

# Using Network Structure to Identify Peer Effects

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## Overview

- Large empirical literature on peer group effects based on the “linear-in-means” model of social interactions (Manski, 1993).
- Quality of research in this area is uneven and has been heavily criticized (e.g., Angrist, 2013).
  - over 20 years since Manski (1993) conditions for identification (and their interpretation) evidently not fully understood by some practitioners.

## Overview (continued)

- Recent work on network games with linear best reply functions (e.g., Jackson and Zenou, 2015; Bramouille, Kranton and D'Amours, 2014).
  - provides micro-foundations for linear-in-means model.
  - facilitates intuitive assessment of conditions for identification.
  - also connected to older literature on input-output models recently explores by Acemoglu and co-authors.

## Key References

- Manski (1993, *Review of Economic Studies*)
- Brock and Durlauf (2001, *Handbook of Econometrics*)
- Bramouille, Djebbari and Fortin (2009, *Journal of Econometrics*)

## Notation

- Let  $G = \text{diag}(\mathbf{D}\iota_N)^{-1} \mathbf{D}$  be the **row-normalized network adjacency matrix**.
  - Note that all rows of this matrix sum to 1 by construction.
  - The matrix is row-stochastic (when graph is connected).
  - Focus on undirected links today, but extension to directed case follows.

## Notation (continued)

- Let  $G_i$  denotes the  $i^{th}$  row of  $G$  and define

$$G_i y = \sum_{j \neq i} G_{ij} y_j \stackrel{def}{=} \bar{y}_{n(i)}$$

$$G_i X = \sum_{j \neq i} G_{ij} X_j \stackrel{def}{=} \bar{X}_{n(i)}.$$

These equal

- the average action of player  $i$ 's peers.
- the average of her peers' attribute vector.

## Utility

- Assume that the utility agent  $i$  receives from action profile  $\mathbf{y}$ , given network structure ( $\mathbf{D}$ ) and agent attributes ( $\mathbf{X}$ ), is

$$\begin{aligned} u_i(\mathbf{y}; \mathbf{D}, \mathbf{X}) &= v_i(\mathbf{D}, \mathbf{X}) y_i - \frac{1}{2} y_i^2 + \beta \bar{y}_{n(i)} y_i \\ &= v_i(\mathbf{D}, \mathbf{X}) y_i - \frac{1}{2} y_i^2 + \beta \mathbf{G}_{i\mathbf{y}} y_i. \end{aligned} \quad (1)$$

## Utility (continued)

- Assume that  $|\beta| < 1$  and define  $v_i(\mathbf{D}, \mathbf{X})$  as

$$\begin{aligned} v_i(\mathbf{D}, \mathbf{X}) &= X_i' \gamma + \bar{X}_{n(i)}' \delta + A + U_i \\ &= X_i' \gamma + (\mathbf{G}_i \mathbf{X})' \delta + A + U_i. \end{aligned}$$

- Comment: alternative is provided by quadratic “conformist” preferences (e.g., Akerlof, 1997).
- Comment: recall our discussion of social multiplier centrality earlier.



## Equilibrium

- The observed action  $\mathbf{Y}$  corresponds to a Nash equilibrium.
  - No agent can increase her utility by changing her action given the actions of all other agents in the network.
- The econometrician observes the triple  $(\mathbf{Y}, \mathbf{X}, \mathbf{D})$ .
  - she does not observe  $A$ , nor does she observe  $\mathbf{U}$ , the  $N \times 1$  vector of individual-level heterogeneity terms.
  - agents *do* observe  $(A, \mathbf{U})$ .

## Endogenous and Exogenous Social Effects

- **endogenous:** the marginal utility associated with an increase in  $y_i$  is increasing in the average action of one's peers,  $\bar{y}_{n(i)}$ :

$$\frac{\partial^2 u_i(y, D, X)}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

- **exogenous or contextual:** the marginal utility associated with an increase in  $y_i$  varies with peer attributes:

$$\frac{\partial^2 u_i(y, D, X)}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

## Endogenous and Exogenous Social Effects (continued)

- Endogenous and exogenous effects have different policy implications (except under special network structures)
  - effects of a “local” intervention may spread across the entire network in the presence of endogenous effects
  - effects are localized if only exogenous effects are present

## Correlated Effects

- **correlated effects:** agents located in networks with high values of  $A$  will choose higher actions.

$$\frac{\partial^2 u_i(\mathbf{y}, \mathbf{D}, \mathbf{X})}{\partial y_i \partial A} = 1.$$

- Endogenous, contextual and correlated effects all cause outcomes across members of a common network to covary.
- Attributing this covariance to true spillovers, whether endogenous or contextual, versus group-level heterogeneity is difficult.

## Policy Implications

- Spillovers raise the possibility that
  - rewirings of the network – the addition or subtraction of links – could improve the distribution of outcomes.
  - intervening at different locations of the network will have different effects on the distribution of outcomes.
- These claims will become clear shortly.

## Linear Best Replies

- F.O.C for optimal behavior generates best response functions of the form

$$Y_i = A + \beta \bar{Y}_{n(i)} + X_i' \gamma + \bar{X}_{n(i)}' \delta + U_i$$

for  $i = 1, \dots, N$ .

- Called the **linear-in-means** model of social interactions (e.g., Brock and Durlauf, 2001).
- Basis of most empirical work on peer effects.

## Linear Best Replies (continued)

An agent's best reply varies with

(i) the average action of those to whom she is directly connected  $\bar{Y}_{n(i)}$  ,

(ii) her own observed attributes  $X_i$ ,

(iii) the average attributes of her direct peers  $\bar{X}_{n(i)}$ ,

(iv) the unobserved network effect,  $A$ , and

(v) unobserved own attributes,  $U_i$ .

## A System of Simultaneous Equations

- The  $N$  best reply functions define an  $N \times 1$  system of (linear) simultaneous equations.
- A least squares fit of  $Y_i$  onto a constant,  $\bar{Y}_{n(i)}$ ,  $X$  and  $\bar{X}_{n(i)}$  will not provide consistent estimates of  $\theta_0 = (A_0, \beta_0, \gamma'_0, \delta'_0)'$ .
- Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.



## Anatomy of the Reflection Problems

- Define the index set

$$\mathcal{N}(i) = \{j : D_{ij} = 1\}$$

with cardinality  $N_i$ .

- $Y_i$  is a component of the best response functions of  $j \in \{j : j \in \mathcal{N}(i)\}$ .
- $U_i$  will be correlated with all  $Y_j \in \{Y_j : j \in \mathcal{N}(i)\}$ .
- $\Rightarrow U_i$  will covary with  $\bar{Y}_{n(i)}$ !

## Reduced Form

- Write the system of best replies in matrix form:

$$\mathbf{Y} = A\iota_N + \mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta + \beta\mathbf{G}\mathbf{Y} + \mathbf{U}. \quad (2)$$

- If  $|\beta| < 1$ , then  $I_N - \beta\mathbf{G}$  is strictly (row) diagonally dominant & hence non-singular.
- Solving for the equilibrium action vector as a function of  $\mathbf{D}$ ,  $\mathbf{X}$ ,  $A$  and  $\mathbf{U}$  alone yields

$$\begin{aligned} \mathbf{Y} = & A(I_N - \beta\mathbf{G})^{-1}\iota_N + (I_N - \beta\mathbf{G})^{-1}(\mathbf{X}\gamma + \mathbf{G}\mathbf{X}\delta) \\ & + (I_N - \beta\mathbf{G})^{-1}\mathbf{U}. \end{aligned}$$

## Reduced Form

It is helpful to simplify the reduced form in a number of ways. First, using the series expansion

$$(I_N - \beta \mathbf{G})^{-1} = \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k,$$

as well as the fact that  $\mathbf{G}\iota_N = \iota_N$  (and hence that  $\mathbf{G}^k\iota_N = \iota_N$  for  $k \geq 1$ ) we get the simplification:

$$\begin{aligned} A(I_N - \beta \mathbf{G})^{-1} \iota_N &= A \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \right] \iota_N \\ &= A (1 + \beta + \beta^2 + \beta^3 + \dots) \iota_N \\ &= \frac{A}{1 - \beta} \iota_N. \end{aligned}$$

## The Social Multiplier REDUX!

- Further manipulation yields a reduced form of

$$\mathbf{Y} = \frac{A}{1-\beta} \iota_N + \mathbf{X}\gamma + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+1} \mathbf{X} \right] (\gamma\beta + \delta) + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \right] \mathbf{U}.$$

- Consider a policy which changes the value of  $X_i$  by  $\Delta$ .
- What is the effect of this intervention on the distribution of outcomes?

## The Social Multiplier (continued)

- We can conceptualize the effect of the intervention as spreading out in a series of “waves”.
- Let  $\mathbf{c}_i$  be a vector with a 1 in the  $i^{th}$  element and zeros elsewhere. For simplicity assume  $\delta = 0$  (i.e., no exogenous effects).
- In the first “wave” the intervention changes agent  $i$ 's action alone. The effect on the distribution of outcomes is

$$\Delta' \gamma \mathbf{c}_i$$

## The Social Multiplier (continued)

- In the second “wave” agent  $i$ 's friends revise their best response in reaction to  $i$ 's initial change in action. The effect on the distribution of outcomes is

$$\Delta' \gamma \beta \mathbf{G} \mathbf{c}_i$$

- In the third “wave” agent  $i$ 's friends' friends revise their best response in reaction to  $i$ 's friends' wave two changes in action. The effect on the distribution of outcomes is

$$\Delta \gamma \beta^2 \mathbf{G}^2 \mathbf{c}_i.$$

## The Social Multiplier (continued)

- In the  $k^{th}$  wave we have a change in the action vector of

$$\Delta\gamma\beta^{k-1}\mathbf{G}^{k-1}\mathbf{c}_i.$$

- The “long-run” or full effect of the change in  $X_i$  on the entire distribution of outcomes is

$$\Delta\gamma(I_N - \beta\mathbf{G})^{-1}\mathbf{c}_i. \tag{3}$$

- The planner can use the form of  $\mathbf{G}$  to efficiently target interventions.

## Reduced Form (continued)

- $\mathbf{GX} = \bar{\mathbf{X}}$  is a matrix consisting of the average of friends' characteristics (with  $i^{th}$  row  $\bar{X}_{n(i)}$ ).
- $\mathbf{G}^2\mathbf{X} = \mathbf{G}\bar{\mathbf{X}}$  is a matrix consisting of an average of your friends' friends' average attributes (with  $i^{th}$  row  $\bar{X}_{n(i)}^{ff}$  ).
- $\mathbf{G}^3\bar{\mathbf{X}}$  is an average of your friends' friends' average of their friends' average attributes (with  $i^{th}$  row  $\bar{X}_{n(i)}^{fff}$  )



## Reduced Form (continued)

- Extra credit: describe  $\mathbf{G}^4 \bar{\mathbf{X}}$  in words.
- Use this notation we get

$$\mathbf{Y} = \frac{A}{1-\beta} \iota_N + \mathbf{X}\gamma + \bar{\mathbf{X}}(\gamma\beta + \delta) + \left[ \sum_{k=1}^{\infty} \beta^k \mathbf{G}^k \bar{\mathbf{X}} \right] (\gamma\beta + \delta) + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \right] \mathbf{U}.$$

- In equilibrium, an agent's action will vary with own attributes, her peers', her peers' peers' and so on.

## Connection to Dynamic Panel Data

- The endogenous effect induces a distributed lag in  $\mathbf{X}$  in the reduced form expression for  $\mathbf{Y}$ .
- In dynamic linear panel data models with strictly exogenous regressors, state dependence induces an analogous structure (Chamberlain, 1984; Arellano, 2003).

## Formulation as an IV Problem

- Bramoulle, Djebbari and Fortin's (2009) propose a linear IV procedure.
- Our **structural equations** are

$$\mathbf{Y} = A\iota_N + \beta\bar{\mathbf{Y}} + \mathbf{X}\gamma + \bar{\mathbf{X}}\delta + \mathbf{U}.$$

## Formulation as an IV Problem (continued)

- Let  $\bar{\mathbf{Y}} = \mathbf{G}\mathbf{Y}$  to be the  $N \times 1$  of peer average actions. Multiplying the reduced form by  $\mathbf{G}$  yields the **first stage equations**

$$\begin{aligned}\bar{\mathbf{Y}} = & \frac{A}{1-\beta} \iota_M + \bar{\mathbf{X}}\gamma + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^{k+1} \bar{\mathbf{X}} \right] (\gamma\beta + \delta) \\ & + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k \right] \bar{\mathbf{U}}.\end{aligned}$$

## Estimation

- The dataset consists of a random sample of networks indexed by  $c$ 
  - with the size of network  $c$  equal to  $N_c$  and
  - with action profile  $\mathbf{Y}_c$ , adjacency matrix  $\mathbf{D}_c$  and attribute matrix  $\mathbf{X}_c$ .

## Estimation (continued)

- Assume that  $\mathbb{E} [\mathbf{U}_c | \mathbf{D}_c, \mathbf{X}_c, N_c] = 0$ .
- This (effectively) restricts the network formation process (in many cases unrealistically).

## Estimation (continued)

- The following moment restriction holds at the population vector  $\theta_0$

$$\mathbb{E} \left[ \begin{pmatrix} \iota_{N_c} & \mathbf{G}_c \bar{\mathbf{X}}_c & \mathbf{X}_c & \bar{\mathbf{X}}_c \end{pmatrix}' \right. \\ \left. \times \left( \mathbf{Y}_c - A_0 \iota_{N_c} - \beta_0 \bar{\mathbf{Y}}_c - \mathbf{X}_c \gamma_0 - \bar{\mathbf{X}}_c \delta_0 \right) \right] = 0$$

- If  $I_{N_c}$ ,  $\mathbf{G}_c$  and  $\mathbf{G}_c^2$  are linearly independent and  $\gamma\beta + \delta \neq 0$ , then a GMM estimator will be consistent (Bramouille, Djebbari and Fortin (2009, Proposition 1)).

## Friends-of-Friends Instrument

- Linear IV fit of  $Y_{ci}$  onto a constant,  $\bar{Y}_{cn(i)}$ ,  $X_{ci}$  and  $\bar{X}_{cn(i)}$  with  $\bar{X}_{cn(i)}^{\text{ff}}$  serving as an excluded instrument for  $\bar{Y}_{cn(i)}$  .
  - consistent estimates of  $\beta$ ,  $\gamma$ , and  $\delta$ ;
  - see Di Giorgi, Pellizzari and Redaelli (2010, *AEJ*) for an illustrative application.



## Non-identification Result of Manski (1993)

- Consider the case where  $\mathbf{G}_c$  equals

$$\mathbf{G}_c = \left( \iota_{N_c} \iota'_{N_c} - I_{N_c} \right) \frac{1}{N_c - 1}.$$

- Often used in economics of education applications.
- Under this network structure we have

$$\mathbf{G}_c^2 = \frac{1}{N_c - 1} I_{N_c} + \frac{N_c - 2}{N_c - 1} \mathbf{G}_c.$$

## Non-identification Result of Manski (1993)

- If groups/networks vary in size, then  $I_{N_c}$ ,  $G_c$  and  $G_c^2$  will be linearly independent (cf., Lee, 2007).
- If groups are equal in size identification fails.
- $N_c \rightarrow \infty$ , which is (essentially) Manski's (1993) case, gives  $G_c^2 = G_c$ .

## Identification via Non-Transitivity

- Bramouille, Djebbari and Fortin (2009) note that if the pair  $(i, j)$  are not connected then  $D_{ij} = 0$ .
- If they share some friends in common, then  $(i, j)^{th}$  element of  $\mathbf{D}^2$ , which equals  $\sum_k D_{ik}D_{kj}$ , will be non-zero.
- The presence of intransitive triads (i.e., two-stars), in at least some networks, guarantees linear independence of  $I_{N_c}$ ,  $\mathbf{G}_c$  and  $\mathbf{G}_c^2$ .

## Network Effects

- One generalization of the model allows the intercept to vary across sampled networks.
- If  $A_c$  varies across networks we get a reduced form of

$$\begin{aligned} \mathbf{Y}_c = & \frac{A_c}{1 - \beta} \iota_{N_c} + \mathbf{X}_c \gamma + \bar{\mathbf{X}}_c (\gamma \beta + \delta) \\ & + \left[ \sum_{k=1}^{\infty} \beta^k \mathbf{G}_c^k \bar{\mathbf{X}}_c \right] (\gamma \beta + \delta) + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}_c^k \right] \mathbf{U}_c. \end{aligned}$$

## Network Effects (continued)

- Subtracting “first stage” from this equation eliminates the “network effect”, yielding

$$\begin{aligned} \mathbf{Y}_c - \bar{\mathbf{Y}}_c &= (\mathbf{X}_c - \bar{\mathbf{X}}_c) \gamma + (I_{N_c} - \mathbf{G}_c) \bar{\mathbf{X}} (\gamma\beta + \delta) \\ &\quad + \left[ \sum_{k=1}^{\infty} \beta^k \mathbf{G}^k (I_{N_c} - \mathbf{G}_c) \bar{\mathbf{X}} \right] (\gamma\beta + \delta) \\ &\quad + \left[ \sum_{k=0}^{\infty} \beta^k \mathbf{G}_c^k \right] (\mathbf{U}_c - \bar{\mathbf{U}}_c). \end{aligned}$$

- If  $I_{N_c}$ ,  $\mathbf{G}_c$ ,  $\mathbf{G}_c^2$  and  $\mathbf{G}_c^3$  are linearly independent  $\theta_0$  is identified (need networks with diameter of at least three).

## Estimation with Network Effects

- Let  $\bar{Y}_{cn(i)}^{ff}$  equal the  $i^{th}$  element of  $\mathbf{G}_c^2 \mathbf{Y}_c$ .
  - equals the average of my friends' averages of their friends behavior.
- Recall that  $\bar{X}_{cn(i)}^{fff}$  is the  $i^{th}$  row of  $\mathbf{G}_c^3 \mathbf{X}$ .
  - equals a (weighted) average of agent characteristics up to three degrees away from  $i$ .

## Estimation with Network Effects (continued)

- A linear IV fit of  $Y_{ci} - \bar{Y}_{cn(i)}$  onto  $\bar{Y}_{cn(i)} - \bar{Y}_{cn(i)}^{\text{ff}}$ ,  $X_{ci} - \bar{X}_{cn(i)}$  and  $\bar{X}_{cn(i)} - \bar{X}_{cn(i)}^{\text{ff}}$  with
  - $\bar{X}_{cn(i)}^{\text{ff}} - \bar{X}_{cn(i)}^{\text{fff}}$  serving as an excluded instrument for  $\bar{Y}_{cn(i)} - \bar{Y}_{cn(i)}^{\text{ff}}$  ;
  - standard errors “clustered” at the network level.
- Yields consistent estimates of  $\theta_0$  and asymptotically valid standard error estimates.

## Empirical Work

- Identification of  $\theta_0$  requires maintaining fairly strong assumptions about the network formation process.
- Condition  $\mathbb{E}[\mathbf{U}_c | \mathbf{D}_c, \mathbf{X}_c, N_c, A_c] = 0$  provides a useful way for assessing the plausibility of empirical work.
- Can I predict the idiosyncratic component of behavior using network structure, agent characteristics and/or network size?