# Overview of Stochastic Calculus Foundations

Ashwin Rao

ICME, Stanford University

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# Continuous and Non-Differentiable Sample Paths

- Sample paths of Brownian motion $z_t$ are continuous
- Sample paths of $z_t$ are almost always non-differentiable, meaning

$$\lim_{h \to 0} \frac{z_{t+h} - z_t}{h} \text{ is almost always infinite}$$

- The intuition is that $\frac{dz_t}{dt}$ has standard deviation of $\frac{1}{\sqrt{dt}}$, which goes to $\infty$ as $dt$ goes to 0
- So, this derivative $\frac{dz_t}{dt}$ is almost always infinite

# Infinite Total Variation of Sample Paths

- Sample paths of Brownian motion are of infinite total variation, i.e.

$$\lim_{h \to 0} \sum_{i=m}^{n-1} |z_{(i+1)h} - z_{ih}| \text{ is almost always infinite}$$

- More succinctly, we write

$$\int_S^T |dz_t| = \infty \text{ (almost always)}$$

# Finite Quadratic Variation of Sample Paths

- Sample paths of Brownian Motion are of finite quadratic variation, i.e.

$$\lim_{h \to 0} \sum_{i=m}^{n-1} (z_{(i+1)h} - z_{ih})^2 = h(n-m)$$

- More succinctly, we write

$$\int_S^T (dz_t)^2 = T - S$$

- This means it's expected value is $T - S$ and it's variance is 0
- This leads to Ito's Lemma and Ito Isometry

# Fokker-Planck equation of a Stochastic Process

- Let's say we are given the following stochastic process

$$dX_t = \mu(X_t, t) \cdot dt + \sigma(X_t, t) \cdot dz_t$$

- The Fokker-Planck equation of this process is the PDE:

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial \{\mu(x,t) \cdot p(x,t)\}}{\partial x} + \frac{\partial^2 \{\frac{\sigma^2(x,t)}{2} \cdot p(x,t)\}}{\partial x^2}$$

  where $p(x,t)$ is the probability density function of $X_t$
- The Fokker-Planck equation is used with problems where the initial distribution is known (Kolmogorov forward equation)
- However, if the problem is to know the distribution at previous times, the Feynman-Kac formula can be used (a consequence of the Kolmogorov backward equation)

# Feynman-Kac Formula

- Consider the partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} + \mu(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\sigma^2(x,t)}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - V(x,t)u(x,t) = f(x,t)$$

  defined for all $x \in \mathbb{R}$, $t \in [0, T]$, subject to boundary $u(x, T) = \psi(x)$, where $\mu, \sigma, \psi, V, f$ are known functions, $T$ is a parameter and $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is the unknown.

- Then the Feynman-Kac formula tells us that the solution $u(x,t)$ can be written as the following conditional expectation:

$$\mathbb{E}^{\mathbb{Q}}[\int_t^T e^{-\int_t^r V(X_\tau,\tau)d\tau} \cdot f(X_r, r) \cdot dr + e^{-\int_t^T V(X_\tau,\tau)d\tau} \cdot \psi(X_T)|X_t = x]$$

  under measure $\mathbb{Q}$ such that $X_t$ is an Ito process defined as:

$$dX_t = \mu(X_t, t) \cdot dt + \sigma(X_t, t) \cdot dz_t^{\mathbb{Q}}$$

  where $z_t^{\mathbb{Q}}$ is Brownian motion under $\mathbb{Q}$, and initial condition $X_t = x$

# Stopping Time $\tau$

- Random variable $\tau$ such that $Pr[\tau \leq t]$ is in $\sigma$-algebra $\mathcal{F}_t$, for all $t$
- Deciding whether $\tau \leq t$ only depends on information up to time $t$
- Hitting time of a Borel set $A$ for a process $X_t$ is the first time $X_t$ takes a value within the set $A$
- Hitting time is an example of stopping time. Formally,

$$T_{X,A} = \underset{t \in \mathbb{R}}{\text{argmin}}\{X_t \in A\}$$

eg: Hitting time of a process to exceed a certain fixed level

# Optimal Stopping Problem

- Optimal Stopping problem:

$$V(x) = \operatorname*{argmax}_{\tau} \mathbb{E}[G(X_\tau) | X_0 = x]$$

where $\tau$ is a set of stopping times of $X_t$, $V(\cdot)$ is called the value function, and $G$ is called the reward (or gain) function.

- Note that sometimes we can have several stopping times that maximize $\mathbb{E}[G(X_\tau)]$ and we say that the optimal stopping time is the smallest stopping time achieving the maximum value.
- Optimal Exercise Time of an American Option is an example
  - $X_t$ is underlying price
  - $x$ is spot price
  - $\tau$ is set of times when option is in-the-money
  - $V(\cdot)$ is American option price as function of spot ptice
  - $G(\cdot)$ is the option payoff function

# Markov Property

- Markov property says that the $\mathcal{F}_t$-conditional PDF of $X_{t+h}$ depends only on the present state $X_t$
- Strong Markov property says that for every stopping time $\tau$, the $\mathcal{F}_\tau$-conditional PDF of $X_{\tau+h}$ depends only on $X_\tau$

# Infinitesimal Generator and Dynkin's Formula

- Infinitesimal Generator of a time homogeneous Ito diffusion $X_t$ is the PDE operator $A$ (operating on functions $f$) defined as

$$A \bullet f(x) = \lim_{t \to 0} \frac{\mathbb{E}[f(X_t)|X_0 = x] - f(x)}{t}$$

- If $dX_t = \mu(X) \cdot dt + \sigma(X) \cdot dz_t$, then

$$A \bullet f(x) = \sum_i \mu_i(x) \frac{\partial f}{\partial x_i}(x) + \sum_{i,j} (\sigma(x)\sigma(x)^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

- Let $\tau$ be a stopping time conditional on $X_0 = x$. Then, Dynkin's formula says:

$$\mathbb{E}[f(X_\tau)|X_0 = x] = f(x) + \mathbb{E}[\int_0^\tau A \bullet f(X_s) \cdot ds | X_0 = x]$$

- Stochastic generalization of 2nd fundamental theorem of calculus