Exchangeable Random Graphs

Econometric Methods for Networks,

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Introduction

- First of two lectures on network nonparametrics

- Rest of today:
  - Aldous-Hoover representation
    - Orbanz and Roy (2015)
  - Nearest neighborhood smoothing for edge probability estimation
    - Zhang, Levina and Zhu (2015)

Introduction (continued)

- Tomorrow:
  - graph limits (e.g., Lovász, 2012)
  - estimation of network moments
    - Holland and Leindhart (1976)
    - Bickel, Chen and Levina (2011)
    - Bhattacharya and Bickel (2015)

Setup

Let $G(\mathcal{V}, \mathcal{E})$ be a finite undirected random graph with

- agents/vertices $\mathcal{V} = \{1, \ldots, N\}$,
- links/edges $\mathcal{E} = \{\{i,j\}, \{k,l\}, \ldots\}$, and
- adjacency matrix $\mathbf{D} = \left[D_{ij}\right]$ with

$$D_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

Setup (continued)

- The expected adjacency matrix equals

$$\mathbf{P} = \left[P_{ij}\right] = \left[\mathbb{E}\left[D_{ij}\middle| U_1, \ldots, U_2, \alpha\right]\right]$$

  for $i < j$.

- Here $\{U_i\}_{i=1}^N$ and α are *latent* random variables introduced (and explained below).

- Form of $\mathbf{P}$ might indicate community structure...

- ...or guide other aspects of model formulation

Exchangeable Networks

- Let π be a permutation of the index set $\{1, \ldots, N\}$.

- In many situations it is natural to assume that
$$\left[D_{ij}\right] \stackrel{d}{=} \left[D_{\pi(i)\pi(j)}\right] \tag{1}$$
for every permutation π and $i < j$, $j = 1, \ldots, N$.

  - $\stackrel{d}{=}$ indicates equality of distribution.

- Condition (1) $\Rightarrow$ our beliefs about the probability of a link between two agents does not depend on their labels.

- Networks with this property are *jointly exchangeable*.

Exchangeable Networks (continued)

- Does exchangeability have any modeling implications?

- Does $\mathbf{D}$ converge to a *graph limit* as $N \to \infty$?

- Dense graph implication:

  - if $\left[D_{ij}\right] \stackrel{d}{=} \left[D_{\pi(i)\pi(j)}\right]$ then $\rho = \text{Pr}\left(D_{ij} = 1\right)$ is either bounded away from zero or zero.

  - exchangeable graphs are either dense or empty!

Exchangeable Sequences

- The sequence $Y_1, Y_2, \ldots$ is said to be **infinitely exchangeable** if, for every $N \geq 2$ and permutation π,

$$(Y_1, Y_2, \ldots, Y_N) \stackrel{d}{=} \left(Y_{\pi(1)}, Y_{\pi(2)}, \ldots, Y_{\pi(N)}\right).$$

- i.i.d. sequences are exchangeable...

- ... but non i.i.d.sequences can be too:

$$Z + Y_1, Z + Y_2, \ldots$$

  for Z some non-trivial random variable, drawn independently of the i.i.d. sequence $Y_1, Y_2, \ldots$

de Finetti Theorem

- de Finetti (1931): the sequence of binary random variable $Y_1, Y_2, \ldots$ is infinitely exchangeable if, and only if,

$$\Pr\left(Y_1 = y_1, Y_2 = y_2, \ldots, Y_N = y_N\right) = \int_0^1 \alpha^{t_N} (1-\alpha)^{N-t_N} \, d\Pi(\alpha)$$

for $t_N = \sum_{i=1}^{N} y_i$, all $N \geq 2$, and Π some measure on $\alpha \in [0,1]$.

- For any infinitely exchangeable sequence we have that – conditional on the random variable α–

$$\Pr\left(Y_1 = y_1, Y_2 = y_2, \ldots, Y_N = y_N \mid \alpha\right) = F_\alpha(y_1) \, F_\alpha(y_2) \times \cdots \times F_\alpha(y_N)$$

for $F_\alpha(y) = \alpha^y (1-\alpha)^{1-y}$ if $y \in \{0,1\}$ and zero otherwise.

de Finetti Theorem (continued)

- Representation result: any exchangeable binary sequence can be modeled ‘as if’ the DGP were:

  1. Draw $\alpha \sim \Pi$
  2. Draw $Y_i \sim F_\alpha$ for $i = 1, \ldots, N$

- *Conditional* on α, $Y_1, Y_2, \ldots$ is an i.i.d. sequence, where each of its members have the same *random* distribution function $F_\alpha(y)$.

- See Orbanz and Roy (2015) for non-technical survey of de Finetti type results

Alternative Formulation

- The right-continuous inverse of $F_\alpha(u)$ (i.e., quantile function) is

$$g_\alpha(u) \stackrel{\text{def}}{=} F_\alpha^{-1}(u) = \begin{cases} 0 & \text{if } 0 < u < 1-\alpha \\ 1 & \text{if } 1-\alpha \le u < 1 \end{cases}.$$

- This gives:

$$(Y_1, Y_2, \ldots) \stackrel{d}{=} (g_\alpha(U_1), g_\alpha(U_2), \ldots)$$

for $\{U_i\}_{i=1}^{\infty}$ a sequence of independent $\mathcal{U}[0,1]$ random variables.

- We further have that

$$\mathbb{E}[Y_i | U_i = u] = g_\alpha(u).$$

Alternative Formulation

- The "*sequon*" (**<u>sequ</u>**ence functi**<u>on</u>**) $g_\alpha(u)$ is not identifiable...

  – consider $g_\alpha(u)$ above with:

$$g_\alpha^*(u) = \begin{cases} 0 & \text{if } 0 < u < \frac{1-\alpha}{2} \\ 1 & \text{if } \frac{1-\alpha}{2} \leq u < \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} \leq u < \frac{2-\alpha}{2} \\ 1 & \text{if } \frac{2-\alpha}{2} \leq u < 1 \end{cases} .$$

- ...but "moments" are identifiable:

  – $\frac{1}{N} \sum Y_i \xrightarrow{p} \mathbb{E}\left[g_\alpha(U) \middle| \alpha \right] = \alpha$

Aldous-Hoover

- Aldous (1981) and Hoover (1979) (essentially) showed that a random graph is jointly exchangeable if, and only if, it admits the representation

$$\left[D_{ij}\right] \stackrel{d}{=} \left[g_\alpha\left(U_i, U_j, V_{ij}\right)\right]$$

  for $\{U_i\}_{i=1}^{\infty}$ and $\left\{V_{ij}\right\}_{i<j}$ sequences of independent $\mathcal{U}\left[0,1\right]$ random variables.

- Here α is a mixing parameter as in de Finetti (1931).

  - $g_\alpha\left(\cdot,\cdot,\cdot\right)$ is a random function

Aldous-Hoover (continued)

- Averaging over V_{ij} yields

$$
\begin{aligned}
h_\alpha\left(u_i, u_j\right) &= \mathbb{E}\left[D_{ij}\middle| U_i = u_i, U_j = u_j, \alpha\right] \\
&= \mathbb{E}\left[g_\alpha\left(u_i, u_j, V_{ij}\right)\middle| \alpha\right] \\
&= \int_0^1 g_\alpha\left(u_i, u_j, v\right) \mathrm{d}v
\end{aligned}
$$

  from which we get the more convenient representation, for $i < j$,

$$
\left[D_{ij}\right] \overset{d}{=} \left[\mathbf{1}\left(V_{ij} \le h_\alpha\left(U_i, U_j\right)\right)\right]
$$

- $h_\alpha\left(U_i, U_j\right)$ is a *graphon*: short for **graph** functi**on**.

Aldous-Hoover (continued)

- The Aldous-Hoover representation theorem implies that we can proceed 'as if' links formed independently conditional on the agent-specific latent variables $\{U_i\}_{i=1}^{\infty}$ and α.

- A network generating process is:

  1. "Draw" α or choose a graphon;

  2. Draw $U_i \sim \mathcal{U}[0,1]$ for agents $i = 1, \ldots, N$;

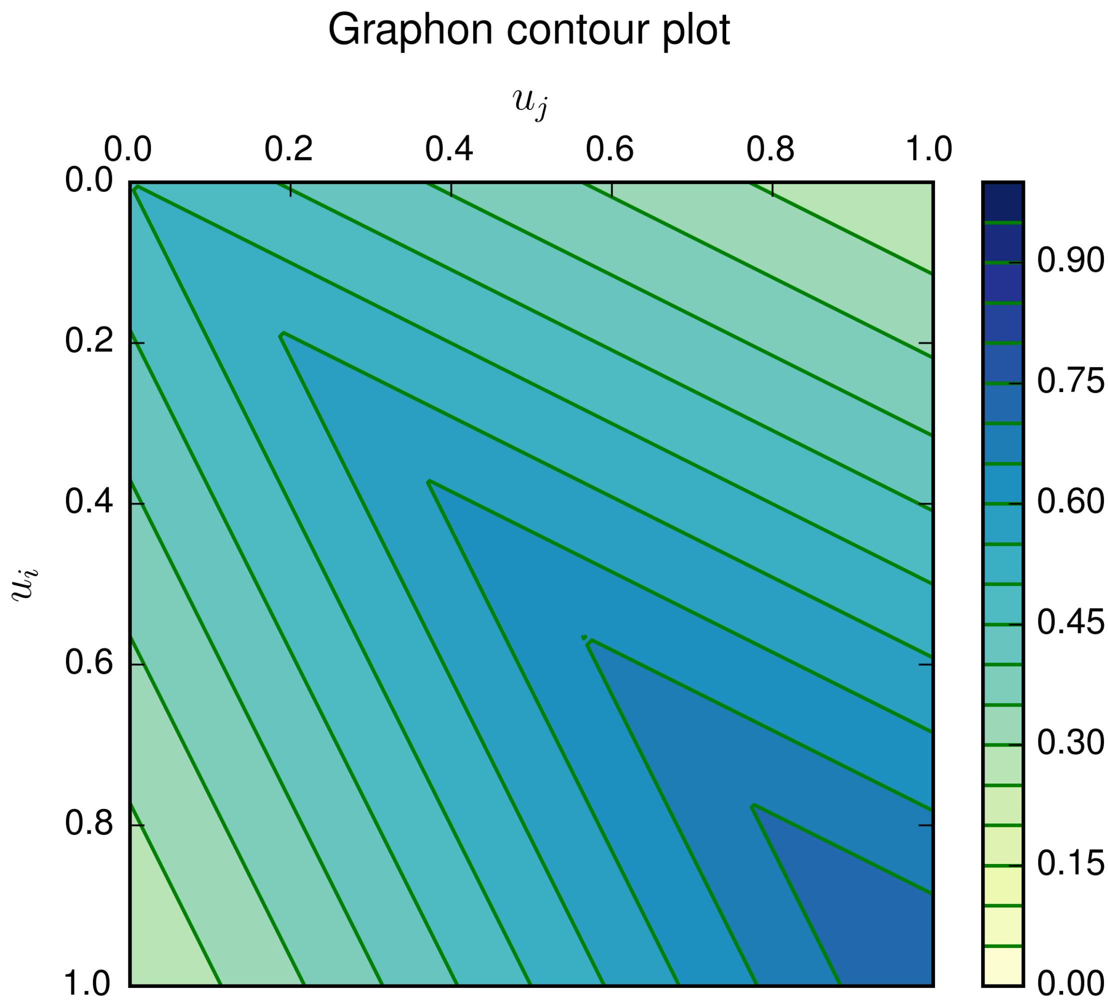
  3. Construct $\mathbf{D}$, by sampling $D_{ij} \Big| h_\alpha(\bullet,\bullet), U_i, U_j \sim \text{Bernoulli}\left(h_\alpha\left(U_i, U_j\right)\right)$ for every dyad $\{i, j\}$ with $i < j$.

Aldous-Hoover (continued)

- *any* exchangeable random graph may be modeled as a mixture conditionally independent edge formation processes

- Conditional independence structure useful for large sample theory

- Representation result: actual network generating process may not coincide with representation (cf., reduced form)



Note: $h\left(u_i, u_j\right) = \frac{\exp(\alpha_0+\alpha_1(u_i+u_j)+\alpha_2|u_i-u_j|)}{1+\exp(\alpha_0+\alpha_1(u_i+u_j)+\alpha_2|u_i-u_j|)}$

Graphon

- The graphon $h_\alpha(u,v)$ is not identifiable...
  - consider the m.p.t. $\varphi(U) = 1 - U$ or $\varphi(U) = 2U \bmod 1$
  - $g_\alpha\left(U_i, U_j, V_{ij}\right)$ and $g_\alpha\left(\varphi(U_i), \varphi\left(U_j\right), V_{ij}\right)$ generate graphs with the same properties

- ...but link/edge probabilities *are* identifiable (under assumptions).
  - $p_{ij} = \mathbb{E}\left[D_{ij} \middle| \mathbf{U}\right] = h_\alpha\left(U_i, U_j\right)$

Graphon (Bickel & Chen, 2009)

- For statistical analysis it is convenient to formulate the graphon somewhat differently.

- Consider the network DGP

$$\Pr\left(D_{ij}=1\middle|U_i,U_j,\alpha\right)=h_\alpha\left(U_i,U_j\right)$$

and define

$$\rho_\alpha=\int_0^1\int_0^1 h_\alpha(u,v)\,\mathrm{d}u\mathrm{d}v$$
$$w_\alpha(u,v)=f_{U_i,U_j|D_{ij},\alpha}\left(u,v\middle|D_{ij}=1,\alpha\right).$$

- Since $f_{U_i,U_j|\alpha}(u,v|\alpha)=1$ on $[0,1]^2$ we get the formulation

$$h_\alpha(u,v)=\rho_\alpha w_\alpha(u,v).$$

Graphon (Bickel & Chen, 2009)

- The Bickel and Chen (2009) formulation is useful for sequences of network GPs where ρ_α, the network density, is indexed by N.
  - i.e., $\rho_{\alpha,N} \to 0$ as $N \to \infty$
  - in practice we ignore any dependence of $w_\alpha(u,v)$ on N

- The rate at which $\rho_{\alpha,N} \to 0$ controls the sparsity links

- If $\lambda_N = (N-1)\,\rho_{\alpha,N} \to \lambda > 0$ as $N \to \infty$ the graph is *sparse*
  - other cases: $\lambda_N = O(N)$ (*dense*) or $\lambda_N = O(\ln N)$ (*semi-dense*)

Edge Probability Estimation

- Define the inner product

$$\langle f, g \rangle = \int f(u)\, g(u)\, \mathrm{d}u$$

  with the associated norm

$$\|f\| = \langle f, f \rangle^{1/2} = \left[\int f(u)^2\, \mathrm{d}u \right]^{1/2}.$$

- Linking behavior of an agent of type u is summarized by the *graphon slice* $\rho w(u, \bullet)$.

- Measure "distance" between agent i, with $U_i = u$, and agent j, with $U_j = v$, by:

$$\begin{aligned} d(u, v) &= \|\rho w(u, \cdot) - \rho w(v, \cdot)\|_2 \qquad (2) \\ &= \rho \left[\int [w(u, t) - w(v, t)]^2\, \mathrm{d}t \right]^{1/2} \end{aligned}$$

Network Neighbors

- $\mathbf{P} = \mathbb{E}\left[\mathbf{D}|\,\mathbf{U}\right]$ denotes the expected adjacency matrix.

- $\mathbf{P}_{i\bullet}$ denotes the i^{th} row of this matrix

- Distance between i and j is

$$\begin{aligned} d_N(i,j) &= \left\| \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet} \right\|_2 \qquad (3) \\ &= \left[\frac{1}{N-2} \sum_{k \neq i,j} \left(P_{ik} - P_{jk} \right)^2 \right]^{1/2} \end{aligned}$$

- i^{th} and j^{th} elements of both $\mathbf{P}_{i\bullet}$ and $\mathbf{P}_{j\bullet}$ are removed prior to calculating $d(i,j)$.

Nearest Network Neighbors

- j *is an exact neighbor of* i if $d_N(i,j) = 0$
  - i and j have identical (expected) adjacency (matrix) slices.
  - i.e., identical *ex ante* linking behavior
  - *realized* links may differ

Nearest Neighbor Averaging

- In a finite network it may be that agent i has no exact neighbors, but we can still find a set of *nearest neighbors*:

$$\mathcal{N}_i = \left\{ j \, : \, \left\| \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet} \right\|_2 \leq q_i \left(h_N \right) \right\} \tag{4}$$

  where $q_i \left(h_N \right)$ is the h_N^{th} sample quantile of $\left\{ \left\| \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet} \right\|_2 \right\}_{j=1, j \neq i}^{N}$.

- If $N = 1,000$ and $h_N = 0.05$, then we would take the 50 nearest neighbors.

- Estimate P_{ij} by the local average

$$\widehat{P}_{ij}^{\text{oracle}} = \frac{1}{2} \left(\underbrace{\frac{\sum_{k \in \mathcal{N}_i} D_{kj}}{|\mathcal{N}_i|}} + \frac{\sum_{l \in \mathcal{N}_j} D_{il}}{\left| \mathcal{N}_j \right|} \right). \tag{5}$$

- Unfortunately $\mathbf{P}$ is not observed!

Finding Network Neighbors

- Can me construct a measure of distance between two agents base on the (observed) adjacency matrix alone?

- Zhang et al. (2015) observe that

$$
\begin{aligned}
d_N^2(i,j) &= \left\|\mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}\right\|_2^2 \\
&= \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet} \right\rangle \\
&= \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} \right\rangle - \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{j\bullet} \right\rangle \\
&\leq \left|\left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} \right\rangle\right| + \left|\left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{j\bullet} \right\rangle\right|
\end{aligned}
$$

- Need estimates of $\langle \mathbf{P}_{i\bullet}, \mathbf{P}_{i\bullet} \rangle$, $\left\langle \mathbf{P}_{j\bullet}, \mathbf{P}_{j\bullet} \right\rangle$ and $\left\langle \mathbf{P}_{i\bullet}, \mathbf{P}_{j\bullet} \right\rangle$ to form estimate of $d_N(i,j)$.

Finding Neighbors (continued)

- $\langle \mathbf{P}_{i\bullet}, \mathbf{P}_{i\bullet} \rangle = \frac{1}{N-1} \sum_{i \neq j} P_{ij}^2$ is hard to estimate...

- ...apparently requires estimate of P_{ij} (which is our target!)

- However the (limit of the) term

$$\left\langle \mathbf{P}_{i\bullet}, \mathbf{P}_{j\bullet} \right\rangle = \frac{1}{N-2} \sum_{k \neq i,j} P_{ik} P_{jk}$$

  is not hard to estimate since

$$\mathbb{E}\left[\frac{1}{N-2} \sum_{k} D_{ik} D_{jk}\right] = \mathbb{E}\left[\frac{1}{N-2} \sum_{k \neq i,j} P_{ik} P_{jk}\right].$$

- Recall edges form independently conditional on $\mathbf{U}$.

Finding Neighbors (continued)

- Assume that $w(u,v)$ is Lipschitz continuous:

$$\rho \left\| w(u,\cdot) - w(v,\cdot) \right\|_2 \leq C \left\| u - v \right\|_2 .$$

- With N large we can find an agent $l \neq i, j$ such that $\left| U_j - U_l \right| \leq \epsilon_N$ for $\epsilon_N = o(1)$.

- We get

$$\begin{aligned}
\left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} \right\rangle \right| &= \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{k\bullet} \right\rangle \right. \\
&\quad \left. + \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} - \mathbf{P}_{k\bullet} \right\rangle \right| \\
\text{(TI)} &\leq \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{k\bullet} \right\rangle \right| \\
&\quad + \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{i\bullet} - \mathbf{P}_{k\bullet} \right\rangle \right| \\
\text{(CS)} &\leq \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{k\bullet} \right\rangle \right| \\
&\quad + \left\| \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet} \right\|_2 \left\| \mathbf{P}_{i\bullet} - \mathbf{P}_{k\bullet} \right\|_2 \\
&\leq \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{k\bullet} \right\rangle \right| + C_{i,j} \epsilon_N
\end{aligned}$$

Finding Neighbors (continued)

- Combining results we have that

  $$d^2(i,j) \leq 2 \max_{l \neq i,j} \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right| + 2C_{i,j}\epsilon_N$$

  - ...if $2 \max_{l \neq i,j} \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right| \approx 0$, then $d^2(i,j) \approx 0$ if N is large.

  - Zhang et al. (2015) estimate

    $$2 \max_{l \neq i,j} \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right|$$

    by $\widehat{d}^2(i,j)$ equal to

    $$2 \max_{l \neq i,j} \left| \frac{1}{N-2} \sum_{k \neq i,j} D_{ik}D_{lk} - \sum_{k \neq i,j} D_{jk}D_{lk} \right|$$

- *Estimated neighborhood* of agent i is then

  $$\widehat{\mathcal{N}}_i = \left\{ j \,:\, \widehat{d}^2(i,j) \leq q_i(h_N) \right\}.$$

Zhang et al. (2015) Estimate

- Estimate P_{ij} by

$$\widehat{P}_{ij} = \frac{1}{2}\left(\frac{\sum_{k\in\widehat{\mathcal{N}}_i} D_{kj}}{\left|\widehat{\mathcal{N}}_i\right|} + \frac{\sum_{l\in\widehat{\mathcal{N}}_j} D_{il}}{\left|\widehat{\mathcal{N}}_j\right|}\right)$$

- Consistency requires that $h_N = C\sqrt{\frac{\ln N}{N}}$ for some C.

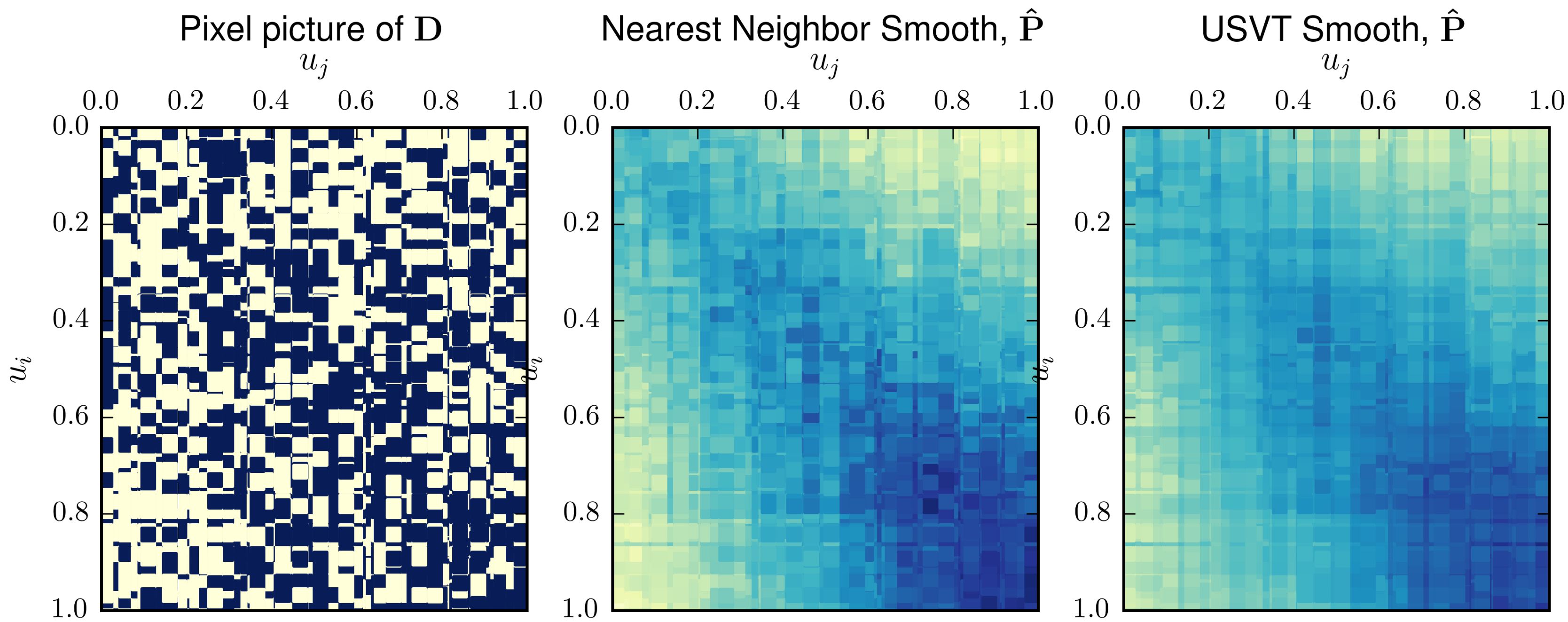
- Zhang et al. (2015) suggest that $C = 0.1$ works well in practice.

  - $K_N = \lfloor N h_N \rfloor = \left\lfloor 0.1\,(N \ln N)^{1/2} \right\rfloor$ or $K_{1000} \approx 8$ and $K_{2000} \approx 12$.

Alternative Distance Measure

$$
\begin{aligned}
&\max_{l \neq i,j} \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right| \\
&\leq \sum_{l \neq i,j} \left| \left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right| \cdot 1 \\
\text{(HI)} &\leq \left[\sum_{l \neq i,j} \left(\left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right)^2 \right]^{1/2} \cdot \left[\sum_{k \neq i,j} 1^2 \right]^{1/2} \\
&= \left[(N-2) \sum_{k \neq i,j} \left(\left\langle \mathbf{P}_{i\bullet} - \mathbf{P}_{j\bullet}, \mathbf{P}_{l\bullet} \right\rangle \right)^2 \right]^{1/2} \\
&= \left[\frac{1}{N-2} \sum_{l \neq i,j} \left(\sum_{k \neq i,l} P_{ik} P_{kl} - \sum_{k \neq j,l} P_{jk} P_{kl} \right)^2 \right]^{1/2} \\
&= d_N^* (i,j)
\end{aligned}
$$

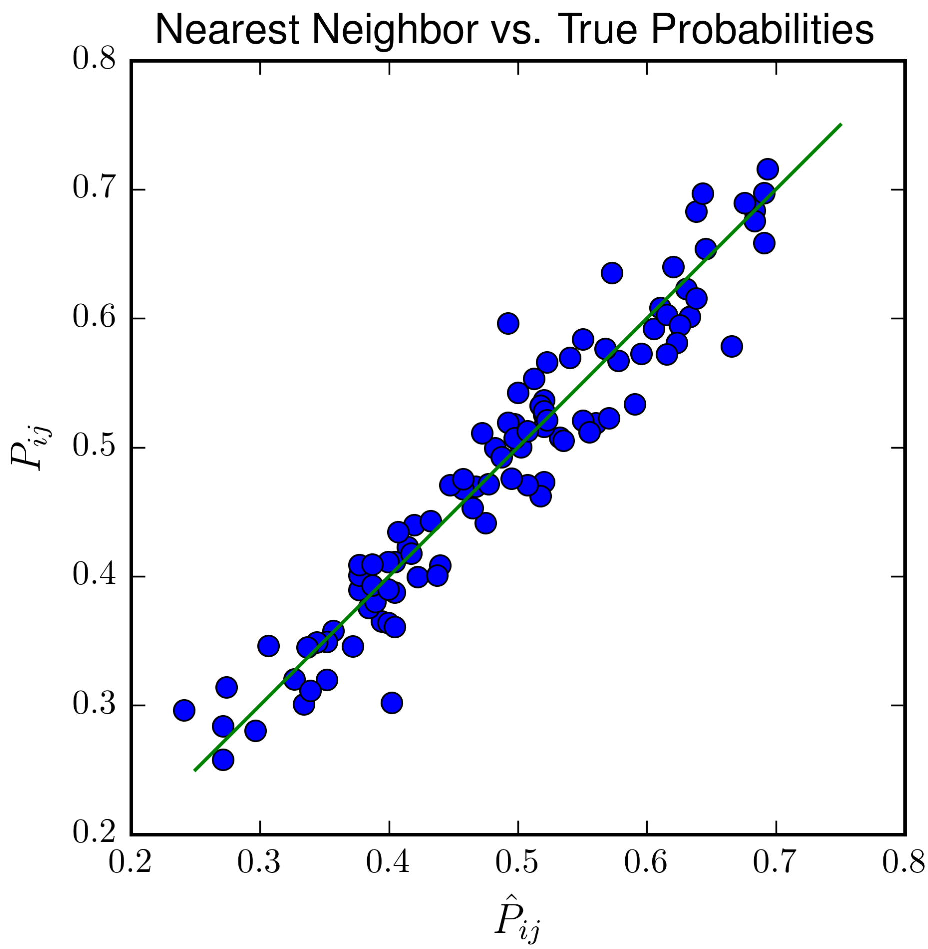
We can use the 'smoother' $\widehat{d}_N^* (i,j)$ to find nearest neighbors instead.



Pixel picture of $\mathbf{D}$
Nearest Neighbor Smooth, $\hat{\mathbf{P}}$
USVT Smooth, $\hat{\mathbf{P}}$
u_j
u_i
0.0
0.2
0.4
0.6
0.8
1.0

Goodness-of-Fit

(N $= 2{,}000$, $h_N = 0.1$)



Practicalities

- In example it is natural to order i by their realized values of U_i

- This information is not available in real world examples

- In practice, we can order agents by degree or its smoothed estimate $\sum_j \widehat{P}_{ij}$
  - should be sufficient to 'see' a block structure (for example) in many cases