, since our panel data do not allow us to observe the complete life cycle of earnings for each household (because of sample attrition, death or the finite sample period of the data), we use an imputation to estimate wages in the years of the household's economic life for which they do not appear in the CWHS. To do this, we estimate the following equation, separately by household type (where household types are single male, single female, couple with male head, or couple with female head):

$$ln(w_{i,t}) = \alpha_i + \beta_1 age_{i,t} + \beta_2 age_{i,t}^2 + \beta_3 * age_{i,t}^3 + \varepsilon_{i,t} \tag{4.1}$$

The parameter estimates, including the household fixed effects, from Equation 4.1 are shown in Table 4.1. These estimates are then used to impute values for log wages in years of each households' economic life for which we do not have data. This creates a balanced panel of log wages of households with heads aged 21 to 80. The actual and imputed wage values are then used to calculate the net present value of lifetime labor endowments per adult for each household. Specifically, we define lifetime income for household i as:

$$LI_i = \sum_{t=21}^{80} \left(\frac{1}{1+r}\right)^{t-21} (w_{i,t} * 4000) \tag{4.2}$$

[1] The CPS survey asks retrospective questions about income in the last year and average hours worked per week (and weeks worked) in the last year). Therefore, these CPS surveys line up with tax years 1991-2009.

[2] This threshold is equivalent to $50 million of wage income in one year at full time (40 hours per week) of work.

Table 4.1: Initial Log Wage Regressions

Dependent variables	Single males	Single females	Married, male head	Married, female head
Age	0.177***	0.143***	0.134***	0.065**
	(0.006)	(0.005)	(0.004)	(0.027)
Age^2	-0.003***	-0.002***	-0.002***	-0.000
	(0.000)	(0.000)	(0.000)	(0.001)
Age^3	0.000***	0.000***	0.000***	0.000
	(0.000)	(0.000)	(0.000)	(0.000)
Constant	-0.839***	-0.648***	-0.042	1.004***
	(0.072)	(0.070)	(0.058)	(0.376)
Adj R^2	-0.007	0.011	-0.032	-0.324
Observations	88,833	96,670	141,564	6,314

Source: CWHS data, 1991 – 2009.
** Significant at the 5 percent level ($p < 0.05$).
*** Significant at the 1 percent level ($p < 0.01$).

Note that households are all have the same time endowment in each year (4000 hours). Thus the amount of the time endowment scales lifetime income up or down, but does not change the lifetime income of one household relative to another. This is not the case with the interest rate, r, which we fix at 4%. Changes in the interest rate differentially impact the lifetime income calculation for different individuals because they may face different earnings profiles. For example, a higher interest rate would reduced the discounted present value of lifetime income for those individuals whose wage profiles peaked later in their economic life by a larger amount than it would reduce the discounted present value of lifetime income for individuals whose wage profiles peaked earlier.

4.3 Profiles by Lifetime Income

With observations of lifetime income for each household, we next sort households and find the percentile of the lifetime income distribution that each household falls in. With these percentiles, we create our lifetime income groupings.

$$\lambda_j = [0.25, 0.25, 0.2, 0.1, 0.1, 0.09, 0.01] \tag{4.3}$$

That is, lifetime income group one includes those in below the 25th percentile, group two includes those from the 25th to the median, group three includes those from the median to the 70th percentile, group four includes those from the 70th to the 80th percentile, group 5 includes those from the 80th to 90th percentile, group 6 includes those from the 90th to 99th percentile, and group 7 consists of the top one percent in the lifetime income distribution. Table 4.2 presents descriptive statistics for each of these groups.

To get a life-cycle profile of effective labor units for each group, we estimate the wage profile for each lifetime income group. We do this by estimating the following regression model separately for each lifetime income group using data on actual (not imputed) wages:

$$ln(w_{i,t}) = \alpha + \beta_1 age_{i,t} + \beta_2 age_{i,t}^2 + \beta_3 * age_{i,t}^3 + \varepsilon_{i,t} \tag{4.4}$$

Table 4.2: Descriptive Statistics by Lifetime Income Category

Lifetime Income Category:	1	2	3	4	5	6	7	All
Percentiles	0-25	25-50	50-70	70-80	80-90	90-99	99-100	0-100
Observations	65,698	101,484	74,253	33,528	31,919	24,370	2,129	333,381
Fraction Single								
Females	0.30	0.24	0.25	0.32	0.38	0.40	0.22	0.28
Males	0.18	0.22	0.30	0.35	0.38	0.37	0.20	0.26
Fraction Married								
Female Head	0.08	0.00	0.00	0.00	0.00	0.00	0.00	0.02
Male Head	0.45	0.53	0.45	0.32	0.23	0.23	0.57	0.39
Mean:								
Age, Primary	51.72	44.15	38.05	34.09	31.53	30.79	40.17	39.10
Hourly Wage	11.60	16.98	20.46	23.04	26.06	40.60	237.80	21.33
Annual Wages	25,178	44,237	54,836	57,739	61,288	92,191	529,522	51,604
Lifetime Income	666,559	1,290,522	1,913,029	2,535,533	3,249,287	5,051,753	18,080,868	2,021,298

* CWHS data, 1991-2009, all nominal values in 2005$.

The estimated parameters from equation (4.4) are given in Table 4.3. The life-cycle earnings profiles implied by these parameters are plotted in Figure 4.1. Note that there are few individuals above age 80 in the data. To extrapolate these estimates for model ages 80-100, we use an arctan function of the following form:

$$y = \left(\frac{-a}{\pi}\right) * arctan(bx + c) + \frac{a}{2} \tag{4.5}$$

where x is age, and a, b, and c are the parameters we search over for the best fit of the function to the following three criteria: 1) the value of the function should match the value of the data at age 80 2) the slope of the arctan should match the slope of the data at age 80 and 3) the value of the function should match the value of the data at age 100 times a constant. This constant is .5 for all lifetime income groups, except the 2nd highest ability is .7 (otherwise, the 2nd highest has a lower income than the 3rd highest ability group in the last few years).

Table 4.3: Log Wage Regressions, by Lifetime Income Group

Lifetime income groups (percentiles)	Constant	*Age*	Age^2	Age^3	Observations
0 to 25	3.41000000***	-0.09720122***	0.00247639***	-0.00001842***	65,698
	(0.08718100)	(0.00543339)	(0.00010901)	(0.00000071)	
25 to 50	0.69689692***	0.05995294***	-0.00004086	-0.00000521***	101,484
	(0.05020758)	(0.00345549)	(0.00007627)	(0.00000054)	
50 to 70	-0.78761958***	0.17654618***	-0.00240656***	0.00001039***	74,253
	(0.04519637)	(0.00338371)	(0.00008026)	(0.00000061)	
70 to 80	-1.11000000***	0.21168263***	-0.00306555***	0.00001438***	33,528
	(0.06838352)	(0.00530190)	(0.00012927)	(0.00000099)	
80 to 90	-0.93939272***	0.21638731***	-0.00321041***	0.00001579***	31,919
	(0.08333727)	(0.00664647)	(0.00016608)	(0.00000130)	
90 to 99	1.60000000***	0.04500235***	0.00094253***	-0.00001470***	24,370
	(0.11723131)	(0.00931334)	(0.00022879)	(0.00000176)	
99 to 100	1.89000000***	0.09229392**	0.00012902	-0.00001169*	2,129
	(0.50501510)	(0.03858202)	(0.00090072)	(0.00000657)	

Source: CWHS data, 1991 – 2009.
* Significant at the 10 percent level ($p < 0.10$).
** Significant at the 5 percent level ($p < 0.05$).
*** Significant at the 1 percent level ($p < 0.01$).

Chapter 5

Households

In this chapter, we describe what is arguably the most important economic agent in the `OG-ITA` model: the household. We model households in `OG-ITA` rather than individuals, because we want to abstract from the concepts of gender, marital status, and number of children. Furthermore, the household is the usual unit of account in tax data. Because `OG-ITA` is primarily a fiscal policy model using Italian data, it is advantageous to have the most granular unit of account be the household.

5.1 Budget Constraint

We described the derivation and dynamics of the population distribution in Chapter 3. A measure $\omega_{1,t}$ of households is born each period, become economically relevant at age $s = E+1$ if they survive to that age, and live for up to $E+S$ periods (S economically active periods), with the population of age-s individuals in period t being $\omega_{s,t}$. Let the age of a household be indexed by $s = \{1, 2, ...E+S\}$.

At birth, each household age $s = 1$ is randomly assigned one of J ability groups, indexed by j. Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ...\lambda_J]$. Once an household is born and assigned to an ability type, it remains that ability type for its entire lifetime. This is deterministic ability heterogeneity as described in Chapter 4. Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, ...e_{j,E+S}]$. The budget constraint for the age-s household in lifetime income group j at time t is the following,

$$c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{BQ_t}{\lambda_j \omega_{s,t}} \quad \forall j,t \quad \text{and} \quad s \geq E+1 \quad \text{where} \quad b_{j,E+1,t} = 0 \quad \forall j,t \tag{5.1}$$

where $c_{j,s,t}$ is consumption, $b_{j,s+1,t+1}$ is savings for the next period, r_t is the interest rate (return on savings), $b_{j,s,t}$ is current period wealth (savings from last period), w_t is the wage, and $n_{j,s,t}$ is labor supply.

The next term on the right-hand-side of the budget constraint (5.1) represents the portion of total bequests BQ_t that go to the age-s, income-group-j household. Let $\zeta_{j,s}$ be the fraction of total bequests BQ_t that go to the age-s, income-group-j household, such that $\sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\zeta_{j,s} = 1$. We must divide that amount by the population of (j,s) households $\lambda_j\omega_{s,t}$. Chapter 6 details how to calibrate the $\zeta_{j,s}$ values from consumer finance data.

5.2 Elliptical Disutility of Labor Supply

In `OG-ITA`, the period utility function of each household is a function of consumption $c_{j,s,t}$, savings $b_{j,s+1,t+1}$, and labor supply $n_{j,s,t}$.[1] We detail this utility function, its justification, and functional form in Section 5.3. With endogenous labor supply $n_{j,s,t}$, we must specify how labor enters an agent's utility function and what are the constraints. Assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labor $n_{j,s,t} \in [0,\tilde{l}]$ or leisure $l_{j,s,t} \in [0,\tilde{l}]$.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \quad \forall s,t \tag{5.2}$$

The functional form for the utility of leisure or the disutility of labor supply has important implications for the computational tractability of the model. One difference of the household's labor supply decision $n_{j,s,t}$ from the consumption decision $c_{j,s,t}$ is that the consumption decision only has a lower bound $c_{j,s,t} \geq 0$ whereas the labor supply decision has both upper and lower bounds $n_{j,s,t} \in [0,\tilde{l}]$. Evans and Phillips (2017) show that many of the traditional functional forms for the disutility of labor—Cobb-Douglas, constant Frisch elasticty, constant relative risk aversion (CRRA)—do not have Inada conditions on both the upper and lower bounds of labor supply. To solve these in a heterogeneous agent model would require occasionally binding constraints, which is a notoriously difficult computational problem.

Evans and Phillips (2017) propose using an equation for an ellipse to match the disutility of labor supply to whatever traditional functional form one wants. Our preferred specification in `OG-ITA` is to fit an elliptical disutility of labor supply function to approximate a linearly separable constant Frisch elasticity (CFE) functional form. Let $v(n)$ be a general disutility of labor function. A CFE disutility of labor function is the following,

$$v(n) \equiv \frac{n^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \quad \theta > 0 \tag{5.3}$$

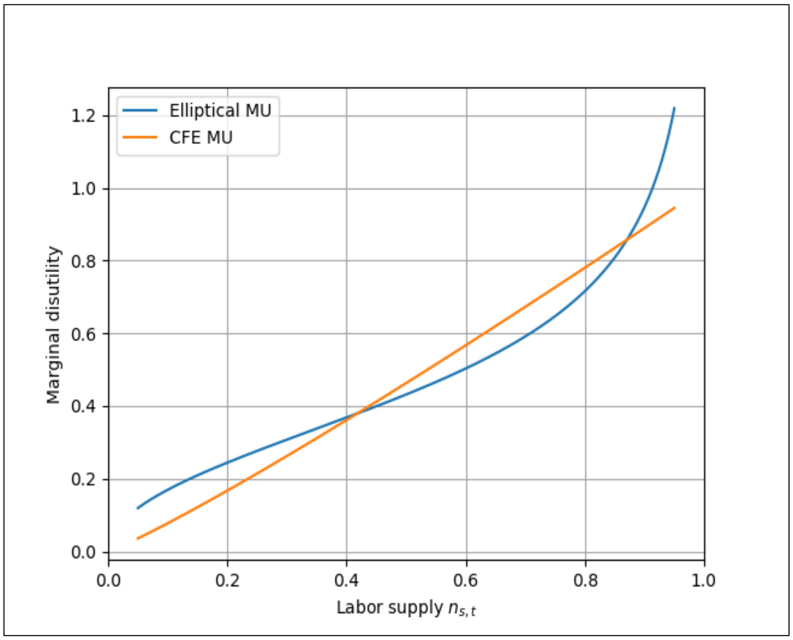
where $\theta > 0$ represents the Frisch elasticity of labor supply. The elliptical disutility of labor supply functional form is the following,

$$v(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}, \quad b,\upsilon > 0 \tag{5.4}$$

where $b > 0$ is a scale parameter and $\upsilon > 0$ is a curvature parameter. This functional form satisfies both $v'(n) > 0$ and $v''(n) > 0$ for all $n \in (0,1)$. Further, it has Inada conditions at both the upper and lower bounds of labor supply $\lim_{n\to 0} v'(n) = 0$ and $\lim_{n\to \tilde{l}} v'(n) = -\infty$.

[1] Savings enters the period utility function to provide a "warm glow" bequest motive.

Figure 5.1: Comparison of CFE marginal disutility of leisure $\theta = 0.9$ to fitted elliptical utility



Because it is the marginal disutility of labor supply that matters for household decision making, we want to choose the parameters of the elliptical disutility of labor supply function (b, υ) so that the elliptical marginal utilities match the marginal utilities of the CFE disutility of labor supply. Figure 5.1 shows the fit of marginal utilities for a Frisch elasticity of $\theta = 0.9$ and a total time endowment of $\tilde{l} = 1.0$. The estimated elliptical utility parameters in this case are $b = 0.527$ and $\upsilon = 1.497$.[2]

5.3 Optimality Conditions

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{j,s,t+s-1}\}_{s=1}^S$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^S$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints. The household period utility function is the following.

$$
\begin{split}
u(c_{j,s,t}, n_{j,s,t}, b_{j,s+1,t+1}) \equiv \frac{(c_{j,s,t})^{1-\sigma} - 1}{1-\sigma} + e^{g_y t(1-\sigma)}\chi^n_s\left(b\left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}\right) + \\
\chi^b_j \rho_s \frac{(b_{j,s+1,t+1})^{1-\sigma} - 1}{1-\sigma} \quad \forall j,t \quad \text{and} \quad E+1 \leq s \leq E+S
\end{split}
\tag{5.5}
$$

[2] Peterman (2016) shows that in a macro-model that has only an intensive margin of labor supply and no extensive margin and represents a broad composition of individuals supplying labor—such as `OG-ITA`—a Frisch elasticity of around 0.9 is probably appropriate. He tests the implied macro elasticity when the assumed micro elasticities are small on the intensive margin but only macro aggregates—which include both extensive and intensive margin agents—are observed.

The period utility function (5.5) is linearly separable in $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s+1,t+1}$. The first term is a constant relative risk aversion (CRRA) utility of consumption. The second term is the elliptical disutility of labor described in Section 5.2. The constant χ^n_s adjusts the disutility of labor supply relative to consumption and can vary by age s, which is helpful for calibrating the model to match labor market moments. See Chapter 15 for a discussion of the calibration.

It is necessary to multiply the disutility of labor in (5.5) by $e^{g_y(1-\sigma)}$ because labor supply $n_{j,s,t}$ is stationary, but both consumption $c_{j,s,t}$ and savings $b_{j,s+1,t+1}$ are growing at the rate of technological progress (see Chapter 12). The $e^{g_y(1-\sigma)}$ term keeps the relative utility values of consumption, labor supply, and savings in the same units.

The final term in the period utility function (5.5) is the "warm glow" bequest motive. It is a CRRA utility of savings, discounted by the mortality rate ρ_s.[3] Intuitively, it signifies the utility a household gets in the event that they don't live to the next period with probability ρ_s. It is a utility of savings beyond its usual benefit of allowing for more consumption in the next period. This utility of bequests also has constant χ^b_j which adjusts the utility of bequests relative to consumption and can vary by lifetime income group j. This is helpful for calibrating the model to match wealth distribution moments. See Chapter 15 for a discussion of the calibration. Note that any bequest before age $E+S$ is unintentional as it was bequeathed due an event of death that was uncertain. Intentional bequests are all bequests given in the final period of life in which death is certain $b_{j,E+S+1,t}$.

The household lifetime optimization problem is to choose consumption $c_{j,s,t}$, labor supply $n_{j,s,t}$, and savings $b_{j,s+1,t+1}$ in every period of life to maximize expected discounted lifetime utility, subject to budget constraints and upper-bound and lower-bound constraints.

$$\max_{\{(c_{j,s,t}),(n_{j,s,t}),(b_{j,s+1,t+1})\}_{s=E+1}^{E+S}} \sum_{s=1}^{S} \beta^{s-1}\left[\Pi_{u=E+1}^{E+s}(1-\rho_u)\right] u(c_{j,s,t+s-1}, n_{j,s,t+s-1}, b_{j,s+1,t+s}) \quad (5.6)$$

$$\text{s.t.} \quad c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{BQ_t}{\lambda_j \omega_{s,t}} \quad (5.1)$$

$$\text{and} \quad c_{j,s,t} \geq 0,\; n_{j,s,t} \in [0,\tilde{l}],\; \text{and}\; b_{j,1,t} = 0 \quad \forall j,t,\; \text{and}\; E+1 \leq s \leq E+S$$

The nonnegativity constraint on consumption does not bind in equilibrium because of the Inada condition $\lim_{c\to 0} u_1(c,n,b') = \infty$, which implies consumption is always strictly positive in equilibrium $c_{j,s,t} > 0$ for all j, s, and t. The warm glow bequest motive in (5.5) also has an Inada condition for savings at zero, so $b_{j,s,t} > 0$ for all j, s, and t. This is an implicit borrowing constraint.[4] And finally, as discussed in Section 5.2, the elliptical disutility of labor supply functional form in (5.5) imposes Inada conditions on both the upper and lower bounds of labor supply such that labor supply is strictly interior in equilibrium $n_{j,s,t} \in (0,\tilde{l})$ for all j, s, and t.

The household maximization problem can be further reduced by substituting in the household budget constraint, which binds with equality. This simplifies the household's

[3] See Section 3.2 of Chapter 3 for a detailed discussion of mortality rates in `OG-ITA`.

[4] It is important to note that savings also has an implicit upper bound $b_{j,s,t} \leq k$ above which consumption would be negative in current period. However, this upper bound on savings in taken care of by the Inada condition on consumption.

problem to choosing labor supply $n_{j,s,t}$ and savings $b_{j,s+1,t+1}$ every period to maximize lifetime discounted expected utility. The $2S$ first order conditions for every type-j household that characterize the its S optimal labor supply decisions and S optimal savings decisions are the following.

$$w_t e_{j,s}(c_{j,s,t})^{-\sigma} = e^{g_y(1-\sigma)}\chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S \tag{5.7}$$

$$(c_{j,s,t})^{-\sigma} = \chi^b_j\rho_s(b_{j,s+1,t+1})^{-\sigma} + \beta\bigl(1-\rho_s\bigr)\bigl(1+r_{t+1}\bigr)(c_{j,s+1,t+1})^{-\sigma} \quad \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1 \tag{5.8}$$

$$(c_{j,E+S,t})^{-\sigma} = \chi^b_j(b_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad \text{and} \quad s=E+S \tag{5.9}$$

5.4 Expectations

To conclude the household's problem, we must make an assumption about how the age-s household can forecast the time path of interest rates, wages, and total bequests $\{r_u,w_u,BQ_u\}_{u=t}^{t+S-s}$ over his remaining lifetime. As we will show in Chapters 13 and 14, the equilibrium interest rate r_t, wage w_t, and total bequests BQ_t will be functions of the state vector $\mathbf{\Gamma}_t$, which turns out to be the entire distribution of savings at in period t.

Define $\mathbf{\Gamma}_t$ as the distribution of household savings across households at time t.

$$\mathbf{\Gamma}_t \equiv \bigl\{b_{j,s,t}\bigr\}_{s=E+2}^{E+S} \quad \forall j,t \tag{5.10}$$

Let general beliefs about the future distribution of capital in period $t+u$ be characterized by the operator $\Omega(\cdot)$ such that:

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{5.11}$$

where the e superscript signifies that $\mathbf{\Gamma}^e_{t+u}$ is the expected distribution of wealth at time $t+u$ based on general beliefs $\Omega(\cdot)$ that are not constrained to be correct.[5]

[5]In Chapter 14 we will assume that beliefs are correct (rational expectations) for the non-steady-state equilibrium in Definition 14.1.

Chapter 6

Calibrated Bequests

This chapter describes how we calibrate the distribution of total bequests BQ_t to each living household of age s and lifetime income group j. The matrix that governs this distribution $\zeta_{j,s}$ is seen in the household budget constraint 5.1.

A large number of papers study the effects of different bequest motives and specifications on the distribution of wealth, though there is no consensus regarding the true bequest transmission process.[1]

[1]See De Nardi and Yang (2014), De Nardi (2004), Nishiyama (2002), Laitner (2001), Gokhale et al. (2000), Gale and Scholz (1994), Hurd (1989), Venti and Wise (1988), Kotlikoff and Summers (1981), and Wolff (2015).

Part III

Firms Theory

Chapter 7

Firms

The production side of the `OG-ITA` model is populated by a unit measure of identical perfectly competitive firms that rent capital K_t and hire labor L_t to produce output Y_t.

7.1 Production Function

Firms produce output Y_t using inputs of capital K_t and labor L_t according to a general constant elasticity (CES) of substitution production function,

$$Y_t = F(K_t, L_t) \equiv Z_t\biggl[(\gamma)^{\frac{1}{\varepsilon}}(K_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(e^{g_y t}L_t)^{\frac{\varepsilon-1}{\varepsilon}}\biggr]^{\frac{\varepsilon}{\varepsilon-1}} \quad \forall t \tag{7.1}$$

where Z_t is an exogenous scale parameter (total factor productivity) that can be time dependent, γ represents the capital share of income, and ε is the constant elasticity of substitution between capital and labor. We have included constant productivity growth g_y as the rate of labor augmenting technological progress.

A nice feature of the CES production function is that the Cobb-Douglas production function is a nested case for $\varepsilon = 1$.

$$Y_t = Z_t(K_t)^\gamma(e^{g_y t}L_t)^{1-\gamma} \quad \text{for} \quad \varepsilon = 1 \quad \forall t \tag{7.2}$$

7.2 Optimality Conditions

The profit function of the representative firm is the following.

$$PR_t = F(K_t, L_t) - w_t L_t - \bigl(r_t + \delta\bigr)K_t \quad \forall t \tag{7.3}$$

Gross income for the firms is given by the production function $F(K, L)$ because we have normalized the price of the consumption good to 1. Labor costs to the firm are $w_t L_t$, and capital costs are $(r_t + \delta)K_t$. The per-period economic depreciation rate is given by δ.

Taking the derivative of the profit function (7.3) with respect to labor L_t and setting it equal to zero and taking the derivative of the profit function with respect to capital K_t and setting it equal to zero, respectively, characterizes the optimal labor and capital demands.

$$w_t = e^{g_y t}(Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{Y_t}{e^{g_y t}L_t}\right]^{\frac{1}{\varepsilon}} \quad \forall t \tag{7.4}$$

$$r_t = (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{Y_t}{K_t}\right]^{\frac{1}{\varepsilon}} - \delta \quad \forall t \tag{7.5}$$

Part IV

Government Theory

Chapter 8

Household Taxes and Data

Chapter 9

Corporate Taxes and Data

Chapter 10

Unbalanced Government Budget Constraint

Part V

Market clearing and Stationarization

Chapter 11

Market Clearing

Three markets must clear in OG-ITA—the labor market, the capital market, and the goods market. By Walras' Law, we only need to use two of those market clearing conditions because the third one is redundant. In the model, we choose to use the labor market clearing condition and the capital market clearing condition, and to ignore the goods market clearing condition. But we present all three market clearing conditions here. Further, the redundant goods market clearing condition—sometimes referred to as the resource constraint—makes for a nice check on the solution method to see if everything worked.

We also characterize here the law of motion for total bequests BQ_t. Although it is not technically a market clearing condition, one could think of the bequests law of motion as the bequests market clearing condition.

11.1 Market Clearing Conditions

Labor market clearing (11.1) requires that aggregate labor demand L_t measured in efficiency units equal the sum of household efficiency labor supplied $e_{j,s}n_{j,s,t}$.

$$L_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{11.1}$$

Capital market clearing (11.2) requires that aggregate capital demand from firms K_t equal the sum of capital savings and investment by households $b_{j,s,t}$.

$$K_t = \sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\omega_{s-1,t-1}\lambda_j b_{j,s,t} + i_s\omega_{s,t-1}\lambda_j b_{j,s,t}\Big) \quad \forall t \tag{11.2}$$

Aggregate consumption C_t is defined as the sum of all household consumptions, and aggregate investment is defined by the resource constraint $Y_t = C_t + I_t$ as shown in (11.3).

$$\begin{split} &Y_t = C_t + K_{t+1} - \left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\omega_{s,t}\lambda_j b_{j,s,t+1}\right) - (1-\delta)K_t \quad \forall t \\ &\text{where}\quad C_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\omega_{s,t}\lambda_j c_{j,s,t} \end{split} \tag{11.3}$$

Note that the extra terms with the immigration rate i_s in the capital market clearing equation (11.2) and the resource constraint (11.3) accounts for the assumption that age-s immigrants in period t bring with them (or take with them in the case of out-migration) the same amount of capital as their domestic counterparts of the same age. Note also that the term in parentheses with immigration rates i_s in the sum acts is equivalent to a net exports term in the standard equation $Y = C + I + G + NX$. That is, if immigration rates are positive, then immigrants are bringing capital into the country and the term in parentheses has a negative sign in front of it. Negative exports are imports.

11.2 Total Bequests Law of Motion

Total bequests BQ_t are the collection of savings of household from the previous period who died at the end of the period. These savings are augmented by the interest rate because they are returned after being invested in the production process.

$$BQ_t = (1+r_t)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\omega_{s-1,t-1}b_{j,s,t}\right) \quad \forall t \tag{11.4}$$

Because the form of the period utility function in (5.5) ensures that $b_{j,s,t} > 0$ for all j, s, and t, total bequests will always be positive $BQ_{j,t} > 0$ for all j and t.

Chapter 12

Stationarization

The previous chapters derive all the equations necessary to solve for the steady-state and nonsteady-state equilibria of this model. However, because labor productivity is growing at rate g_y as can be seen in the firms' production function (7.1) and the population is growing at rate $\tilde{g}_{n,t}$ as defined in (3.5), the model is not stationary. Different endogenous variables of the model are growing at different rates.

Table 12.1 lists the definitions of stationary versions of these endogenous variables. Variables with a "ˆ" signify stationary variables. The first column of variables are growing at the productivity growth rate g_y. These variables are most closely associated with individual variables. The second column of variables are growing at the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with population values. The third column of variables are growing at both the productivity growth rate g_y and the population growth rate $\tilde{g}_{n,t}$. These variables are most closely associated with aggregate variables. The last column shows that the interest rate r_t and household labor supply $n_{j,s,t}$ are already stationary.

Table 12.1: Stationary variable definitions

Sources of growth			Not
$e^{g_y t}$	$\tilde{N}_t$	$e^{g_y t}\tilde{N}_t$	growing[a]
$\hat{c}_{j,s,t} \equiv \frac{c_{j,s,t}}{e^{g_y t}}$	$\hat{\omega}_{s,t} \equiv \frac{\omega_{s,t}}{\tilde{N}_t}$	$\hat{Y}_t \equiv \frac{Y_t}{e^{g_y t}\tilde{N}_t}$	$n_{j,s,t}$
$\hat{b}_{j,s,t} \equiv \frac{b_{j,s,t}}{e^{g_y t}}$	$\hat{L}_t \equiv \frac{L_t}{\tilde{N}_t}$	$\hat{K}_t \equiv \frac{K_t}{e^{g_y t}\tilde{N}_t}$	r_t
$\hat{w}_t \equiv \frac{w_t}{e^{g_y t}}$		$\hat{BQ}_{j,t} \equiv \frac{BQ_{j,t}}{e^{g_y t}\tilde{N}_t}$	
$\hat{y}_{j,s,t} \equiv \frac{y_{j,s,t}}{e^{g_y t}}$		$\hat{C}_t \equiv \frac{C_t}{e^{g_y t}\tilde{N}_t}$	

[a] The interest rate r_t in (7.5) is already stationary because Y_t and K_t grow at the same rate. Household labor supply $n_{j,s,t} \in [0,\tilde{l}]$ is stationary.

The usual definition of equilibrium would be allocations and prices such that households optimize (5.7), (5.8), and (5.9), firms optimize (7.4) and (7.5), and markets clear (11.1) and (11.2), and (11.4). In this chapter, we show how to stationarize each of these characterizing equations so that we can use our fixed point methods described in Sections 13.2 and 14.2 to solve for the equilibria in Definitions 13.1 and 14.1.

12.1 Stationarized Household Equations

The stationary version of the household budget constraint (5.1) is found by dividing both sides of the equation by $e^{g_y t}$. For the savings term $b_{j,s+1,t+1}$, we must multiply and divide by $e^{g_y(t+1)}$, which leaves an $e^{g_y} = \frac{e^{g_y(t+1)}}{e^{g_y t}}$ in front of the stationarized variable.

$$\begin{split}\hat{c}_{j,s,t} + e^{g_y}\hat{b}_{j,s+1,t+1} = (1+r_t)\hat{b}_{j,s,t} + \hat{w}_t e_{j,s} n_{j,s,t} + \zeta_{j,s}\frac{\hat{BQ}_t}{\lambda_j \hat{\omega}_{s,t}} \\ \forall j,t \quad \text{and} \quad s\geq E+1 \quad \text{where} \quad b_{j,E+1,t}=0 \quad \forall j,t\end{split} \tag{12.1}$$

Because total bequests BQ_t grows at both the labor productivity growth rate and the population growth rate, we have to multiply and divide that term by the economically relevant population $\tilde{N}_t$. This stationarizes total bequests $\hat{BQ}_t$ and the population level in the denominator $\hat{\omega}_{s,t}$.

We stationarize the Euler equations for labor supply (5.7) by dividing both sides by $e^{g_y(1-\sigma)}$. On the left-hand-side, e^{g_y} stationarizes the wage $\hat{w}_t$ and $e^{-\sigma g_y}$ goes inside the parentheses and stationarizes consumption $\hat{c}_{j,s,t}$. On the right-and-side, the $e^{g_y(1-\sigma)}$ terms cancel out.

$$\begin{split}\hat{w}_t e_{j,s}(\hat{c}_{j,s,t})^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{j,s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \\ \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S\end{split} \tag{12.2}$$

We stationarize the Euler equations for savings (5.8) and (5.9) by dividing both sides of the respective equations by $e^{-\sigma g_y t}$. On the right-hand-side of the equation, we then need to multiply and divide both terms by $e^{-\sigma g_y(t+1)}$, which leaves a multiplicative coefficient $e^{-\sigma g_y}$.

$$\begin{split}(\hat{c}_{j,s,t})^{-\sigma} = e^{-\sigma g_y}\left[\chi^b_j\rho_s(\hat{b}_{j,s+1,t+1})^{-\sigma} + \beta\big(1-\rho_s\big)\big(1+r_{t+1}\big)(\hat{c}_{j,s+1,t+1})^{-\sigma}\right] \\ \forall j,t, \quad \text{and} \quad E+1\leq s\leq E+S-1\end{split} \tag{12.3}$$

$$(\hat{c}_{j,E+S,t})^{-\sigma} = e^{-\sigma g_y}\chi^b_j(\hat{b}_{j,E+S+1,t+1})^{-\sigma} \quad \forall j,t \quad \text{and} \quad s=E+S \tag{12.4}$$

12.2 Stationarized Firms Equations

The nonstationary production function (7.1) can be stationarized by dividing both sides by $e^{g_y t}\tilde{N}$. This stationarizes output $\hat{Y}_t$ on the left-hand-side. Because the general CES production function is homogeneous of degree 1, $F(xK,xL)=xF(K,L)$, which means the right-hand-side of the production function is stationarized by dividing by $e^{g_y t}\tilde{N}_t$.

$$\hat{Y}_t = F(\hat{K}_t,\hat{L}_t) \equiv Z_t\left[(\gamma)^{\frac{1}{\varepsilon}}(\hat{K}_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\hat{L}_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \quad \forall t \tag{12.5}$$

Notice that the growth term multiplied by the labor input drops out in this stationarized version of the production function. We stationarize the nonstationary profit function (7.3) in the same way, by dividing both sides by $e^{g_y t}\tilde{N}_t$.

$$\hat{PR}_t = F(\hat{K}_t,\hat{L}_t) - \hat{w}_t\hat{L}_t - \big(r_t+\delta\big)\hat{K}_t \quad \forall t \tag{12.6}$$

The firms' first order equation for labor demand (7.4) is stationarized by dividing both sides by $e^{g_y t}$. This stationarizes the wage $\hat{w}_t$ on the left-hand-side and cancels out the $e^{g_y t}$ term in front of the right-hand-side. To complete the stationarization, we multiply and divide the $\frac{Y_t}{e^{g_y t}L_t}$ term on the right-hand-side by $\tilde{N}_t$.

$$\hat{w}_t = (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{\hat{Y}_t}{\hat{L}_t}\right]^{\frac{1}{\varepsilon}} \quad \forall t \tag{12.7}$$

It can be seen from the firms' first order equation for capital demand (7.5) that the interest rate is already stationary. If we multiply and divide the $\frac{Y_t}{K_t}$ term on the right-hand-side by $e^{t_y t}\tilde{N}_t$, those two aggregate variables become stationary. In other words, Y_t and K_t grow at the same rate and $\frac{Y_t}{K_t} = \frac{\hat{Y}_t}{\hat{K}_t}$.

$$\begin{aligned} r_t &= (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\hat{Y}_t}{\hat{K}_t}\right]^{\frac{1}{\varepsilon}} - \delta \quad \forall t \\ &= (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{Y_t}{K_t}\right]^{\frac{1}{\varepsilon}} - \delta \quad \forall t \end{aligned} \tag{7.5}$$

12.3 Stationarized Market Clearing Equations

The labor market clearing equation (11.1) is stationarized by dividing both sides by $\tilde{N}_t$.

$$\hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s}n_{j,s,t} \quad \forall t \tag{12.8}$$

The capital market clearing equation (11.2) is stationarized by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the right-hand-side has population levels from the previous period $\omega_{s,t-1}$, we have to multiply and divide both terms inside the parentheses by $\tilde{N}_{t-1}$ which leaves us with the term in front of $\frac{1}{1+\tilde{g}_{n,t}}$.

$$\hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\hat{\omega}_{s-1,t-1}\lambda_j\hat{b}_{j,s,t} + i_s\hat{\omega}_{s,t-1}\lambda_j\hat{b}_{j,s,t}\Big) \quad \forall t \tag{12.9}$$

We stationarize the goods market clearing (11.3) condition by dividing both sides by $e^{g_y t}\tilde{N}_t$. On the right-hand-side, we must multiply and divide the K_{t+1} term by $e^{g_y(t+1)}\tilde{N}_{t+1}$ leaving the coefficient $e^{g_y}(1+\tilde{g}_{n,t+1})$. And the term that subtracts the sum of imports of next period's immigrant savings we must multiply and divide by $e^{g(t+1)}$, which leaves the term e^{g_y}.

$$\begin{split}\hat{Y}_t = \hat{C}_t + e^{g_y}(1+\tilde{g}_{n,t+1})\hat{K}_{t+1} - e^{g_y}\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J} i_s\hat{\omega}_{s,t}\lambda_j\hat{b}_{j,s,t+1}\right) - (1-\delta)\hat{K}_t \quad \forall t \\ \text{where} \quad \hat{C}_t \equiv \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j\hat{c}_{j,s,t}\end{split} \tag{12.10}$$

We stationarize the law of motion for total bequests BQ_t in (11.4) by dividing both sides by $e^{g_y t}\tilde{N}_t$. Because the population levels in the summation are from period $t-1$, we must multiply and divide the summed term by $\tilde{N}_{t-1}$ leaving the term in the denominator of $1+\tilde{g}_{n,t}$.

$$\hat{BQ}_t = \left(\frac{1+r_t}{1+\tilde{g}_{n,t}}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\hat{\omega}_{s-1,t-1}\hat{b}_{j,s,t}\right) \quad \forall t \tag{12.11}$$

Part VI

Equilibrium Definitions and Solution Methods

Chapter 13

Stationary Steady-state Equilibrium

In this chapter, we define the stationary steady-state equilibrium of the `OG-ITA` model. Chapters 3 through 11 derive the equations that characterize the equilibrium of the model. However, we cannot solve for any equilibrium of the model in the presence of nonstationarity in the variables. Nonstationarity in `OG-ITA` comes from productivity growth g_y in the production function (7.1) and population growth $\tilde{g}_{n,t}$ as described in Chapter 3. We showed in Chapter 12 how to stationarize all the characterizing equations.

13.1 Stationary Steady-State Equilibrium Definition

With the stationarized model, we can now define the stationary steady-state equilibrium. This equilibrium will be long-run values of the endogenous variables that are constant over time. In a perfect foresight model, the steady-state equilibrium is the state of the economy at which the model settles after a finite amount of time, regardless of the initial condition of the model. Once the model arrives at the steady-state, it stays there indefinitely unless it receives some type of shock or stimulus.

These stationary values have all the growth components from productivity growth and population growth removed as defined in Table 12.1. Because the productivity growth rate g_y and population growth rate series $\tilde{g}_{n,t}$ are exogenous, we can transform the stationary equilibrium values of the variables back to their nonstationary values by reversing the identities in Table 12.1.

Definition 13.1 (Stationary steady-state equilibrium). A non-autarkic stationary steady-state equilibrium in the `OG-ITA` model is defined as constant allocations of stationary household labor supply $n_{j,s,t} = \bar{n}_{j,s}$ and savings $\hat{b}_{j,s+1,t+1} = \bar{b}_{j,s+1}$ for all j, t, and $E+1 \leq s \leq E+S$, and constant prices $\hat{w}_t = \bar{w}$ and $r_t = \bar{r}$ for all t such that the following conditions hold:

i. the population has reached its stationary steady-state distribution $\hat{\omega}_{s,t} = \bar{\omega}_s$ for all s and t as characterized in Section 3.4,

ii. households optimize according to (12.2), (12.3), and (12.3),

iii. firms optimize according to (12.7) and (7.5),

iv. markets clear according to (12.8), (12.9), and (12.11).

13.2 Stationary Steady-state Solution Method

This section describes the solution method for the stationary steady-state equilibrium described in Definition 13.1. The steady-state is characterized by $2JS$ equations and $2JS$ unknowns. However, because some of the other equations cannot be solved for analytically and substituted into the Euler equations, we must take a two-stage approach to the equilibrium solution. We first make a guess at steady-state interest rate $\bar{r}$, wage $\bar{w}$, and total bequests $\overline{BQ}$. We call these three steady-state guesses the "outer loop" of the steady-state solution method. They are the macroeconomic variables necessary to solve the household's problem.

The "inner loop" of the steady-state solution method is to solve for the steady-state household decisions $\bar{b}_{j,s}$ and labor supply $\bar{n}_{j,s}$ for all j and $E+1 \leq s \leq E+S$ given the values of the outer-loop variables. Because the lifetime optimization problem of each household of type j is a highly nonlinear system of $2S$ equations and $2S$ unknowns, we break the inner loop problem into two stages, the first of which is a univariate convex optimization problem and the second of which is a serial series of univariate convex optimization problems.

The first stage of the inner loop is to guess an initial steady-state consumption $\bar{c}_{j,1}$ for each household of type j. The second stage of the inner loop is to solve for each period household optimization problem recursively given the initial consumption guess from the first stage. We update the first stage guess for $\bar{c}_{j,1}$ until the implied consumption in the last period $\bar{c}_{j,S}$ and savings in the last period $\bar{b}_{j,S+1}$ satisfy the last period savings Euler equation (12.3). We outline this algorithm in the following steps.

1. Use the techniques from Section 3.4 to solve for the steady-state population distribution vector $\bar{\boldsymbol{\omega}}$ and steady-state growth rate $\bar{g}_n$ of the exogenous population process.

2. Choose an initial guess for the values of the steady-state interest rate $\bar{r}^i$, wage $\bar{w}^i$, and total bequests $\overline{BQ}^i$, where superscript i is the index of the iteration number of the guess.

    (a) Note that if the production function is Cobb-Douglas ($\varepsilon = 1$), then you only have to guess the steady-state values of the steady-state interest rate $\bar{r}^i$ and total bequests $\overline{BQ}^i$. In this case, the steady-state wage $\bar{w}$ is determined by the interest rate using equations (7.5) and (12.7). In this Cobb-Douglas case ($\varepsilon = 1$), choosing both $\bar{r}$ and $\bar{w}$ in the outer loop can cause the solution method to not converge.

3. Given guesses for $\bar{r}^i$, $\bar{w}^i$, and $\overline{BQ}^i$, solve for the steady-state household labor supply $\bar{n}_{j,s}$ and savings $\bar{b}_{j,s}$ decisions for all j and $E+1 \leq s \leq E+S$ using two-stage approach.

    (a) Given $\bar{r}^i$, $\bar{w}^i$, and $\overline{BQ}^i$, guess an initial steady-state consumption $\bar{c}^m_{j,E+1}$ for each type-j household, where m is the index of the inner-loop iteration.

i. Given $\bar{r}^i$ and $\bar{w}^i$, $\overline{BQ}^i$, and $\bar{c}^m_{j,E+1}$, and the fact that $\bar{b}_{j,E+1} = 0$, we can use the household labor supply Euler equation (12.2) to solve for $\bar{n}_{j,E+1}$ for all j. This problem is a univariate root finder in $\bar{n}_{j,E+1}$.

$$\bar{w}^i e_{j,s}(\bar{c}_{j,s})^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_{j,s}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\forall j,\quad\text{and}\quad E+1\leq s\leq E+S$$

ii. Given $\bar{c}^m_{j,E+1}$, $\bar{b}_{j,E+1} = 0$, and $\bar{n}_{j,E+1}$, we can use the household budget constraint (12.1) to solve analytically for $\bar{b}_{j,E+2}$ for all j.

$$\bar{b}_{j,s+1} = e^{-g_y}\left[(1+\bar{r}^i)\bar{b}_{j,s} + \bar{w}^i e_{j,s}\bar{n}_{j,s} + \zeta_{j,s}\frac{\overline{BQ}^i}{\lambda_j\bar{\omega}_s} - \bar{c}_{j,s}\right]$$
$$\forall j\quad\text{and}\quad E+1\leq s\leq E+S$$

iii. Given $\bar{c}^m_{j,E+1}$ and $\bar{b}_{j,E+2}$, use the household labor supply Euler equation (12.3) to solve for $\bar{c}_{j,E+2}$ for all j. This problem is a univariate root finder in $\bar{c}_{j,E+2}$

$$(\bar{c}_{j,s})^{-\sigma} = e^{-\sigma g_y}\left[\chi^b_j\rho_s(\bar{b}_{j,s+1})^{-\sigma} + \beta\big(1-\rho_s\big)\big(1+\bar{r}^i\big)(\bar{c}_{j,s+1})^{-\sigma}\right]$$
$$\forall j,\quad\text{and}\quad E+1\leq s\leq E+S-1$$

iv. Repeat in serial steps (i) through (iii) until solved for the all households' steady-state lifetime decisions $\{\bar{c}_{j,s},\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j.

(b) Given household lifetime decisions $\{\bar{c}_{j,s},\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ for all j based on guesses for initial period consumption $\bar{c}^m_{j,E+1}$ for all j and outer loop guesses $\bar{r}^i$, $\bar{w}^i$, and $\overline{BQ}^i$, check the error in the last period savings Euler equation (12.4) based on $\bar{c}_{j,E+S}$ and $\bar{b}_{j,E+S+1}$.

$$error_j \equiv e^{-\sigma g_y}\chi^b_j(\bar{b}_{j,E+S+1,t+1})^{-\sigma} - (\bar{c}_{j,E+S})^{-\sigma}\quad\forall j$$

(c) If the error is greater than some small positive tolerance $error_j > toler_c$ for some j, then update the guesses for initial consumption $\bar{c}^{m+1}_{j,1}$ and repeat steps (a) and (b).

(d) If the error is less than some small positive tolerance $error_j \leq toler_c$ for all j, then $\{\bar{c}_{j,s},\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ is the full set of partial equilibrium household steady-state solutions given guesses for $\bar{r}^i$, $\bar{w}^i$, and $\overline{BQ}^i$.

4. Given partial equilibrium household steady-state solutions $\{\bar{c}_{j,s},\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ based on macroeconomic variable guesses $\bar{r}^i$, $\bar{w}^i$, and $\overline{BQ}^i$, check the errors in the five equations that characterize each of the macroeconomic variable guesses.

(a) If we substitute the two market clearing conditions (12.8) and (12.9), and the firm's production function (12.5) into the firm's first order condition for capital demand (7.5), we get an expression in which household decisions $\{\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the interest rate $\bar{r}^{i'}$.

$$\bar{r}^{i'} = (\bar{Z})^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\bar{Y}}{\bar{K}}\right]^{\frac{1}{\varepsilon}} - \delta$$

$$\text{where}\quad \bar{Y} = \bar{Z}\left[(\gamma)^{\frac{1}{\varepsilon}}(\bar{K})^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\bar{L})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\text{and}\quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$

$$\text{and}\quad \bar{K} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

The error for this variable is the percent difference between the initial guess for the interest rate $\bar{r}^i$ and the steady-state interest rate implied by household optimization based on the initial guess $\bar{r}^{i'}$.

$$error_r = \frac{\bar{r}^{i'} - \bar{r}^i}{\bar{r}^i}$$

(b) If we substitute the two market clearing conditions (12.8) and (12.9), and the firm's production function (12.5) into the firm's first order condition for labor demand (12.7), we get an expression in which household decisions $\{\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the wage $\bar{w}^{i'}$.

$$\bar{w}^{i'} = (\bar{Z})^{\frac{\varepsilon-1}{\varepsilon}}\left[(1-\gamma)\frac{\bar{Y}}{\bar{L}}\right]^{\frac{1}{\varepsilon}}$$

$$\text{where}\quad \bar{Y} = \bar{Z}\left[(\gamma)^{\frac{1}{\varepsilon}}(\bar{K})^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\bar{L})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\text{and}\quad \bar{L} = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\bar{\omega}_s\lambda_j e_{j,s}\bar{n}_{j,s}$$

$$\text{and}\quad \bar{K} = \frac{1}{1+\bar{g}_n}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\bar{\omega}_{s-1}\lambda_j\bar{b}_{j,s} + i_s\bar{\omega}_s\lambda_j\bar{b}_{j,s}\Big)$$

The error for this variable is the percent difference between the initial guess for the wage $\bar{w}^i$ and the steady-state wage implied by household optimization based on the initial guess $\bar{w}^{i'}$.

$$error_w = \frac{\bar{w}^{i'} - \bar{w}^i}{\bar{w}^i}$$

(c) The stationarized law of motion for total bequests (12.11) provides the expression in which household decisions $\{\bar{n}_{j,s},\bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$ imply a value for the wage $\overline{BQ}^{i'}$.

Note that we need all the household decisions here because $\bar{r}^{i'}$ enters the equation on the right-hand-side.

$$\overline{BQ}^{i'} = \left(\frac{1+\bar{r}^{i'}}{1+\bar{g}_n}\right)\left(\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\rho_{s-1}\lambda_j\bar{\omega}_{s-1}\bar{b}_{j,s}\right)$$

The error for this variable is the percent difference between the initial guess for total bequests $\overline{BQ}^{i}$ and the steady-state total bequests implied by household optimization based on the initial guess $\overline{BQ}^{i'}$.

$$error_{bq} = \frac{\overline{BQ}^{i'} - \overline{BQ}^{i}}{\overline{BQ}^{i}}$$

5. If the maximum absolute error among the three outer loop error terms is greater than some small positive tolerance $toler_{out}$,

$$\max\Big|\left(error_r, error_w, error_{bq}\right)\Big| > toler_{out}$$

then update the guesses for the outer loop variables as a convex combination governed by $\xi_{ss}\in(0,1]$ of the respective initial guesses and the new implied values and repeat steps (3) through (5).

$$\left[\bar{r}^{i+1}, \bar{w}^{i+1}, \overline{BQ}^{i+1}\right] = \xi_{ss}\left[\bar{r}^{i'}, \bar{w}^{i'}, \overline{BQ}^{i'}\right] + (1-\xi_{ss})\left[\bar{r}^{i}, \bar{w}^{i}, \overline{BQ}^{i}\right]$$

6. If the maximum absolute error among the three outer loop error terms is less-than-or-equal-to some small positive tolerance $toler_{ss,out}$,

$$\max\Big|\left(error_r, error_w, error_{bq}\right)\Big| \leq toler_{ss,out}$$

then the steady-state has been found.

   (a) Make sure that the resource constraint (goods market clearing) (12.10) is satisfied. It is redundant, but this is a good check as to whether everything worked correctly.

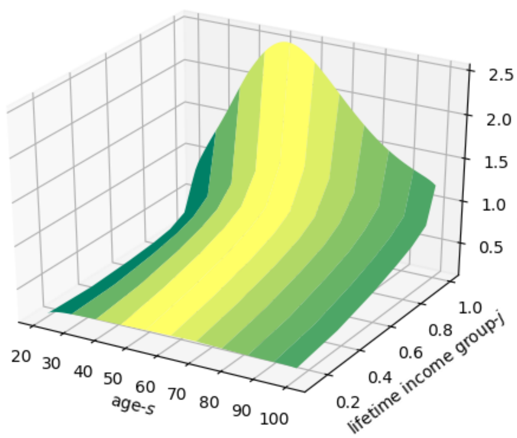
   (b) Make sure that all the $2JS$ household Euler equations are solved to a satisfactory tolerance.

13.3 Baseline Steady-state Results

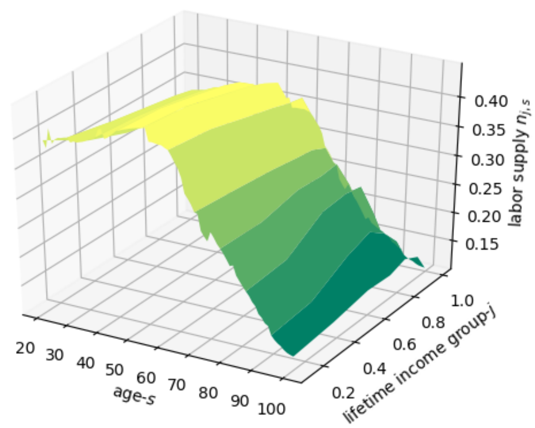
[TODO: Update results in this section. These results are from OG-USA model runs.] In this section, we use the baseline calibration described in Chapter 15 to show some steady-state results from `OG-ITA`. Figure 13.1 shows the household steady-state variables by age s and lifetime income group j.

Table 13.1 lists the steady-state prices and aggregate variable values along with some of the maximum error values from the characterizing equations.

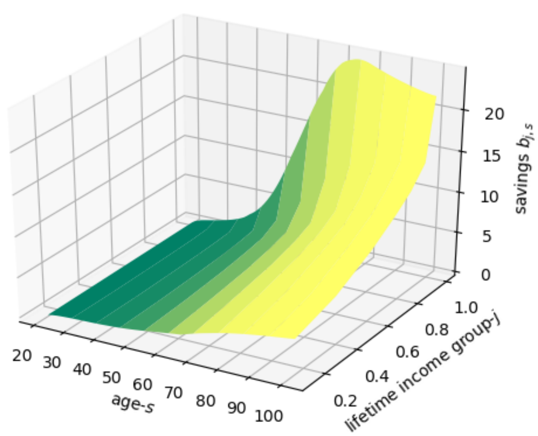
Figure 13.1: Steady-state distributions of household consumption $\bar{c}_{j,s}$, labor supply $\bar{n}_{j,s}$, and savings $\bar{b}_{j,s+1}$



(a) Consumption $\bar{c}_{j,s}$



(b) Labor supply $\bar{n}_{j,s}$



(c) Savings $\bar{b}_{j,s+1}$

Table 13.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Variable	Value
$\bar{r}$	0.058	$\bar{w}$	1.148
$\bar{Y}$	0.630	$\bar{C}$	0.462
$\bar{I}$	0.144	$\bar{K}$	1.810
$\bar{L}$	0.357	$\bar{B}$	2.440
$\overline{BQ}$	0.106		
Max. abs. labor supply Euler error	4.57e-13	Max. abs. savings Euler error	8.52e-13
Resource constraint error	-4.39e-15	Serial computation time	1 hr. 25.9 sec.*

* The steady-state computation time does not include any of the exogenous parameter computation processes, the longest of which is the estimation of the baseline tax functions which computation takes 1 hour and 15 minutes.

Chapter 14

Stationary Non Steady-state Equilibrium

In this chapter, we define the stationary nonsteady-state equilibrium of the `OG-ITA` model. Chapters 3 through 11 derive the equations that characterize the equilibrium of the model. We also need the steady-state solution from Chapter 13 to solve for the nonsteady-state equilibrium transition path. As with the steady-state equilibrium, we must use the stationarized version of the characterizing equations from Chapter 12.

14.1 Stationary Nonsteady-State Equilibrium Definition

We define a stationary nonsteady-state equilibrium as the following.

Definition 14.1 (Stationary Nonsteady-state functional equilibrium). A non autarkic nonsteady-state functional equilibrium in the `OG-ITA` model is defined as stationary allocation functions of the state $\left\{n_{j,s,t} = \phi_s\big(\hat{\boldsymbol{\Gamma}}_t\big)\right\}_{s=E+1}^{E+S}$ and $\left\{\hat{b}_{j,s+1,t+1} = \psi_s\big(\hat{\boldsymbol{\Gamma}}_t\big)\right\}_{s=E+1}^{E+S}$ for all j and t and stationary price functions $\hat{w}(\hat{\boldsymbol{\Gamma}}_t)$ and $r(\hat{\boldsymbol{\Gamma}}_t)$ for all t such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (5.11), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\hat{\boldsymbol{\Gamma}}_{t+u} = \hat{\boldsymbol{\Gamma}}^e_{t+u} = \Omega^u\left(\hat{\boldsymbol{\Gamma}}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (12.2), (12.3), and (12.3),

iii. firms optimize according to (12.7) and (7.5),

iv. markets clear according to (12.8), (12.9), and (12.11).

14.2 Stationary Nonsteady-state Solution Method

This section describes the solution method for the stationary nonsteady-state equilibrium described in Definition 14.1. We use the time path iteration (TPI) method. This method was originally outlined in a series of papers between 1981 and 1985[1] and in the seminal book Auerbach and Kotlikoff (1987, ch. 4) for the perfect foresight case and in Nishiyama and Smetters (2007, Appendix II) and Evans and Phillips (2014, Sec. 3.1) for the stochastic case. The intuition for the TPI solution method is that the economy is infinitely lived, even though the agents that make up the economy are not. Rather than recursively solving for equilibrium policy functions by iterating on individual value functions, one must recursively solve for the policy functions by iterating on the entire transition path of the endogenous objects in the economy (see Stokey et al. (1989, ch. 17)).

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 13.1 in a finite number of periods $T < \infty$ regardless of the initial state $\hat{\boldsymbol{\Gamma}}_1$. The first step in solving for the nonsteady-state equilibrium transition path is to solve for the steady-state using the method described in Section 13.2. The next step is a transition path "outer loop" step, analogous to the outer loop described in the steady-state solution method. Guess transition paths for aggregate variables $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i\}$, where $\boldsymbol{r}^i = \{r_1^i, r_2^i, ...r_T^i\}$, $\hat{\boldsymbol{w}}^i = \{\hat{w}_1^i, \hat{w}_2^i, ...\hat{w}_T^i\}$, and $\hat{\boldsymbol{BQ}}^i = \left\{\hat{BQ}_1^i, \hat{BQ}_2^i, ...\hat{BQ}_T^i\right\}$. The only requirement on these transition paths is that the initial total bequests $\hat{BQ}_1^i$ conform to the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$, and that the economy has reached the steady-state by period $t = T$ $\{r_T^i, \hat{w}_T^i, \hat{BQ}_T^i\} = \{\bar{r}, \bar{w}, \overline{BQ}\}$.

The "inner loop" of the nonsteady-state transition path solution method is to solve for the full set of lifetime savings decisions $\bar{b}_{j,s+1,t+1}$ and labor supply decisions $\bar{n}_{j,s,t}$ for every household that will be alive between periods $t = 1$ and $t = T$. Because we know the initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ in the transition path and we know the long-run steady-state $\bar{\boldsymbol{\Gamma}}$, we do not have to use the two-stage inner-loop method for solving the households' problems that we used in Section 13.2. Because we know the neighborhood where the solutions live, we can simply solve for the $2JS$ equations and unknowns for each household's lifetime decisions using a multivariate root finder. This is much faster than the two-stage method describe in Section 13.2. We outline this algorithm in the following steps.

1. Compute the steady-state solution $\{\bar{n}_{j,s}, \bar{b}_{j,s}\}_{s=E+1}^{E+S}$ corresponding to Definition 13.1.

2. Given initial state of the economy $\hat{\boldsymbol{\Gamma}}_1$ and steady-state solutions $\{\bar{n}_{j,s}, \bar{b}_{j,s+1}\}_{s=E+1}^{E+S}$, guess transition paths of outer loop macroeconomic variables $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i\}$ such that $\hat{BQ}_1^i$ is consistent with $\hat{\boldsymbol{\Gamma}}_1$ and $\{r_t^i, \hat{w}_t^i, \hat{BQ}_t^i\} = \{\bar{r}, \bar{w}, \overline{BQ}\}$ for all $t \geq T_1$.

    (a) We choose two long-run time periods, T_1 and T_2. The first time period $t = T_1$ is the period in which the time paths of all the macroeconomic guesses hit their steady-state and stay at their steady-state thereafter. The second time period $t = T_2 > T_1$ is the period after which all the endogenous inner loop household variables hit their steady-state and stay at their steady-state thereafter. These

[1] See Auerbach et al. (1981, 1983), Auerbach and Kotlikoff (1983c,b,a), and Auerbach and Kotlikoff (1985).

two periods should be different because it requires time periods for the endogenous variables to hit the steady-state after the macroeconomic time path guesses have hit their steady-state.

3. Given initial condition $\hat{\boldsymbol{\Gamma}}_1$, outer-loop guesses for the aggregate time paths $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i\}$, solve for the inner loop lifetime decisions of every household that will be alive across the time path $\{n_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$.

   (a) Given time path guesses $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i\}$, solve for each household's lifetime decisions $\{\hat{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j, $E+1 \leq s \leq E+S$, and $1 \leq t \leq T_2 + S - 1$.

      i. In the transition path equilibrium solution method, the household problem can be solved with a multivariate root finder solving the $2S$ equations and unknowns at once for all j and $1 \leq t \leq T_2 + S - 1$, as opposed to the two-stage method for the steady-state solution described in Section 13.2. Use $2S$ household Euler equations (12.2), (12.3), and (12.4) to solve for each household's $2S$ lifetime decisions.

      ii. If one solves for each household's problem serially from the oldest households alive in period $t = 1$ to the youngest and then for every household born in period $t = 1, 2, ...T_2 - 1$, one can use the equilibrium guesses of the previous generation as initial guesses for the solver. This speeds up computation further and makes the initial guess for the highly nonlinear system of equations start closer to the solution value.

4. Given partial equilibrium household nonsteady-state solutions $\{n_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ for all j and $1 \leq t \leq T_2$ based on macroeconomic variable time path guesses $\{\boldsymbol{r}^i, \hat{\boldsymbol{w}}^i, \hat{\boldsymbol{BQ}}^i\}$, check the errors across the three time paths in the four equations that characterize each of the macroeconomic variable guesses.

   (a) If we substitute the two market clearing conditions (12.8) and (12.9), and the firm's production function (12.5) into the firm's first order condition for capital demand (7.5), we get an expression in which household decisions $\{\bar{n}_{j,s,t}, \hat{b}_{j,s+1,t+1}\}_{s=E+1}^{E+S}$ imply a values for the interest rate $\boldsymbol{r}^{i'}$.

$$r_t^{i'} = (Z_t)^{\frac{\varepsilon-1}{\varepsilon}}\left[\gamma\frac{\hat{Y}_t}{\hat{K}_t}\right]^{\frac{1}{\varepsilon}} - \delta$$

$$\text{where}\quad \hat{Y}_t = Z_t\left[(\gamma)^{\frac{1}{\varepsilon}}(\hat{K}_t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)^{\frac{1}{\varepsilon}}(\hat{L}_t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\text{and}\quad \hat{L}_t = \sum_{s=E+1}^{E+S}\sum_{j=1}^{J}\hat{\omega}_{s,t}\lambda_j e_{j,s} n_{j,s}$$

$$\text{and}\quad \hat{K}_t = \frac{1}{1+\tilde{g}_{n,t}}\sum_{s=E+2}^{E+S+1}\sum_{j=1}^{J}\Big(\hat{\omega}_{s-1,t-1}\lambda_j\bar{b}_{j,s} + i_s\hat{\omega}_{s,t-1}\lambda_j\bar{b}_{j,s}\Big)$$

The error for this variable is the percent difference between the initial guess for the interest rate $\bar{r}^i$ and the steady-state interest rate implied by household optimization based on the initial guess $\bar{r}^{i'}$.

$$error_r = \frac{\bar{r}^{i'} - \bar{r}^i}{\bar{r}^i}$$

14.3 Baseline Nonsteady-state Results

[TODO: Put baseline TPI results here.]

Part VII

Calibration and International Options

Chapter 15

Calibration

Chapter 16

Small Open Economy Option

Appendices

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