MDP FOR UNDER-/OVER-CAPACITY OPTIMIZATION

1. Introduction

$\mathbb{Z}$ refers to the set of integers, $\mathbb{R}$ refers to the set of real numbers. We will subscript $\mathbb{Z}$ and $\mathbb{R}$ to denote appropriate subsets of $\mathbb{Z}$ and $\mathbb{R}$. $\mathbb{I}_b$ is the indicator function (for $b \in Boolean$) whose value is 1 if b is True and 0 if b is False.

We consider a single store and single item served inventory from a supplier with infinite inventory and lead time of $L \in \mathbb{Z}_{\geq 0}$ epochs. Review period is assumed to be 1 epoch. There is a fixed space capacity of $P \in \mathbb{Z}_{>0}$ units for the item at the store (P refers to the Planogram size). The item can only be replenished in multiples of $C \in \mathbb{Z}_{>0}$ units (C refers to the casepack size). Our goal is to identify a replenishment policy that minimizes the expected cost of inventory movements (defined by fixed cost $K \in \mathbb{R}_{\geq 0}$ and variable cost $J \in \mathbb{R}_{\geq 0}$), of store inventory going below the capacity of P (defined by a convex under-capacity cost function f) and of store inventory going above the capacity of P (defined by a convex over-capacity cost function g).

2. Costs

- If the on-hand inventory in the store is $x < P$, there is a cost of $f(P - x)$ where $f(0) = 0$ and $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a convex function (f provided as input to this problem).
- If the on-hand inventory in the store is $x > P$, there is a cost of $g(x - P)$ where $g(0) = 0$ and $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a convex function (g provided as input to this problem).
- The cost of moving $y \in \mathbb{Z}_{\geq 0}$ casepacks of inventory from supplier to store is $K\mathbb{I}_{y>0} + Jy$ for $K, J \in \mathbb{R}_{\geq 0}$.

Note that this formulation doesn't have any notion of holding costs (it can be absorbed into definition of functions f and g), doesn't have any notion of stockout cost (it can be absorbed into definition of function f, noting that on-hand inventory can go negative due to unmet demand in the below formulation), and doesn't have any notion of presentation-minimum (it can be absorbed into definition of function f).

3. Inventory

- Denote on-hand inventory (a.k.a. Inventory Level) at the store at the start of epoch t as: $IL_t \in \mathbb{Z}$ (note: IL_t is allowed to go negative if demand is unmet at the store, leading to back-ordering).

- Denote on-order inventory arriving in k epochs ($1 \leq k \leq L$) at the start of epoch t as $OO_{t,k} \in \mathbb{Z}_{\geq 0}$

4. Inventory Movements

Denote number of casepacks of inventory ordered in epoch t as $q_t \in \mathbb{Z}_{\geq 0}$. The store will receive that inventory of $q_t C$ in epoch $t + L$. Denote $R_t \in \mathbb{Z}_{\geq 0}$ as the inventory received in epoch t. Following the epoch t of inventory ordering and until the epoch $t + L$ of inventory receipt, this quantity $q_t C$ will appear in the flow equations (see below) as on-order $OO_{t+j,L-j+1}, 1 \leq j \leq L$. For the special case where $L = 0$, $R_t = q_t C$ (Sequence of Events below illustrates that within an epoch, receipt of inventory happens after ordering of inventory).

Demand at store in epoch t is denoted by random variable D_t.

5. States and Actions

The MDP *State* in epoch t is defined by the vector:

$$[IL_t, OO_{t,1}, \ldots OO_{t,L}]$$

The MDP *Action* in epoch t is the number of casepacks ordered, i.e., q_t

6. Sequence of events in an epoch

(1) Observe *State* (observation of the inventory level IL_t and of the on-orders $OO_{t,1}, \ldots, OO_{t,L}$).
(2) Perform *Action* (ordering of inventory as number of casepacks q_t).
(3) Calculate movement cost $K\mathbb{I}_{q_t>0} + Jq_t$.
(4) Receipt of inventory R_t at the store.
(5) Calculate over-capacity cost $g(\max(0, IL_t + R_t - P))$.
(6) Occurrence of demand at the store (including missed sales, i.e., stockouts at the store).
(7) Calculate under-capacity cost $f(\max(0, P - IL_{t+1}))$.

7. Equations defining Inventory Flow

The following equations define the inventory flow in any epoch t:

$$R_t = \begin{cases} OO_{t,1} & \text{if } L > 0 \\ q_t C & \text{if } L = 0 \end{cases} \text{ for all } t$$

$$IL_{t+1} = IL_t + R_t - D_t \text{ for all } t$$

$$OO_{t+1,k} = OO_{t,k+1} \text{ for all } t, \text{ for all } 1 \leq k < L$$

$$OO_{t+1,L} = q_t C \text{ for all } t$$

8. Cost Equations

The MDP *Reward* in epoch t is defined by the following cost components:

- Movement Cost:
$$K\mathbb{I}_{q_t>0} + Jq_t$$
- Over-capacity Cost:
$$g(\max(0, IL_t + R_t - P)$$
- Under-capacity Cost:
$$f(\max(0, P - IL_{t+1}))$$

9. A Presentation-Minimum Model

Presentation-Minimum is the idea that we'd like to have at least $PM \in \mathbb{Z}_{>0}$ units of inventory on our shelf with high probability. So one possible model that accommodates the *Presentation-Minimum* idea is to set $f(x) = 0$ if $x < P - PM$ and otherwise $f(x)$ is a high enough value (to prohibit inventory from going below *Presentation-Minimum*). Also, we set $g(x) = x$. This way we ensure that the inventory will not fall below the *Presentation-Minimum* with high probability while minimizing the inventory in the backroom.