Describing Social Networks

Econometric Methods for Networks,

CORE, Dec 12th - 14th, 2016

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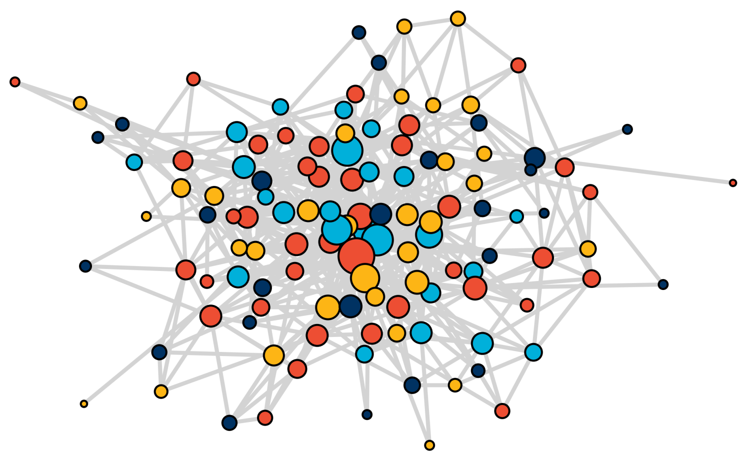
Nyakatoke Risk-Sharing Network



Note: node sizes are proportional to household degree

Wealth $<$ 150,000 TSh	300,000 TSh Wealth $<$ 600,000 TSh
150,000 TSh Wealth $<$ 300,000 TSh	Wealth $\geq$ 600,000 TSh

(N = 119, n = 7,021)

Questions

- How do the number, structure and characteristics of an agent's ties influence her behaviors and outcomes?

- How are ties formed? Are externalities involved?

- What configuration of ties would a social planner choose?

  - How does this idealized network compare with the observed one?

  - Are observed networks efficient?

Questions (continued)

- Can we identify "important" agents in the network? Why is this interesting?

- What policies influence network structure (and outcomes)?

- How does network structure influence the diffusion of disease, ideas and new technologies?

- Are there optimal locations on a network in which to intervene?

Applications...

- Buyer-seller networks (Industrial Organization)
- Friendship networks (Education, Labor)
- Criminal networks (Urban)
- Trading networks (Industrial Organization and International Trade)
- Political networks (Political Economy)
- Bank networks (Finance)
- Online networks

...and Funding!

- SBE Directorate of the National Science Foundation (NSF) recently identified network analysis as one of five key "cross cutting themes" with special grant opportunities.

Literatures

- Psychology, sociology, anthropology, political science and economics all have empirical and theoretical literatures on "networks".

  - Wasserman & Faust (1994)

  - Jackson (2008)

- Networks are widely-studied in Physics.

  - Newman (2010)

Literatures

- The mathematical representation of networks as graphs makes discrete math (esp. graph theory), matrix analysis, and computer science highly useful.

- The statistical/econometric literature *very* underdeveloped (cf., Goldenberg *et al*. 2009)

- ...but growing rapidly (e.g., Bickel & Chen, 2009; Bickel, Chen & Levina, 2011; Graham, 2015; de Paula, 2016).

Outline of Course

- Lecture 1 (12/12/16): Describing social networks
  - introduction to network data
  - definition and computation of basic summary network statistics
- Lecture 2 (12/12/16): Nonparametrics
  - graphons, graph limits
  - nonparametric estimation of link probabilities
- Lecture 3 (12/13/16): Inference
  - network moments
  - network subsampling/bootstrap

Outline of Course (continued)

- Lecture 4 (12/13/16): Link formation #1
  - importance sampling from networks w/ fixed degree
  - Dyadic models of link formation
- Lecture 5 (12/14/16): Link formation #2
  - network formation w/ interdepedencies
  - strategic models
- Lecture 6 (12/14/16): Peer effects
  - network structure & peer effects
  - neighborhood effects

Computation

- Computational illustrations in class

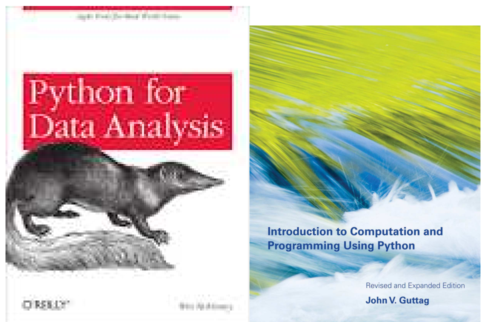
- All code is available on the course GitHub repository (`https://github.com/bryangraham/short_courses`)

- If you want to follow along (recommend, but not required) use the *Anaconda* distribution of Python v 2.7.12 `https://www.continuum.io/downloads`

- Includes key packages for data analysis & scientific computing (e.g., numpy, scipy, pandas, networkx)

- Also useful: Graphviz (visualation), Yhat Rodeo (IDE)

Computation (continued)



https://github.com/wesm/pydata-book

QuantEcon

http://quant-econ.net/

Basic Terms & Notation

- An **undirected graph** $G(\mathcal{N}, \mathcal{E})$ consists of a set of **nodes** $\mathcal{N} = \{1, \ldots, N\}$ and a list of unordered pairs of nodes called **edges** $\mathcal{E} = \{\{i,j\}, \{k,l\}, \ldots\}$ for $i, j, k, l \in \mathcal{N}$.

- A graph is conveniently represented by its **adjacency matrix** $\mathbf{D} = \left[D_{ij}\right]$ where

$$D_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

- No self-ties & unordered edges $\Rightarrow \mathbf{D}$ is a symmetric binary matrix with a diagonal of so-called structural zeros.

- vertex: node, agent or player.

- edges: links, friendships, connections or ties.

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 1 is connected to agents 2 and 5.
- Agent 2 is connected to agent 1.
- Agent 3 is connected to no one.
- Agent 4 is connected to agent 5.

Basic Terms & Notation (continued)

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

- Agent 5 is connected to agents 1 and 4.
- Agents 2 and 5 are indirectly connected through agent 1 (i.e., share her as a common friend).
- Agents 2 and 4 are indirectly connected through agents 1 and 5.
- 3 out of 10 possible ties are present in the network.

Agents, Dyads and Triads

- A network consists of
  - N agents
  - $\binom{N}{2} = \frac{1}{2}N(N-1) = O\left(N^2\right)$ pairs of agents or **dyads.**
  - $\binom{N}{3} = \frac{1}{6}N(N-1)(N-2) = O\left(N^3\right)$ triples of agents of **triads**.
  - $\binom{N}{4} = \frac{1}{24}N(N-1)(N-2)(N-3) = O\left(N^4\right)$ triples of agents of **tetrads**.

- In summarizing a network adjacency matrix it is convenient to conceptualize statistics as measures of agent-, dyad-, triad- or p-subgraph-level attributes.

Agent-level Statistics: Degree

- The total number of links belonging to agent i, or her **degree** is $D_{i+} = \sum_j D_{ij}$.

- The **degree sequence** of a network is $\mathbf{D}_+ = \left(D_{1+}, \ldots, D_{N+}\right)'$.

- The **degree distribution** gives the frequency of each possible agent-level degree count $\{0, 1, \ldots, N\}$ in the network.

Degree (continued)

- Some researchers take the degree distribution as their primary object of interest (e.g., Barabási and Albert, 1999).
  - Other key topological features of a network are fundamentally constrained by its degree distribution.

- Some datasets report agent degrees with no other network information

Dyad-level Statistics: Density

- Dyads are either linked or unlinked.

- The **density** of a network equals the frequency with which any randomly drawn dyad is linked:

$$P_N = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} D_{ij}. \qquad (2)$$

- Note that $\lambda_N = (N-1)\, P_N$ coincides with average degree.

- The density of the Nyakatoke network is 0.0698.

- Low density and skewed degree distributions (with fat tails) are common features of real world social networks.

Paths

$$\mathbf{D}^2 = \begin{pmatrix} D_{1+} & \sum_i D_{1i}D_{2i} & \cdots & \sum_i D_{1i}D_{Ni} \\ \sum_i D_{1i}D_{2i} & D_{2+} & \cdots & \sum_i D_{2i}D_{Ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_i D_{1i}D_{Ni} & \sum_i D_{2i}D_{Ni} & \cdots & D_{N+} \end{pmatrix}$$

- The i^{th} diagonal element of $\mathbf{D}^2$ equals the number of agent i's links or her degree.

- The $\{i, j\}^{th}$ element of $\mathbf{D}^2$ gives the number of links agent i has in common with agent j (i.e., the number of "friends in common").

Paths (continued)

- graph theory: the $\{i,j\}^{th}$ element of $\mathbf{D}^2$ gives the number of **paths** of length two from agent i to agent j.

- if i and j share the common friend k, then a length two path from i to j is given by $i \to k \to j$.

Paths (continued)

$$\mathbf{D}^3 = \begin{pmatrix} \sum_{i,j} D_{1i}D_{ij}D_{j1} & \cdots & \sum_{i,j} D_{1i}D_{ij}D_{jN} \\ \vdots & \ddots & \vdots \\ \sum_{i,j} D_{1i}D_{ij}D_{jN} & \cdots & \sum_{i,j} D_{Ni}D_{ij}D_{jN} \end{pmatrix}$$

- $\{i,j\}^{th}$ element gives the number of paths of length 3 from i to j.

- If both i and j are connected to k as well as to each other, then the $\{i,j,k\}$ triad is transitive (i.e., "the friend of my friend is also my friend").

Paths (continued)

- The i^{th} diagonal element $\mathbf{D}^3$ is a count of the number of transitive triads or **triangles** to which i belongs (with $i{-}j{-}k$ and $i{-}k{-}j$ counted separately).
  - If $\{i, j, k\}$ is a closed triad it is counted twice each in the i^{th}, j^{th} and k^{th} diagonal elements of $\mathbf{D}^3$.
  - $\mathrm{Tr}\left(\mathbf{D}^3\right)/6$ equals the number of *unique* triangles in the network.

K-Length Paths

- The $\{i,j\}^{th}$ element of $\mathbf{D}^K$ gives the number of paths of length K from agent i to agent j.

- Let $D_{ij}^{(K)}$ denote the $\{i,j\}^{th}$ element of $\mathbf{D}^K$.

- $\mathbf{D}^0 = I_N$, the only zero length walks in the network are from each agent to herself.

- Under the maintained hypothesis, $D_{ij}^{(K)}$ equals the number of K-length paths from i to j. The number of $K+1$ length paths from i to j then equals
$$\sum_{k=1}^{N} D_{ik}^{(K)} D_{kj},$$
which equals the $\{i,j\}^{th}$ element of $\mathbf{D}^{K+1}$. The claim follows by induction.

Distance

- The **distance** between agents i and j equals the minimum length path connecting them.

- If there is no path connecting i to j, then the distance between them is infinite.

- We can use powers of the adjacency matrix to calculate these distances:

$$M_{ij} = \min_k \left\{ k : D_{ij}^{(k)} > 0 \right\}$$

- If the network consists of a single, giant, connected component, we can compute average path length as

$$\overline{M} = \binom{N}{2}^{-1} \sum_{i=1}^{N} \sum_{j<i} M_{ij}. \qquad (3)$$

Small World Problem

Frequency of minimum path lengths in the Nyakatoke network

	1	2	3	4	5
Count	490	2666	3298	557	10
Frequency	0.0698	0.3797	0.4697	0.0793	0.0014

Source: de Weerdt (2004) and author's calculations.

- Less than 7 percent of all *pairs* of households are directly connected in Nykatoke.
- ...but over 40 percent dyads are no more than two degrees apart.
- ..and over 80 percent are separated by three or fewer degrees.

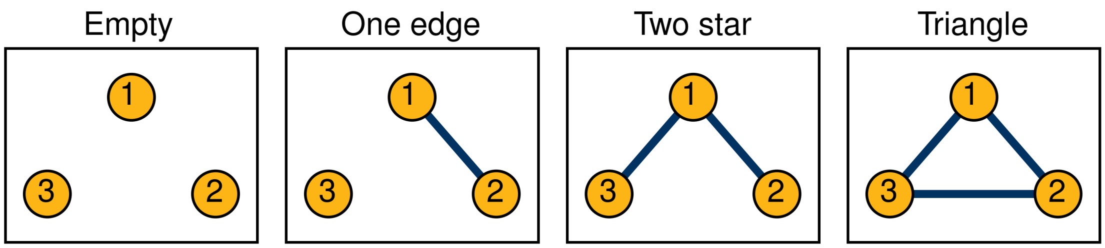
Small World Problem (continued)

- **diameter:** largest distance between two agents.

- The diameter of the Nyakatoke network is 5.

- Small-world problem: why do we see sparsity and low diameter together (Milgram, 1967)?

Triad Census



- **Triads**, a set of three unique agents, come in four types:
  - no connections or **empties**
    - one connection or **one-edges**
    - two connections or **two-stars**
    - three connections or **triangles**
  - A complete enumeration of them into their four possible types constitutes a *triad census.*

Triad Census: Triangles

- Each agent can belong to as many as

$$(N-1)(N-2)$$

triangles.

- The counts of these triangles are contained in the N diagonal elements of $\mathbf{D}^3$.

- However each such triangle appears 6 times in these counts: as $\{i,j,k\}$, $\{i,k,j\}$, $\{j,i,k\}$, $\{j,k,i\}$, $\{k,i,j\}$ and $\{k,j,i\}$. Thus

$$T_T = \frac{\mathrm{Tr}\left(\mathbf{D}^3\right)}{6} \tag{4}$$

equals the total number of unique triangles in the network.

Triad Census: Two-Stars

- Each dyad can share of up to $N-2$ links in common.

- These counts are contained in the lower (or upper) off-diagonal elements of $\mathbf{D}^2$.

- Each triad appears three times in these counts: as $\{i, j, k\}$, $\{i, k, j\}$ and $\{j, k, i\}$. If it is a
  - two star, then only one of $D_{ji}D_{ki}$, $D_{ij}D_{kj}$, or $D_{ik}D_{jk}$ quantities will equal one,
  - triangle, then all three will equal one.

Two-Stars (continued)

- $\Rightarrow$ vech $\left(\mathbf{D}^2\right)'\iota$ gives the network count of *three times* the number triangles *plus* the number of two-stars, hence

$$T_{TS} = \text{vech}\left(\mathbf{D}^2\right)'\iota - \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2} \qquad (5)$$

equals the number of two-star triads in the network

Triad Census: One-Edges & Empties

- If *all* triads are empty or have only one edge, then there will be $(N-2)\,\text{vech}\,(\mathbf{D})\,\iota$ one edge triads.

- If some triads are two-stars or triangles this count will be incorrect.

- Subtracting twice the number of two stars and three times the number of triangles gives the correct answer:

$$T_{OE} = (N-2)\,\text{vech}\,(\mathbf{D})'\,\iota - 2\text{vech}\left(\mathbf{D}^2\right)'\iota + \frac{\text{Tr}\left(\mathbf{D}^3\right)}{2} \tag{6}$$

- The number of empty triads, T_E, equals $\binom{N}{3}$ minus the total number of other triad types.

Triad Census: Nyakatoke Network

	empty	one-edge	two-star	triangle
Count	221,189	48,245	4,070	315
Proportion	0.8078	0.1762	0.0149	0.0012
Random	0.8049	0.1812	0.0136	0.0003

Transitivity

- The **Transitivity Index**, sometimes called the clustering coefficient, is

$$
\begin{aligned}
\text{TI} &= \frac{3T_T}{T_{TS} + 3T_T} \\
&= \frac{1}{2}\frac{\text{Tr}\left(\mathbf{D}^3\right)}{\text{vech}\left(\mathbf{D}^2\right)'\iota}
\end{aligned}
$$

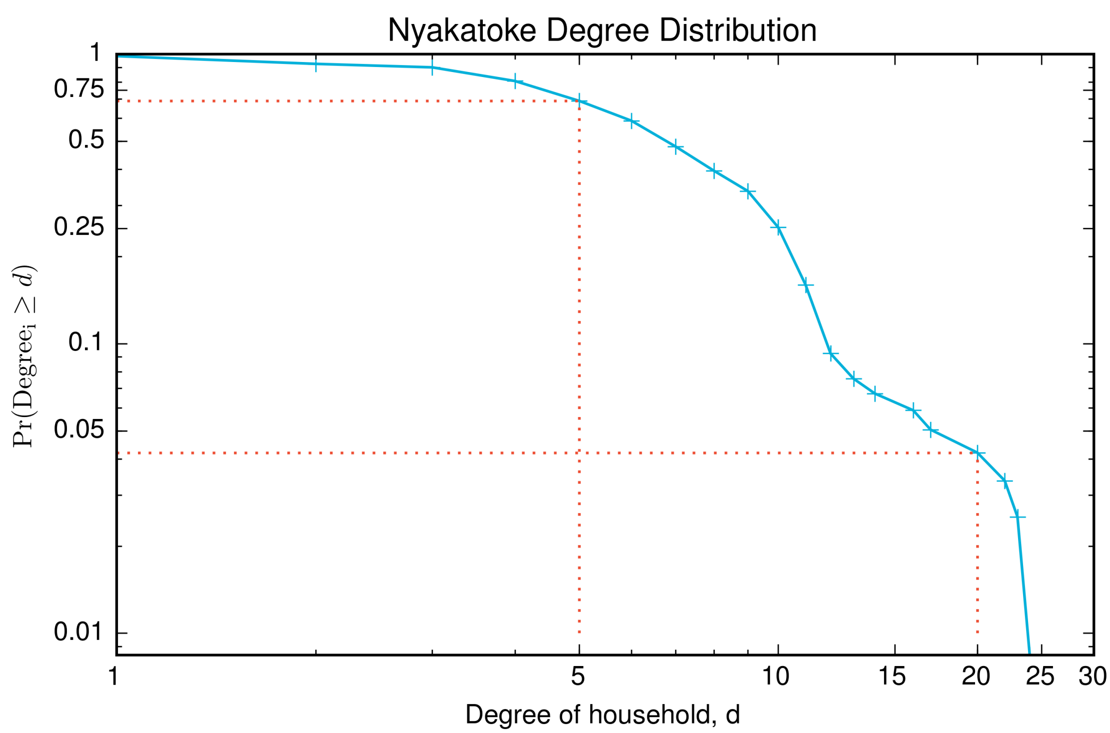
- In random graphs TI should be close to network density.

- For the Nyakatoke network TI = 0.1884 and $P_N = 0.0698$.

- Network transitivity *may*

  – facilitate risk sharing and other activities which require monitoring (cf., Jackson et al., 2012).

Nyakatoke Degree Distribution



Degree Distribution Redux

- Average degree equals $\lambda_N = \left(\frac{2T_{OE}+4T_{TS}+6T_T}{N(N-2)}\right)$.

- Degree variance equals

$$S_N^2 = \frac{2}{N}\left(T_{TS}+3T_T\right) - \lambda_N\left[1-\lambda_N\right].$$

- Knowledge of mean degree, degree variance and the number of triangles is equivalent to knowledge of the triad census.

- The degree distribution constrains other features of the network.
  - models of network formation should allow for arbitrary degree distributions.

Power Laws

- Barabási and Albert (1999) assert that the degree distribution of many networks, at least over some range, follow discrete Pareto or 'power law' distributions:

$$F\left(d_{+}\right) = 1 - \frac{A}{1-\alpha} d_{+}^{1-\alpha}$$

for $d_{+} = \underline{d}_{+}, \ldots, N$ and $F\left(d_{+}\right) = \Pr\left(D_{i+} \leq d_{+}\right)$.

- Here $\underline{d}_{+} > 0$ is some threshold degree level below which the power law distribution may not apply.

- Taking logs yields the linear relationship

$$\ln\left(1 - F\left(D_{i+}\right)\right) = \ln\left(\frac{A}{1-\alpha}\right) + (1-\alpha)\ln D_{i+}.$$

The coefficient, $1-\alpha$, may be estimated by OLS (cf., Clauset, Shalizi and Newman, 2009).

Centrality

- Questions:
  - removal of what agent would reduce crime the most in a criminal network?
  - "where" should a policy-maker introduce new technologies/innovations?

- For some policy questions it is useful to have a measure of an agent's "centrality" in a network.

Eigenvector Centrality

- Bonacich (1972) recursively defined an agent's **centrality,** power, or importance within a network, $c_i^{\mathsf{EC}}(\mathbf{D},\phi)$, to be proportional to the sum of her links to other agents, weighted by their own centralities.

- Letting $\mathbf{c}^{\mathsf{EC}}(\mathbf{D},\phi)$ be the N vector of centrality measures this gives

$$\mathbf{c}^{\mathsf{EC}}(\mathbf{D},\phi) = \phi\mathbf{D}\mathbf{c}^{\mathsf{EC}}(\mathbf{D},\phi).$$

- Since $\left(\mathbf{D} - \frac{1}{\phi}I_N\right)\mathbf{c}^{\mathsf{EC}}(\mathbf{D},\phi) = 0$ Bonacich's measure corresponds to a normalized eigenvector of $\mathbf{D}$.

- Typically $\phi = 1/\lambda_{\mathsf{max}}$, with λ_{max} the largest eigenvalue of $\mathbf{D}$, is used for normalization (this ensures positive centrality measures).

Katz-Bonacich Centrality

- This measure is increasing in the number of direct friends and indirect friends, with weights discounted according to the degree of separation.

- The $N \times 1$ vector of centrality measures for each agent is:

$$
\begin{aligned}
\mathbf{c}^{\mathsf{KB}}(\mathbf{D}, \phi) &= \phi \mathbf{D} \iota_N + \phi^2 \mathbf{D}^2 \iota_N + \phi^3 \mathbf{D}^3 \iota_N + \cdots \\
&= \left(I_N + \phi \mathbf{D} + \phi^2 \mathbf{D}^2 + \cdots \right) (\phi \mathbf{D} \iota_N) \\
&= \sum_{k=0}^{\infty} \phi^k \mathbf{D}^k \cdot (\phi \mathbf{D} \iota_N)
\end{aligned}
$$

Katz-Bonacich Centrality (continued)

- For $\phi < 1/\lambda_{\mathsf{max}}$ the sequence converges that so:

$$\mathbf{c}^{\mathsf{KB}}(\mathbf{D}, \phi) = (I_N - \phi \mathbf{D})^{-1} (\phi \mathbf{D} \iota_N).$$

- For $\phi \to 1/\lambda_{\mathsf{max}}$ from below $\mathbf{c}^{\mathsf{KB}}(\mathbf{D}, \phi) \to \mathbf{c}^{\mathsf{EC}}(\mathbf{D}, \phi)$.

- Related to equilibrium effort in quadratic complementarity games on networks (e.g., Jackson and Zenou (2015)).

Diffusion Centrality

- Consider the following diffusion process (cf., Banerjee *et al.*, 2013):

  1. An "idea" is injected at node i.

  2. In period 1 i shares this idea with her friends with probability ϕ.

  3. In period 2 i again shares with probability ϕ, any friends with knowledge of the idea share with their friends with probability ϕ, etc.

Diffusion Centrality (continued)

- After T periods the expected *total* number of time *all* nodes hear about a new idea originating from agent i (including repetitions), is given by the i^{th} element of

$$\begin{aligned} \mathbf{c}^{\mathsf{DC}}(\mathbf{D}, \phi, T) &= \left[\sum_{t=1}^{T} (\phi\mathbf{D})^t\right] \iota_N \\ &= \left[\sum_{t=0}^{T-1} (\phi\mathbf{D})^t\right] (\phi\mathbf{D})\, \iota_N. \end{aligned}$$

- So that as $T \to \infty$, we have $\mathbf{c}^{\mathsf{DC}}(\mathbf{D}, \phi, T) \to \mathbf{c}^{\mathsf{KB}}(\mathbf{D}, \phi)$ for $\phi < 1/\lambda_{\mathsf{max}}$.

Wrapping Up

- Network data, as encapsulated by adjacency matrices are complex
  - rich combinatoric structure
  - strong dependencies across different statistics of $\mathbf{D}$
- Researchers have motivated the various statistics reviewed here both formally and heuristically.
- ....methods of (frequentist) inference associated with the statistics reviewed here are still under development.
  - how do we compute a asymptotic standard error for average degree?