

# **Beta Model & Network Simulation**

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## Introduction

- In practice methods of inference used in “network science” are decidedly heuristic/approximate.
- Widespread awareness among researchers about the lack of a coherent body of large network distribution theory.

## Introduction (continued)

- Is the network in hand especially “transitive”?
  - Compare with Erdos-Renyi random graph; but are measured differences statistically significant? (how do we conceptualize “sampling error”)
  - Compare with a large set of (empirical) reference graphs. Is the graph of interest usual?
  - Combine an ad hoc and/or approximate variance estimate with a normal reference distribution (w/o limit theory it is difficult to evaluate this approach; see last lecture)

## Inference: Exact

- Blitzstein and Diaconis (2011) – additional work in both machine learning and statistics
- Look at a reference set of graphs (e.g., all graphs with degree sequences identical to the graph of interest)
  - Is transitivity (for example) in the graph *in hand* high relative to this reference group? (exact p-value approach)
  - Computational challenge: how to enumerate, or draw uniformly, from reference graph distribution

## Inference: Asymptotic

- Last lecture: Bickel, Chen & Levina and Bhattacharya and Bickel (2015):
  - Derive limit theory for network statistics (specifically normalized subgraph counts)
  - Challenge is also computational – both statistics and their variance estimates hard to construct

## Beta Model

- Models with network externalities are attractive because
  - they capture what is believed to be an *a priori* important feature of link formation,
  - they generate clustering, which we observe in real word networks.

## Beta Model (continued)

- An alternative (ideally complementary) way to generate clustering is to introduce unobserved, agent-level, heterogeneity.
  - beta model:  $D_{ij} = 1 \left( A_i + A_j - U_{ij} \leq 0 \right)$
  - $A_i$  measures attractiveness, trustworthiness, productivity etc.
  - Distribution of  $\mathbf{A}$  is unrestricted; components of  $\mathbf{U}$  are i.i.d. (logistic).
- cf. “state dependence vs. heterogeneity” in dynamic discrete choice analysis (Heckman, 1978; 1981a-c; Chamberlain, 1985).

## Beta Model (continued)

- Assuming  $U_{ij}$  i.i.d. logistic yields a link probability of

$$\Pr(D_{ij} = 1 | \mathbf{A}) = \frac{\exp(A_i + A_j)}{1 + \exp(A_i + A_j)} = \frac{\exp(W'_{ij}\mathbf{A})}{1 + \exp(W'_{ij}\mathbf{A})}$$

with  $W_{ij}$  the  $N \times 1$  vector with a one for its  $i^{th}$  and  $j^{th}$  elements and zeros elsewhere.

- Choosing  $A_i = -\frac{1}{2} \ln\left(\frac{p}{1-p}\right)$  for  $i = 1, \dots, N$  yields the Erdos-Renyi random graph model.



## Beta Model (continued)

- The likelihood,  $\Pr(\mathbf{D} = \mathbf{d} | \mathbf{A})$ , includes  $\binom{N}{2}$  conditional independent components

$$\prod_{i=1}^N \prod_{j < i} \left[ \frac{\exp(W'_{ij} \mathbf{A})}{1 + \exp(W'_{ij} \mathbf{A})} \right]^{d_{ij}} \left[ \frac{1}{1 + \exp(W'_{ij} \mathbf{A})} \right]^{1-d_{ij}}.$$

- ...but “only”  $N$  parameters
- model is non-standard since the dimension of the parameter space grows with  $N$

## Beta Model (continued)

- Manipulating the likelihood gives the exponential family representation

$$\Pr(\mathbf{D} = \mathbf{d} | \mathbf{A}) = c(\mathbf{Z}; \mathbf{A}) \exp(T(\mathbf{d})' \mathbf{A}) \quad (1)$$

where

$$T(\mathbf{d}) = \left( d_{1+} \cdots d_{N+} \right)' = \mathbf{d}_+.$$

- The network's *degree sequence*, is a sufficient statistic for  $\mathbf{A}$ .

## **Beta Model (continued)**

- The beta model allows for networks with arbitrary degree distributions.
- Despite its simplicity it is reasonable flexible ( $N$  parameters) and provides a useful benchmark.

## Beta Model (continued)

- Let  $\mathbb{D}_{N,\mathbf{d}_+}$  denote the set of all networks with  $N$  agents and degree sequence  $\mathbf{D}_+ = \mathbf{d}_+$ .
- Let  $|\mathbb{D}_{N,\mathbf{d}_+}|$  denote the cardinality of  $\mathbb{D}_{N,\mathbf{d}_+}$ .
  - $|\mathbb{D}_{N,\mathbf{d}_+}|$  is generally huge, even for small  $N$ .
- Under the  $\beta$ -model the probability distribution of networks conditional on their degree sequence is uniform:

$$\Pr(\mathbf{D} = \mathbf{d} | \mathbf{d} \in \mathbb{D}_{N,\mathbf{d}_+}) = \frac{1}{|\mathbb{D}_{N,\mathbf{d}_+}|}.$$

## Testing

- Let  $f(\mathbf{D})$  be some function of the adjacency matrix
  - examples: transitivity index, diameter, number of K-length paths etc.
- Let  $f(d)$  be the value of the statistic in the observed network.
- We seek to evaluate

$$\Pr(f(\mathbf{D}) \leq f(d)) = \frac{\sum_{\mathbf{v} \in \mathbb{D}_{N,d_+}} \mathbf{1}(f(\mathbf{v}) \leq f(d))}{|\mathbb{D}_{N,d_+}|}. \quad (2)$$

## Testing: Intuition

- If the probability that measured transitivity, in a network randomly drawn from the null distribution, lies above observed transitivity is very low, we take that as evidence against the  $\beta$ -model.

## Testing

- This approach to testing is
  - very precise about its description of the null hypothesis
  - exact
- no alternative hypothesis is specified...
- ...however the choice of statistic should be guided by researcher intuitions about what departures from the null model are of concern.

## Sampling from $\mathbb{D}_{N,d_+}$

- Direct enumeration of all the elements of  $\mathbb{D}_{N,d_+}$  is generally not feasible.
- Need a method of sampling from  $\mathbb{D}_{N,d_+}$  uniformly and also estimating its size (implement an approximation of the ideal test).



## Sampling from $\mathbb{D}_{N,\mathbf{d}_+}$ (continued)

- Blitzstein and Diaconis (2010) develop a sequential importance sampling algorithm for uniformly sampling from  $\mathbb{D}_{N,\mathbf{d}_+}$
- Two challenges:
  - how to generate a random draw from  $\mathbb{D}_{N,\mathbf{d}_+}$
  - how to do so uniformly (importance weights)

## Graphical Integer Sequences

- To construct  $\mathbf{D}$  we begin with a matrix of zeros and sequentially add links to it until its rows and columns sum to the target degree sequence.
- Problem is that unless links are added carefully it is easy to get “stuck” (cf., Snijders, 1991).
- The key is to check whether residual degree sequences are graphical as you add links (avoid dead ends).
- $\mathbf{D}_+ = (2, 2, 1)$  is not graphic

## Graphical Integer Sequences (continued)

- Erdos and Gallai (1961) showed  $\mathbf{D}_+$  is graphical if and only if  $\sum_{i=1}^N D_{i+}$  is even and

$$\sum_{i=1}^k D_{i+} \leq k(k-1) + \sum_{i=k+1}^N \min(k, D_{i+}) \text{ for each } k \in \{1, \dots, N\}.$$

## Graphical Integer Sequences (continued)

*Necessity:*

- even: if  $i$  is linked to  $j$ , then the link is counted in both  $D_{i+}$  and  $D_{j+}$
- For any set  $S$  of  $k$  agents, there can be at most  $\binom{k}{2} = \frac{1}{2}k(k-1)$  links between them (first term).
- For the  $N - k$  agents  $i \notin S$ , then can be at most  $\min(k, D_{i+})$  links from  $i$  to agents in  $S$ .

## A Recursive Test

**Theorem:** (Havel-Hakimi) Let  $D_{i+} > 0$ , if  $\mathbf{D}_+$  does not have at least  $D_{i+}$  positive entries other than  $i$  it is not graphical. Assume this condition holds. Let  $\tilde{\mathbf{D}}_+$  be a degree sequence of length  $N - 1$  obtained by

- [i] deleting the  $i^{th}$  entry of  $\mathbf{D}_+$  and
- [ii] subtracting 1 from each of the  $D_{i+}$  highest elements in  $\mathbf{D}_+$  (aside from the  $i^{th}$  one).

$\mathbf{D}_+$  is graphical if and only if  $\tilde{\mathbf{D}}_+$  is graphical. If  $\mathbf{D}_+$  is graphical, then it has a realization where agent  $i$  is connected to any of the  $D_{i+}$  highest degree agents (other than  $i$ ).

## Blitzstein and Diaconis Procedure

- Start with lowest degree agent (with at least one link).
- (Randomly) Link this agent with high degree agents.
- A one is subtracted from the lowest degree agent's and her chosen partner's degree counts.
- Continue until the **residual degree sequence** is zero.

## Blitzstein and Diaconis Procedure (continued)

- Consider the example

$$\begin{aligned} (3, 2, 2, 2, 1) &\rightarrow (3, 1, 2, 2, 0) \rightarrow (2, 0, 2, 2, 0) \\ &\rightarrow (1, 0, 2, 1, 0) \rightarrow (0, 0, 1, 1, 0) \rightarrow (0, 0, 0, 0, 0) \end{aligned}$$

- Now imagine that in the fourth step instead of linking agent 1 with agent 3, agents 1 and 4 were linked.
  - this would have resulted in a residual degree sequence of  $(0, 0, 2, 0, 0)$ , which is not graphic.
  - algorithm doesn't allow this to occur by checking for whether the residual degree sequence associated with a candidate link is graphical.

## Blitzstein and Diaconis Procedure (continued)

- Let  $(\oplus_{i_1, \dots, i_k} \mathbf{D}_+)$  be the vector obtained by adding a one to the  $i_1, \dots, i_k$  elements of  $\mathbf{D}_+$ :

$$(\oplus_{i_1, \dots, i_k} \mathbf{D}_+)_j = \begin{cases} D_{j+} + 1 & \text{for } j \in \{i_1, \dots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$

- Let  $(\ominus_{i_1, \dots, i_k} \mathbf{D}_+)$  be the vector obtained by subtracting one from the  $i_1, \dots, i_k$  elements of  $\mathbf{D}_+$ :

$$(\ominus_{i_1, \dots, i_k} \mathbf{D}_+)_j = \begin{cases} D_{j+} - 1 & \text{for } j \in \{i_1, \dots, i_k\} \\ D_{j+} & \text{otherwise} \end{cases}$$



## Blitzstein and Diaconis Procedure (continued)

**Algorithm:** A sequential algorithm for constructing a random graph with degree sequence  $\mathbf{D}_+ = (D_{1+}, \dots, D_{N+})'$  is

1. Let  $\mathbf{G}$  be an empty adjacency matrix.
2. If  $\mathbf{D}_+ = \mathbf{0}$  terminate with output  $\mathbf{G}$
3. Choose the agent  $i$  with minimal positive degree  $D_{i+}$ .
4. Construct a list of candidate partners

$$J = \{j \neq i : \mathbf{G}_{ij} = \mathbf{G}_{ji} = 0 \text{ and } \ominus_{i,j} \mathbf{D}_+ \text{ graphical}\}$$

5. Pick a partner  $j \in J$  with probability proportional to its degree in  $\mathbf{D}_+$ .
6. Set  $\mathbf{G}_{ij} = \mathbf{G}_{ji} = 1$  and update  $\mathbf{D}_+$  to  $\Theta_{i,j}\mathbf{D}_+$ .
7. Repeat steps 4 to 6 until the degree of agent  $i$  is zero.
8. Return to step 2.

The input for the algorithm is the target degree sequence  $\mathbf{D}_+$  and the output is an undirected adjacency matrix  $\mathbf{G}$  with  $\mathbf{G}'\iota = \mathbf{D}_+$ .

## Importance Weights

- The Blitzstein and Diaconis (2010) procedure delivers a random draw from  $\mathbb{D}_{N, \mathbf{d}_+}$ , but not a *uniform* random draw.
- Construct importance weights in order to compute expectations using the correct reference distribution.
- Let  $\mathcal{Y}_{N, \mathbf{d}_+}$  denote the set of all possible sequences of links generated by the algorithm given input  $\mathbf{D}_+ = \mathbf{d}_+$ .

## Importance Weights(continued)

- Let  $\mathcal{G}(Y)$  be the adjacency matrix induced by link sequence  $Y$ .
  - Let  $Y$  and  $Y'$  are equivalent if  $\mathcal{G}(Y) = \mathcal{G}(Y')$ .
- We can partition  $\mathcal{Y}_{N, \mathbf{d}_+}$  into a set of equivalence classes whose number coincides with the cardinality of  $\mathbb{D}_{N, \mathbf{d}_+}$ .

### Importance Weights(continued)

- Let  $c(Y)$  denote the number of possible link sequences produced by the algorithm that produce  $Y$ 's end point adjacency matrix.
- Let  $i_1, i_2, \dots, i_M$  be the sequence of agents chosen in step 3 of the algorithm in which  $Y$  is the output.

## Importance Weights(continued)

- Let  $a_1, \dots, a_m$  be the degree sequences of  $i_1, \dots, i_M$  at the time when each agent was *first* selected in step 3.
- Blitzstein and Diaconis show that:

$$c(Y) = \prod_{k=1}^M a_k!$$

## Importance Weights(continued)

- Let  $\sigma(Y)$  be the probability that the algorithm produces link sequence  $Y$ .
- $\sigma(Y)$  is easy to compute:
  - each time a link in step 5 is chosen record the probability with which it was chosen.
  - this equals the residual degree of the chosen agent divided by the sum of the residual degrees of all agents in the choice set.
  - the product of all these probabilities equals  $\sigma(Y)$ .

## Importance Weights(continued)

- Let  $f(\mathbf{G})$  be some function of the adjacency matrix and consider the expected value

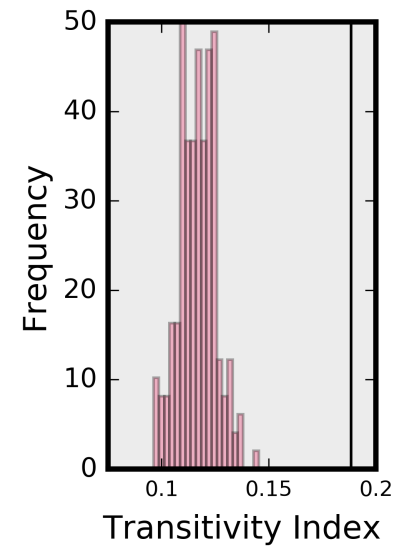
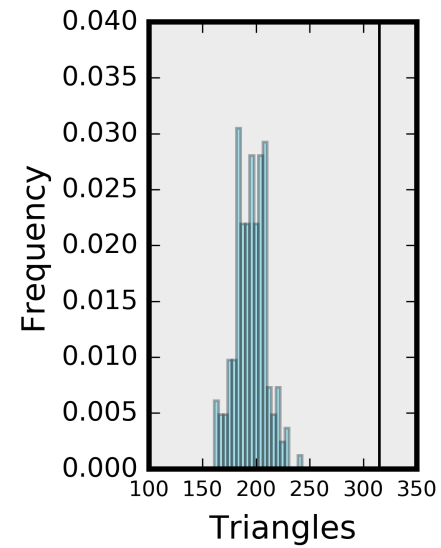
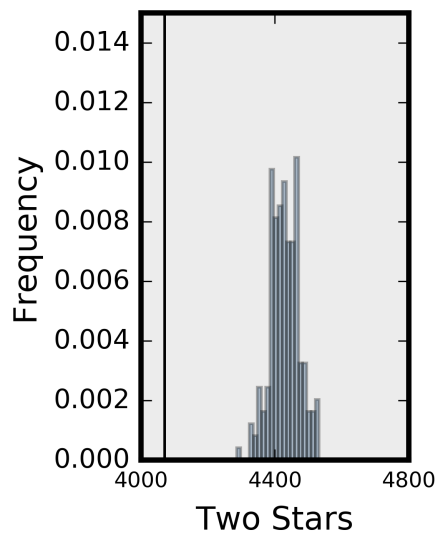
$$\begin{aligned}\mathbb{E} \left[ \frac{\pi(\mathcal{G}(Y))}{c(Y) \sigma(Y)} f(\mathcal{G}(Y)) \right] &= \sum_{y \in \mathbb{Y}_{N,d}} \frac{\pi(\mathcal{G}(y))}{c(y) \sigma(y)} f(\mathcal{G}(y)) \sigma(y) \\ &= \sum_{y \in Y_{N,d}} \frac{\pi(\mathcal{G}(y))}{c(y)} f(\mathcal{G}(y)) \\ &= \sum_{\mathbf{g} \in \mathbb{D}_{N,d}} \sum_{\{y: \mathcal{G}(y)=\mathbf{g}\}} \frac{\pi(\mathbf{g})}{c(y)} f(\mathbf{g}) \\ &= \sum_{\mathbf{g} \in \mathbb{D}_{N,d}} \pi(\mathbf{g}) f(\mathbf{g}) \\ &= \mathbb{E}_{\pi} [f(\mathbf{G})].\end{aligned}$$



## Importance Weights (continued)

- The ratio  $\pi(G(Y_t)) / c(Y_t) \sigma(Y_t)$  is called the likelihood ratio or the **importance weight**.
- Our random network draws are not uniform from the set of interest
- The importance weights correct for the fact that we are sampling from the wrong distribution

## Nyakatoke Example



## Blitzstein and Diaconis Wrap-Up

- While using the  $\beta$ -model as a reference model is restrictive it
  - is a natural starting point for hypothesis testing.
  - suggests that an investment in computation skills is likely to be valuable to anyone doing empirical work.
- It might be of interest to condition on additional features of the network in hand...