Refresher on Stochastic Calculus Foundations

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Continuous and Non-Differentiable Sample Paths

- Sample paths of Brownian motion z_t are continuous
- Sample paths of z_t are almost always non-differentiable, meaning

$$\lim_{h \to 0} \frac{z_{t+h} - z_t}{h} \text{ is almost always infinite}$$

- The intuition is that $\frac{dz_t}{dt}$ has standard deviation of $\frac{1}{\sqrt{dt}}$, which goes to ∞ as dt goes to 0

Infinite Total Variation of Sample Paths

- Sample paths of Brownian motion are of infinite total variation, i.e.

$$\lim_{h \to 0} \sum_{i=m}^{n-1} |z_{(i+1)h} - z_{ih}| \text{ is almost always infinite}$$

- More succinctly, we write

$$\int_S^T |dz_t| = \infty \text{ (almost always)}$$

Finite Quadratic Variation of Sample Paths

- Sample paths of Brownian Motion are of finite quadratic variation, i.e.

$$\lim_{h \to 0} \sum_{i=m}^{n-1} (z_{(i+1)h} - z_{ih})^2 = h(n - m)$$

- More succinctly, we write

$$\int_S^T (dz_t)^2 = T - S$$

- This means it's expected value is $T - S$ and it's variance is 0
- This leads to Ito's Lemma (Taylor series with $(dz_t)^2$ replaced with dt)
- This also leads to Ito Isometry (next slide)

Ito Isometry

- Let $X_t : [0, T] \times \Omega \rightarrow \mathbb{R}$ be a stochastic process adapted to filtration $\mathcal{F}$ of brownian motion z_t.
- Then, we know that the Ito integral $\int_0^T X_t \cdot dz_t$ is a martingale
- Ito Isometry tells us about the variance of $\int_0^T X_t \cdot dz_t$

$$\mathbb{E}[(\int_0^T X_t \cdot dz_t)^2] = \mathbb{E}[\int_0^T X_t^2 \cdot dt]$$

- Extending this to two Ito integrals, we have:

$$\mathbb{E}[(\int_0^T X_t \cdot dz_t)(\int_0^T Y_t \cdot dz_t)] = \mathbb{E}[\int_0^T X_t \cdot Y_t \cdot dt]$$

Fokker-Planck equation for PDF of a Stochastic Process

- We are given the following stochastic process:

$$dX_t = \mu(X_t, t) \cdot dt + \sigma(X_t, t) \cdot dz_t$$

- The Fokker-Planck equation of this process is the PDE:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial\{\mu(x,t) \cdot p(x,t)\}}{\partial x} + \frac{\partial^2\{\frac{\sigma^2(x,t)}{2} \cdot p(x,t)\}}{\partial x^2}$$

  where $p(x,t)$ is the probability density function of X_t
- The Fokker-Planck equation is used for problems where the initial distribution is known (Kolmogorov forward equation)
- However, if the problem is to know the distribution at previous times, the Feynman-Kac formula can be used (a consequence of the Kolmogorov backward equation)

Feynman-Kac Formula (PDE-SDE linkage)

- Consider the partial differential equation for $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$:

$$\frac{\partial u(x,t)}{\partial t} + \mu(x,t) \frac{\partial u(x,t)}{\partial x} + \frac{\sigma^2(x,t)}{2} \frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t) u(x,t) = f(x,t)$$

subject to $u(x, T) = \psi(x)$, where μ, σ, V, f, ψ are known functions.

- Then the Feynman-Kac formula tells us that the solution $u(x,t)$ can be written as the following conditional expectation:

$$\mathbb{E}[(\int_t^T e^{-\int_t^u V(X_s,s)ds} \cdot f(X_u, u) \cdot du) + e^{-\int_t^T V(X_u,u)du} \cdot \psi(X_T) | X_t = x]$$

such that X_u is the following Ito process with initial condition $X_t = x$:

$$dX_u = \mu(X_u, u) \cdot du + \sigma(X_u, u) \cdot dz_u$$

Stopping Time

- Stopping time τ is a "random time" (random variable) interpreted as time at which a given stochastic process exhibits certain behavior
- Stopping time often defined by a "stopping policy" to decide whether to continue/stop a process based on present position and past events
- Random variable τ such that $Pr[\tau \leq t]$ is in σ-algebra $\mathcal{F}_t$, for all t
- Deciding whether $\tau \leq t$ only depends on information up to time t
- Hitting time of a Borel set A for a process X_t is the first time X_t takes a value within the set A
- Hitting time is an example of stopping time. Formally,

$$T_{X,A} = \min\{t \in \mathbb{R} | X_t \in A\}$$

eg: Hitting time of a process to exceed a certain fixed level

Optimal Stopping Problem

- Optimal Stopping problem for Stochastic Process X_t:

$$V(x) = \max_{\tau} \mathbb{E}[G(X_\tau)|X_0 = x]$$

  where τ is a set of stopping times of X_t, $V(\cdot)$ is called the value function, and G is called the reward (or gain) function.
- Note that sometimes we can have several stopping times that maximize $\mathbb{E}[G(X_\tau)]$ and we say that the optimal stopping time is the smallest stopping time achieving the maximum value.
- Example of Optimal Stopping: Optimal Exercise of American Options
  - X_t is stochastic process for underlying security's price
  - x is underlying security's current price
  - τ is set of exercise times corresponding to various stopping policies
  - $V(\cdot)$ is American option price as function of underlying's current price
  - $G(\cdot)$ is the option payoff function

Markov Property

- Markov property says that the $\mathcal{F}_t$-conditional PDF of X_{t+h} depends only on the present state X_t
- Strong Markov property says that for every stopping time τ, the $\mathcal{F}_\tau$-conditional PDF of $X_{\tau+h}$ depends only on X_τ

Infinitesimal Generator and Dynkin's Formula

- Infinitesimal Generator of a time-homogeneous $\mathbb{R}^n$-valued diffusion X_t is the PDE operator A (operating on functions $f : \mathbb{R}^n \to \mathbb{R}$) defined as

$$A \bullet f(x) = \lim_{t \to 0} \frac{\mathbb{E}[f(X_t)|X_0 = x] - f(x)}{t}$$

- For $\mathbb{R}^n$-valued diffusion X_t given by: $dX_t = \mu(X_t) \cdot dt + \sigma(X_t) \cdot dz_t$,

$$A \bullet f(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \sum_{i,j} (\sigma(x)\sigma(x)^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

- If τ is stopping time conditional on $X_0 = x$, Dynkin's formula says:

$$\mathbb{E}[f(X_\tau)|X_0 = x] = f(x) + \mathbb{E}[\int_0^\tau A \bullet f(X_s) \cdot ds | X_0 = x]$$

- Stochastic generalization of 2nd fundamental theorem of calculus