CONSTRAINED DYNAMIC PROGRAM FOR BACKROOM MINIMIZATION ENSURING ADEQUATE SHELF INVENTORY

1. Introduction

$\mathbb{Z}$ refers to the set of integers, $\mathbb{R}$ refers to the set of real numbers. We will subscript $\mathbb{Z}$ and $\mathbb{R}$ to denote appropriate subsets of $\mathbb{Z}$ and $\mathbb{R}$.

We consider a single store and single item served inventory from a supplier with infinite inventory and lead time of $L \in \mathbb{Z}_{\geq 0}$ epochs. Review period is assumed to be 1 epoch. The customer demand (as units of the item) experienced at the store in epoch t is denoted by random variable $D_t \in \mathbb{Z}_{\geq 0}$.There is a fixed capacity of $P \in \mathbb{Z}_{>0}$ units for the item on the shelf (planogram) at the store. The item can only be replenished in multiples of $C \in \mathbb{Z}_{>0}$ units (C refers to the casepack size). Our goal is to identify an ordering policy that minimizes the "average backroom inventory" (backroom inventory refers to the store inventory that is in excess of P) while ensuring that the shelf inventory in every epoch is at least a specified fraction $\alpha \in [0, 1]$ of P with probability at least $\beta \in [0, 1]$ (PresMin condition).

2. Inventory

- Denote on-hand inventory (a.k.a. Inventory Level) at the store at the start of epoch t as: $IL_t \in \mathbb{Z}$ (note: IL_t is allowed to go negative if demand is unmet at the store, leading to back-ordering).
- Denote on-order inventory at the start of epoch t to arrive in k epochs ($1 \leq k \leq L$) as $OO_{t,k} \in \mathbb{Z}_{\geq 0}$

3. Inventory Movements

Denote number of casepacks of inventory ordered in epoch t as $q_t \in \mathbb{Z}_{\geq 0}$. The store will receive that inventory of $q_t C$ in epoch $t + L$. Denote $R_t \in \mathbb{Z}_{\geq 0}$ as the inventory received in epoch t. Following the epoch t of inventory ordering and until the epoch $t + L$ of inventory receipt, this quantity $q_t C$ will appear in the flow equations (see below) as on-order $OO_{t+j,L-j+1}, 1 \leq j \leq L$. For the special case where $L = 0$, $R_t = q_t C$ (Sequence of Events below illustrates that within an epoch, inventory receipt is after inventory ordering).

4. Constrained Dynamic Program

The *State* in epoch t is defined by the vector:

$$[IL_t, OO_{t,1}, \dots OO_{t,L}]$$

The *Action* in epoch t is the number of casepacks ordered, i.e., q_t.

The *Cost* in epoch t is defined as the backroom inventory upon receipt of inventory at the store, i.e., $\max(0, IL_t + R_t - P)$. We set up the problem as an Average-Cost Dynamic Program with the requirements (constraints) that on-hand inventory $IL_t \geq \alpha P$ with probability $\geq \beta$ for all epochs t.

5. Sequence of events in an epoch

(1) Observe *State* (observation of the inventory level IL_t and of the on-orders $OO_{t,1}, \ldots, OO_{t,L}$).
(2) Check if $IL_t \geq \alpha P$.
(3) Perform *Action* (ordering of inventory as number of casepacks q_t).
(4) Receipt of inventory R_t at the store.
(5) Calculate *Cost* as the backroom inventory, i.e., $\max(0, IL_t + R_t - P)$.
(6) Occurrence of demand D_t at the store (including missed sales, i.e., stockouts at the store).

6. Equations defining Inventory Flow

The following equations define the inventory flow in any epoch t:

$$R_t = \begin{cases} OO_{t,1} & \text{if } L > 0 \\ q_t C & \text{if } L = 0 \end{cases} \text{ for all } t$$

$$IL_{t+1} = \max(0, IL_t + R_t - D_t) \text{ for all } t$$

$$OO_{t+1,k} = OO_{t,k+1} \text{ for all } t, \text{ for all } 1 \leq k < L$$

$$OO_{t+1,L} = q_t C \text{ for all } t$$

7. A Heuristic Policy

We want to order "enough" in epoch t so that epoch $t+L+1$ on-hand inventory $IL_{t+L+1} \geq \alpha P$ with probability $\geq \beta$. If we assume backordering, then this can be written as:

$$Pr[IL_t + \sum_{k=1}^{L} OO_{t,k} + q_t C - \sum_{k=0}^{L} D_{t+k} \geq \alpha P] \geq \beta$$

.

If we also assume that ordering can be in continuous quantities (rather than casepacks), then ordering "just enough" in epoch t to satisfy the constraint $Pr[IL_{t+L+1} \geq \alpha P] \geq \beta$ would automatically minimize expected backroom inventory in epoch $t + L$. This "just enough" order quantity is given by:

$$\max(0, \alpha P + F_{t,L}^{-1}(\beta) - (IL_t + \sum_{k=1}^{L} OO_{t,k}))$$

where $F_{t,L}(\cdot)$ is the inverse cumulative mass function of the discrete random variable $\sum_{k=0}^{L} D_{t+k}$

So we could start with this heuristic as an approximation to the optimal policy for our case of "no-backordering" and "ordering in casepacks" as follows:

$$q_t = \max(0, \lceil \frac{\alpha P + F_{t,L}^{-1}(\beta) - (IL_t + \sum_{k=1}^{L} OO_{t,k})}{C} \rceil)$$

8. Continuous-Time, Continuous-Inventory, Deterministic Demand

Now we consider a simple setting where demand happens at a constant deterministic rate in continuous time and in continuous quantity (call the demand rate $\lambda \in \mathbb{R}$ units per time). In this setting, we can adapt the lead time, POG and casepack to be continuous, i.e. $L, P, C \in \mathbb{R}$.

In this setting, the inventory level at the store follows an EOQ-like sawtooth function with the height of the sawtooth equal to C and the width of the sawtooth equal to $\frac{C}{\lambda}$. Our task is to determine the bottom of the sawtooth (denoted B) so that the fraction of time the inventory level IL_t goes below αP is no more than $1 - \beta$ (adaption of the PresMin condition). Once we establish the bottom of the sawtooth B, we will immediately have the top of the sawtooth (denoted T) as $B + C$, and consequently, we will have the backroom units per time, and the Order Point.

Due to the constant rate of demand, ensuring that the fraction of time the inventory level IL_t goes below αP is no more than $1 - \beta$ is equivalent to ensuring that inventory consumed in a cycle when below the level αP is no more than $(1 - \beta)C$ (because inventory consumed in the entire cycle is C). So, the PresMin condition can be written as: $B \geq \alpha P - (1 - \beta)C$.

We consider two cases:

- $C \leq (1 - \alpha)P + (1 - \beta)C$ meaning the casepack is small enough to avoid backroom: if we set $T = P$ (hence, no backroom), $B = P - C \geq \alpha P - (1 - \beta)C$ (PresMin condition satisfied)
- $C > (1 - \alpha)P + (1 - \beta)C$ meaning the casepack is not small enough to avoid backroom: if we set $B = \alpha P - (1 - \beta)C$, $T = C + \alpha P - (1 - \beta)C = \alpha P + \beta C > P$ which means we will have some backroom inventory.

Combining the two cases, we can succinctly write B and T as follows:

$$B = \max(P - C, \alpha P - (1 - \beta)C)$$
$$T = \max(P, \alpha P + \beta C)$$

The maximum backroom inventory

$$E = T - P = \max(0, \beta C - (1 - \alpha)P)$$

In a given cycle, the time for which there is backroom inventory is $\frac{E}{\lambda}$ and so, the aggregate backroom units in any cycle is:

$$\frac{1}{2} \cdot E \cdot \frac{E}{\lambda}$$

Since, each cycle is of duration $\frac{C}{\lambda}$, the backroom units per time is:

$$\frac{\frac{1}{2} \cdot E \cdot \frac{E}{\lambda}}{\frac{C}{\lambda}} = \frac{E^2}{2C} = \frac{(\max(0, \beta C - (1-\alpha)P))^2}{2C}$$

Since the lead time is L, we need to place the order exactly L units of time before the inventory level $IL_t = B$. Given the cyclic and regular nature of the sawtooth, we can simply consider the nearest ordering time point before $IL_t = B$. The amount of time between this ordering time point and the time when $IL_t = B$ is:

$$L - \lfloor \frac{L\lambda}{C} \rfloor \cdot \frac{C}{\lambda}$$

and hence, the inventory level at the time of ordering (denoted as Inventory Level Order Point $ILOP$) is:

$$B + \lambda(L - \lfloor \frac{L\lambda}{C} \rfloor \cdot \frac{C}{\lambda}) = \max(P - C, \alpha P - (1-\beta)C) + \lambda(L - \lfloor \frac{L\lambda}{C} \rfloor \cdot \frac{C}{\lambda})$$

At the time of ordering, the number of casepacks on-order is $\lfloor \frac{L\lambda}{C} \rfloor$ and so, the number of units of inventory on-order is: $\lfloor \frac{L\lambda}{C} \rfloor \cdot C$. So, we can say that at the time of ordering, the inventory position IP_t (defined as IL_t plus the number of units of inventory on-order) is:

$$\max(P - C, \alpha P - (1-\beta)C) + \lambda(L - \lfloor \frac{L\lambda}{C} \rfloor \cdot \frac{C}{\lambda}) + \lfloor \frac{L\lambda}{C} \rfloor \cdot C$$

We can refer to the above expression as the Inventory Position Order Point $IPOP$.