An Incomplete Overview of Strategic Models of Network Formation

Econometric Methods for Networks,

University of Warwick, May 30th to June 1st, 2018

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Overview

- Why study models of network formation?
  - equilibrium $\mathbf{D}$ may be inefficient.
  - planner may have preferences over $\mathbf{D}$ and hence is interested in policies which influence it.
  - we might view network manipulations as a mechanism for influencing other outcomes.
  - correct for "endogenous network formation bias" (cf., Auerbach, 2016)

Existing Approaches

- "Applied theory approach": posit generative models of network formation that match "stylized facts" (Albert and Barabasi, 2002).

- ERGM: directly write down likelihoods for $\Pr(\mathbf{D} = \mathbf{d})$ and try to maximize them (cf., Shalizi and Rinaldo, 2013, *Annals of Statistics*):

  - generally no micro-foundations...

  - ...but see Mele (2017, *Econometrica*)

Existing Approaches

- Random Utility Models (RUM): Sheng (2012), Christakis *et al.* (2012), Imbens and Goldsmith-Pinkham (2013), Graham (2013, 2016, 2017), de Paula et al. (2018), Menzel (2016).

- Specialized structural modes: Banerjee *et al.* (2012).

Random Utility Approach

- This approach is both natural, and familiar, to economists.

- Provides a framework for modeling the effect on link surplus (i.e., utility) of

  - observed agent/dyad covariates;

  - unobserved agent attributes (heterogeneity);

  - preference interdependencies (e.g., a taste for transitivity).

A Simple Model of Network Formation

- Consider a network of three agents: i, j, k
  - Link formation: $D_{ij} = \mathbf{1}\left(\alpha + \beta D_{ik} D_{jk} - U_{ij} \geq 0\right)$ with $\beta \geq 0$ (returns to transitivity).
  - Three "types" of U_{ij} draws: $\mathbb{U}_L = (-\infty, \alpha]$, $\mathbb{U}_M = (\alpha, \alpha + \beta]$ or $\mathbb{U}_H = (\alpha + \beta, \infty)$.
  - Positive measure on the subset of the support of $\mathbf{U} = \left(U_{ij}, U_{ik}, U_{jk}\right)'$ with multiple NE networks.
- The model is *incomplete* (cf., Bresnahan and Reiss, 1991; Tamer, 2003).

A Simple Model of Network Formation (continued)

- There are $3^3 = 27$ "configurations" of $\mathbf{U}$...

- ...but only $\frac{(3+3-1)!}{3!(3-1)!} = 10$ non-isomorphic ones.

- Two of these configurations admit multiple NE networks:
  - if $U_{ij} \in \mathbb{U}_L$, $U_{ik} \in \mathbb{U}_M$ and $U_{jk} \in \mathbb{U}_M$;
  - if $U_{ij} \in \mathbb{U}_M$, $U_{ik} \in \mathbb{U}_M$ and $U_{jk} \in \mathbb{U}_M$.

A Simple Model of Network Formation (continued)

- Only one realization (out of 4 possible realizations of $\mathbf{D}$) is uniquely predicted:
  - if $U_{ij} \in \mathbb{U}_L$, $U_{ik} \in \mathbb{U}_L$ and $U_{jk} \in \mathbb{U}_H$, then links $\{i, j\}$ and $\{i, k\}$ form and $\{j, k\}$ does not.

- cf., Ciliberto and Tamer (2009), Sheng (2012), de Paula et al. (2018) provide methods for analyzing incomplete models of network formation.

- serious challenges to implementation at scale.

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Dyads form links *sequentially* and *myopically.*

- If the linking order is ij, ik and jk we have:

  - $D_{ij} = \mathbf{1}\left(\alpha - U_{ij} \geq 0\right)$;

  - $D_{ik} = \mathbf{1}\,(\alpha - U_{ik} \geq 0)$;

  - $D_{jk} = \mathbf{1}\left(\alpha + \beta \mathbf{1}\left(\alpha - U_{ij} \geq 0\right) \mathbf{1}\,(\alpha - U_{ik} \geq 0) - U_{jk} \geq 0\right)$.

- Conditional on the ij, ik and jk the realization of $\mathbf{U}$ delivers a unique prediction of $\mathbf{D}$.

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Since we don't observe the order of link formation we
  - assign a (prior) distribution to it;
  - work with an integrated likelihood.

- With three dyads there are $3! = 6$ possible link orderings. Let $O \in \mathbb{O} = \{1, 2, 3, 4, 5, 6\}$ be the possible orderings. Our integrated likelihood is

$$\Pr(\mathbf{D} = \mathbf{d}) = \sum_{o \in \mathbb{O}} \Pr(\mathbf{D} = \mathbf{d} | O = o) \Pr(O = o).$$

Christakis, Fowler, Imbens and Kalyanaraman (2010)

- Christakis et al. (2010) use Bayesian MCMC methods
  - provides a method of inference as well;
  - (no large sample theory for their estimator).

- Simulation methods, and assumptions about the timing of link formation, are also central to work by Goldsmith-Imbens and Pinkham (2013), Hsieh & Lee (2016), and Mele (2017).

Pairwise Stability (for NTU models)

Let $\nu_i : \mathbb{D}_N \rightarrow \mathbb{R}$ be a utility function for agent i, which maps adjacency matrices into utils.

The marginal utility for agent i associated with (possible) edge (i, j) is

$$MU_{ij}(\mathbf{D}) = \begin{cases} \nu_i(\mathbf{D}) - \nu_i(\mathbf{D} - ij) & \text{if } D_{ij} = 1 \\ \nu_i(\mathbf{D} + ij) - \nu_i(\mathbf{D}) & \text{if } D_{ij} = 0 \end{cases} \tag{1}$$

where $\mathbf{D} - ij$ is the adjacency matrix associated with the network obtained after deleting edge (i, j) and $\mathbf{D} + ij$ the one obtained via link addition.

Pairwise Stability (continued)

The network G is pairwise stable if (i) no agent wishes to dissolve a link

$$\forall (i,j) \in \mathcal{E}(G),\ MU_{ij}(\mathbf{D}) \geq 0 \text{ and } MU_{ji}(\mathbf{D}) \geq 0 \tag{2}$$

and (ii) no pair of agents wishes to form a link

$$\forall (i,j) \notin \mathcal{E}(G),\ MU_{ij}(\mathbf{D}) > 0 \Rightarrow MU_{ji}(\mathbf{D}) < 0. \tag{3}$$

1. utility is *nontransferable* across agents;

2. the strategic moniker aside, pairwise stability is a really non-strategic/myopic notion of equilibrium.

Miyauchi (2016)

Nice example of simple, clear and very elegant applied econometric research.

Draws on insights from the theory of supermodular games (and their estimation).

Inference requires many independent networks, but could be generalized to allow for a single large network.

Miyauchi (continued)

Consider the mapping $\varphi(\mathbf{D}) : \mathbb{D}_N \to \mathbb{I}_{\binom{N}{2}}$:

$$\varphi(\mathbf{D}) \equiv \begin{bmatrix} \mathbf{1}(MU_{12}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{21}(\mathbf{D}) \geq 0) \\ \mathbf{1}(MU_{13}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{31}(\mathbf{D}) \geq 0) \\ \vdots \\ \mathbf{1}(MU_{N-1N}(\mathbf{D}) \geq 0)\,\mathbf{1}(MU_{NN-1}(\mathbf{D}) \geq 0) \end{bmatrix}. \tag{4}$$

Observe that $\mathbf{1}\left(MU_{ij}(\mathbf{D}) \geq 0\right)\mathbf{1}\left(MU_{ji}(\mathbf{D}) \geq 0\right)$ equals 1 if condition (i) of pairwise stability holds (which implies edge (i,j) is present)...

...and zero otherwise (which implies condition (ii) and hence the absence of edge (i,j)).

Miyauchi (continued)

If the observed network is pairwise stable, its adjacency matrix is the fixed point

$$\mathbf{D} = \text{vech}^{-1}\left[\varphi\left(\mathbf{D}\right)\right]. \tag{5}$$

There may, of course, be many $\mathbf{d} \in \mathbb{D}_N$ such that $\mathbf{d} = \text{vech}^{-1}\left[\varphi\left(\mathbf{d}\right)\right]$.

Miyauchi (continued)

If the preference profile $\{\nu_i\}_{i=1}^N$ implies that the marginal utilities $MU_{ij}(\mathbf{d})$ are weakly increasing in $\mathbf{d}$ for all i and j (restrictive), then one can invoke Tarski's fixed point theorem.

The set of pairwise stable networks corresponds to a complete lattice with a *maximum* and *minimum equilibrium*.

Minimum equilibrium, say $\underline{d}$, can be computed by fixed point iteration starting from the empty adjacency matrix.

Maximum equilibrium, say $\overline{\mathbf{d}}$, from the complete graph K_N.

Miyauchi: Parametric Example

Let

$$\nu_i (\mathbf{d}, \mathbf{U}; \theta_0) = \sum_j d_{ij} \left[\alpha_0 + \beta_0 \left[\sum_k d_{ik} d_{jk} \right] - U_{ij} \right], \qquad (6)$$

with $\mathbf{U} = \left[U_{ij} \right]$, $\theta = (\alpha, \beta)'$.

The elements of $\left\{ U_{ij} \right\}_{i<j}$, as is common in discrete choice analysis, are assumed to be i.i.d random draws from some known distribution (e.g, the standard Normal or Logistic distribution).

The marginal utility agent i gets from a link with j is

$$MU_{ij} (\mathbf{d}, \mathbf{U}; \theta_0) = \alpha_0 + \beta_0 \left[\sum_k d_{ik} d_{jk} \right] - U_{ij} \qquad (7)$$

Miyauchi: Parametric Example (continued)

For a given draw of $\mathbf{U}$ and value of θ we can compute minimum and maximum equilibria, respectively $\underline{d}(\mathbf{U};\theta)$ and $\overline{\mathbf{d}}(\mathbf{U};\theta)$, by fixed point iteration.

Let $\underline{G}_N(\mathbf{U};\theta)$ and $\overline{G}_N(\mathbf{U};\theta)$ be the graphs corresponding to these adjacency matrices.

Using these graphs we can compute, for example, the injective homomorphism frequencies $t_{\text{hom}}(S, \underline{G}_N(\mathbf{U};\theta))$ and $t_{\text{hom}}\left(S, \overline{G}_N(\mathbf{U};\theta)\right)$ for $S = \wedge, \triangle$ etc.

Miyauchi: Parametric Example (continued)

The homomorphism frequencies,

$$t_{\text{hom}}\left(S, \underline{G}_N\left(\mathbf{U};\theta\right)\right) \;\&\; t_{\text{hom}}\left(S, \overline{G}_N\left(\mathbf{U};\theta\right)\right)$$

correspond to specific draws of $\mathbf{U}$ and values of θ.

Using simulation to integrate out $\mathbf{U}$, yields the vector

$$\underline{\pi}\left(\theta\right) = \frac{1}{B}\sum_{b=1}^{N}\begin{pmatrix} t_{\text{hom}}\left(S_1, \underline{G}_N\left(\mathbf{U}^{(b)};\theta\right)\right) \\ \vdots \\ t_{\text{hom}}\left(S_J, \underline{G}_N\left(\mathbf{U}^{(b)};\theta\right)\right) \end{pmatrix}$$

for $\mathbf{U}^{(1)}, \mathbf{U}^{(2)}\ldots, \mathbf{U}^{(B)}$ a sequence of independent random utility shifter profiles and $S_1, \ldots, S_J$ a set of J identifying motifs of interest.

Similar calculation gives $\overline{\pi}\left(\theta\right)$.

Miyauchi: Parametric Example (continued)

The econometrician observes of $c = 1, \ldots, C$ independent networks, with, in a slight change relative to earlier notation, G_c denoting the c^{th} network/graph.

Let $\pi(G_c)$ be the vector of $S_1, \ldots, S_J$ homomorphism frequencies as observed in the c^{th} network.

Let $\underline{\pi}_c(\theta)$ and $\overline{\pi}^c(\theta)$ be the corresponding expected frequencies at the minimum and maximum pairwise stable equilibria for that network at parameter θ.

Miyauchi: Parametric Example (continued)

Consider adding to the set-up the assumption that agents select the maximum equilibrium (cf., Jia, 2008).

In that case

$$\mathbb{E}\left[\bar{\pi}_c\left(\theta_0\right) - \pi\left(G_c\right)\right] = 0, \tag{8}$$

is a valid moment condition.

If the set of chosen motifs is sufficiently rich so as to point identify θ, then consistent estimation of θ_0 by the method of simulated moments is straightforward.

Miyauchi: Set Identification

Without an equilibrium selection assumption we have the pair of moment inequalities

$$\mathbb{E}\left[\bar{\pi}_c(\theta_0) - \pi(G_c)\right] \geq 0 \qquad (9)$$
$$\mathbb{E}\left[\underline{\pi}_c(\theta_0) - \pi(G_c)\right] \leq 0.$$

Confidence intervals which asymptotically cover θ_0 with probability at least $1-\alpha$ can be constructed using the approach outlined by, for example Andrews and Soares (2010).

Wrap-Up

Identification and estimation of strategic models of network formation is an active area: Leung (2015), Sheng (2013), Menzel (2015) etc.

Authors take various modeling approaches; no master theory or framework yet.

Key challenge is to connect economic models with exchangeable random graph theory discussed earlier.

Undertaking empirical work is key to making progress...