Testing for Externalities in Network Formation by Simulation

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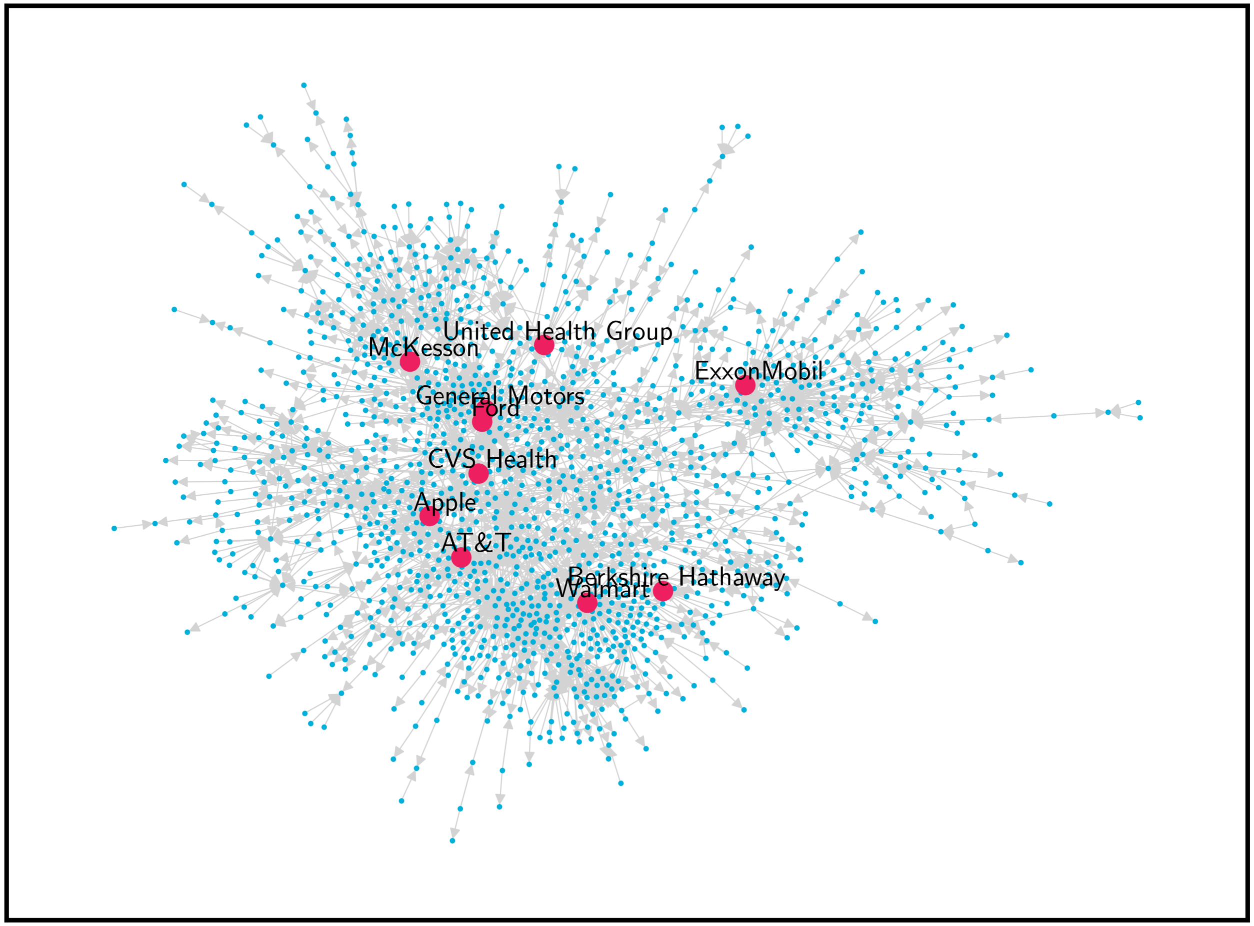
US Buyer-Supplier Network, 2015



<u>Two classes of network formation models</u>

<u>Null model</u>: the utility i generates by linking with j depends upon ego (i) and alter (j) attributes alone (attributes may be observed or unobserved).

- Stochastic Block Model

- β-Model

<u>Alternative model</u>: the utility generated by an i to j link *additionally varies* with the presence or absence of *other links* in the network.

- Strategic models

Research question

Can we determine whether the network in hand was generated according to null or alternative model?

Very little prior work in this space.

Why I care and you should (might?) too

With strategic behavior:

1. There may be multiple equilibrium network configurations.

2. The observed configuration may not maximize welfare.

3. Vertex removal (and/or local re-wirings) can trigger a process of link revision global in scope.

The effect of policies on the form of a network are very different under the null vs. the alternative.

Utility

Random utility framework *a la* McFadden (1973).

Let $\mathbf{d} \in \mathbb{D}$ be an undirected adjacency matrix. The utility agent i gets from some feasible network wiring $\mathbf{d}$ is

$$\nu_i\left(\mathbf{d}, \mathbf{U}; \gamma_0, \delta_0\right) = \sum_j d_{ij}\left[A_i + B_j + \gamma_0 s_{ij}\left(\mathbf{d}\right) - U_{ij}\right],$$

where:

1. A_i is a “extroversion effect”;

2. B_j is a “popularity effect”;

Utility (continued)

1. $s_{ij}(\mathbf{d}) = s_{ij}(\mathbf{d} - ij) = s_{ij}(\mathbf{d} + ij)$ is a network/strategic effect; can be used to model:

   (a) rich-get-richer: $s_{ij}(\mathbf{d}) = d_{ji}$;

   (b) transitivity: $s_{ij}(\mathbf{d}) = \sum_k d_{ik} d_{jk}$;

2. $\left\{U_{ij}\right\}_{i \neq i}$ idiosyncratic utility shifter (i.i.d. logistic)

Pelican and Graham (2019) work with a much more general model.

Utility (continued)

The marginal utility for agent i associated with (possible) edge (i, j) is

$$MU_{ij}(\mathbf{D}) = \begin{cases} \nu_i(\mathbf{D}) - \nu_i(\mathbf{D} - ij) & \text{if } D_{ij} = 1 \\ \nu_i(\mathbf{D} + ij) - \nu_i(\mathbf{D}) & \text{if } D_{ij} = 0 \end{cases} \quad (1)$$

recalling that $\mathbf{D} - ij$ is the adjacency matrix associated with the network obtained after deleting edge (i, j)...

...and $\mathbf{D} + ij$ the one obtained via link addition.

Equilibrium

Network is undirected.

It is convenient to assume utility is transferable.

Use *pairwise stable with transfers* equilibrium concept from Bloch and Jackson (2006).

(Pairwise stability with Transfers) The network $G(\mathcal{V}, \mathcal{E})$ is pairwise stable with transfers if
(i) $\forall (i, j) \in \mathcal{E}(G), MU_{ij}(\mathbf{D}) + MU_{ji}(\mathbf{D}) \geq 0$
(ii) $\forall (i, j) \notin \mathcal{E}(G), MU_{ij}(\mathbf{D}) + MU_{ji}(\mathbf{D}) < 0$

Equilibrium (continued)

The marginal utility agent i gets from a link with j is

$$MU_{ij}(\mathbf{d}, \mathbf{U}; \gamma_0, \delta_0) = A_i + B_j + \gamma_0 s_{ij}(\mathbf{d}) - U_{ij}.$$

Pairwise stability then implies that, conditional on the realizations of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{U}$ and the value of externality parameter γ_0, the observed network must satisfy, for $i = 1, \ldots, N-1$ and $j = i+1, \ldots, N$

$$D_{ij} = \mathbf{1}\left(\tilde{A}_i + \tilde{A}_j + \gamma_0 \tilde{s}_{ij}(\mathbf{d}) \geq \tilde{U}_{ij}\right) \tag{2}$$

with $\tilde{A}_i = A_i + B_i$, $\tilde{s}_{ij}(\mathbf{d}) = s_{ij}(\mathbf{d}) + s_{ji}(\mathbf{d})$ and $\tilde{U}_{ij} = U_{ij} + U_{ji}$.

Defines a system of $\binom{N}{2} = \frac{1}{2}N(N-1)$ nonlinear simultaneous equations

Equilibrium: Fixed-Point Representation

Consider, similar to Miyauchi (2016), the mapping $\varphi(\mathbf{D}) : \mathbb{D}_N \to \mathbb{I}_{\binom{N}{2}}$:

$$\varphi(\mathbf{d}) \equiv \begin{bmatrix} \mathbf{1}\left(\tilde{A}_1 + \tilde{A}_2 + \gamma_0 \tilde{s}_{12}(\mathbf{d}) \geq U_{12}\right) \\ \mathbf{1}\left(\tilde{A}_1 + \tilde{A}_3 + \gamma_0 \tilde{s}_{13}(\mathbf{d}) \geq U_{13}\right) \\ \vdots \\ \mathbf{1}\left(\tilde{A}_{N-1} + \tilde{A}_N + \gamma_0 \tilde{s}_{N-1N}(\mathbf{d}) \geq U_{N-1N}\right) \end{bmatrix}. \qquad (3)$$

The observed adjacency matrix corresponds to the fixed point

$$\mathbf{D} = \mathrm{vech}^{-1}\left[\varphi(\mathbf{D})\right].$$

There may be other $\mathbf{d} \in \mathbb{D}_N$ such that $\mathbf{d} = \mathrm{vech}^{-1}\left[\varphi(\mathbf{d})\right]$.

Existence using Tarski's fixed point theorem (for many $s_{ij}(\mathbf{d})$).

Testing goal: challenges

Goal is to construct a test of the no strategic interaction ($\gamma_0 = 0$) null.

Three key challenges:

1. null is composite – nuisance parameter $\delta = \tilde{\mathbf{A}}$ is high dimensional (worry: size distortion);

2. can't evaluate likelihood under the alternative (worry: how to maximize power?);

3. characterizing/simulating null distribution (worry: feasibility).

Testing goal: solutions

1. Apply exponential family theory (Ferguson, 1967; Lehmann & Romano, 2005).

2. Find *locally* best test:

   (a) derivative of likelihood w.r.t to γ difficult to compute (incompleteness);

   (b) exploit insights from the econometrics of games (e.g., Tamer, 2003; Bajari *et al*. 2010a,b).

3. Use methods for (constrained) network simulation (e.g., Sinclair, 1993)

Constructing the Test

Under the null we have, for $i = 1, \ldots, N-1$ and $j = i+1, \ldots, N$,

$$\Pr\left(D_{ij} = 1\right) = \frac{\exp\left(\tilde{A}_i + \tilde{A}_j\right)}{1 + \exp\left(\tilde{A}_i + \tilde{A}_j\right)},$$

which corresponds to the β-model of network formation.

Probability of $\mathbf{D} = \mathbf{d}$ takes the exponential family form

$$P_0\left(\mathbf{d}; \tilde{\mathbf{A}}\right) = c\left(\tilde{\mathbf{A}}\right) \exp\left(\mathbf{d}'_+ \tilde{\mathbf{A}}\right)$$

with $\mathbf{d}_+ = \left(d_{1+}, \ldots, d_{N+}\right)$ equal to the degree sequence of the network.

Constructing the Test (continued)

Let $\mathbb{D}_{N,\mathbf{d}_+}$ denote the set of all undirected $N \times N$ adjacency matrices with degree counts also equal to $\mathbf{d}_+$.

$\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ denotes the size, or cardinality, of this set.

Constructing the Test (continued)

Under H_0 the conditional likelihood of $\mathbf{D} = \mathbf{d}$ given $\mathbf{D}_+ = \mathbf{d}_+$ is

$$P_0\left(\mathbf{d} \middle| \mathbf{D}_+ = \mathbf{d}_+\right) = \frac{1}{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|}.$$

Under the null of no externalities *all networks with identical degree sequences are equally probable*.

This insight will form the basis of our test.

Constructing the Test (continued)

Let $T(\mathbf{d})$ be some statistic of the adjacency matrix $\mathbf{D} = \mathbf{d}$, say its transitivity index.

Test critical function equals

$$\phi(\mathbf{d}) = \begin{cases} 1 & T(\mathbf{d}) > c_\alpha\left(\mathbf{d}_+\right) \\ g_\alpha\left(\mathbf{d}_+\right) & T(\mathbf{d}) = c_\alpha\left(\mathbf{d}_+\right) \\ 0 & T(\mathbf{d}) < c_\alpha\left(\mathbf{d}_+\right) \end{cases}.$$

We will reject the null if our statistic exceeds some critical value, $c_\alpha\left(\mathbf{d}_+\right)$ and accept it – or fail to reject it – if our statistic falls below this critical value.

Constructing the Test (continued)

The critical value $c_\alpha\left(\mathbf{d}_+\right)$ is chosen to set the rejection probability of our test under the null equal to α (i.e., to control size).

In order to find the appropriate value of $c_\alpha\left(\mathbf{d}_+\right)$ we need to know the distribution of $T\left(\mathbf{D}\right)$ under the null.

This distribution is straightforward to characterize if we proceed conditional on the degree sequence observed in the network in hand.

Constructing the Test (continued)

Under the null all possible adjacency matrices with degree sequence $\mathbf{d}_+$ are equally probable.

The null distribution of $T(\mathbf{D})$ therefore equals its distribution across all these matrices.

By enumerating all the elements of $\mathbb{D}_{N,\mathbf{d}_+}$ and calculating $T(\mathbf{d})$ for each one, we could directly – and exactly – compute this distribution.

In practice this is not (generally) computationally feasible.

Constructing the Test (continued)

If we could efficiently enumerate the elements of $\mathbb{D}_{N,\mathbf{d}_+}$ we would find $c_\alpha\left(\mathbf{d}_+\right)$ by solving

$$\Pr\left(T\left(\mathbf{D}\right) \geq c_\alpha\left(\mathbf{d}_+\right) \middle| \mathbf{D} \in \mathbb{D}_{\mathbf{N},\mathbf{d}_+}\right) = \frac{\sum_{\mathbf{D}\in\mathbb{D}_{\mathbf{N},\mathbf{d}_+}} \mathbf{1}\left(T\left(\mathbf{D}\right) \geq c_\alpha\left(\mathbf{d}_+\right)\right)}{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|}$$

Alternatively we might instead calculate the p-value:

$$\Pr\left(T\left(\mathbf{D}\right) \geq T\left(\mathbf{d}_+\right) \middle| \mathbf{D} \in \mathbb{D}_{\mathbf{N},\mathbf{d}_+}\right) = \frac{\sum_{\mathbf{D}\in\mathbb{D}_{\mathbf{N},\mathbf{d}_+}} \mathbf{1}\left(T\left(\mathbf{D}\right) \geq T\left(\mathbf{d}_+\right)\right)}{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|}$$

Choosing $T(\mathbf{d})$

Pelican and Graham (2019) show how to choose $T(\mathbf{d})$ to maximize power against local alternatives.

This is hard because one must work with the likelihood of the network under the alternative (which is incomplete).

In practice – as is common with randomization tests – can pick a test statistic intuitively.

For example $T(\mathbf{d})$ might be the transitivity index.

Testing: Intuition

If the probability that measured transitivity, in a network randomly drawn from the null distribution, lies above observed transitivity is very low...

...we take that as evidence against the β-model and "reject".

Testing

- This approach to testing is
  - very precise about its description of the null hypothesis;
  - exact.
- We have motivated this test via a particular alternative (and can optimize power vis-a-vis it), but rejection may occur for many reasons.
- ...at minimum the choice of statistic should be guided by researcher intuitions about what departures from the null model are of particular concern.

Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$

- Direct enumeration of all the elements of $\mathbb{D}_{N,\mathbf{d}_+}$ is generally not feasible.

- Need a method of sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ *uniformly* and also estimating its size.

- We will implement an approximation of the ideal test.

Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ (continued)

- Blitzstein and Diaconis (2011) develop a sequential importance sampling algorithm for (effectively) uniformly sampling from $\mathbb{D}_{N,\mathbf{d}_+}$

- Two challenges:

  - how to generate a random draw from $\mathbb{D}_{N,\mathbf{d}_+}$;

  - how to do so uniformly (importance weights).

Graphical Integer Sequences

- To construct $\mathbf{D}$ we begin with a matrix of zeros and sequentially add links to it until its rows and columns sum to the target degree sequence.

- Problem is that unless links are added carefully it is easy to get "stuck" (cf., Snijders, 1991).

- The key is to check whether residual degree sequences are graphical as you add links (avoid dead ends).

- $\mathbf{D}_+ = (2, 2, 1)$ is not graphic

Graphical Integer Sequences (continued)

- Erdos and Gallai (1961) showed $\mathbf{D}_+$ is graphical if and only if $\sum_{i=1}^{N} D_{i+}$ is even and

$$\sum_{i=1}^{k} D_{i+} \leq k\,(k-1) + \sum_{i=k+1}^{N} \text{min}\Big(k, D_{i+}\Big) \text{ for each } k \in \{1, \ldots, N\}\,.$$

Graphical Integer Sequences (continued)

Necessity:

- even: if i is linked to j, then the link is counted in both D_{i+} and D_{j+}.

- For any set S of k agents, there can be at most $\binom{k}{2} = \frac{1}{2}k\,(k-1)$ links between them (first term).

- For the $N-k$ agents $i \notin S$, there can be at most $\min\left(k, D_{i+}\right)$ links from i to agents in S.

Graphical Integer Sequences (continued)

Sufficiency of the condition is (evidently) much harder to show.

Erdos and Gallai Theorem provides a simple test for graphicality of a degree sequence.

The next theorem, due to Havel (1955) and Hakimi (1962), shows that this test may be applied recursively.

A Recursive Test

Theorem: *(Havel-Hakimi) Let $D_{i+} > 0$, if $\mathbf{D}_+$ does not have at least D_{i+} positive entries other than i it is not graphical. Assume this condition holds. Let $\tilde{\mathbf{D}}_+$ be a degree sequence of length $N-1$ obtained by*

[i] deleting the i^{th} entry of $\mathbf{D}_+$ and

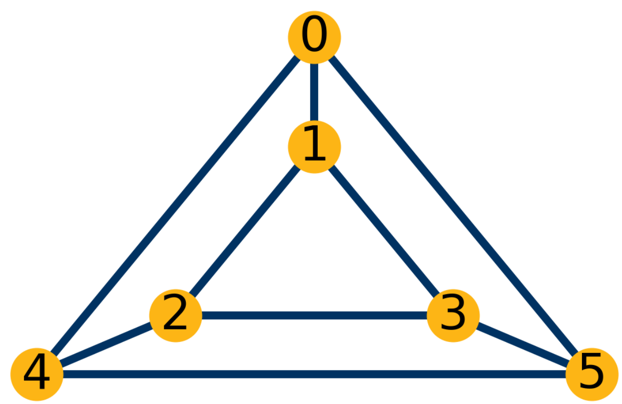
[ii] subtracting 1 from each of the D_{i+} highest elements in $\mathbf{D}_+$ (aside from the i^{th} one).

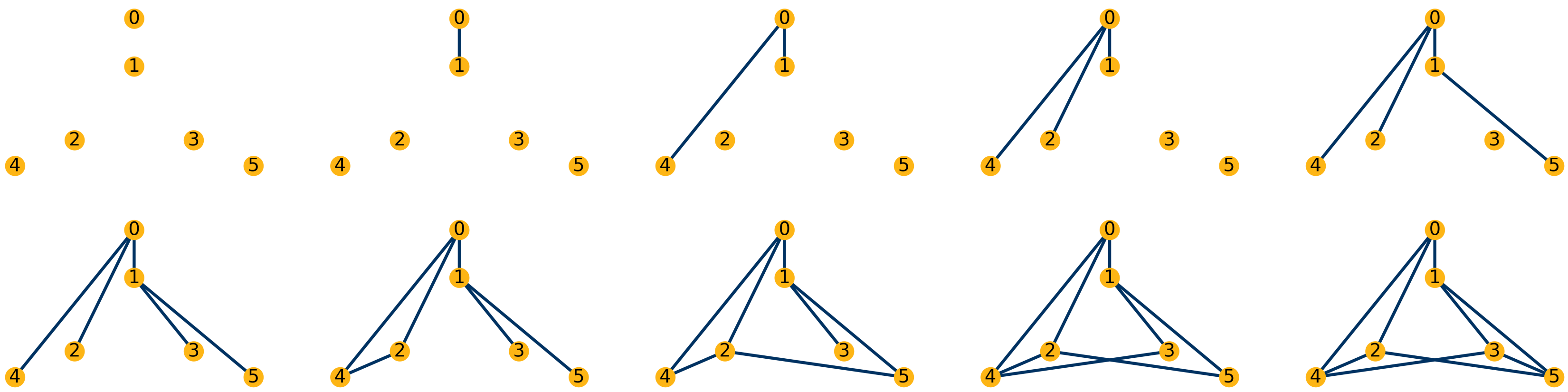
$\mathbf{D}_+$ is graphical if and only if $\tilde{\mathbf{D}}_+$ is graphical. If $\mathbf{D}_+$ is graphical, then it has a realization where agent i is connected to any of the D_{i+} highest degree agents (other than i).

Blitzstein and Diaconis Procedure

- Start with lowest degree agent (with at least one link).
- (Randomly) Link this agent with high degree agents.
- A one is subtracted from the lowest degree agent's, as well as her chosen partners', degree counts.
- Continue until the **residual degree sequence** is zero.

3-regular (i.e., cubic graph)




0
1
2
3
4
5

Blitzstein and Diaconis Procedure (continued)

- Consider the example

$$
\begin{aligned}
(3,3,3,3,3,3) &\to (2,2,3,3,3,3) \to (1,2,3,3,2,3) \to (0,2,2,3,2,3) \\
&\to (0,1,2,3,2,2) \to (0,0,2,2,2,2) \to (0,0,1,2,1,2) \\
&\to (0,0,0,2,1,1) \to (0,0,0,1,0,1) \to (0,0,0,0,0,0)\,.
\end{aligned}
$$

- Now imagine that in the 8th step instead of linking agent 3 with agent 4, agents 4 and 5 were linked.

Blitzstein and Diaconis Procedure (continued)

- This would have resulted in a residual degree sequence of $(0, 0, 0, 2, 0, 0)$, which is not graphic.

- Algorithm doesn't allow this to occur by checking for whether the residual degree sequence associated with a candidate link is graphical.

Blitzstein and Diaconis Procedure (continued)

- Let $\left(\oplus_{i_1,\ldots,i_k}\mathbf{D}_+\right)$ be the vector obtained by adding a one to the $i_1, \ldots, i_k$ elements of $\mathbf{D}_+$:

$$\left(\oplus_{i_1,\ldots,i_k}\mathbf{D}_+\right)_j = \begin{cases} D_{j+} + 1 & \text{for } j \in \{i_1, \ldots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

- Let $\left(\ominus_{i_1,\ldots,i_k}\mathbf{D}_+\right)$ be the vector obtained by subtracting one from the $i_1, \ldots, i_k$ elements of $\mathbf{D}_+$:

$$\left(\ominus_{i_1,\ldots,i_k}\mathbf{D}_+\right)_j = \begin{cases} D_{j+} - 1 & \text{for } j \in \{i_1, \ldots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

Blitzstein and Diaconis Procedure (continued)

Algorithm: A sequential algorithm for constructing a random graph with degree sequence $\mathbf{d}_+ = \left(d_{1+}, \ldots, d_{N+}\right)'$ is

1. Let $\mathbf{D}$ be an empty adjacency matrix.

2. If $\mathbf{D}_+ = \mathbf{0}$ terminate with output $\mathbf{D}$

3. Choose the agent i with minimal positive degree d_{i+}.

4. Construct a list of candidate partners

$$J = \left\{ j \neq i \,:\, \mathbf{D}_{ij} = \mathbf{D}_{ji} = 0 \text{ and } \ominus_{i,j} \mathbf{d}_+ \text{ graphical} \right\}.$$

5. Pick a partner $j \in J$ with probability proportional to its degree in $\mathbf{d}_+$.

6. Set $\mathbf{D}_{ij} = \mathbf{D}_{ji} = 1$ and update $\mathbf{d}_+$ to $\ominus_{i,j}\mathbf{d}_+$.

7. Repeat steps 4 to 6 until the degree of agent i is zero.

8. Return to step 2.

The input for the algorithm is the target degree sequence $\mathbf{d}_+$ and the output is an undirected adjacency matrix $\mathbf{D}$ with $\mathbf{D}'\iota = \mathbf{d}_+$.

Importance Weights

- The Blitzstein and Diaconis (2010) procedure delivers a random draw from $\mathbb{D}_{N,\mathbf{d}_+}$, but not a *uniform* random draw.

- Construct importance weights in order to compute expectations using the correct reference distribution.

- Let $\mathbb{Y}_{N,\mathbf{d}_+}$ denote the set of all possible sequences of links generated by the algorithm given input $\mathbf{d}_+$.

Importance Weights (continued)

- Let $\mathcal{D}(Y)$ be the adjacency matrix induced by link sequence Y.
  - Let Y and Y' are equivalent if $\mathcal{D}(Y) = \mathcal{D}(Y')$.

- We can partition $\mathbb{Y}_{N,\mathbf{d}_+}$ into a set of equivalence classes whose number coincides with the cardinality of $\mathbb{D}_{N,\mathbf{d}_+}$.

Importance Weights (continued)

- Let $c(Y)$ denote the number of possible link sequences produced by the algorithm that produce Y's end point adjacency matrix.

- Let $i_1, i_2, \ldots, i_M$ be the sequence of agents chosen in step 3 of the algorithm in which Y is the output.

Importance Weights (continued)

- Let $a_1, \ldots, a_m$ be the degrees of $i_1, \ldots, i_M$ at the time when each agent was *first* selected in step 3.

- Blitzstein and Diaconis show that:

$$c(Y) = \prod_{k=1}^{M} a_k!$$

Importance Weights (continued)

Consider two equivalent link sequences Y and Y'.

Because links are added to vertices by minimal degree (see Step 3), the sequences $i_1, i_2, \ldots, i_M$ coincide for Y and Y'.

This means that *the exact same links*, albeit perhaps in a different order, are added at each "stage" of the algorithm (i.e., when the algorithm iterates through steps 4 to 7 repeatedly for a given agent).

The number of different ways to add agent i_k's links during such a "stage" is simply $a_k!$ and hence $c(Y) = \prod_{k=1}^{M} a_k!$

Importance Weights (continued)

- Let $\sigma(Y)$ be the probability that the algorithm produces link sequence Y.

- $\sigma(Y)$ is easy to compute:
  - each time a link in step 5 is chosen we record the probability with which it was chosen.
  - this equals the residual degree of the chosen agent divided by the sum of the residual degrees of all agents in the choice set.
  - the product of all these probabilities equals $\sigma(Y)$.

Importance Weights (continued)

Let $T(\mathbf{D})$ be some statistic the adjacency. We have that $\mathbb{E}\left[\frac{\pi(\mathcal{D}(Y))}{c(Y)\sigma(Y)}\mathbf{1}\left(T\left(\right.\right.\right.$
equals

$$\begin{aligned}
&= \sum_{y \in \mathbb{Y}_{N,\mathbf{d}}} \frac{\pi(\mathcal{D}(y))}{c(y)\,\sigma(y)} \mathbf{1}\left(T(\mathcal{D}(Y)) > T(\mathbf{d})\right)\sigma(y) \\
&= \sum_{y \in \mathbb{Y}_{N,\mathbf{d}}} \frac{\pi(\mathcal{D}(y))}{c(y)} \mathbf{1}\left(T(\mathcal{D}(Y)) > T(\mathbf{d})\right) \\
&= \sum_{\mathbf{D} \in \mathbb{D}_{N,\mathbf{d}_+}} \sum_{\{y\,:\,\mathcal{D}(y) = \mathbf{D}\}} \frac{\pi(\mathbf{D})}{c(y)} \mathbf{1}\left(T(\mathbf{D}) > T(\mathbf{d})\right) \\
&= \sum_{\mathbf{D} \in \mathbb{D}_{N,\mathbf{d}_+}} \pi(\mathbf{D})\,\mathbf{1}\left(T(\mathbf{D}) > T(\mathbf{d})\right) \\
&= \mathbb{E}_\pi\left[\mathbf{1}\left(T(\mathbf{D}) > T(\mathbf{d})\right)\right].
\end{aligned}$$

Importance Weights (continued)

Here $\pi(\mathbf{D})$ is the probability attached to the adjacency matrix $\mathbf{D} \in \mathbb{D}_{N,\mathbf{d}_+}$ in the target distribution over $\mathbb{D}_{N,\mathbf{d}_+}$.

The ratio $\pi(\mathcal{D}(Y))/c(Y)\sigma(Y)$ is called the likelihood ratio or the *importance weight.*

We would like $\pi(\mathbf{D}) = 1/\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ for all $\mathbf{D} \in \mathbb{D}_{N,\mathbf{d}_+}$.

If we set $\pi(\mathbf{D}) = 1$ we see that $\mathbb{E}\left[\frac{1}{c(Y)\sigma(Y)}\right] = \left|\mathbb{D}_{N,\mathbf{d}_+}\right|$. This suggests the analog estimator for $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ of

$$\widehat{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|} = \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c(Y_b)\,\sigma(Y_b)}\right] \tag{4}$$

Importance Weights (continued)

These results suggest we estimate a p-value for our test by

$$\widehat{\rho}_{T(\mathbf{G})} = \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c\left(Y_b\right)\sigma\left(Y_b\right)}\right]^{-1} \times \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c\left(Y_b\right)\sigma\left(Y_b\right)}\mathbf{1}\left(T\left(\mathbf{D}_b\right) > T\left(\mathbf{d}\right)\right)\right]$$

An attractive feature is that the importance weights need only be estimated up to a constant.

This feature is useful when dealing with numerical overflow issues that can arise when $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ is too large to estimate.

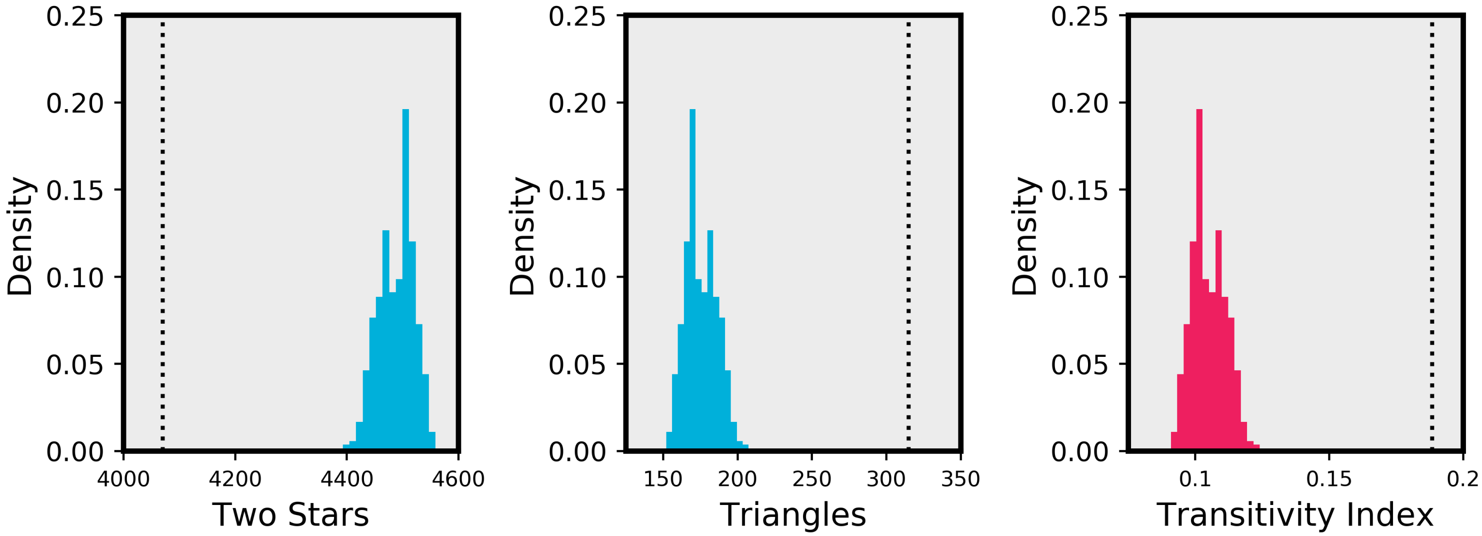
Importance Weights (continued)

- The ratio $\pi\left(\mathbf{D}\left(Y\right)\right)/c\left(Y\right)\sigma\left(Y\right)$ is called the likelihood ratio or the **importance weight.**

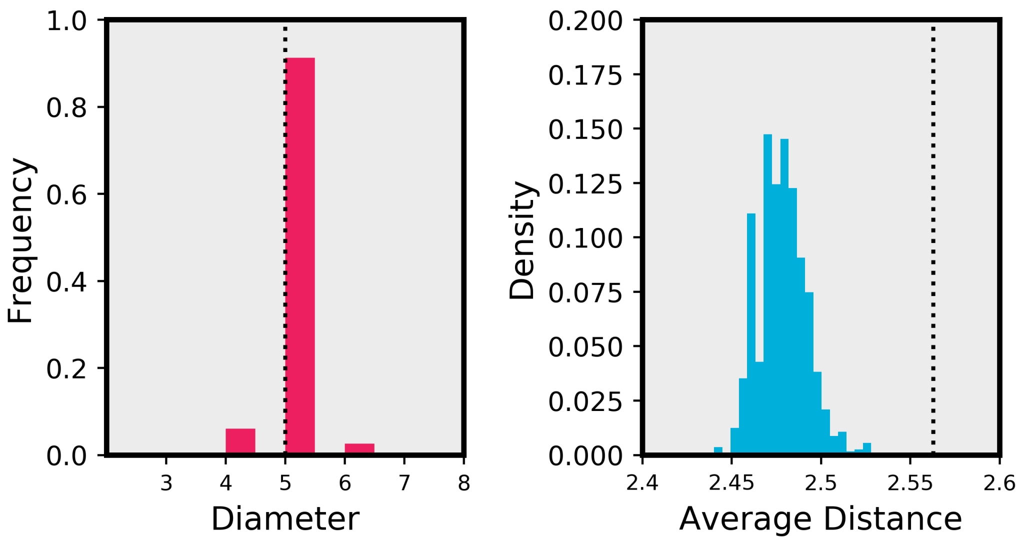
- Our random network draws are not uniform from the set of interest.

- The importance weights correct for the fact that we are sampling from the wrong distribution.

Nyakatoke Example



Nyakatoke Example (continued)



Wrap-Up

- While using the β-model as a reference model is restrictive it
  - is a natural starting point for hypothesis testing;
  - suggests that an investment in computation skills is likely to be valuable to anyone doing empirical work.

- It might be of interest to condition on additional features of the network in hand...

- ...see Pelican and Graham (2019).