# Using Network Structure to Identify Peer Effects

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# Overview

- Large empirical literature on peer group effects based on the "linear-in-means" model of social interactions (Manski, 1993).

- Quality of research in this area is uneven and has been heavily criticized (e.g., Angrist, 2013).

  - over 20 years since Manski (1993) conditions for identification (and their interpretation) evidently not fully understood by some practitioners.

# Overview (continued)

- Recent work on network games with linear best reply functions (e.g., Jackson and Zenou, 2015; Bramoulle, Kranton and D'Amours, 2014).

  – provides micro-foundations for linear-in-means model.

  – facilitates intuitive assessment of conditions for identification.

  – also connected to older literature on input-output models recently explored by Acemoglu and co-authors.

# Key References

- Manski (1993, *Review of Economic Studies*)

- Brock and Durlauf (2001, *Handbook of Econometrics*)

- Bramoulle, Djebbari and Fortin (2009, *Journal of Economet-rics*)

# Notation

- Let $\mathbf{G} = \operatorname{diag}(\mathbf{D}\iota_{\mathsf{N}})^{-1}\mathbf{D}$ be the **row-normalized network adjacency matrix.**

    - Note that all rows of this matrix sum to 1 by construction.

    - The matrix is row-stochastic (when graph is connected).

    - Focus on undirected links today, but extension to directed case follows.

# Notation (continued)

- Let $\mathbf{G}_i$ denotes the $i^{th}$ row of $\mathbf{G}$ and define

$$\mathbf{G}_i\mathbf{y} \;=\; \sum_{j \neq i} G_{ij} y_j \;\overset{def}{\equiv}\; \bar{y}_{n(i)}$$

$$\mathbf{G}_i\mathbf{X} \;=\; \sum_{j \neq i} G_{ij} X_j \;\overset{def}{\equiv}\; \bar{X}_{n(i)}.$$

These equal

    — the average action of player $i$'s peers.

    — the average of her peers' attribute vector.

## Utility

Assume that the utility agent $i$ receives from action profile $\mathbf{y}$, given network structure $(\mathbf{D})$ and agent attributes $(\mathbf{X})$, is

$$
\begin{aligned}
u_i\left(\mathbf{y}; \mathbf{D}, \mathbf{X}\right) &= v_i\left(\mathbf{D}, \mathbf{X}\right) y_i - \frac{1}{2}y_i^2 + \beta \bar{y}_{n(i)} y_i \\
&= v_i\left(\mathbf{D}, \mathbf{X}\right) y_i - \frac{1}{2}y_i^2 + \beta \mathbf{G}_i \mathbf{y} y_i. \qquad (1)
\end{aligned}
$$

## Utility (continued)

Assume that $|\beta| < 1$ and define $v_i \left(\mathbf{D}, \mathbf{X}\right)$ as

$$
\begin{aligned}
v_i \left(\mathbf{D}, \mathbf{X}\right) &= X_i'\gamma + \bar{X}_{n(i)}'\delta + A + U_i \\
&= X_i'\gamma + \left(\mathbf{G}_i \mathbf{X}\right)'\delta + A + U_i.
\end{aligned}
$$

Comment: alternative is provided by quadratic "conformist" preferences (e.g., Akerlof, 1997).

Comment: recall our discussion of social multiplier centrality earlier.

# Equilibrium

- The observed action $\mathbf{Y}$ corresponds to a Nash equilibrium.

  - No agent can increase her utility by changing her action given the actions of all other agents in the network.

- The econometrician observes the triple $(\mathbf{Y}, \mathbf{X}, \mathbf{D})$.

  - she does not observe $A$, nor does she observe $\mathbf{U}$, the $N \times 1$ vector of individual-level heterogeneity terms.

  - agents *do* observe $(A, \mathbf{U})$.

# Endogenous and Exogenous Social Effects

- **endogenous**: the marginal utility associated with an increase in $y_i$ is increasing in the average action of one's peers, $\bar{y}_{n(i)}$:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{y}_{n(i)}} = \beta.$$

- **exogenous or contextual:** the marginal utility associated with an increase in $y_i$ varies with peer attributes:

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial \bar{X}'_{n(i)}} = \delta.$$

## Endogenous and Exogenous Social Effects (continued)

- Endogenous and exogenous effects have different policy implications (except under special network structures)

  - effects of a "local" intervention may spread across the entire network in the presence of endogenous effects

  - effects are localized if only exogenous effects are present

# Correlated Effects

- **correlated effects**: agents located in networks with high values of $A$ will choose higher actions.

$$\frac{\partial^2 u_i\left(\mathbf{y}, \mathbf{D}, \mathbf{X}\right)}{\partial y_i \partial A} = 1.$$

- Endogenous, contextual and correlated effects all cause outcomes across members of a common network to covary.

- Attributing this covariance to true spillovers, whether endogenous or contextual, versus group-level heterogeneity is difficult.

# Policy Implications

- Spillovers raise the possibility that

    - rewirings of the network − the addition or subtraction of links − could improve the distribution of outcomes.

    - intervening at different locations of the network will have different effects on the distribution of outcomes.

- These claims will become clear shortly.

# Linear Best Replies

- F.O.C for optimal behavior generates best response functions of the form

$$Y_i = A + \beta \bar{Y}_{n(i)} + X_i'\gamma + \bar{X}_{n(i)}'\delta + U_i$$

for $i = 1, \ldots, N$.

- Called the **linear-in-means** model of social interactions (e.g., Brock and Durlauf, 2001).

- Basis of most empirical work on peer effects.

# Linear Best Replies (continued)

An agent's best reply varies with

(i) the average action of those to whom she is directly connected $\bar{Y}_{n(i)}$ ,

(ii) her own observed attributes $X_i$,

(iii) the average attributes of her direct peers $\bar{X}_{n(i)}$,

(iv) the unobserved network effect, $A$, and

(v) unobserved own attributes, $U_i$.

# A System of Simultaneous Equations

- The $N$ best reply functions define an $N \times 1$ system of (linear) simultaneous equations.

- A least squares fit of $Y_i$ onto a constant, $\bar{Y}_{n(i)}$, $X$ and $\bar{X}_{n(i)}$ will not provide consistent estimates of $\theta_0 = \left(A_0, \beta_0, \gamma_0', \delta_0'\right)'$.

- Manski (1993) calls this feature of the linear-in-means model the **reflection problem**.

# Anatomy of the Reflection Problems

- Define the index set

$$\mathcal{N}(i) = \left\{ j \; : \; D_{ij} = 1 \right\}$$

  with cardinality $N_i$.

  - $Y_i$ is a component of the best response functions of $j \in \{j \; : \; j \in \mathcal{N}(i)\}$.

  - $U_i$ will be correlated with all $Y_j \in \left\{ Y_j \; : \; j \in \mathcal{N}(i) \right\}$.

  - $\Rightarrow U_i$ will covary with $\bar{Y}_{n(i)}$!

# Reduced Form

- Write the system of best replies in matrix form:

$$\mathbf{Y} = A\iota_N + \mathbf{X}\gamma + \mathbf{GX}\delta + \beta\mathbf{GY} + \mathbf{U}. \qquad (2)$$

- If $|\beta| < 1$, then $I_N - \beta\mathbf{G}$ is strictly (row) diagonally dominant & hence non-singular.

- Solving for the equilibrium action vector as a function of $\mathbf{D}$, $\mathbf{X}$, $A$ and $\mathbf{U}$ alone yields

$$\mathbf{Y} = A\left(I_N - \beta\mathbf{G}\right)^{-1}\iota_N + \left(I_N - \beta\mathbf{G}\right)^{-1}\left(\mathbf{X}\gamma + \mathbf{GX}\delta\right) \\ + \left(I_N - \beta\mathbf{G}\right)^{-1}\mathbf{U}.$$

# Reduced Form

It is helpful to simplify the reduced form in a number of ways.
First, using the series expansion

$$(I_N - \beta\mathbf{G})^{-1} = \sum_{k=0}^{\infty} \beta^k \mathbf{G}^k,$$

as well as the fact that $\mathbf{G}\iota_N = \iota_N$ (and hence that $\mathbf{G}^k\iota_N = \iota_N$ for $k \geq 1$) we get the simplification:

$$
\begin{aligned}
A\left(I_N - \beta\mathbf{G}\right)^{-1}\iota_N &= A\left[\sum_{k=0}^{\infty} \beta^k \mathbf{G}^k\right]\iota_N \\
&= A\left(1 + \beta + \beta^2 + \beta^3 + \cdots\right)\iota_N \\
&= \frac{A}{1-\beta}\iota_N.
\end{aligned}
$$

# The Social Multiplier REDUX!

- Further manipulation yields a reduced from of

$$\mathbf{Y} = \frac{A}{1-\beta}\iota_N + \mathbf{X}\gamma + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^{k+1}\mathbf{X}\right](\gamma\beta+\delta)$$

$$+ \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U}.$$

- Consider a policy which changes the value of $X_i$ by $\triangle$.

- What is the effect of this intervention on the distribution of outcomes?

# The Social Multiplier (continued)

- We can conceptualize the effect of the intervention as spreading out in a series of "waves".

- Let $\mathbf{c}_i$ be a vector with a 1 in the $i^{th}$ element and zeros elsewhere. For simplicity assume $\delta = 0$ (i.e., no exogenous effects).

- In the first "wave" the intervention changes agent $i's$ action alone. The effect on the distribution of outcomes is

$$\triangle' \gamma \mathbf{c}_i$$

# The Social Multiplier (continued)

- In the second "wave" agent $i's$ friends revise their best response in reaction to $i's$ initial change in action. The effect on the distribution of outcomes is

$$\triangle' \gamma \beta \mathbf{G} \mathbf{c}_i$$

- In the third "wave" agent $i's$ friends' friends revise their best response in reaction to $i's$ friends' wave two changes in action. The effect on the distribution of outcomes is

$$\triangle \gamma \beta^2 \mathbf{G}^2 \mathbf{c}_i.$$

# The Social Multiplier (continued)

- In the $k^{th}$ wave we have a change in the action vector of

$$\triangle \gamma \beta^{k-1} \mathbf{G}^{k-1} \mathbf{c}_i.$$

- The "long-run" or full effect of the change in $X_i$ on the entire distribution of outcomes is

$$\triangle \gamma \left( I_N - \beta \mathbf{G} \right)^{-1} \mathbf{c}_i. \tag{3}$$

- The planner can use the form of $\mathbf{G}$ to efficiently target interventions.

# Reduced Form (continued)

- $\mathbf{GX} = \overline{\mathbf{X}}$ is a matrix consisting of the average of friends' characteristics (with $i^{th}$ row $\bar{X}_{n(i)}$).

- $\mathbf{G}^2\mathbf{X} = \mathbf{G}\overline{\mathbf{X}}$ is a matrix consisting of an average of your friends' friends' average attributes (with $i^{th}$ row $\bar{X}^{\mathsf{ff}}_{n(i)}$ ).

- $\mathbf{G}^3\overline{\mathbf{X}}$ is an average of your friends' friends' average of their friends' average attributes (with $i^{th}$ row $\bar{X}^{\mathsf{fff}}_{n(i)}$ )

# Reduced Form (continued)

- <u>Extra credit:</u> describe $\mathbf{G}^4\bar{\mathbf{X}}$ in words.

- Use this notation we get

$$
\begin{aligned}
\mathbf{Y} \;=\;\; & \frac{A}{1-\beta}\iota_N + \mathbf{X}\gamma + \bar{\mathbf{X}}\left(\gamma\beta + \delta\right) + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}^k\bar{\mathbf{X}}\right]\left(\gamma\beta + \delta\right) \\
& + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}^k\right]\mathbf{U}.
\end{aligned}
$$

- In equilibrium, an agent's action will vary with own attributes, her peers', her peers' peers' and so on.

# Connection to Dynamic Panel Data

- The endogenous effect induces a distributed lag in $\mathbf{X}$ in the reduced form expression for $\mathbf{Y}$.

- In dynamic linear panel data models with strictly exogenous regressors, state dependence induces an analogous structure (Chamberlain, 1984; Arellano, 2003).

# Formulation as an IV Problem

- Bramoulle, Djebbari and Fortin's (2009) propose a linear IV procedure.

- Our **structural equations** are

$$\mathbf{Y} = A\iota_N + \beta\bar{\mathbf{Y}} + \mathbf{X}\gamma + \bar{\mathbf{X}}\delta + \mathbf{U}.$$

## Formulation as an IV Problem (continued)

Let $\bar{Y} = GY$ to be the $N \times 1$ of peer average actions. Multiplying the reduced form by $G$ yields the **first stage equations**

$$
\bar{Y} = \frac{A}{1-\beta} \iota_M + \bar{X}\gamma + \left[ \sum_{k=0}^{\infty} \beta^k G^{k+1} \bar{X} \right] (\gamma\beta + \delta)
$$
$$
+ \left[ \sum_{k=0}^{\infty} \beta^k G^k \right] \bar{U}.
$$

# Estimation

The dataset consists of a random sample of networks indexed by $c$

- with the size of network $c$ equal to $N_c$ and

- with action profile $\mathbf{Y}_c$, adjacency matrix $\mathbf{D}_c$ and attribute matrix $\mathbf{X}_c$.

# Estimation (continued)

- Assume that $\mathbb{E}\left[\mathbf{U}_c \middle| \mathbf{D}_c, \mathbf{X}_c, N_c\right] = 0$.

- This (effectively) restricts the network formation process (in many cases unrealistically).

# Estimation (continued)

- The following moment restriction holds at the population vector $\theta_0$

$$\mathbb{E}\left[\left(\begin{array}{cccc} \iota_{N_c} & \mathbf{G}_c\bar{\mathbf{X}}_c & \mathbf{X}_c & \bar{\mathbf{X}}_c \end{array}\right)' \right.$$
$$\left. \times \left(\mathbf{Y}_c - A_0\iota_{N_c} - \beta_0\bar{\mathbf{Y}}_c - \mathbf{X}_c\gamma_0 - \bar{\mathbf{X}}_c\delta_0\right)\right] = 0$$

- If $I_{N_c}$, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ are linearly independent and $\gamma\beta + \delta \neq 0$, then a GMM estimator will be consistent (Bramoulle, Djebbari and Fortin (2009, Proposition 1)).

# Friends-of-Friends Instrument

- Linear IV fit of $Y_{ci}$ onto a constant, $\bar{Y}_{cn(i)}$, $X_{ci}$ and $\bar{X}_{cn(i)}$ with $\bar{X}^{\text{ff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)}$ .

  – consistent estimates of $\beta$, $\gamma$, and $\delta$;

  – see Di Giorgi, Pellizzari and Redaelli (2010, *AEJ*) for an illustrative application.

# Non-identification Result of Manski (1993)

- Consider the case where $\mathbf{G}_c$ equals

$$\mathbf{G}_c = \left( \iota_{N_c} \iota'_{N_c} - I_{N_c} \right) \frac{1}{N_c - 1}.$$

- Often used in economics of education applications.

- Under this network structure we have

$$\mathbf{G}_c^2 = \frac{1}{N_c - 1} I_{N_c} + \frac{N_c - 2}{N_c - 1} \mathbf{G}_c.$$

# Non-identification Result of Manski (1993)

- If groups/networks vary in size, then $I_{N_c}$, $\mathbf{G}_c$ and $\mathbf{G}_c^2$ will be linearly independent (cf., Lee, 2007).

- If groups are equal in size identification fails.

- $N_c \rightarrow \infty$, which is (essentially) Manski's (1993) case, gives $\mathbf{G}_c^2 = \mathbf{G}_c$.

# Identification via Non-Transitivity

- Bramoulle, Djebbari and Fortin (2009) note that if the pair $(i, j)$ are not connected then $D_{ij} = 0$.

- If they share some friends in common, then $(i, j)^{th}$ element of $\mathbf{D}^2$, which equals $\sum_k D_{ik} D_{kj}$, will be non-zero.

- The presence of intransitive triads (i.e., two-stars), in at least some networks, guarantees linear independence of $I_{N_c}$, $\mathbf{G}_c$ and $\mathbf{G}_c^2$.

# Network Effects

- One generalization of the model allows the intercept to vary across sampled networks.

- If $A_c$ varies across networks we get a reduced form of

$$
\mathbf{Y}_c = \frac{A_c}{1-\beta}\iota_{N_c} + \mathbf{X}_c\gamma + \bar{\mathbf{X}}_c\left(\gamma\beta + \delta\right)
$$

$$
+ \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}_c^k\bar{\mathbf{X}}_c\right]\left(\gamma\beta + \delta\right) + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G}_c^k\right]\mathbf{U}_c.
$$

# Network Effects (continued)

- Subtracting "first stage" from this equation eliminates the "network effect", yielding

$$
\begin{aligned}
\mathbf{Y}_c - \bar{\mathbf{Y}}_c \;=\; & \left(\mathbf{X}_c - \bar{\mathbf{X}}_c\right)\gamma + \left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\left(\gamma\beta + \delta\right) \\
& + \left[\sum_{k=1}^{\infty}\beta^k\mathbf{G}^k\left(I_{N_c} - \mathbf{G}_c\right)\bar{\mathbf{X}}\right]\left(\gamma\beta + \delta\right) \\
& + \left[\sum_{k=0}^{\infty}\beta^k\mathbf{G_c}^k\right]\left(\mathbf{U}_c - \bar{\mathbf{U}}_c\right).
\end{aligned}
$$

- If $I_{N_c}$, $\mathbf{G}_c$, $\mathbf{G}_c^2$ and $\mathbf{G}_c^3$ are linearly independent $\theta_0$ is identified (need networks with diameter of at least three).

# Estimation with Network Effects

- Let $\bar{Y}^{\text{ff}}_{cn(i)}$ equal the $i^{th}$ element of of $\mathbf{G}_c^2 \mathbf{Y}_c$.

  - equals the average of my friends' averages of their friends behavior.

- Recall that $\bar{X}^{\text{fff}}_{cn(i)}$ is the $i^{th}$ row of $\mathbf{G}_c^3 \mathbf{X}$.

  - equals a (weighted) average of agent characteristics up to three degrees away from $i$.

# Estimation with Network Effects (continued)

- A linear IV fit of $Y_{ci} - \bar{Y}_{cn(i)}$ onto $\bar{Y}_{cn(i)} - \bar{Y}^{\mathsf{ff}}_{cn(i)}$, $X_{ci} - \bar{X}_{cn(i)}$ and $\bar{X}_{cn(i)} - \bar{X}^{\mathsf{ff}}_{cn(i)}$ with

  - $\bar{X}^{\mathsf{ff}}_{cn(i)} - \bar{X}^{\mathsf{fff}}_{cn(i)}$ serving as an excluded instrument for $\bar{Y}_{cn(i)} - \bar{Y}^{\mathsf{ff}}_{cn(i)}$ ;

  - standard errors "clustered" at the network level.

- Yields consistent estimates of $\theta_0$ and asymptotically valid standard error estimates.

# Empirical Work

- Identification of $\theta_0$ requires maintaining fairly strong assumptions about the network formation process.

- Condition $\mathbb{E}\left[\mathbf{U}_c|\,\mathbf{D}_c, \mathbf{X}_c, N_c, A_c\right] = 0$ provides a useful way for assessing the plausibility of empirical work.

- Can I predict the idiosyncratic component of behavior using network structure, agent characteristics and/or network size?