HJB Equation and Merton's Portfolio Problem

Ashwin Rao

ICME, Stanford University

November 23, 2018

Overview

1. Problem Statement
2. HJB Equation as Optimal Discounted Value Function PDE
3. Reducing the PDE to an ODE
4. Optimal Allocation and Consumption
5. Insights and Real-World Adaptation

Informal Problem Statement

- You will live for (deterministic) T more years
- Current Wealth + PV of Future Income (less Debt) is $W_0 > 0$.
- You can invest in (allocate to) n risky assets and a riskless asset
- Each asset has known normal distribution of returns
- Allowed to long or short any fractional quantities of assets
- Trading in continuous time $0 \leq t < T$, with no transaction costs
- You can consume any fractional amount of wealth at any time
- Dynamic Decision: Optimal Allocation and Consumption at each time
- To maximize lifetime-aggregated utility of consumption
- Consumption Utility assumed to have constant Relative Risk-Aversion

Problem Notation

For simplicity, we state and solve the problem for 1 risky asset but the solution generalizes easily to n risky assets.

- Riskless asset: $dR_t = r \cdot R_t \cdot dt$
- Risky asset: $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$ (i.e. Geometric Brownian)
- $\mu > r > 0, \sigma > 0$ (for n assets, we work with a covariance matrix)
- Wealth at time t is denoted by $W_t > 0$
- Fraction of wealth allocated to risky asset denoted by $\pi(t, W_t)$
- Fraction of wealth in riskless asset will then be $1 - \pi(t, W_t)$
- Wealth consumption denoted by $c(t, W_t) \geq 0$
- Utility of Consumption function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $0 < \gamma \neq 1$
- Utility of Consumption function $U(x) = \log(x)$ for $\gamma = 1$
- $\gamma =$ (constant) Relative Risk-Aversion $\frac{-x \cdot U''(x)}{U'(x)}$

Problem Statement

- We write π_t, c_t instead of $\pi(t, W_t), c(t, W_t)$ to lighten notation
- Balance constraint implies the following process for Wealth W_t

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$$

- At any time t, determine optimal $[\pi(t, W_t), c(t, W_t)]$ to maximize:

$$E[\int_t^T \frac{e^{-\rho(s-t)} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds + \frac{e^{-\rho(T-t)} \cdot B(T) \cdot W_T^{1-\gamma}}{1-\gamma} \mid W_t]$$

- where $\rho \geq 0$ is the utility discount rate, $B(T)$ is the bequest function
- We can solve this problem for arbitrary bequest $B(T)$ but for simplicity, will consider $B(T) = \epsilon^\gamma$ where $0 < \epsilon \ll 1$, meaning "no bequest" (we need this ϵ-formulation for technical reasons).
- We will solve this problem for $\gamma \neq 1$ ($\gamma = 1$ is easier, hence omitted)

Continuous-Time Stochastic Control

- Think of this as a continuous-time Stochastic Control problem
- The *State* is (t, W_t)
- The *Action* is $[\pi_t, c_t]$
- The *Reward* per unit time is $U(c_t)$
- The *Return* is the usual accumulated discounted *Reward*
- Find *Policy* : $(t, W_t) \rightarrow [\pi_t, c_t]$ that maximizes the *Expected Return*
- Note: $c_t \geq 0$, but π_t is unconstrained

Optimal Discounted Value Function

- Instead of the usual Value Function (*Expected Return* from a given *State*), we consider the Discounted Value Function
- Discounted Value Function is simply the Value Function further discounted to time 0
- We focus on the Optimal Discounted Value Function $V^*(t, W_t)$

$$V^*(t, W_t) = \max_{\pi_t, c_t} E[\int_t^T \frac{e^{-\rho s} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds + \frac{e^{-\rho T} \cdot \epsilon^\gamma \cdot W_T^{1-\gamma}}{1-\gamma}]$$

- $V^*(t, W_t)$ satisfies a simple recursive formulation for $0 \leq t < t_1 < T$.

$$V^*(t, W_t) = \max_{\pi_t, c_t} E[V^*(t_1, W_{t_1}) + \int_t^{t_1} \frac{e^{-\rho s} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds]$$

HJB Equation for Optimal Discounted Value Function

Rewriting in stochastic differential form, we have the HJB formulation

$$\max_{\pi_t, c_t} E[dV^*(t, W_t) + \frac{e^{-\rho t} \cdot c_t^{1-\gamma}}{1-\gamma} \cdot dt] = 0$$

Use Ito's Lemma on dV^*, remove the dz_t term since it's a martingale, and divide throughout by dt to produce the HJB Equation in PDE form:

$$\max_{\pi_t, c_t}[\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}((\pi_t(\mu - r) + r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W_t^2}\frac{\pi_t^2 \sigma^2 W_t^2}{2} + \frac{e^{-\rho t} \cdot c_t^{1-\gamma}}{1-\gamma}] = 0$$

Let us write the above equation more succinctly as:

$$\max_{\pi_t, c_t} \Phi(t, W_t; \pi_t, c_t) = 0$$

Note: we are working with the constraints $W_t > 0, c_t \geq 0$ for $0 \leq t < T$

Optimal Allocation and Consumption

Find optimal π_t^*, c_t^* by taking partial derivatives of $\Phi(t, W_t; \pi_t, c_t)$ with respect to π_t and c_t, and equate to 0 (first-order conditions for Φ).

- Partial derivative of Φ with respect to π_t:

$$(\mu - r) \cdot \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \pi_t \cdot \sigma^2 \cdot W_t = 0$$

$$\Rightarrow \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}$$

- Partial derivative of Φ with respect to c_t:

$$-\frac{\partial V^*}{\partial W_t} + e^{-\rho t} \cdot (c_t^*)^{-\gamma} = 0$$

$$\Rightarrow c_t^* = (\frac{\partial V^*}{\partial W_t} \cdot e^{\rho t})^{\frac{-1}{\gamma}}$$

Optimal Discounted Value Function PDE

Now substitute π_t^* and c_t^* in $\Phi(t, W_t; \pi_t, c_t)$ and set it to 0, which gets us the Optimal Discounted Value Function PDE:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{(\frac{\partial V^*}{\partial W_t})^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} \cdot r \cdot W_t + \frac{\gamma}{1-\gamma} \cdot e^{\frac{-\rho t}{\gamma}} \cdot (\frac{\partial V^*}{\partial W_t})^{\frac{\gamma-1}{\gamma}} = 0$$

The boundary condition is:

$$V^*(T, W_T) = e^{-\rho T} \cdot \epsilon^\gamma \cdot \frac{W_T^{1-\gamma}}{1-\gamma}$$

The second-order conditions for Φ are satisfied **under the assumptions** $c_t^* > 0, W_t > 0, \frac{\partial^2 V^*}{\partial W_t^2} < 0$ for all $0 \leq t < T$ (we will later show that these are all satisfied in the solution we derive), and for concave $U(\cdot)$, i.e., $\gamma > 0$

Solving the PDE with a guess solution

We surmise with a guess solution

$$V^*(t, W_t) = f(t)^\gamma \cdot e^{-\rho t} \cdot \frac{W_t^{1-\gamma}}{1-\gamma}$$

Then,

$$\frac{\partial V^*}{\partial t} = (\gamma \cdot f(t)^{\gamma-1} \cdot f'(t) - \rho \cdot f(t)^\gamma) \cdot e^{-\rho t} \cdot \frac{W_t^{1-\gamma}}{1-\gamma}$$

$$\frac{\partial V^*}{\partial W_t} = f(t)^\gamma \cdot e^{-\rho t} \cdot W_t^{-\gamma}$$

$$\frac{\partial^2 V^*}{\partial W_t^2} = -f(t)^\gamma \cdot e^{-\rho t} \cdot \gamma \cdot W_t^{-\gamma-1}$$

PDE reduced to an ODE

Substituting the guess solution in the PDE, we get the simple ODE:

$$f'(t) = \nu \cdot f(t) - 1$$

where

$$\nu = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu - r)^2}{2\sigma^2\gamma} + r\right)}{\gamma}$$

with boundary condition $f(T) = \epsilon$.
The solution to this ODE is:

$$f(t) = \begin{cases} \frac{1 + (\nu\epsilon - 1) \cdot e^{-\nu(T-t)}}{\nu} & \text{for } \nu \neq 0 \\ T - t + \epsilon & \text{for } \nu = 0 \end{cases}$$

Optimal Allocation and Consumption

Putting it all together (substituting the solution for $f(t)$), we get:

$$\pi^*(t, W_t) = \frac{\mu - r}{\sigma^2 \gamma}$$

$$c^*(t, W_t) = \frac{W_t}{f(t)} = \begin{cases} \frac{\nu \cdot W_t}{1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)}} & \text{for } \nu \neq 0 \\ \frac{W_t}{T-t+\epsilon} & \text{for } \nu = 0 \end{cases}$$

$$V^*(t, W_t) = \begin{cases} e^{-\rho t} \cdot \frac{(1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)})^\gamma}{\nu^\gamma} \cdot \frac{W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu \neq 0 \\ e^{-\rho t} \cdot \frac{(T-t+\epsilon)^\gamma \cdot W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu = 0 \end{cases}$$

- $f(t) > 0$ for all $0 \leq t < T$ (for all ν) ensures $W_t, c_t^* > 0, \frac{\partial^2 V^*}{\partial W_t^2} < 0$. This ensures the constraints $W_t > 0$ and $c_t \geq 0$ are satisfied and the second-order conditions for Φ are also satisfied.
- The HJB Formulation was key and this solution approach provides a template for similar continuous-time stochastic control problems.

Gaining Insights into the Solution

- Optimal Allocation $\pi^*(t, W_t)$ is constant (independent of t and W_t)
- Optimal Fractional Consumption $\frac{c^*(t,W_t)}{W_t}$ depends only on t $(= \frac{1}{f(t)})$
- With Optimal Allocation & Consumption, the Wealth process is:

$$\frac{dW_t}{W_t} = (r + \frac{(\mu - r)^2}{\sigma^2 \gamma} - \frac{1}{f(t)}) \cdot dt + \frac{\mu - r}{\sigma \gamma} \cdot dz_t$$

- Expected Portfolio Return is constant over time $(= r + \frac{(\mu-r)^2}{\sigma^2\gamma})$
- Assuming $\epsilon < \frac{1}{\nu}$, Fractional Consumption $\frac{1}{f(t)}$ increases over time
- Expected Rate of Wealth Growth $r + \frac{(\mu-r)^2}{\sigma^2\gamma} - \frac{1}{f(t)}$ decreases over time
- If $r + \frac{(\mu-r)^2}{\sigma^2\gamma} > \frac{1}{f(0)}$, we start by Consuming $<$ Expected Portfolio Growth and over time, we Consume $>$ Expected Portfolio Growth
- Wealth Growth Volatility is constant $(= \frac{\mu-r}{\sigma\gamma})$

Porting this to Real-World Portfolio Optimization

- Analytical tractability in Merton's formulation was due to:
  - Normal distribution of asset returns
  - Constant Relative Risk-Aversion
  - Frictionless, continuous trading
- However, real-world situation involves:
  - Discrete amounts of assets to hold and discrete quantities of trades
  - Transaction costs
  - Locked-out days for trading
  - Non-stationary/arbitrary/correlated processes of multiple assets
  - Changing/uncertain risk-free rate
  - Consumption constraints
  - Arbitrary Risk-Aversion/Utility specification
- $\Rightarrow$ Approximate Dynamic Programming or Reinforcement Learning
- Large Action Space points to Policy Gradient Algorithms