

Beta Model & Network Simulation

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Introduction

- In practice methods of inference used in “network science” are decidedly heuristic/approximate.
- Widespread awareness among researchers of the lack of a coherent body of large network distribution theory.

Introduction (continued)

- Is the network in hand especially “transitive”?
 - Compare with Erdos-Renyi random graph; but are measured differences statistically significant? (how do we conceptualize “sampling error”)
 - Compare with a large set of (empirical) reference graphs. Is the graph of interest usual (cf., Milo et al., 2002)?
 - Combine an ad hoc and/or approximate variance estimate with a normal reference distribution (w/o limit theory it is difficult to evaluate this approach; but see earlier lectures)

Inference: Exact w/ Strong Null

- Blitzstein and Diaconis (2011) – additional work in both machine learning and statistics.
- Look at a reference set of graphs (e.g., all graphs with degree sequences identical to the graph of interest)
 - Is transitivity (for example) in the graph *in hand* high relative to this reference group? (exact p-value approach);
 - Computational challenge: how to enumerate, or draw uniformly, from reference graph distribution.

Inference: Asymptotic

- Earlier lecture: Bickel, Chen & Levina and Bhattacharya and Bickel (2015):
 - Derive limit theory for network statistics (specifically normalized subgraph counts);
 - Challenge is also computational – both statistics and their variance estimates hard to construct.

Beta Model

- Models with network externalities are attractive because
 - they capture what is believed to be an *a priori* important feature of link formation;
 - they generate clustering, which we observe in real word networks.

Beta Model (continued)

- An alternative (ideally complementary) way to generate clustering is to introduce unobserved, *agent-level*, heterogeneity.
 - beta model: $D_{ij} = 1 (A_i + A_j - U_{ij} \geq 0)$;
 - A_i measures attractiveness, trustworthiness, productivity etc;
 - Distribution of \mathbf{A} is unrestricted; components of \mathbf{U} are i.i.d. (logistic).
- cf. “state dependence vs. heterogeneity” in dynamic discrete choice analysis (Heckman, 1978; 1981a-c; Chamberlain, 1985).

Beta Model (continued)

- Assuming U_{ij} i.i.d. logistic yields a link probability of

$$\Pr(D_{ij} = 1 | \mathbf{A}) = \frac{\exp(A_i + A_j)}{1 + \exp(A_i + A_j)} = \frac{\exp(W'_{ij}\mathbf{A})}{1 + \exp(W'_{ij}\mathbf{A})}$$

with W_{ij} the $N \times 1$ vector with a one for its i^{th} and j^{th} elements and zeros elsewhere.

- Choosing $A_i = -\frac{1}{2} \ln\left(\frac{p}{1-p}\right)$ for $i = 1, \dots, N$ yields the Erdos-Renyi random graph model.

Beta Model (continued)

The likelihood, $\Pr(\mathbf{D} = \mathbf{d} | \mathbf{A})$, includes $\binom{N}{2}$ conditional independent components:

$$\Pr(\mathbf{D} = \mathbf{d} | \mathbf{A}) = \prod_{i=1}^N \prod_{j < i} \left[\frac{\exp(W'_{ij}\mathbf{A})}{1 + \exp(W'_{ij}\mathbf{A})} \right]^{d_{ij}} \left[\frac{1}{1 + \exp(W'_{ij}\mathbf{A})} \right]^{1-d_{ij}}.$$

...but “only” N parameters.

Model is non-standard since the dimension of the parameter space grows with N .

Beta Model (continued)

- Manipulating the likelihood gives the exponential family representation

$$\Pr(\mathbf{D} = \mathbf{d} | \mathbf{A}) = c(\mathbf{A}) \exp(T(\mathbf{d})' \mathbf{A}) \quad (1)$$

where

$$T(\mathbf{d}) = \left(d_{1+} \cdots d_{N+} \right)' = \mathbf{d}_+.$$

- The network's *degree sequence*, is a sufficient statistic for \mathbf{A} .

Beta Model (continued)

- The beta model allows for networks with arbitrary degree distributions.
- Despite its simplicity it is reasonable flexible (N parameters) and provides a useful benchmark model for hypothesis testing purposes.

Beta Model (continued)

- Let $\mathbb{D}_{N,\mathbf{d}_+}$ denote the set of all networks with N agents and degree sequence $\mathbf{D}_+ = \mathbf{d}_+$.
- Let $|\mathbb{D}_{N,\mathbf{d}_+}|$ denote the cardinality of $\mathbb{D}_{N,\mathbf{d}_+}$.
 - $|\mathbb{D}_{N,\mathbf{d}_+}|$ is generally huge, even for small N .
- Under the β -model the probability distribution of networks conditional on their degree sequence is uniform:

$$\Pr(\mathbf{D} = \mathbf{d} | \mathbf{d} \in \mathbb{D}_{N,\mathbf{d}_+}) = \frac{1}{|\mathbb{D}_{N,\mathbf{d}_+}|}.$$

Testing

- Let $f(\mathbf{D})$ be some function of the adjacency matrix
 - examples: transitivity index, diameter, number of K-length paths etc.
- Let $f(\mathbf{d})$ be the value of the statistic in the observed network.
- We seek to evaluate

$$\Pr(f(\mathbf{D}) \leq f(\mathbf{d}) | \mathbf{D} \in \mathbb{D}_{N, \mathbf{d}_+}) = \frac{\sum_{\mathbf{v} \in \mathbb{D}_{N, \mathbf{d}_+}} \mathbf{1}(f(\mathbf{v}) \leq f(\mathbf{d}))}{|\mathbb{D}_{N, \mathbf{d}_+}|}. \quad (2)$$

Testing: Intuition

- If the probability that measured transitivity, in a network randomly drawn from the null distribution, lies above observed transitivity is very low, we take that as evidence against the β -model.

Testing

- This approach to testing is
 - very precise about its description of the null hypothesis;
 - exact.
- no alternative hypothesis is specified...
- ...however the choice of statistic should be guided by researcher intuitions about what departures from the null model are of particular concern.

Sampling from \mathbb{D}_{N,d_+}

- Direct enumeration of all the elements of \mathbb{D}_{N,d_+} is generally not feasible.
- Need a method of sampling from \mathbb{D}_{N,d_+} uniformly and also estimating its size (implement an approximation of the ideal test).

Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ (continued)

- Blitzstein and Diaconis (2010) develop a sequential importance sampling algorithm for uniformly sampling from $\mathbb{D}_{N,\mathbf{d}_+}$
- Two challenges:
 - how to generate a random draw from $\mathbb{D}_{N,\mathbf{d}_+}$;
 - how to do so uniformly (importance weights).

Graphical Integer Sequences

- To construct \mathbf{D} we begin with a matrix of zeros and sequentially add links to it until its rows and columns sum to the target degree sequence.
- Problem is that unless links are added carefully it is easy to get “stuck” (cf., Snijders, 1991).
- The key is to check whether residual degree sequences are graphical as you add links (avoid dead ends).
- $\mathbf{D}_+ = (2, 2, 1)$ is not graphic

Graphical Integer Sequences (continued)

- Erdos and Gallai (1961) showed \mathbf{D}_+ is graphical if and only if $\sum_{i=1}^N D_{i+}$ is even and

$$\sum_{i=1}^k D_{i+} \leq k(k-1) + \sum_{i=k+1}^N \min(k, D_{i+}) \text{ for each } k \in \{1, \dots, N\}.$$

Graphical Integer Sequences (continued)

Necessity:

- even: if i is linked to j , then the link is counted in both D_{i+} and D_{j+} .
- For any set S of k agents, there can be at most $\binom{k}{2} = \frac{1}{2}k(k-1)$ links between them (first term).
- For the $N - k$ agents $i \notin S$, then can be at most $\min(k, D_{i+})$ links from i to agents in S .

A Recursive Test

Theorem: (Havel-Hakimi) Let $D_{i+} > 0$, if \mathbf{D}_+ does not have at least D_{i+} positive entries other than i it is not graphical. Assume this condition holds. Let $\tilde{\mathbf{D}}_+$ be a degree sequence of length $N - 1$ obtained by

- [i] deleting the i^{th} entry of \mathbf{D}_+ and
- [ii] subtracting 1 from each of the D_{i+} highest elements in \mathbf{D}_+ (aside from the i^{th} one).

\mathbf{D}_+ is graphical if and only if $\tilde{\mathbf{D}}_+$ is graphical. If \mathbf{D}_+ is graphical, then it has a realization where agent i is connected to any of the D_{i+} highest degree agents (other than i).

Blitzstein and Diaconis Procedure

- Start with lowest degree agent (with at least one link).
- (Randomly) Link this agent with high degree agents.
- A one is subtracted from the lowest degree agent's, as well as her chosen partners', degree counts.
- Continue until the **residual degree sequence** is zero.

Blitzstein and Diaconis Procedure (continued)

- Consider the example

$$\begin{aligned} (3, 2, 2, 2, 1) &\rightarrow (3, 1, 2, 2, 0) \rightarrow (2, 0, 2, 2, 0) \\ &\rightarrow (1, 0, 2, 1, 0) \rightarrow (0, 0, 1, 1, 0) \rightarrow (0, 0, 0, 0, 0) \end{aligned}$$

- Now imagine that in the fourth step instead of linking agent 1 with agent 3, agents 1 and 4 were linked.
 - this would have resulted in a residual degree sequence of $(0, 0, 2, 0, 0)$, which is not graphic.
 - algorithm doesn't allow this to occur by checking for whether the residual degree sequence associated with a candidate link is graphical.

Blitzstein and Diaconis Procedure (continued)

- Let $(\oplus_{i_1, \dots, i_k} \mathbf{D}_+)$ be the vector obtained by adding a one to the i_1, \dots, i_k elements of \mathbf{D}_+ :

$$(\oplus_{i_1, \dots, i_k} \mathbf{D}_+)_j = \begin{cases} D_{j+} + 1 & \text{for } j \in \{i_1, \dots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

- Let $(\ominus_{i_1, \dots, i_k} \mathbf{D}_+)$ be the vector obtained by subtracting one from the i_1, \dots, i_k elements of \mathbf{D}_+ :

$$(\ominus_{i_1, \dots, i_k} \mathbf{D}_+)_j = \begin{cases} D_{j+} - 1 & \text{for } j \in \{i_1, \dots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

Blitzstein and Diaconis Procedure (continued)

Algorithm: A sequential algorithm for constructing a random graph with degree sequence $\mathbf{D}_+ = (D_{1+}, \dots, D_{N+})'$ is

1. Let \mathbf{G} be an empty adjacency matrix.
2. If $\mathbf{D}_+ = \mathbf{0}$ terminate with output \mathbf{G}
3. Choose the agent i with minimal positive degree D_{i+} .
4. Construct a list of candidate partners

$$J = \left\{ j \neq i : \mathbf{G}_{ij} = \mathbf{G}_{ji} = 0 \text{ and } \ominus_{i,j} \mathbf{D}_+ \text{ graphical} \right\}.$$

5. Pick a partner $j \in J$ with probability proportional to its degree in \mathbf{D}_+ .
6. Set $\mathbf{G}_{ij} = \mathbf{G}_{ji} = 1$ and update \mathbf{D}_+ to $\Theta_{i,j}\mathbf{D}_+$.
7. Repeat steps 4 to 6 until the degree of agent i is zero.
8. Return to step 2.

The input for the algorithm is the target degree sequence \mathbf{D}_+ and the output is an undirected adjacency matrix \mathbf{G} with $\mathbf{G}'\iota = \mathbf{D}_+$.

Importance Weights

- The Blitzstein and Diaconis (2010) procedure delivers a random draw from $\mathbb{D}_{N, \mathbf{d}_+}$, but not a *uniform* random draw.
- Construct importance weights in order to compute expectations using the correct reference distribution.
- Let $\mathcal{Y}_{N, \mathbf{d}_+}$ denote the set of all possible sequences of links generated by the algorithm given input $\mathbf{D}_+ = \mathbf{d}_+$.

Importance Weights(continued)

- Let $\mathcal{G}(Y)$ be the adjacency matrix induced by link sequence Y .
 - Let Y and Y' are equivalent if $\mathcal{G}(Y) = \mathcal{G}(Y')$.
- We can partition $\mathcal{Y}_{N, \mathbf{d}_+}$ into a set of equivalence classes whose number coincides with the cardinality of $\mathbb{D}_{N, \mathbf{d}_+}$.

Importance Weights(continued)

- Let $c(Y)$ denote the number of possible link sequences produced by the algorithm that produce Y 's end point adjacency matrix.
- Let i_1, i_2, \dots, i_M be the sequence of agents chosen in step 3 of the algorithm in which Y is the output.

Importance Weights(continued)

- Let a_1, \dots, a_m be the degrees of i_1, \dots, i_M at the time when each agent was *first* selected in step 3.
- Blitzstein and Diaconis show that:

$$c(Y) = \prod_{k=1}^M a_k!$$

Importance Weights(continued)

- Let $\sigma(Y)$ be the probability that the algorithm produces link sequence Y .
- $\sigma(Y)$ is easy to compute:
 - each time a link in step 5 is chosen we record the probability with which it was chosen.
 - this equals the residual degree of the chosen agent divided by the sum of the residual degrees of all agents in the choice set.
 - the product of all these probabilities equals $\sigma(Y)$.

Importance Weights(continued)

Let $f(\mathbf{G})$ be some function of the adjacency matrix and consider the expected value

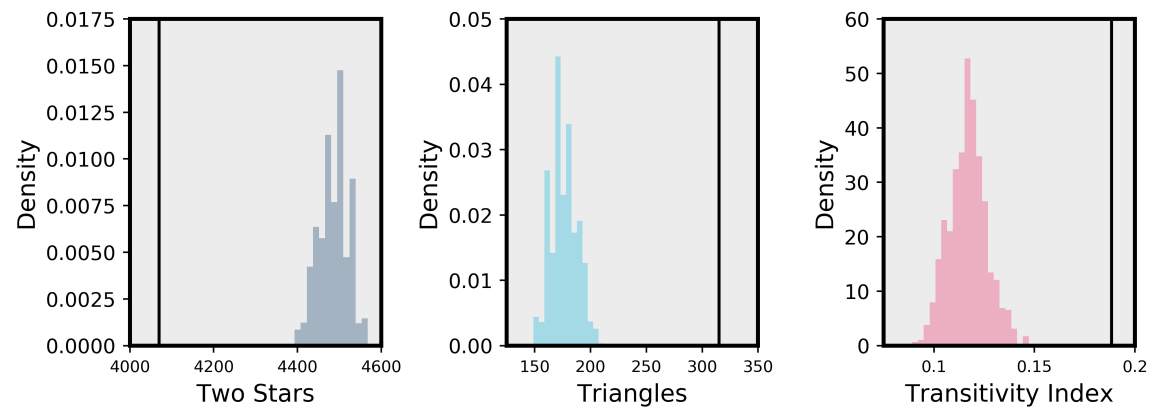
$$\begin{aligned}\mathbb{E} \left[\frac{\pi(\mathcal{G}(Y))}{c(Y) \sigma(Y)} f(\mathcal{G}(Y)) \right] &= \sum_{y \in \mathbb{Y}_{N,d}} \frac{\pi(\mathcal{G}(y))}{c(y) \sigma(y)} f(\mathcal{G}(y)) \sigma(y) \\ &= \sum_{y \in \mathbb{Y}_{N,d}} \frac{\pi(\mathcal{G}(y))}{c(y)} f(\mathcal{G}(y)) \\ &= \sum_{\mathbf{g} \in \mathbb{D}_{N,d}} \sum_{\{y: G(y)=\mathbf{g}\}} \frac{\pi(\mathbf{g})}{c(y)} f(\mathbf{g}) \\ &= \sum_{\mathbf{g} \in \mathbb{D}_{N,d}} \pi(\mathbf{g}) f(\mathbf{g}) = \mathbb{E}_{\pi} [f(\mathbf{G})].\end{aligned}$$

Set $\pi(\mathbf{g}) = f(\mathbf{g}) = 1$ to estimate $|\mathbb{D}_{N,d}|$.

Importance Weights (continued)

- The ratio $\pi(G(Y)) / c(Y) \sigma(Y)$ is called the likelihood ratio or the **importance weight**.
- Our random network draws are not uniform from the set of interest.
- The importance weights correct for the fact that we are sampling from the wrong distribution.

Nyakatoke Example



Blitzstein and Diaconis Wrap-Up

- While using the β -model as a reference model is restrictive it
 - is a natural starting point for hypothesis testing;
 - suggests that an investment in computation skills is likely to be valuable to anyone doing empirical work.
- It might be of interest to condition on additional features of the network in hand...