Describing Social Networks

Econometric Methods for Networks,

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Nyakatoke Risk-Sharing Network

Note: node sizes are proportional to household degree

Wealth $<$ 150,000 TSh

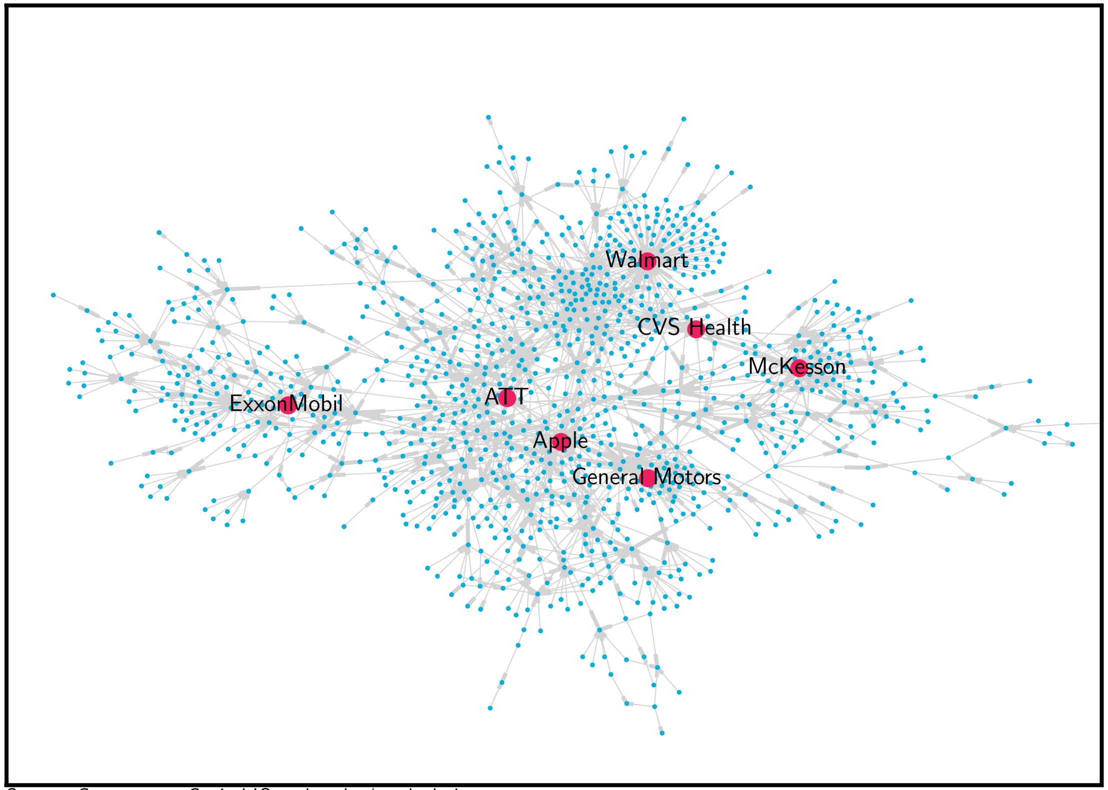
150,000 TSh Wealth $<$ 300,000 TSh

300,000 TSh Wealth $<$ 600,000 TSh

Wealth $\geq$ 600,000 TSh

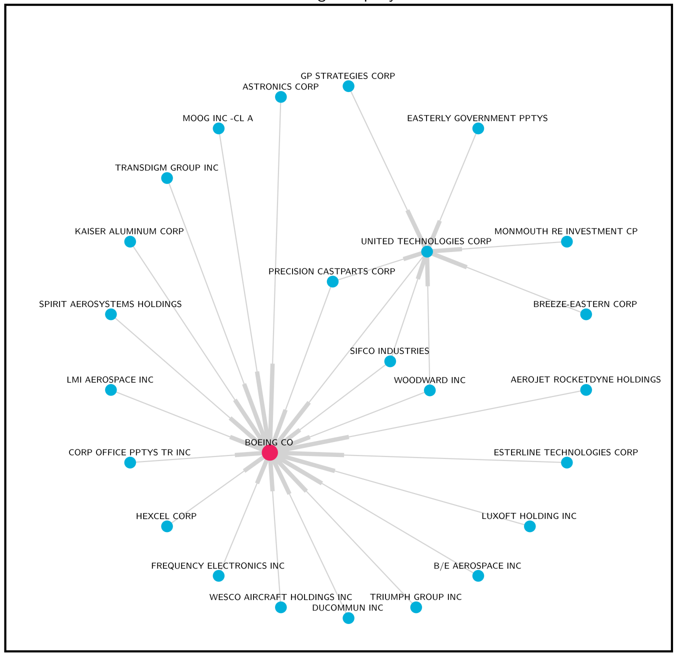
(N = 119, n = 7,021)

United States Inter-Firm Buyer-Seller Network, 2015

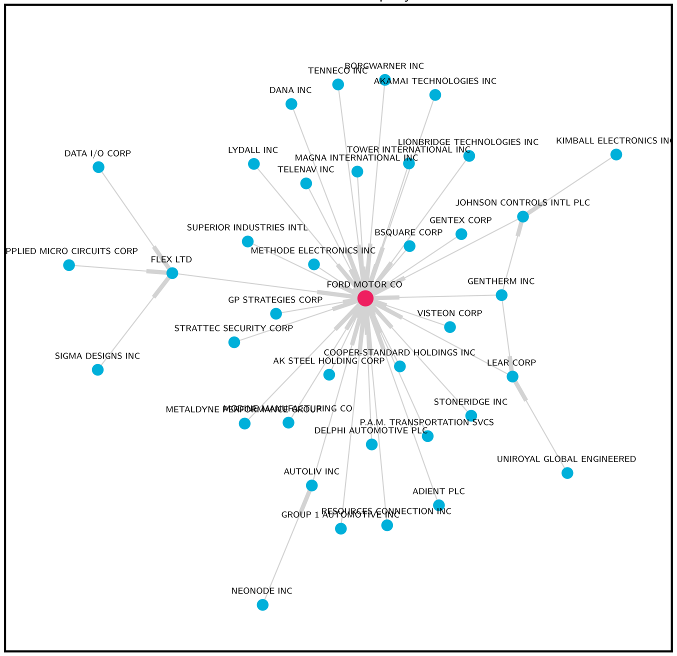


Source: Compustat - Capital IQ and author's calculations.
Raw data available at https://wrds-web.wharton.upenn.edu/wrds/ (Accessed January 2017)

Boeing Company



Ford Motor Company



Questions

- How do the number, structure and characteristics of an agent's ties influence her behaviors and outcomes?

- How are ties formed? Are externalities involved?

- What configuration of ties would a social planner choose?

  – How does this idealized network compare with the observed one?

  – Are observed networks efficient?

Questions (continued)

- Can we identify "important" agents in the network? Why is this interesting?
- What policies influence network structure (and outcomes)?
- How does network structure influence the diffusion of disease, ideas and new technologies?
- Are there optimal locations on a network in which to intervene?

Applications...

- Buyer-supplier networks (Industrial Organization)

- Friendship networks (Education, Labor)

- Criminal networks (Urban)

- Trading networks (Industrial Organization and International Trade)

Applications... (continued)

- Political networks (Political Economy)
- Bank networks (Finance)
- Online networks

...and Funding!

- SBE Directorate of the National Science Foundation (NSF) recently identified network analysis as one of five key "cross cutting themes" with special grant opportunities.

Literatures

- Psychology, sociology, anthropology, political science and economics all have empirical and theoretical literatures on “networks”.
  - Wasserman & Faust (1994)
  - Jackson (2008)
- Networks are widely-studied in Physics.
  - Newman (2010)

Literatures

- The mathematical representation of networks as graphs makes discrete math (esp. graph theory), matrix analysis, and computer science highly useful.

- The statistical/econometric literature *very* underdeveloped (cf., Goldenberg *et al*. 2009).

- ...but growing rapidly (e.g., Bickel & Chen, 2009; Bickel, Chen & Levina, 2011; Graham, 2017; de Paula, 2016).

- A older and rich applied probability literature.

Outline of Course

- Lecture 1 (5/8/17, AM): Describing social networks
  - introduction to network data
  - definition and computation of basic summary network statistics
- Lecture 2 (5/8/17, AM): Centrality
  - centrality, PageRank, social multiplier

Outline of Course (continued)

- Lecture 3 (5/8/17, PM): Shock propagation
  - networks and aggregate volatility
- Lecture 4 (5/8/17, PM): Nonparametrics
  - graphons, graph limits
  - nonparametric estimation of link probabilities

Outline of Course (continued)

- Lecture 5 (5/9/17, AM): Inference
  - network moments
  - network subsampling/bootstrap
- Lecture 6 (5/9/17, AM): Link formation #1
  - importance sampling from networks w/ fixed degree
  - dyadic models of link formation

Outline of Course (continued)

- Lecture 7 (5/9/17, PM): Link formation #2
  - network formation w/ interdependencies
  - strategic models
- Lecture 8 (5/9/17, PM): Peer effects
  - network structure & peer effects

Computation

- Some computational illustrations in class (maybe tomorrow)

- Most code is available on the course GitHub repository (`https://github.com/bryangraham/short_courses`)

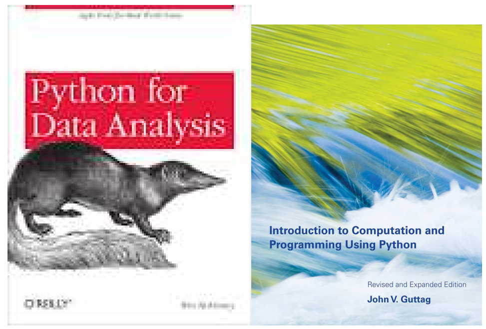
- If you want to follow along (recommended, but not required) use the *Anaconda* distribution of Python v 2.7.12 `https://www.continuum.io/downloads`

Computation (continued)

- *Anaconda* includes key packages for data analysis & scientific computing (e.g., numpy, scipy, pandas, networkx)

- Also useful: Graphviz (visualization), Yhat Rodeo (IDE)

Computation (continued)



https://github.com/wesm/pydata-book

QuantEcon

http://quant-econ.net/

Basic Terms & Notation

- An **undirected graph** $G(\mathcal{N}, \mathcal{E})$ consists of a set of **nodes** $\mathcal{N} = \{1, \ldots, N\}$ and a list of unordered pairs of nodes called **edges** $\mathcal{E} = \{\{i, j\}, \{k, l\}, \ldots\}$ for $i, j, k, l \in \mathcal{N}$.

- A graph is conveniently represented by its **adjacency matrix** $\mathbf{D} = \left[D_{ij}\right]$ where

$$D_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

- No self-ties & unordered edges $\Rightarrow$ $\mathbf{D}$ is a symmetric binary matrix with a diagonal of so-called structural zeros.

Basic Terms & Notation (continued)

- vertex: node, agent or player.

- edges: links, friendships, connections or ties.

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 1 is connected to agents 2 and 5.
- Agent 2 is connected to agent 1.
- Agent 3 is connected to no one, etc.

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 5 is connected to agents 1 and 4.

- Agents 2 and 5 are indirectly connected through agent 1 (i.e., share her as a common friend).

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- 3 out of 10 possible ties are present in the network.

Agents, Dyads and Triads

- A network consists of
  - N agents
  - $\binom{N}{2} = \frac{1}{2}N\left(N-1\right) = O\left(N^2\right)$ pairs of agents or **dyads.**
  - $\binom{N}{3} = \frac{1}{6}N\left(N-1\right)\left(N-2\right) = O\left(N^3\right)$ triples of agents of **triads**.
  - $\binom{N}{4} = \frac{1}{24}N\left(N-1\right)\left(N-2\right)\left(N-3\right) = O\left(N^4\right)$ quadruples of agents of **tetrads**.

Agents, Dyads and Triads (continued)

In summarizing a network adjacency matrix it is convenient to conceptualize statistics as measures of

1. agent-,

2. dyad-,

3. triad- or

4. p-subgraph-level attributes.

Agent-level Statistics: Degree

- The total number of links belonging to agent i, or her **degree** is $D_{i+} = \sum_j D_{ij}$.

- The **degree sequence** of a network is $\mathbf{D}_+ = \left(D_{1+}, \ldots, D_{N+}\right)'$.

- The **degree distribution** gives the frequency of each possible agent-level degree count $\{0, 1, \ldots, N\}$ in the network.

Degree (continued)

- Some researchers take the degree distribution as their primary object of interest (e.g., Barabási and Albert, 1999).
  - Other key topological features of a network are fundamentally constrained by its degree distribution.

- Some datasets report agent degrees with no other network information

Dyad-level Statistics: Density

- Dyads are either linked or unlinked.

- The **density** of a network equals the frequency with which any randomly drawn dyad is linked:

$$P_N = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} D_{ij}. \tag{2}$$

- Note that $\lambda_N = (N-1)\, P_N$ coincides with average degree.

Dyad-level Statistics: Density (continued)

- The density of the Nyakatoke network is 0.0698.

- Low density and skewed degree distributions (with fat tails) are common features of real world social networks.

Paths

$$\mathbf{D}^2 = \begin{pmatrix} D_{1+} & \sum_i D_{1i}D_{2i} & \cdots & \sum_i D_{1i}D_{Ni} \\ \sum_i D_{1i}D_{2i} & D_{2+} & \cdots & \sum_i D_{2i}D_{Ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i D_{1i}D_{Ni} & \sum_i D_{2i}D_{Ni} & \cdots & D_{N+} \end{pmatrix}$$

- The i^{th} diagonal element of $\mathbf{D}^2$ equals the number of agent i's links or her degree.

- The $\{i,j\}^{th}$ element of $\mathbf{D}^2$ gives the number of links agent i has in common with agent j (i.e., the number of "friends in common").

Paths (continued)

- graph theory: the $\{i, j\}^{th}$ element of $\mathbf{D}^2$ gives the number of **paths** of length two from agent i to agent j.

- if i and j share the common friend k, then a length two path from i to j is given by $i \to k \to j$.

Paths (continued)

$$\mathbf{D}^3 = \begin{pmatrix} \sum_{i,j} D_{1i}D_{ij}D_{j1} & \cdots & \sum_{i,j} D_{1i}D_{ij}D_{jN} \\ \vdots & \ddots & \vdots \\ \sum_{i,j} D_{1i}D_{ij}D_{jN} & \cdots & \sum_{i,j} D_{Ni}D_{ij}D_{jN} \end{pmatrix}$$

- $\{i,j\}^{th}$ element gives the number of paths of length 3 from i to j.

- If both i and j are connected to k as well as to each other, then the $\{i,j,k\}$ triad is transitive (i.e., "the friend of my friend is also my friend").

Paths (continued)

- The i^{th} diagonal element $\mathbf{D}^3$ is a count of the number of transitive triads or **triangles** to which i belongs (with $i-j-k$ and $i-k-j$ counted separately).
  - If $\{i,j,k\}$ is a closed triad it is counted twice each in the i^{th}, j^{th} and k^{th} diagonal elements of $\mathbf{D}^3$.
  - $\text{Tr}\left(\mathbf{D}^3\right)/6$ equals the number of *unique* triangles in the network.

K-Length Paths

- The $\{i, j\}^{th}$ element of $\mathbf{D}^K$ gives the number of paths of length K from agent i to agent j.

- Let $D_{ij}^{(K)}$ denote the $\{i, j\}^{th}$ element of $\mathbf{D}^K$.

- $\mathbf{D}^0 = I_N$, the only zero length walks in the network are from each agent to herself.

K-Length Paths (continued)

- Under the maintained hypothesis, $D_{ij}^{(K)}$ equals the number of K-length paths from i to j. The number of $K+1$ length paths from i to j then equals

$$\sum_{k=1}^{N} D_{ik}^{(K)} D_{kj},$$

which equals the $\{i,j\}^{th}$ element of $\mathbf{D}^{K+1}$.

- The claim follows by induction.

Distance

- The **distance** between agents i and j equals the minimum length path connecting them.

- If there is no path connecting i to j, then the distance between them is infinite.

Distance (continued)

- We can use powers of the adjacency matrix to calculate these distances:

$$M_{ij} = \min_k \left\{ k \, : \, D_{ij}^{(k)} > 0 \right\}$$

- If the network consists of a single, giant, connected component, we can compute average path length as

$$\overline{M} = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} M_{ij}. \tag{3}$$

Small World Problem

Frequency of minimum path lengths in the Nyakatoke network

	1	2	3	4	5
Count	490	2666	3298	557	10
Frequency	0.0698	0.3797	0.4697	0.0793	0.0014

Source: de Weerdt (2004) and author's calculations.

Small World Problem (continued)

- Less than 7 percent of all *pairs* of households are directly connected in Nyakatoke.

- ...but over 40 percent dyads are no more than two degrees apart.

- ..and over 90 percent are separated by three or fewer degrees.

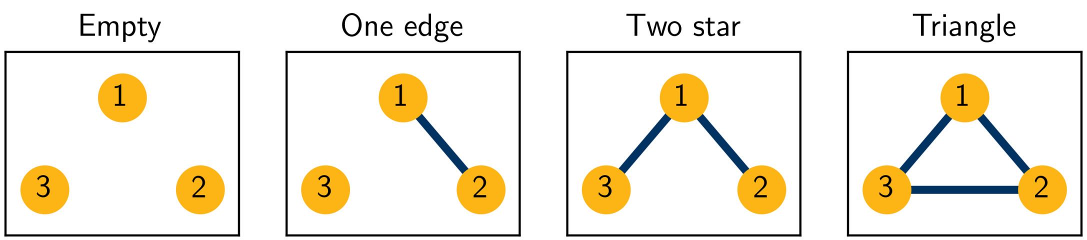
Small World Problem (continued)

- **diameter:** largest distance between two agents.

- The diameter of the Nyakatoke network is 5.

- Small-world problem: why do we see sparsity and low diameter together (Milgram, 1967)?

Triad Census



Triad Census (continued)

- **Triads**, a set of three unique agents, come in four types (isomorphisms):
  - no connections or **empties**
  - one connection or **one-edges**
  - two connections or **two-stars**
  - three connections or **triangles**
- A complete enumeration of them into their four possible types constitutes a *triad census.*

Triad Census: Triangles

- Each agent can belong to as many as $(N-1)(N-2)$ triangles.

- The counts of these triangles are contained in the N diagonal elements of $\mathbf{D}^3$.

- However each such triangle appears 6 times in these counts: as $\{i,j,k\}$, $\{i,k,j\}$, $\{j,i,k\}$, $\{j,k,i\}$, $\{k,i,j\}$ and $\{k,j,i\}$. Thus

$$T_T = \frac{\mathrm{Tr}\left(\mathbf{D}^3\right)}{6} \tag{4}$$

equals the total number of unique triangles in the network.

Triad Census: Two-Stars

- Each dyad can share of up to $N-2$ links in common.

- These counts are contained in the lower (or upper) off-diagonal elements of $\mathbf{D}^2$.

- Each triad appears three times in these counts: as $\{i,j,k\}$, $\{i,k,j\}$ and $\{j,k,i\}$. If it is a
  - two star, then only one of $D_{ji}D_{ki}$, $D_{ij}D_{kj}$, or $D_{ik}D_{jk}$ quantities will equal one,
  - triangle, then all three will equal one.

Two-Stars (continued)

- We have that $\text{vech}\left(\mathbf{D}^2\right)'\iota$ gives the network count of *three times* the number triangles *plus* the number of two-stars.

- Therefore

$$T_{TS} = \text{vech}\left(\mathbf{D}^2\right)'\iota - \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2} \tag{5}$$

equals the number of two-star triads in the network.

Triad Census: One-Edges

- If *all* triads are empty or have only one edge, then there will be $(N-2)\,\text{vech}\,(\mathbf{D})\,\iota$ one edge triads.

- If some triads are two-stars or triangles this count will be incorrect.

- Subtracting twice the number of two stars and three times the number of triangles gives the correct answer:

$$T_{OE} = (N-2)\,\text{vech}\,(\mathbf{D})'\,\iota - 2\text{vech}\left(\mathbf{D}^2\right)'\iota + \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2} \tag{6}$$

Triad Census: Two-Stars

- The number of empty triads, T_E, equals $\binom{N}{3}$ minus the total number of other triad types.

Triad Census: Nyakatoke Network

	empty	one-edge	two-star	triangle
Count	221,189	48,245	4,070	315
Proportion	0.8078	0.1762	0.0149	0.0012
Random	0.8049	0.1812	0.0136	0.0003

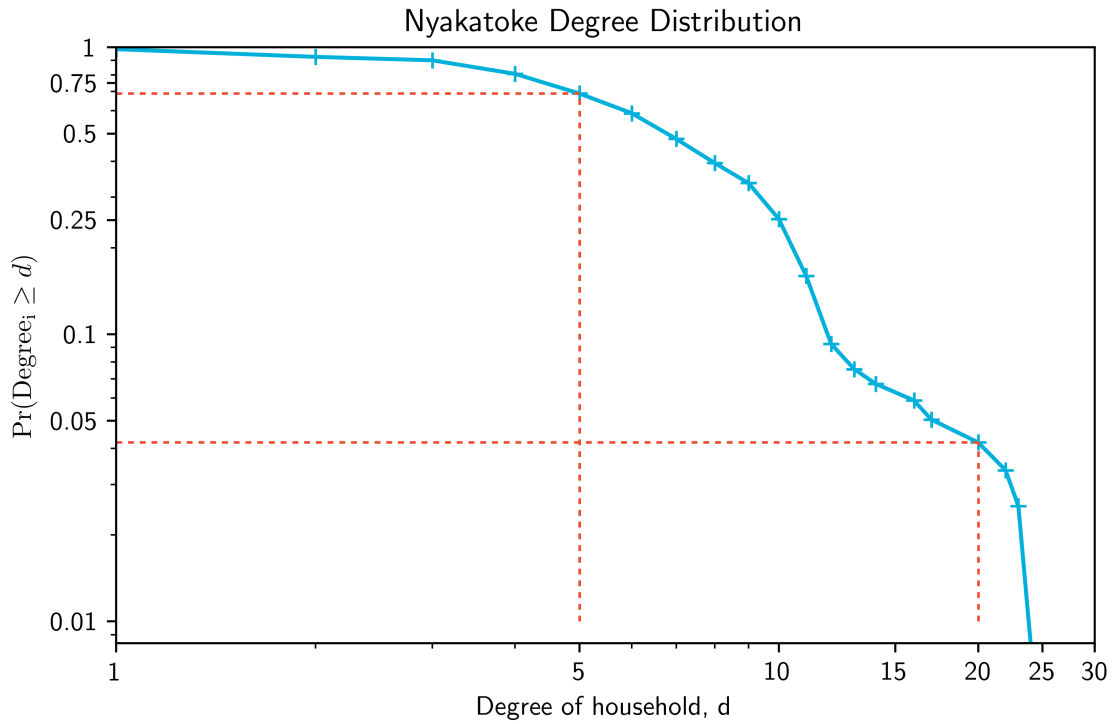
Transitivity

- The **Transitivity Index** (a.k.a. clustering coefficient) is

$$\text{TI} = \frac{3T_T}{T_{TS} + 3T_T}$$

- In random graphs TI should be close to network density.
- For the Nyakatoke network $\text{TI} = 0.1884$ and $P_N = 0.0698$.
- Network transitivity *may* facilitate risk sharing and other activities which require monitoring (cf., Jackson et al., 2012).

Nyakatoke Degree Distribution



Degree Distribution Redux

- Average degree equals $\lambda_N = \left(\frac{2T_{OE}+4T_{TS}+6T_T}{N(N-2)}\right)$.

- Degree variance equals

$$S_N^2 = \frac{2}{N}(T_{TS} + 3T_T) - \lambda_N [1 - \lambda_N].$$

- Knowledge of mean degree, degree variance and the number of triangles is equivalent to knowledge of the triad census.

Degree Distribution Redux (continued)

- The degree distribution constrains other features of the network.

- Models of network formation should allow for arbitrary degree distributions.

Power Laws

- Barabási and Albert (1999) assert that the degree distributions of many networks, at least over some range, follow discrete Pareto or 'power law' distributions (cf., Yule, 1929):

$$F\left(d_{+}\right) = 1 - \left(\frac{\underline{d}_{+}}{d_{+}}\right)^{\alpha}$$

for $d_{+} = \underline{d}_{+}, \ldots, N$ and $F\left(d_{+}\right) = \Pr\left(D_{i+} \leq d_{+}\right)$.

- Here $\underline{d}_{+} > 0$ is some threshold degree level below which the power law distribution may not apply.

Power Laws (continued)

- Taking logs yields the linear relationship

$$\ln\left(1 - F\left(D_{i+}\right)\right) = \alpha \ln\left(\underline{d}_{+}\right) + \alpha \ln D_{i+}.$$

- The coefficient, α, may be estimated by OLS (cf., Clauset, Shalizi and Newman, 2009).

Wrapping Up

- Network data, as encapsulated by adjacency matrices are complex
  - rich combinatoric structure
  - strong dependencies across different statistics of $\mathbf{D}$
- Researchers have motivated the various statistics reviewed here both formally and heuristically.
- ....methods of (frequentist) inference associated with the statistics reviewed here are still under development.