Stochastic Control for Optimal Market-Making

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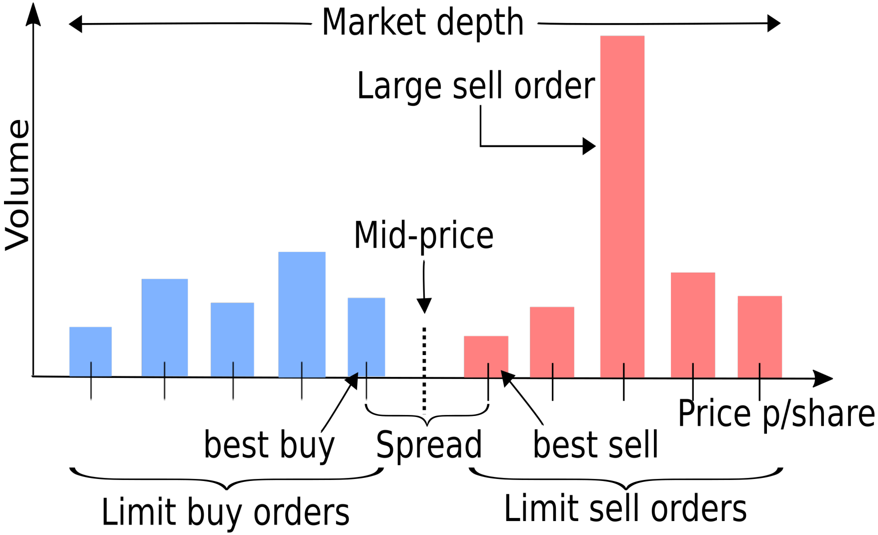
Overview

Trading Order Book (TOB)



Basics of Trading Order Book (TOB)

- Buyers/Sellers express their intent to trade by submitting bids/asks
- These are Limit Orders (LO) with a price P and size N
- Buy LO (P, N) states willingness to buy N shares at a price $\leq P$
- Sell LO (P, N) states willingness to sell N shares at a price $\geq P$
- Trading Order Book aggregates order sizes for each unique price
- So we can represent with two sorted lists of (Price, Size) pairs

$$\text{Bids: } [(P_i^{(b)}, N_i^{(b)}) \mid 1 \leq i \leq m], P_i^{(b)} > P_j^{(b)} \text{ for } i < j$$

$$\text{Asks: } [(P_i^{(a)}, N_i^{(a)}) \mid 1 \leq i \leq n], P_i^{(a)} < P_j^{(a)} \text{ for } i < j$$

- We call $P_1^{(b)}$ as simply *Bid*, $P_1^{(a)}$ as *Ask*, $\frac{P_1^{(a)} + P_1^{(b)}}{2}$ as *Mid*
- We call $P_1^{(a)} - P_1^{(b)}$ as *Spread*, $P_n^{(a)} - P_m^{(b)}$ as *Market Depth*
- A Market Order (MO) states intent to buy/sell N shares at the *best possible price(s)* available on the TOB at the time of MO submission

Trading Order Book (TOB) Activity

- A new Sell LO (P, N) potentially removes best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid (i : P_i^{(b)} \geq P)]$$

- After this removal, it will add the following to the asks side of the TOB

$$(P, \max(0, N - \sum_{i:P_i^{(b)} \geq P} N_i^{(b)}))$$

- A new Buy MO operates analogously (on the other side of the TOB)
- A Sell Market Order N will remove the best bid prices on the TOB

$$\text{Removal: } [(P_i^{(b)}, \min(N_i^{(b)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(b)}))) \mid 1 \leq i \leq m]$$

- A Buy Market Order N will remove the best ask prices on the TOB

$$\text{Removal: } [(P_i^{(a)}, \min(N_i^{(a)}, \max(0, N - \sum_{j=1}^{i-1} N_j^{(a)}))) \mid 1 \leq i \leq n]$$

TOB Dynamics and Market-Making

- Modeling TOB Dynamics involves predicting arrival of MOs and LOs
- Market-makers are liquidity providers (providers of Buy and Sell LOs)
- Other market participants are typically liquidity takers (MOs)
- But there are also other market participants that trade with LOs
- Complex interplay between market-makers & other mkt participants
- Hence, TOB Dynamics tend to be quite complex
- We view the TOB from the perspective of a single market-make who aims to gain with Buy/Sell LOs of appropriate width/size
- By anticipating TOB Dynamics & dynamically adjusting Buy/Sell LOs
- Goal is to maximize *Utility of Gains* at end of a suitable horizon
- If Buy/Sell LOs are too narrow, more frequent but small gains
- If Buy/Sell LOs are too wide, less frequent but large gains
- Market-maker also needs to manage potential unfavorable inventory (long or short) buildup and consequent unfavorable liquidation

Notation for Optimal Market-Making Problem

- We simplify the setting for ease of exposition
- Assume finite time steps indexed by $t = 0, 1, \ldots, T$
- Denote $W_t \in \mathbb{R}$ as Market-maker's wealth at time t
- Denote $I_t \in \mathbb{Z}$ as Market-maker's inventory of shares at time t ($I_0 = 0$)
- $S_t \in \mathbb{R}^+$ is the TOB Mid Price at time t (assume stochastic process)
- $P_t^{(b)} \in \mathbb{R}^+, N_t^{(b)} \in \mathbb{Z}^+$ are market maker's Bid Price, Bid Size at time t
- $P_t^{(a)} \in \mathbb{R}^+, N_t^{(a)} \in \mathbb{Z}^+$ are market-maker's Ask Price, Ask Size at time t
- Assume market-maker can add or remove bids/asks costlessly
- Denote $\delta_t^{(b)} = S_t - P_t^{(b)}$ as Bid Spread, $\delta_t^{(a)} = P_t^{(a)} - S_t$ as Ask Spread
- Random var $X_t^{(b)} \in \mathbb{Z}_{\geq 0}$ denotes bid-shares "hit" *up to* time t
- Random var $X_t^{(a)} \in \mathbb{Z}_{\geq 0}$ denotes ask-shares "lifted" *up to* time t

$$W_{t+1} = W_t + P_t^{(a)} \cdot (X_{t+1}^{(a)} - X_t^{(a)}) - P_t^{(b)} \cdot (X_{t+1}^{(b)} - X_t^{(b)}) \text{ , } I_t = X_t^{(b)} - X_t^{(a)}$$

- Goal to maximize $E[U(W_T + I_T \cdot S_T)]$ for appropriate concave $U(\cdot)$

Markov Decision Process (MDP) Formulation

- Order of MDP activity in each time step $0 \leq t \leq T-1$:
  - Observe *State* $:= (t, S_t, W_t, I_t)$
  - Perform *Action* $:= (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)})$
  - Experience TOB Dynamics resulting in:
    - random bid-shares hit $= X_{t+1}^{(b)} - X_t^{(b)}$ and ask-shares lifted $= X_{t+1}^{(a)} - X_t^{(a)}$
    - update of W_t to W_{t+1}, update of I_t to I_{t+1}
    - stochastic evolution of S_t to S_{t+1}
  - Receive next-step $(t+1)$ *Reward* R_{t+1}

$$R_{t+1} := \begin{cases} 0 & \text{for } 1 \leq t+1 \leq T-1 \\ U(W_{t+1} + I_{t+1} \cdot S_{t+1}) & \text{for } t+1 = T \end{cases}$$

- Goal is to find an *Optimal Policy* π^*:

$$\pi^*(t, S_t, W_t, I_t) = (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)}) \text{ that maximizes } \mathbb{E}[\sum_{t=1}^{T} R_t]$$

- Note: Discount Factor when aggregating Rewards in the MDP is 1

Avellaneda-Stoikov Continuous Time Formulation

- We go over the landmark paper by Avellaneda and Stoikov in 2006
- They derive a simple, clean and intuitive analytical solution
- We adapt our discrete-time notation to their continuous-time setting
- $X_t^{(b)}, X_t^{(a)}$ are *Poisson processes* with *arrival-rate* means $\lambda_t^{(b)}, \lambda_t^{(a)}$

$$dX_t^{(b)} \sim Poisson(\lambda_t^{(b)} \cdot dt) \text{ , } dX_t^{(a)} \sim Poisson(\lambda_t^{(a)} \cdot dt)$$

$$\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)}) \text{ , } \lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)}) \text{ for decreasing functions } f^{(b)}, f^{(a)}$$

$$dW_t = P_t^{(a)} \cdot dX_t^{(a)} - P_t^{(b)} \cdot dX_t^{(b)} \text{ , } I_t = X_t^{(b)} - X_t^{(a)} \text{ (note: } I_0 = 0)$$

- Since infinitesimal Poisson random variables $dX_t^{(b)}$ (shares hit in time dt) and $dX_t^{(a)}$ (shares lifted in time dt) have zero probability mass for values greater than 1, choices of $N_t^{(b)}$ and $N_t^{(a)}$ are irrelevant
- This simplifies the *Action* at time t to be just the pair: $(\delta_t^{(b)}, \delta_t^{(a)})$
- TOB Mid Price Dynamics: $dS_t = \sigma \cdot dz_t$ (scaled brownian motion)
- Utility function $U(x) = -e^{-\gamma x}$ where γ is coefficient of risk-aversion

Real-world Market-Making and Reinforcement Learning

- Arbitrary Price Dynamics $f_t(\cdot)$ and Temporary Price Impact $g_t(\cdot)$
- Non-stationarity/non-linear dynamics/impact require (Numerical) DP
- Frictions: Discrete Prices/Sizes, Constraints on Prices/Sizes, Fees
- Large State space incorporate various external factors in the State
- We could also represent the entire TOB within the State
- So then we'd have to develop a simulator capturing all of the above
- Simulator is a *Data-learnt Sampling Model* of TOB Dynamics
- In practice, we'd need to also capture *Cross-Asset Market Impact*
- Using this simulator and neural-networks func approx, we can do RL
- References: Nevmyvaka, Feng, Kearns; 2006 and Vyetrenko, Xu; 2019
- Exciting area for Future Research as well as Engineering Design