Beta Model & Network Simulation

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Introduction

- In practice methods of inference used in "network science" are decidedly heuristic/approximate.

- Widespread awareness among researchers of the lack of a coherent body of large network distribution theory.

Introduction (continued)

- Is the network in hand especially "transitive"?
  - Compare with Erdos-Renyi random graph; but are measured differences statistically significant? (how do we conceptualize "sampling error")
  - Compare with a large set of (empirical) reference graphs. Is the graph of interest unusual (cf., Milo et al., 2002)?
  - Combine an ad hoc and/or approximate variance estimate with a normal reference distribution (w/o limit theory it is difficult to evaluate this approach; but see earlier lectures)

Inference: Exact w/ Strong Null

- Blitzstein and Diaconis (2011) – additional work in both machine learning and statistics.

- Look at a reference set of graphs (e.g., all graphs with degree sequences identical to the graph of interest)
  - Is transitivity (for example) in the graph *in hand* high relative to this reference group? (exact p-value approach);
  - Computational challenge: how to enumerate, or draw uniformly, from reference graph distribution.

Inference: Asymptotic

- Earlier lecture: Bickel, Chen & Levina and Bhattacharya and Bickel (2015):
  - Derive limit theory for network statistics (specifically normalized subgraph counts);
  - Challenge is also computational – both statistics and their variance estimates are hard to construct.

<u>Beta Model</u>

- Models with network externalities are attractive because
  - they capture what is believed to be an *a priori* important feature of link formation;
  - they generate clustering, which we observe in real word networks.

Beta Model (continued)

- An alternative (ideally complementary) way to generate clustering is to introduce unobserved, *agent-level*, heterogeneity.
  - beta model: $D_{ij} = \mathbf{1}\left(A_i + A_j - U_{ij} \geq 0\right)$;
  - A_i measures attractiveness, trustworthiness, productivity etc;
  - Distribution of $\mathbf{A}$ is unrestricted; components of $\mathbf{U}$ are i.i.d. (logistic).
- cf. "state dependence vs. heterogeneity" in dynamic discrete choice analysis (Heckman, 1978; 1981a-c; Chamberlain, 1985).

Beta Model (continued)

- Assuming U_{ij} i.i.d. logistic yields a link probability of

$$\Pr\left(D_{ij} = 1 \middle| \mathbf{A}\right) = \frac{\exp\left(A_i + A_j\right)}{1 + \exp\left(A_i + A_j\right)} = \frac{\exp\left(W_{ij}'\mathbf{A}\right)}{1 + \exp\left(W_{ij}'\mathbf{A}\right)}$$

with W_{ij} the $N \times 1$ vector with a one for its i^{th} and j^{th} elements and zeros elsewhere.

- Choosing $A_i = -\frac{1}{2}\ln\left(\frac{p}{1-p}\right)$ for $i = 1, \ldots, N$ yields the Erdos-Renyi random graph model.

Beta Model (continued)

The likelihood, $\Pr(\mathbf{D} = \mathbf{d}|\,\mathbf{A})$, includes $\binom{N}{2}$ conditional independent components:

$$\Pr(\mathbf{D} = \mathbf{d}|\,\mathbf{A}) = \prod_{i=1}^{N} \prod_{j<i} \left[\frac{\exp\left(W'_{ij}\mathbf{A}\right)}{1+\exp\left(W'_{ij}\mathbf{A}\right)}\right]^{d_{ij}} \left[\frac{1}{1+\exp\left(W'_{ij}\mathbf{A}\right)}\right]^{1-d_{ij}}.$$

...but "only" N parameters.

Model is non-standard since the dimension of the parameter space grows with N.

Beta Model (continued)

- Manipulating the likelihood gives the exponential family representation

$$\Pr\left(\mathbf{D}=\mathbf{d}\,|\,\mathbf{A}\right)=c\left(\mathbf{A}\right)\exp\left(T\left(\mathbf{d}\right)'\mathbf{A}\right) \tag{1}$$

where

$$T\left(\mathbf{d}\right) = \begin{pmatrix} d_{1+} & \cdots & d_{N+} \end{pmatrix}' = \mathbf{d}_{+}.$$

- The network's *degree sequence*, is a sufficient statistic for $\mathbf{A}$.

Beta Model (continued)

- The beta model allows for networks with arbitrary degree distributions.

- Despite its simplicity it is reasonable flexible (N parameters) and provides a useful benchmark model for hypothesis testing purposes.

Beta Model (continued)

- Let $\mathbb{D}_{N,\mathbf{d}_+}$ denote the set of all networks with N agents and degree sequence $\mathbf{D}_+ = \mathbf{d}_+$.

- Let $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ denote the cardinality of $\mathbb{D}_{N,\mathbf{d}_+}$.

  – $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ is generally *huge*, even for small N.

- Under the β-model the probability distribution of networks conditional on their degree sequence is uniform:

$$\Pr\left(\mathbf{D} = \mathbf{d} \middle|\, \mathbf{d} \in \mathbb{D}_{N,\mathbf{d}_+}\right) = \frac{1}{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|}.$$

Testing

- Let $S(\mathbf{D})$ be some statistic of the adjacency matrix
  - examples: transitivity index, diameter, number of K-length paths etc.

- Let $S(\mathbf{d})$ be the value of the statistic in the observed network.

- We seek to evaluate

$$\Pr\left(S(\mathbf{D}) \leq S(\mathbf{d}) \middle| \, \mathbf{D} \in \mathbb{D}_{\mathbf{N},\mathbf{d}_+} \right) = \frac{\sum_{\mathbf{v} \in \mathbb{D}_{\mathbf{N},\mathbf{d}_+}} \mathbf{1}\left(S(\mathbf{v}) \leq S(\mathbf{d})\right)}{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|}. \tag{2}$$

Testing: Intuition

If the probability that measured transitivity, in a network randomly drawn from the null distribution, lies above observed transitivity is very low...

...we take that as evidence against the β-model and "reject".

Testing

- This approach to testing is
  - very precise about its description of the null hypothesis;
  - exact.
- no alternative hypothesis is specified...
- ...however the choice of statistic should be guided by researcher intuitions about what departures from the null model are of particular concern.

Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$

- Direct enumeration of all the elements of $\mathbb{D}_{N,\mathbf{d}_+}$ is generally not feasible.

- Need a method of sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ *uniformly* and also estimating its size (implement an approximation of the ideal test).

Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ (continued)

- Blitzstein and Diaconis (2010) develop a sequential importance sampling algorithm for uniformly sampling from $\mathbb{D}_{N,\mathbf{d}_+}$

- Two challenges:
  - how to generate a random draw from $\mathbb{D}_{N,\mathbf{d}_+}$;
  - how to do so uniformly (importance weights).

Graphical Integer Sequences

- To construct $\mathbf{D}$ we begin with a matrix of zeros and sequentially add links to it until its rows and columns sum to the target degree sequence.

- Problem is that unless links are added carefully it is easy to get "stuck" (cf., Snijders, 1991).

- The key is to check whether residual degree sequences are graphical as you add links (avoid dead ends).

- $\mathbf{D}_{+} = (2, 2, 1)$ is not graphic

Graphical Integer Sequences (continued)

- Erdos and Gallai (1961) showed $\mathbf{D}_+$ is graphical if and only if $\sum_{i=1}^{N} D_{i+}$ is even and

$$\sum_{i=1}^{k} D_{i+} \leq k\,(k-1) + \sum_{i=k+1}^{N} \min\Big(k, D_{i+}\Big) \text{ for each } k \in \{1, \ldots, N\}\,.$$

Graphical Integer Sequences (continued)

Necessity:

- even: if i is linked to j, then the link is counted in both D_{i+} and D_{j+}.

- For any set S of k agents, there can be at most $\binom{k}{2} = \frac{1}{2}k\,(k-1)$ links between them (first term).

- For the $N-k$ agents $i \notin S$, then can be at most $\min\left(k, D_{i+}\right)$ links from i to agents in S.

Graphical Integer Sequences (continued)

Sufficiency of the condition is (evidently) much harder to show.

Erdos and Gallai Theorem provides a simple test for graphicality of a degree sequence.

The next theorem, due to Havel (1955) and Hakimi (1962), shows that this test may be applied recursively.

A Recursive Test

Theorem: *(Havel-Hakimi) Let $D_{i+} > 0$, if $\mathbf{D}_+$ does not have at least D_{i+} positive entries other than i it is not graphical. Assume this condition holds. Let $\tilde{\mathbf{D}}_+$ be a degree sequence of length $N-1$ obtained by*

[i] deleting the i^{th} entry of $\mathbf{D}_+$ and

[ii] subtracting 1 from each of the D_{i+} highest elements in $\mathbf{D}_+$ (aside from the i^{th} one).

$\mathbf{D}_+$ is graphical if and only if $\tilde{\mathbf{D}}_+$ is graphical. If $\mathbf{D}_+$ is graphical, then it has a realization where agent i is connected to any of the D_{i+} highest degree agents (other than i).

Blitzstein and Diaconis Procedure

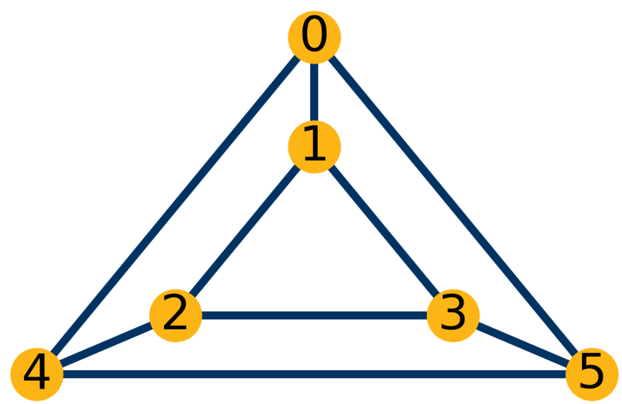
- Start with lowest degree agent (with at least one link).

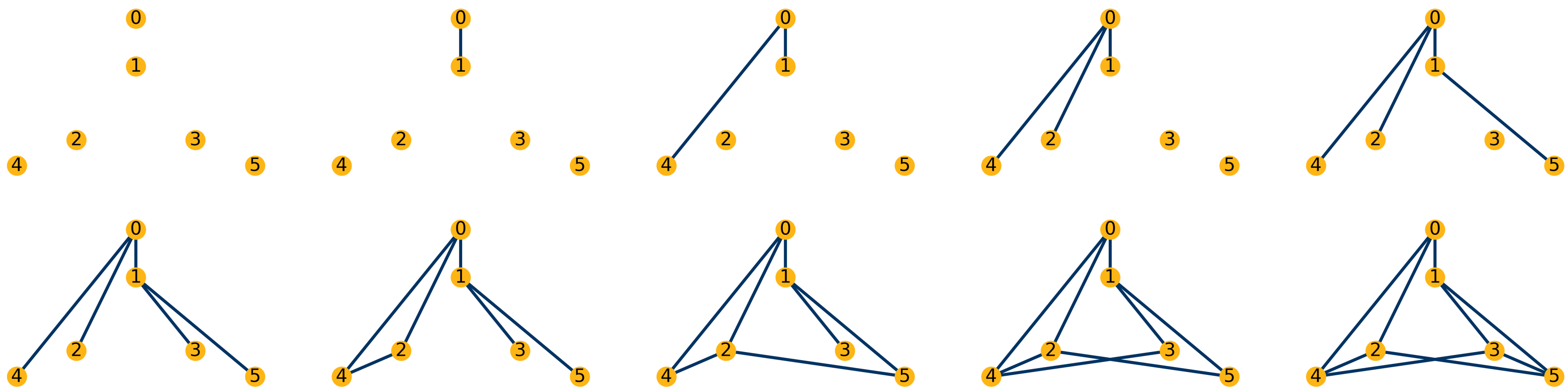
- (Randomly) Link this agent with high degree agents.

- A one is subtracted from the lowest degree agent's, as well as her chosen partners', degree counts.

- Continue until the **residual degree sequence** is zero.

3-regular (i.e., cubic graph)





Blitzstein and Diaconis Procedure (continued)

- Consider the example

$$
\begin{aligned}
(3,3,3,3,3,3) &\to (2,2,3,3,3,3) \to (1,2,3,3,2,3) \to (0,2,2,3,2,3) \\
&\to (0,1,2,3,2,2) \to (0,0,2,2,2,2) \to (0,0,1,2,1,2) \\
&\to (0,0,0,2,1,1) \to (0,0,0,1,0,1) \to (0,0,0,0,0,0)\,.
\end{aligned}
$$

- Now imagine that in the 8th step instead of linking agent 3 with agent 4, agents 4 and 5 were linked.

Blitzstein and Diaconis Procedure (continued)

- This would have resulted in a residual degree sequence of $(0,0,0,2,0,0)$, which is not graphic.

  Algorithm doesn't allow this to occur by checking for whether the residual degree sequence associated with a candidate link is graphical.

Blitzstein and Diaconis Procedure (continued)

- Let $\left(\oplus_{i_1,\ldots,i_k}\mathbf{D}_+\right)$ be the vector obtained by adding a one to the $i_1,\ldots,i_k$ elements of $\mathbf{D}_+$:

$$\left(\oplus_{i_1,\ldots,i_k}\mathbf{D}_+\right)_j = \begin{cases} D_{j+}+1 & \text{for } j \in \{i_1,\ldots,i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

- Let $\left(\ominus_{i_1,\ldots,i_k}\mathbf{D}_+\right)$ be the vector obtained by subtracting one from the $i_1,\ldots,i_k$ elements of $\mathbf{D}_+$:

$$\left(\ominus_{i_1,\ldots,i_k}\mathbf{D}_+\right)_j = \begin{cases} D_{j+}-1 & \text{for } j \in \{i_1,\ldots,i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

Blitzstein and Diaconis Procedure (continued)

Algorithm: A sequential algorithm for constructing a random graph with degree sequence $\mathbf{D}_+ = \left(D_{1+}, \ldots, D_{N+}\right)'$ is

1. Let $\mathbf{G}$ be an empty adjacency matrix.

2. If $\mathbf{D}_+ = \mathbf{0}$ terminate with output $\mathbf{G}$

3. Choose the agent i with minimal positive degree D_{i+}.

4. Construct a list of candidate partners

$$J = \left\{ j \neq i \,:\, \mathbf{G}_{ij} = \mathbf{G}_{ji} = 0 \text{ and } \ominus_{i,j} \mathbf{D}_+ \text{ graphical} \right\}.$$

5. Pick a partner $j \in J$ with probability proportional to its degree in $\mathbf{D}_+$.

6. Set $\mathbf{G}_{ij} = \mathbf{G}_{ji} = 1$ and update $\mathbf{D}_+$ to $\ominus_{i,j}\mathbf{D}_+$.

7. Repeat steps 4 to 6 until the degree of agent i is zero.

8. Return to step 2.

The input for the algorithm is the target degree sequence $\mathbf{D}_+$ and the output is an undirected adjacency matrix $\mathbf{G}$ with $\mathbf{G}'\iota = \mathbf{D}_+$.

Importance Weights

- The Blitzstein and Diaconis (2010) procedure delivers a random draw from $\mathbb{D}_{N,\mathbf{d}_+}$, but not a *uniform* random draw.

- Construct importance weights in order to compute expectations using the correct reference distribution.

- Let $\mathbb{Y}_{N,\mathbf{d}_+}$ denote the set of all possible sequences of links generated by the algorithm given input $\mathbf{D}_+ = \mathbf{d}_+$.

Importance Weights (continued)

- Let $\mathcal{G}(Y)$ be the adjacency matrix induced by link sequence Y.
  - Let Y and Y' are equivalent if $\mathcal{G}(Y) = \mathcal{G}(Y')$.

- We can partition $\mathbb{Y}_{N,\mathbf{d}_+}$ into a set of equivalence classes whose number coincides with the cardinality of $\mathbb{D}_{N,\mathbf{d}_+}$.

Importance Weights (continued)

- Let $c(Y)$ denote the number of possible link sequences produced by the algorithm that produce Y's end point adjacency matrix.

- Let $i_1, i_2, \ldots, i_M$ be the sequence of agents chosen in step 3 of the algorithm in which Y is the output.

Importance Weights (continued)

- Let $a_1, \ldots, a_m$ be the degrees of $i_1, \ldots, i_M$ at the time when each agent was *first* selected in step 3.

- Blitzstein and Diaconis show that:

$$c(Y) = \prod_{k=1}^{M} a_k!$$

Importance Weights (continued)

Consider two equivalent link sequences Y and Y'.

Because links are added to vertices by minimal degree (see Step 3), the sequences $i_1, i_2, \ldots, i_M$ coincide for Y and Y'.

This means that *the exact same links*, albeit perhaps in a different order, are added at each "stage" of the algorithm (i.e., when the algorithm iterates through steps 4 to 7 repeatedly for a given agent).

The number of different ways to add agent i_k's links during such a "stage" is simply $a_k!$ and hence $c(Y) = \prod_{k=1}^{M} a_k!$

Importance Weights (continued)

- Let $\sigma(Y)$ be the probability that the algorithm produces link sequence Y.

- $\sigma(Y)$ is easy to compute:

  – each time a link in step 5 is chosen we record the probability with which it was chosen.

  – this equals the residual degree of the chosen agent divided by the sum of the residual degrees of all agents in the choice set.

  – the product of all these probabilities equals $\sigma(Y)$.

Importance Weights (continued)

Let $S(\mathbf{G})$ be some statistic the adjacency matrix and consider the expected value

$$\begin{aligned}
\mathbb{E}\left[\frac{\pi(\mathcal{G}(Y))}{c(Y)\,\sigma(Y)} S(\mathcal{G}(Y))\right] &= \sum_{y \in \mathbb{Y}_{N,\mathbf{d}}} \frac{\pi(\mathcal{G}(y))}{c(y)\,\sigma(y)} S(\mathcal{G}(y))\,\sigma(y) \\
&= \sum_{y \in \mathbb{Y}_{N,\mathbf{d}}} \frac{\pi(\mathcal{G}(y))}{c(y)} S(\mathcal{G}(y)) \\
&= \sum_{\mathbf{g} \in \mathbb{D}_{N,\mathbf{d}_+}} \sum_{\{y\,:\,G(y)=\mathbf{g}\}} \frac{\pi(\mathbf{g})}{c(y)} S(\mathbf{g}) \\
&= \sum_{\mathbf{g} \in \mathbb{D}_{N,\mathbf{d}_+}} \pi(\mathbf{g})\, S(\mathbf{g}) \\
&= \mathbb{E}_\pi[S(\mathbf{G})]\,.
\end{aligned}$$

Importance Weights (continued)

Here $\pi(\mathbf{G})$ is the probability attached to the adjacency matrix $\mathbf{G} \in \mathbb{D}_{N,\mathbf{d}_+}$ in the target distribution over $\mathbb{D}_{N,\mathbf{d}_+}$.

The ratio $\pi(\mathcal{G}(Y))/c(Y)\sigma(Y)$ is called the likelihood ratio or the *importance weight.*

We would like $\pi(\mathbf{G}) = 1/\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ for all $\mathbf{G} \in \mathbb{D}_{N,\mathbf{d}_+}$.

If we set $\pi(\mathbf{G}) = S(\mathbf{G}) = 1$ we see that $\mathbb{E}\left[\frac{1}{c(Y)\sigma(Y)}\right] = \left|\mathbb{D}_{N,\mathbf{d}_+}\right|$. This suggests the analog estimator for $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ of

$$\widehat{\left|\mathbb{D}_{N,\mathbf{d}_+}\right|} = \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c(Y_b)\sigma(Y_b)}\right]^{-1} \qquad (3)$$

Importance Weights (continued)

These results suggest we estimate the average of $S(\mathbf{G})$ with respect to uniform draws from $\mathbb{D}_{N,\mathbf{d}_+}$ by

$$\widehat{\mu}_{S(\mathbf{G})} = \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c(Y_b)\,\sigma(Y_b)}\right]^{-1} \times \left[\frac{1}{B}\sum_{b=1}^{B}\frac{1}{c(Y_b)\,\sigma(Y_b)}S(\mathbf{G}_b)\right] \quad (4)$$

An attractive feature of (4) is that the importance weights need only be estimated up to a constant.

This feature is useful when dealing with numerical overflow issues that can arise when $\left|\mathbb{D}_{N,\mathbf{d}_+}\right|$ is too large to estimate.

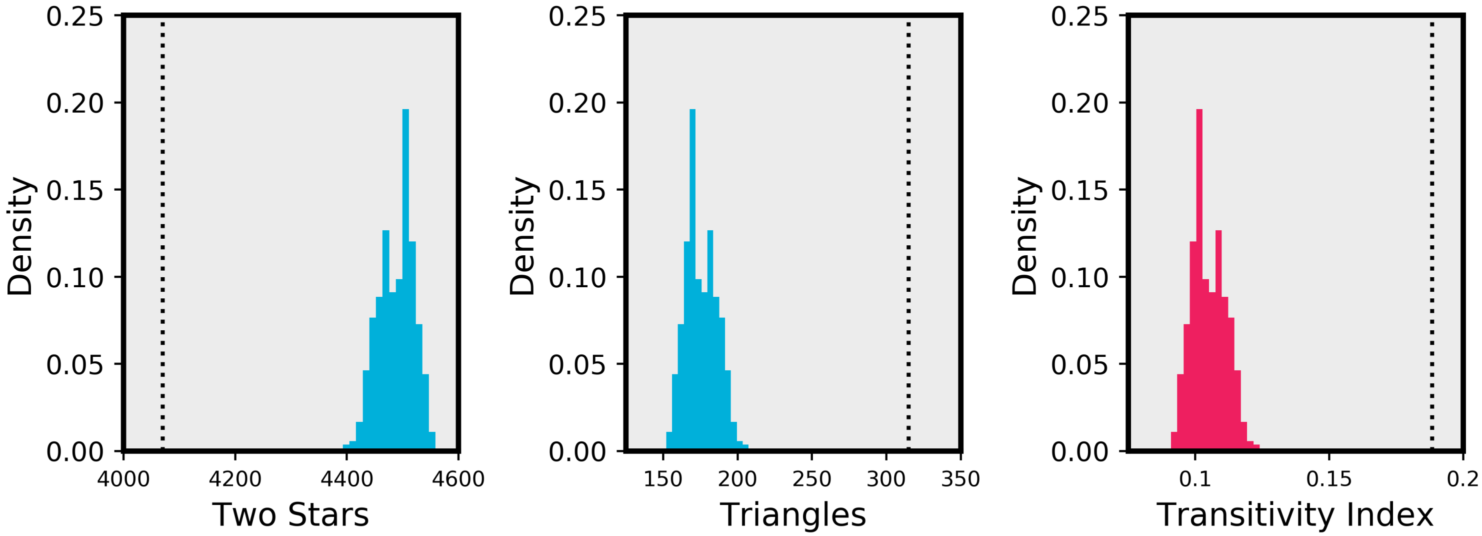
Importance Weights (continued)

- The ratio $\pi\left(\mathbf{G}\left(Y\right)\right)/c\left(Y\right)\sigma\left(Y\right)$ is called the likelihood ratio or the **importance weight.**

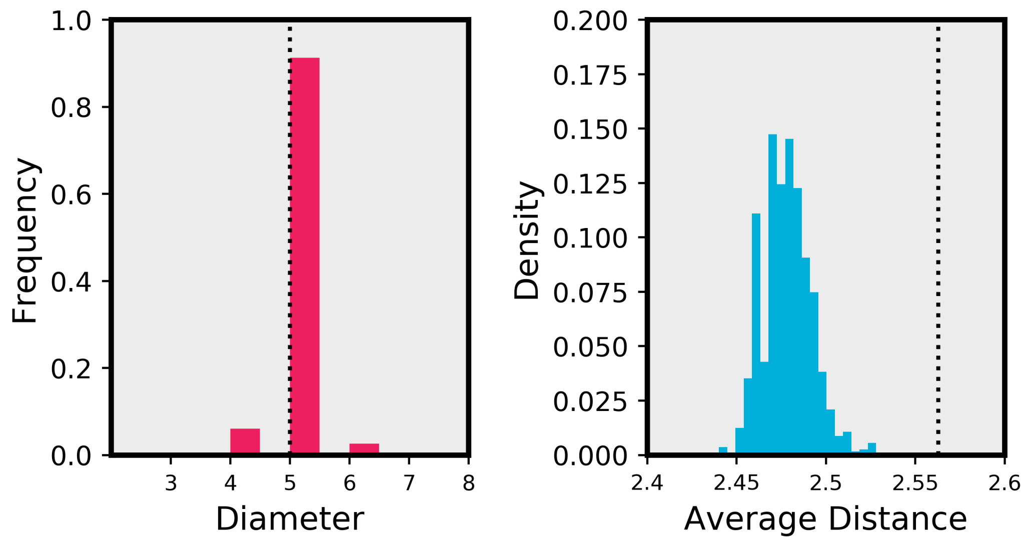
- Our random network draws are not uniform from the set of interest.

- The importance weights correct for the fact that we are sampling from the wrong distribution.

Nyakatoke Example



Nyakatoke Example (continued)



Blitzstein and Diaconis Wrap-Up

- While using the β-model as a reference model is restrictive it
  - is a natural starting point for hypothesis testing;
  - suggests that an investment in computation skills is likely to be valuable to anyone doing empirical work.

- It might be of interest to condition on additional features of the network in hand...