2 Oct 2018

Gravity Models

$\{X_i, U_i\}_{i=1}^{N}$ is as iid random sequence

X_i : Latitude, longitude, GDP, Democracy, WTO etc.
U_i : $U_i = (A_i, B_i)$ A_i propensity to export
latent or unobserved by econometrician B_i propensity to import

For each $N(N-1)$ ordered dyads we obs.

$$Y_{ij} = \tilde{h}(\underbrace{X_i, X_j}_{\text{obs.}}, \underbrace{U_i, U_j, V_{ij}}_{\text{unobs.}})$$

$V_{ij} \sim$ iid and independent of every thing else.

exports from i to j

Gravity Model

R_{ij} is a vector of functions of X_i, X_j

$$Y_{ij} = \exp(R_{ij}'\theta)\underline{A_i B_j} V_{ij}$$

how to deal there?

Head + Mayer (2014)
Handbook chapter...

Assume:

$$\mathbb{E}[Y_{ij} | X_i, X_j] = \exp(R_{ij}'\theta_0) \cdot E[A_i B_j V_{ij} | X_i, X_j]$$

$$\Rightarrow \quad \mathbb{E}[A_i B_j V_{ij} | X_i, X_j] =$$

$$\mathbb{E}[A_i | X_i, X_j] \cdot \mathbb{E}[B_j | X_i, X_j] \cdot \mathbb{E}[V_{ij} | X_i, X_j]$$

$$= \underset{\overset{?}{=}\ \mathbb{E}[A_i] = 1}{\mathbb{E}[A_i | X_i]} \cdot \underset{\overset{?}{=}\ \mathbb{E}[B_j] = 1}{\mathbb{E}[B_j | X_j]} \cdot 1 \qquad \text{(A)}$$

for now assume: (A) s.t.

$$\boxed{\mathbb{E}[Y_{ij} | X_i, X_j] = \exp(R_{ij}'\theta_0)}$$

→ focus on inference issues raised by dyadic data

→ more on identification tomorrow

Example: Santos Silva + Tenreyro (2006, RESTAT)
"Log of Gravity"
Classic Paper in Economics according to Google Scholar.

Recipe: $$\boxed{\mathbb{E}[Y_{ij} \mid X_i, X_j] = \exp(R_{ij}'\theta_0)} \quad (*)$$

cf., Tinbergen (1962)

$$\mathbb{E}[\log Y_{ij} \mid X_i, X_j] = R_{ij}'\theta_0$$

① $$l_{ij}(\theta) = Y_{ij} R_{ij}'\theta - \exp(R_{ij}'\theta)$$

↳ log-likelihood of a poisson R.V. w/ mean $\exp(R_{ij}'\theta)$

② choose $\hat{\theta}$ to maximize the pseudo-composite log-likelihood

$$L_N(\theta) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} l_{ij}(\theta)$$

in practice in Stata: poisson Y R , r

Agenda

1.) consistency not so hard to show

2.) open questions

a) asymptotic normality ?

b) limiting variance ?

} how do we calculate standard errors

Observations:

$$l_N(\theta) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} l_{ij}(\theta)$$

dependence across any pair of summands sharing at least one index

Mean value Theorem:

$$\sqrt{N}(\hat{\theta} - \theta_0) = \underbrace{\left[-H_N(\bar{\theta})\right]^{+}}_{\text{inverse Hessian}} \underbrace{\sqrt{N}\, S_N(\theta_0)}_{\text{"score"}}$$

1   2

$$S_N(\theta) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq i} S_{ij}(\theta) \qquad S_{ij}(\theta) = \frac{\partial l_{ij}(\theta)}{\partial \theta}$$

① LLN $\xrightarrow{p} \Gamma_0^{-1}$

② CLT $\xrightarrow{D}$? $\frac{1}{N}\frac{1}{N-1}\sum_{i=1}^{N}\sum_{j\neq i} S_{ij}(\theta_0)$

$$\sqrt{N}(\hat{\theta}-\theta) = \Gamma_0^{-1}\underbrace{\sqrt{N}\, S_N(\theta_0)}_{*} + o_p(1)$$

(*) is NOT the sum of N independent random variables, so we cannot directly apply a CLT.

(**) it IS the sum of $N(N-1)$ R.V. w/ dependence across them (e.g. $S_{ij}(\theta_0)$ covaries w/ $S_{ik}(\theta_0)$). → existing practice ignores this.

Theory of U-Statistics

- Hoeffding (1948)
- van der Vaart (1998)

Kendall's Tau

$$K_N = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} \operatorname{sgn}\{(X_i - X_j)(Y_i - Y_j)\}$$

$\binom{N}{2} = \frac{1}{2} N(N-1)$

non-parametric, scale free, measure of correlation

$$\operatorname{sgn} = \begin{cases} 1 & \text{if pos} \\ 0 & \text{if zero} \\ -1 & \text{if neg} \end{cases}$$

Classic example of a U-statistic.

$$U_N = \binom{N}{m}^{-1} \sum_{i \in C_{m,N}} h(X_{i_1}, \ldots, X_{i_m})$$

$\{X_i\}_{i=1}^{N}$ is a random sample

$C_{m,N}$ denotes the set of all combinations of indices of size m drawn from $\{1, 2, \ldots, N\}$

m is the <u>order</u> of the statistic

Big Picture

U_N constructed from random sample....

... but dependence across summands ....

... can't directly apply a CLT to U_N

Idea: approximate U_N w/ $\hat{U}_N$ where

$\hat{U}_N$ is a sum of iid R.V. to which

we can apply a CLT.

① $\sqrt{N}\,\hat{U}_N \xrightarrow{D} N(0, \Lambda)$ CLT

② $N \cdot \mathbb{E}[(U_N - \hat{U}_N)^2] \to 0$ convergence in mean square

$\Rightarrow \sqrt{N}\, U_N \xrightarrow{D} N(0, \Lambda)$

Hajek Projection

$$U_N = \binom{N}{2}^{-1} \sum_{i<j} h(X_i, X_j)$$

$$\mathbb{E}[U_N] = \theta_0$$

Hajek Projection is (approx. U_N w/ $\sum_{i=1}^{N} g_i(X_i)$)

$$\boxed{\hat{U}_N - \theta_0 = \frac{2}{N} \sum_{i=1}^{N} \{ \bar{h}_1(X_i) - \theta_0 \}}$$

w/ $\bar{h}_1(x) = E[h(x, X_j)]$

$$\Omega_1 = \mathbb{E}[(\bar{h}_1(X_i) - \theta_0)(\bar{h}_1(X_i) - \theta_0)]$$

$$= E[(h(X_i, X_j) - \theta_0)(h(X_i, X_k) - \theta_0)]$$

CLT

$$\sqrt{N}(\hat{U}_N - \theta_0) \xrightarrow{D} N(0, 4\Omega_1)$$

We can show that $N\,\mathbb{E}[(\hat{U}_N - U_N)^2] \to 0$
as $N \to \infty$

$$\sqrt{N}(U_N - \theta_0) \xrightarrow{D} N(0, 4\Omega_1)$$

<u>Variance estimation</u>

$$\hat{h}_1(X_i) = \frac{1}{N-1} \sum_{j \neq i} h(X_i, X_j)$$

$$\hat{\Omega}_1 = \frac{1}{N} \sum_{i=1}^{N} (\hat{h}_1(X_i) - \hat{\theta})(\hat{h}_1(X_i) - \hat{\theta})'$$

$$\hat{\theta} = U_N$$

Application to Dyadic Regression. <u>Recall</u>:

$$S_N(\theta_0) = \frac{1}{N}\frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} s_{ij}(\theta_0)$$

$$= \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} \underbrace{\left\{ \frac{s_{ij}(\theta_0) + s_{ji}(\theta_0)}{2} \right\}}_{\bar{s}_{ij}(\theta_0)}$$

now symmetric in its index

in the Poisson case $S_{ij}(\theta_0)$ is

$$S_{ij}(\theta_0) = \left(Y_{ij} - \exp(R_{ij}'\theta_0)\right) R_{ij}$$

$R_{ij} = r(X_i, X_j)$, but Y_{ij} varies at the dyad level $\Rightarrow$ $S_N(\theta_0)$ is <u>NOT</u> a U-statistic

<u>Recall</u>: $Y_{ij} = \tilde{h}(X_i, X_j, U_i, U_j, V_{ij})$

$$\Rightarrow \mathbb{E}[Y_{ij} \mid X_i, X_j, U_i, U_j] = h(X_i, X_j, U_i, U_j)$$

$$S_{ij}(\theta_0) = \left(h(X_i, X_j, U_i, U_j) - \exp(R_{ij}'\theta_0)\right) R_{ij} \quad ①$$

$$+ \big(\underbrace{Y_{ij} - h(X_i, X_j, U_i, U_j)}_{\varepsilon_{ij}}\big) R_{ij} \quad ②$$

① depends on X_i, U_i and X_j, U_j alone ... now dyadic-specific R.V.

② not $\mathbb{E}[\varepsilon_{ij}\varepsilon_{ik}] = 0$ (hooray!)

High Level Summary

$$S_N(\theta_0) = \binom{N}{2}^{-1} \sum_{i<j} \bar{s}_{ij}(\theta_0)$$

$$= \underbrace{V_N(\theta_0)}_{\text{U-Statistic}} + \underbrace{T_N(\theta_0)}_{\text{sum of } \binom{N}{2} \text{ conditionally independent R.V.}}$$

$Cov(T_N, V_N) = 0$

Skipping steps, but don't worry!

$$\boxed{\begin{aligned} \sqrt{N}\, S_N(\theta_0) &\xrightarrow{D} N(0, 4\Omega_1) \\ \Omega_1 &= \mathbb{E}\left[\bar{s}_{ij}(\theta_0)\, \bar{s}_{ik}(\theta_0)'\right] \end{aligned}}$$

$$\mathbb{V}(S_N(\theta_0)) = O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right)$$

Recipe

① pseudo composite poisson MLE of Y_{ij} onto R_{ij} to get $\hat{\theta}$ (as in Santos Silva + Tenreyro paper.

② $\hat{\Gamma}$ w/ $H_N(\hat{\theta})$ (numerical Hessian)
(most poisson regression programs w/ provide this)

③ $$\hat{\bar{S}}_{1i}(\hat{\theta}) = \frac{1}{N-1} \sum_{j \neq i} \bar{S}_{ij}(\hat{\theta})$$

$$\bar{S}_{ij}(\hat{\theta}) = (Y_{ij} - \exp(R_{ij}'\hat{\theta}))R_{ij} + (Y_{ji} - \exp(R_{ji}'\hat{\theta}))R_{ji}$$

compute $$\hat{\Omega}_1 = \frac{1}{N} \sum_{i=1}^{N} \hat{\bar{S}}_{1i} \hat{\bar{S}}_{1i}'$$

④ Base inference on

$$\hat{\theta} \sim N\left(\theta_0, \frac{\hat{\Gamma}^{-1} \hat{\Omega}_1 \hat{\Gamma}^{-1\prime}}{N}\right)$$