Towards Improved Pricing and Hedging of Agency Mortgage-Backed Securities (MBS)

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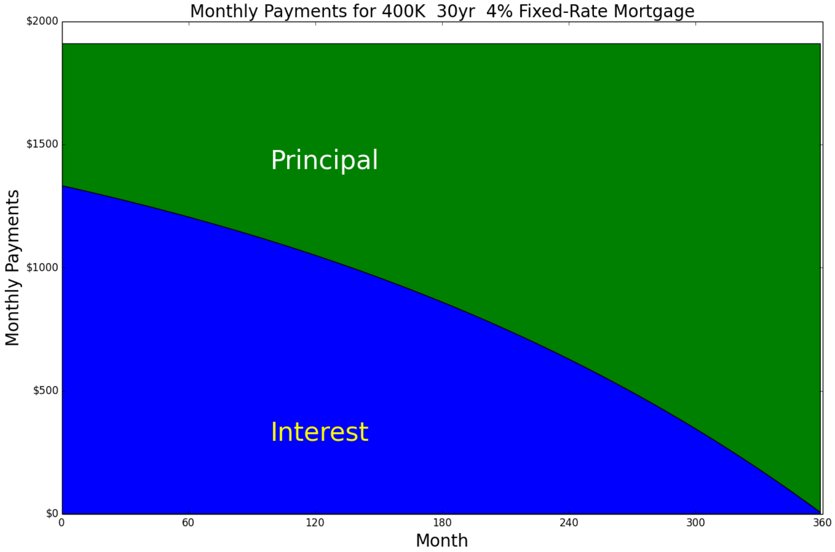
Overview

1. Mortgage Basics
2. Agency Securitization
3. Structuring
4. Industry-Standard approach to Pricing & Hedging
   - Challenges with Trading based on "OAS"
5. Alternative Formulations (to overcome challenges in trading)
   - Foundation 1: Continuous-Cashflow Pricing PDE
   - Foundation 2: Expected Discounted Cashflow (EDC) Pricing Formula
   - Closed-Form Approximations for Price and Sensiivities
   - Markovian Pricing (on a Grid) - simpler, cleaner, faster
   - **Proposal: Pricing/Hedging with "risk-neutral prepayments"**

Mortgage Basics: Borrower's Perspective

- A family wants to buy a house worth \$500K
- Can make down-payment of \$100K $\Rightarrow$ Loan to Value (LTV) of 80%
- Borrower has a good FICO score of 750
- Monthly family income is \$6000, Current monthly debt is only \$100
- So qualifies for a good mortgage rate (30 year, Fixed Rate) of 4%
- Monthly payment (Interest + Principal) amounts to about \$1900
- Debt to Income (DTI) is $\frac{1900+100}{6000} = 33\%$
- **Psychology less about debt mgmt, more about cash flows (DTI)**



Monthly Payments for 400K 30yr 4% Fixed-Rate Mortgage
Principal
Interest
Monthly Payments
$2000
$1500
$1000
$500
$0
0
60
120
180
240
300
360
Month

Mortgage Basics: Lender's Perspective

- The lender requires documentation on income, other assets/debts
- The lender is long a "Bond" receiving monthly Principal & Interest
- Lender's biggest risk is the borrower defaulting on monthly payments
- Think of this as the Borrower's (American) Put option on the House, with Strike = Remaining Principal (Default $\Rightarrow$ House taken away)
- **Lender will offer a rate commensurate with LTV, DTI, FICO**

Lender also exposed to *Voluntary Prepayment* Risk

- *Prepayment* refers to Principal being paid off before maturity
- Borrower Defaulting is refered to as *Involuntary Prepayment*
- Voluntary Prepayment mainly due to Refinancing or Home Sales
- Think of this as the Borrower's (American) Call option on the Loan, with Strike = Remaining Principal
- In-the-money when offered rate is below rate on borrower's mortgage
- Refinancing: Borrowers often exercise deep in-the-money (suboptimal)
- HomeSales: Often happens when option is out-of-the-money
- **So Prepayment Risk is a "hard to model" Risk**

Value, Balance and Price

- The following terms reference the Bond (loan) owned by the lender
- *Value*: Present Value of future Principal and Interest payments
- *Balance*: Principal that is remaining to be paid
- *Price* (Quantity of Interest): *Value* divided by *Balance*
- **Prepayment option is a call option on Price with strike = 1** (ignoring transaction costs)
- Later, we will derive the PDE for *Price* and assess *Price Sensitivity*

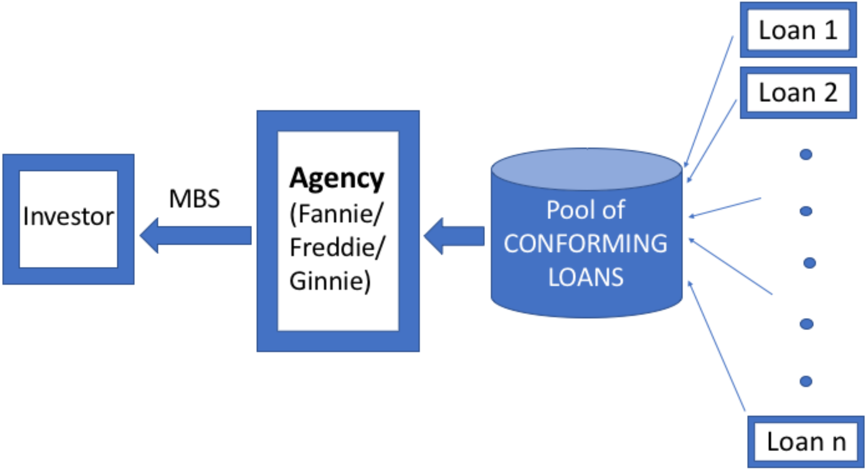
Agency Securitization of a Pool of Mortgages

- "Agency" refers to the Government-sponsored entities (GSE)
- Fannie Mae, Freddie Mac, Ginnie Mae
- Pool of *Conforming* loans (conservative loan size, LTV, DTI, docs)
- Agency receives collective Principal & Interest from the pool
- In Exchange for cash extended to banks to originate mortgages
- Agency issues a Mortgage-Backed Security (backed by mortgage pool)
- Passthrough MBS simply forwards mortgage cash flows to investors

Some details of Agency MBS

- "Servicer" responsible for collection of payments from borrowers
- Servicer receives a servicing fee for the service it provides
- Conforming loans $\Rightarrow$ risk of borrower delinquencies/default is low
- Agency guarantees investors principal even when a borrower defaults
- Defaulted loan is taken out of pool, investor receives **full principal**
- So Agency MBS investor has **no credit-risk** (government backing!)
- Agency receives a guarantee fee for this credit protection
- Guarantee fee + Servicing fee $\approx$ 50bp of Pool Balance
- So MBS coupon $\approx$ 50bp less than Pool's Average Mortgage Rate

Agency MBS securitized from a Conforming Pool of Loans

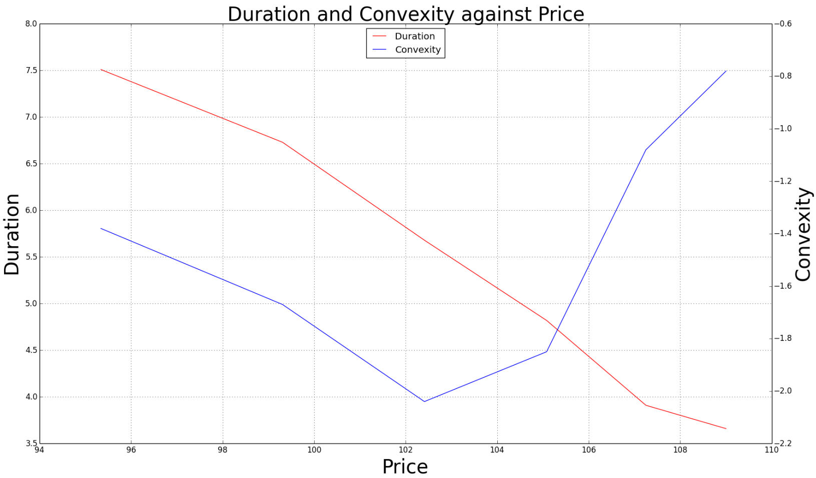


Investor Perspective of Prepayment Risk for Passthroughs

- Investor requires compensation for being short the Prepayment Option
- Prepayments for Premiums (Price > 1) are mainly Refinancings
- Prepayments for Discounts (Price < 1) are mainly HomeSales
    - Price for a Premium reduces when Prepayments rise
    - Price for a Discount increases when Prepayments rise
    - Price for the Par Passthrough is insensitive to Prepayments

Investor Perspective of Hedging Passthroughs

- Duration Definition: $-\frac{1}{P}\frac{\partial P}{\partial r}$
- Duration viewed as cashflows-PV-weighted average time of cash flows
- Duration is lower when Price is higher
- Duration reduces when Prepayments rise
- Short Prepayment Option $\Rightarrow$ Convexity and Vega are negative


Duration and Convexity against Price
Duration
Convexity
Price
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
94
96
98
100
102
104
106
108
110
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
−1.8
−2.0
−2.2

Structuring

- Pro Rata Structuring
- Sequential Pay Tranches
- Planned Amortization Class (PAC) Tranches

The key rule in structuring is that the sum of principal cash flows to the tranches equals the principal cash flows in the collateral (even if one or more of the tranches gets "negative" principal payments, eg: Z-Tranche), and likewise for the sum of interest cash flows.

Pro Rata Structuring

- Proportional Principal flows for each security but different coupons
- Floaters
- Inverse Floaters
- Interest Only (IO)
- Principal Only (PO)
- Inverse Interest Only (IIO)

Sequential Pay Tranches

- All Principal flows to Senior-most tranche until it is paid down
- After which all principal flows to the next tranche and so on ...
- Duration of tranches sorted by tranche seniority
- A particularly exotic structure: Z-Accrual Tranching
  - Junior Tranche is known as the Z-Accrual Tranche
  - Senior Tranche receives interest of Junior Tranche as principal
  - Meanwhile, Junior Tranche accrues its interest as principal
  - When Senior is paid off, Junior receives both principal & interest

Planned Amortization Class (PAC) Tranches

- Simplest Structure: 1 PAC Tranche and 1 Support Tranche
- We set two deterministic prepayment schedules S_H and S_L
- PAC receives minimum of principal payments of S_H and S_L
- Any excess principal flow will go to the Support Tranche
- Thus, PAC principal payments are deterministic and so, low-risk
- Except when prepayments get extremely low or extremely high
- Extremely high prepayments can result in Support getting paid down
- Support Tranche absorbs most of the prepayment risk

Industry-Standard approach to Pricing & Hedging

The components of a typical Agency MBS Pricing/Hedging System

- Stochastic interest rate model calibrated to liquid rate/vol instruments
- Monte Carlo simulation of interest rate model
- Econometric model of prepayments
- Regression model of mortgage rates (function of interest rates)
- Cash flow generator for various MBS Structures
- Pricing/Sensitivities based on notion of "Option-Adjusted Spread"

Interest Rate Model

- Calibration to liquid rate/vol instruments
- Multi-factor model needed to capture non-parallel curve moves
- Capturing Vol Skew/Smile very important for MBS
- So need to also calibrate to OTM swaptions' market vol
- Model Choices: Local Vol, Stochastic Vol, CEV, Jump Diffusion
- Short rate or Markovian HJM models typically prefered

Monte-Carlo Simulation

- Two reasons for needing to do non-Markovian Pricing of MBS:
  - Burnout modeling (can be made Markovian, as explained later)
  - Many CMOs have cash flows that depend on history of cash flows
- Sequence of draws of Brownian increments gives a path of states
- This provides discount factors at each time step
- For cash flows, need analytic/grid mapping from State to Rates/Vols
- Variance reduction:
  - Antithetic variables ($E[dz_t] = 0$ for all t)
  - Ensure $E[(dz_t)^2] = dt, E[(dz_s)(dz_t)] = 0$ using Orthonormalization
  - Control Variates, Quasi Random Numbers, Brownian Bridge

Prepayment Modeling

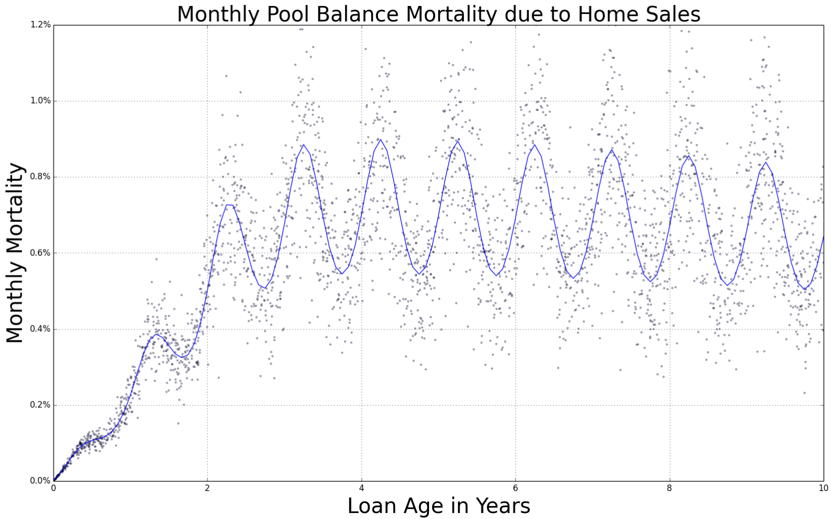
- Goal: Estimating the probability of loan termination
- Using information on loan/borrower/collateral/economic conditions
- Prepayment (Loan Termination) can be due to:
  - HomeSales (typically job-related or buying more expensive house)
  - Refinancing (to get a lower rate, or for cash-out refinance)
  - Default ($\Rightarrow$ loan paid off and removed from pool)
  - Curtailment (partial prepayment to curtail the life of the loan)
- Most prepayments for agency loans are home sales and refinancings

Homogeneity of Loans in Agency Pool $\Rightarrow$ Pool-Level Model

- Homogeneous in loan age, coupon, loan size, underwriting standards
- So, industry-standard is pool-level modeling (with some adjustments)
- Predicts fraction of pool terminating under given economic scenario
- Statistical model with modeler-imposed regressors & functional forms
- Recent advances in Deep Learning technology **should be exploited**
- Loan-level data has been made available recently that is useful to address non-homogeneity of credit quality of loans
- Though investor has no credit risk, borrower credit affects incentive to refinance/sell, and likelihood of default-based termination

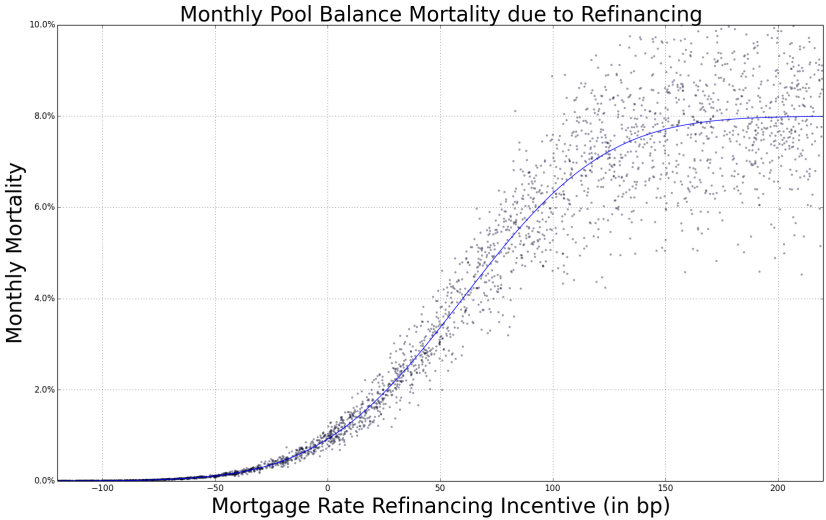
Economic Intuition of HomeSales Model

- HomeSales volume varies significantly by different seasons of the year
- Ramp up in home sales as a function of loan age
- Higher home prices leads to homeowners "trading up"
- Higher credit borrowers face less frictions
- "Lock-in" effect: Borrowers with low mortgage rate (relative to offered rate) reluctant to move as they like to hold on to their low rate


Monthly Pool Balance Mortality due to Home Sales
Monthly Mortality
Loan Age in Years
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0
2
4
6
8
10

Economic Intuition of Refinancing Model

- Incentive is typically the % improvement in monthly payments
- Mortgage Rate difference ("moneyness") is used as an approximation
- Lower credit quality diminishes the ability to refinance
- Credit quality indicators are SATO, FICO, LTV, DTI, Home Prices
- Higher home prices lead to cash-out refinancing
- Burnout Effect (Heterogeneity in borrower refinancing efficiency)
- Opportunity to make Refinancing model Markovian by creating cohorts homogeneous in their refinancing efficiency


Monthly Pool Balance Mortality due to Refinancing
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
Monthly Mortality
−100
−50
0
50
100
150
200
Mortgage Rate Refinancing Incentive (in bp)

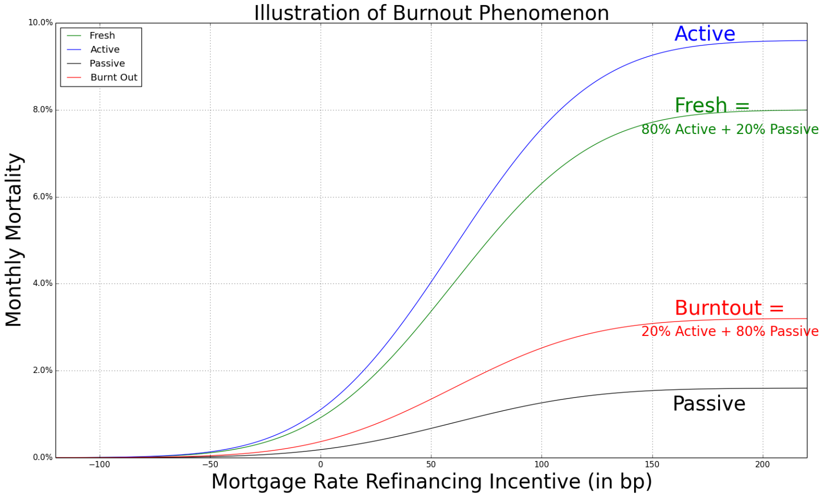


Illustration of Burnout Phenomenon
Fresh
Active
Passive
Burnt Out
Active
Fresh =
80% Active + 20% Passive
Burntout =
20% Active + 80% Passive
Passive
Monthly Mortality
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
−100
−50
0
50
100
150
200
Mortgage Rate Refinancing Incentive (in bp)

Regression model of mortgage rates

- Prepayment model typically based on mortgage rate “moneyness”
- Current Coupon (CC) model and Primary-Secondary Spread Model
- Historical CC regressed against historical rates and vol
- Instead, one can regress pricing-generated CC against rates and vol
- But then this CC model is input to pricing
- This fixed point is resolved by iterating to convergence
- Alternative approach: Price Moneyness and Backward Induction

Pricing/Sensitivities based on "Option-Adjusted Spread" (OAS)

If T is maturity in months and N is number of MC paths,

$$Price = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} (Principal_{i,t} + Interest_{i,t}) \cdot e^{-\sum_1^t (r_{i,u} + OAS)}$$

- OAS is a constant spread across time steps and MC paths
- **Risk-Premium for all risk factors other than interest rates**
- Includes non-interest-rates prepayment risk, credit and liquidity risks

Pricing/Hedging with this OAS-based (industry-standard) approach:

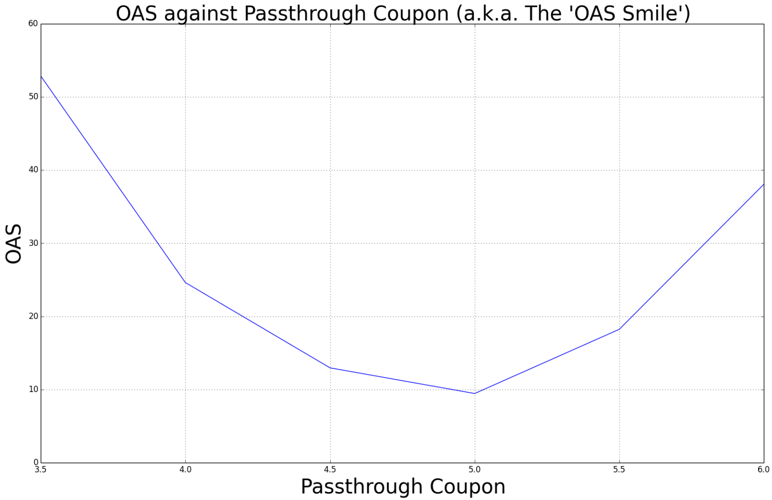
- For MBS with market prices, assess rich/cheap based on implied OAS
- For illiquid MBS, compute Price from trader-defined(!) OAS
- Price Sensitivities (duration, convexity, vega) with constant OAS

Significant challenges in Trading based on OAS

- Passthrough MBS exhibit an "OAS smile"
- Empirical Duration performs better than Model Duration
- Ill-formed theories on "OAS Directionality"
- IOs/POs from same collateral have vastly different OAS
- IO Duration $+$ PO Duration $\neq$ Underlying Pool Duration
- Unclear what OAS to set to price illiquid MBS

We discuss an alternative approach for MBS Pricing based on:

- a continuous-cashflow PDE (and martingale) foundation
- Price of prepayment risk *modulo interest rate dependency*
- separating out residual non-prepayment risks (credit, liquidity)


OAS against Passthrough Coupon (a.k.a. The 'OAS Smile')
OAS
Passthrough Coupon
0
10
20
30
40
50
60
3.5
4.0
4.5
5.0
5.5
6.0

Derivation of Continuous-Cashflow Pricing PDE

Assume continuous-time cash flows and a 1-factor short-rate model.

$$dr(t) = \alpha(r,t) \cdot dt + \sigma(r,t) \cdot dz(t)$$

- $B(t)$ is Balance, $P(t)$ is Price
- $V(t) = P(t) \cdot B(t)$ is "Value"
- $c(r,t)$ is Coupon (interest paid per unit balance per unit time)
- $\pi(r,t)$ is principal paid per unit balance per unit time

$$\pi(r,t) = sched(t) + sale(r,t) + refi(r,t) + curtail(r,t) + defaults(r,t)$$

Cashflow over time dt is $(c(r,t) + \pi(r,t)) \cdot B(t) \cdot dt$

$$dB(t) = -\pi(r,t) \cdot B(t) \cdot dt$$

$\pi(r,t)$ is uncertain for fixed r and t, but first assume it is certain

Derivation of Continuous-Cashflow Pricing PDE

Consider two different MBS with notation subscripted with "1" and "2".

$$dV_1 = B_1 dP_1 + P_1 dB_1$$

Ito's Lemma for dP_1 and substituting $dB_1 = -\pi_1 B_1 dt$ gives:

$$dV_1 = B_1(\frac{\partial P_1}{\partial t}dt + \frac{\partial P_1}{\partial r}dr + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}dt) - P_1\pi_1 B_1 dt$$

$$= B_1(\frac{\partial P_1}{\partial t} + \alpha\frac{\partial P_1}{\partial r} - \pi P_1 + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2})dt + B_1\sigma\frac{\partial P_1}{\partial r}dz$$

$$\frac{dV_1}{V_1} = (\frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2})dt + (\frac{\sigma}{P_1}\frac{\partial P_1}{\partial r})dz \qquad (1)$$

Derivation of Continuous-Cashflow Pricing PDE

Denote the Ito drift and dispersion of $\frac{dV_1}{V_1}$ as α_1 and σ_1.

$$\frac{dV_1}{V_1} = \alpha_1 dt + \sigma_1 dz \tag{2}$$

$$\alpha_1 = \frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2} \tag{3}$$

$$\sigma_1 = \frac{\sigma}{P_1}\frac{\partial P_1}{\partial r} \tag{4}$$

Consider a portfolio $W = V_1 + V_2 = P_1 B_1 + P_2 B_2$ with values of B_1 and B_2 such that we can eliminate the dz term in the expression for dW

$$B_1 \frac{\partial P_1}{\partial r} = -B_2 \frac{\partial P_2}{\partial r} \tag{5}$$

Since we have eliminated the dz term, portfolio W together with the cashflow it generates should grow at the risk-free rate r. In other words,

$$dV_1 + dV_2 + B_1(c_1 + \pi_1)dt + B_2(c_2 + \pi_2)dt = r(V_1 + V_2)dt$$

Derivation of Continuous-Cashflow Pricing PDE

Expanding this out,

$$B_1(\frac{\partial P_1}{\partial t} + \alpha_1 \frac{\partial P_1}{\partial r} - \pi_1 P_1 + (c_1 + \pi_1) + \frac{\sigma^2}{2}\frac{\partial^2 P_1}{\partial r^2})dt +$$

$$B_2(\frac{\partial P_2}{\partial t} + \alpha_2 \frac{\partial P_2}{\partial r} - \pi_2 P_2 + (c_2 + \pi_2) + \frac{\sigma^2}{2}\frac{\partial^2 P_2}{\partial r^2})dt = r(B_1 P_1 + B_2 P_2)dt$$

Combining this with equation 5, we get:

$$\frac{\frac{\partial P_1}{\partial t} + \alpha_1 \frac{\partial P_1}{\partial r} - (\pi_1 + r) P_1 + (c_1 + \pi_1) + \frac{\sigma^2}{2}\frac{\partial^2 P_1}{\partial r^2}}{\frac{\partial P_1}{\partial r}} =$$

$$\frac{\frac{\partial P_2}{\partial t} + \alpha_2 \frac{\partial P_2}{\partial r} - (\pi_2 + r) P_2 + (c_2 + \pi_2) + \frac{\sigma^2}{2}\frac{\partial^2 P_2}{\partial r^2}}{\frac{\partial P_2}{\partial r}}$$

Derivation of Continuous-Cashflow Pricing PDE

Using equations 3 and 4, the above equation can be expressed as:

$$\frac{\alpha_1 + \frac{c_1+\pi_1}{P_1} - r}{\sigma_1} = \frac{\alpha_2 + \frac{c_2+\pi_2}{P_2} - r}{\sigma_2} \tag{6}$$

- Note the numerator of LHS of Eq 6 is the expected *excess return* per unit time of investing in MBS 1 (expected growth rate of process V_1 together with its P & I cash flows, less r).
- The denominator is the standard deviation of the return per unit time.
- Their ratio is the familiar **Price of Interest-Rate Risk** λ_r (which is the same for every security exposed to only interest-rate risk).
- This is what equation 6 is telling us, which can be re-expressed as:

$$\frac{\alpha_1 + \frac{c_1+\pi_1}{P_1} - r}{\sigma_1} = \frac{\alpha_2 + \frac{c_2+\pi_2}{P_2} - r}{\sigma_2} = \lambda_r$$

Derivation of Continuous-Cashflow Pricing PDE

Substituting in the above equation for α_1 from equation 3 and for σ_1 from equation 4, we get:

$$\frac{\frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2} + \frac{c_1+\pi_1}{P_1} - r}{\frac{\sigma}{P_1}\frac{\partial P_1}{\partial r}} = \lambda_r$$

Reorganize and drop the subscript 1 in P_1, c_1, π_1 to arrive at the PDE:

$$\frac{\partial P}{\partial t} + (\alpha - \lambda_r \sigma)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r)P$$

Derivation of Continuous-Cashflow Pricing PDE

If we shift to the risk-neutral measure (denoted as Q), V_1 together with the cash flows it generates should grow at risk-free rate r. Hence,

$$\frac{dV_1}{V_1} = (r - \frac{c_1 + \pi_1}{P_1})dt + \sigma_1 dz^{(Q)} = (\alpha_1 - \lambda_r \sigma_1)dt + \sigma_1 dz^{(Q)}$$

Since, $dz^{(Q)} = \lambda_r dt + dz$, the Ito process for the short-rate r in the risk-neutral measure can be written as:

$$dr = (\alpha - \lambda_r \sigma)dt + \sigma dz^{(Q)} = \alpha^{(Q)} dt + \sigma dz^{(Q)}$$

where $\alpha^{(Q)} = \alpha - \lambda_r \sigma$ is the risk-neutral drift for r.
So, the PDE can also be expressed in terms of the risk-neutral drift of r:

$$\frac{\partial P}{\partial t} + \alpha^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r)P$$

Derivation of Continuous-Cashflow Pricing PDE

- But in reality, π depends on stochastic (risk) factors other than r
- Each of which deserve a *Price of Risk* (hence, a return spread)
- We cannot model/capture *Price of Risk* of all these factors. So,

$$\frac{\partial P}{\partial t} + \alpha^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r + s)P$$

  where s is our "good-old" Option-Adjusted Spread (OAS).
- To unravel s, define stochastic factors for prepayments *other than r*
- Not economic factors, but (multiplicative) **forecast-error risk factors**

$$refi'(x, r, t) = refi(r, t) \cdot x(t)$$

$$sale'(y, r, t) = sale(r, t) \cdot y(t)$$

Derivation of Continuous-Cashflow Pricing PDE

$$dx(t) = \beta_x(1-x)dt + \sigma_x dz_x(t) \text{ with } x(0) = 1$$

$$dy(t) = \beta_y(1-y)dt + \sigma_y dz_y(t) \text{ with } y(0) = 1$$

Analogous to rate risk, shifting to risk-neutral measure, we have:

$$dx(t) = \alpha_x^{(Q)} dt + \sigma_x dz_x^{(Q)}(t) \text{ where } \alpha_x^{(Q)} = \beta_x(1-x) - \lambda_x \sigma_x$$

$$dy(t) = \alpha_y^{(Q)} dt + \sigma_y dz_y^{(Q)}(t) \text{ where } \alpha_y^{(Q)} = \beta_y(1-y) - \lambda_y \sigma_y$$

For consistency of notation, we define the risk-neutral process for r as:

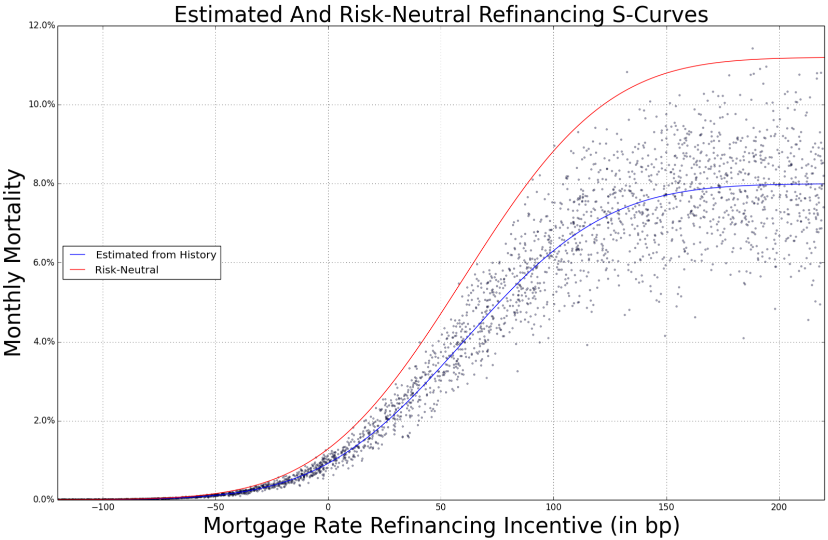
$$dr(t) = \alpha_r^{(Q)} dt + \sigma_y dz_r^{(Q)}(t)$$

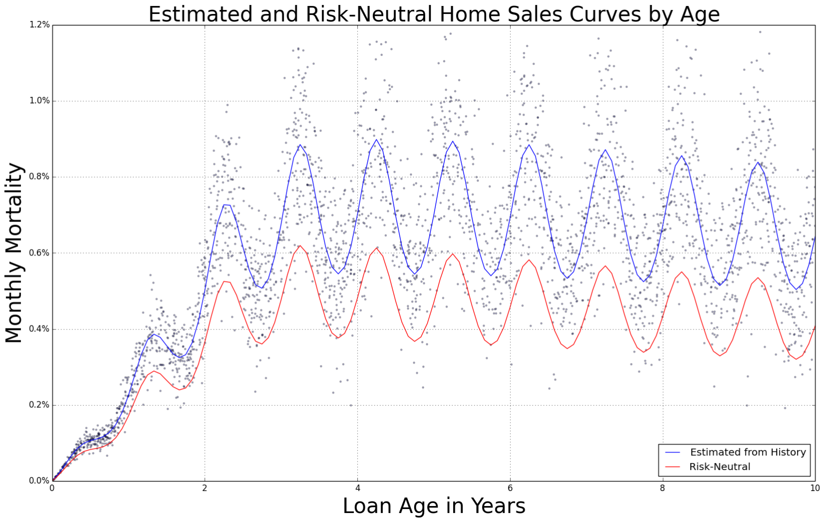
Assume that the covariance of $dz_x^{(Q)}(t)$ and $dz_y^{(Q)}(t)$ is $\rho \cdot dt$ and that each of $dz_x^{(Q)}(t)$ and $dz_y^{(Q)}(t)$ are independent of $dz_r^{(Q)}(t)$.

Derivation of Continuous-Cashflow Pricing PDE

$$\frac{\partial P}{\partial t} + \alpha_r^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2 P}{\partial r^2} + \alpha_x^{(Q)} \frac{\partial P}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 P}{\partial x^2} + \alpha_y^{(Q)} \frac{\partial P}{\partial y} + \frac{\sigma_y^2}{2} \frac{\partial^2 P}{\partial y^2}$$
$$+ \rho \sigma_x \sigma_y \frac{\partial^2 P}{\partial x \partial y} + c + \pi^{(Q)} = (\pi^{(Q)} + r + rs) P \quad (7)$$

- Q now refers to risk-neutral measure with respect to joint $\{r, x, y\}$
- $\sigma_x, \sigma_y, \rho, \beta_x, \beta_y$ estimated from errors in prepayment forecasts
- λ_x, λ_y calibrated to liquid MBS market prices (shifting x, y drifts)
- We refer to Q-measure-shifted x, y (**and** $\pi^{(Q)}$) as "risk-neutral"
- rs is the spread due to any residual risk such as liquidity or credit
- $s - rs$ **due to prepayment risk modulo interest rates dependency**


Estimated And Risk-Neutral Refinancing S-Curves
Monthly Mortality
12.0%
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
−100
−50
0
50
100
150
200
Mortgage Rate Refinancing Incentive (in bp)
Estimated from History
Risk-Neutral


Estimated and Risk-Neutral Home Sales Curves by Age
Monthly Mortality
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0
2
4
6
8
10
Loan Age in Years
Estimated from History
Risk-Neutral

Martingale-based (Expected Discounted Cashflow) Pricing

Define the money-market process $M(t)$ as:

$$dM(t) = r(t) \cdot M(t) \cdot dt, M(0) = 1$$

Consider a process $\theta(t)$ derived from $V(t)$ and the cash flows, as follows:

$$d\theta(t) = dV(t) + (c(r,t) + \pi(r,t))B(t)dt$$

As before, assume $\pi(r,t)$ is certain for fixed r. Then, $\frac{\theta(t)}{M(t)}$ is a martingale in risk-neutral measure Q. So, for any $T > 0$,

$$\frac{\theta(0)}{M(0)} = E_Q[\frac{\theta(T)}{M(T)}]$$

Since $\theta(0) = V(0), M(0) = 1$, and without loss of generality, $B(0) = 1$,

$$P(0) = \frac{\theta(0)}{M(0)} = E_Q[\frac{V(T) + \int_0^T (c(r,t) + \pi(r,t)) \cdot B(t) \cdot dt}{M(T)}]$$

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If we set T to MBS maturity, $V(T) = B(T) = 0$. Then,

$$
\begin{aligned}
P(0) &= E_Q[\int_0^T \frac{(c(r,t) + \pi(r,t)) \cdot B(t)}{M(t)} dt] \\
&= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot B(t) \cdot e^{-\int_0^u r(u) du} dt] \\
&= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot e^{-\int_0^u (r(u) + \pi(r,u)) du} dt]
\end{aligned}
\tag{8}
$$

Conceptualize as cash flows discounted at rate $r + \pi$ in Q-measure

Martingale-based (Expected Discounted Cashflow) Pricing

- But in reality, π depends on stochastic (risk) factors other than r
- Each of which deserve a *Price of Risk* (hence, a return spread)
- We cannot model/capture *Price of Risk* of all these factors. So,

$$\begin{aligned} P(0) &= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot B(t) \cdot e^{-\int_0^u (s+r(u))du} dt] \\ &= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot e^{-\int_0^u (s+r(u)+\pi(r,u))du} dt] \end{aligned} \tag{9}$$

Introducing multipliers x, y and calibrating λ_x, λ_y to liquid MBS prices, we denote the resulting principal payment rate as $\pi^{(Q)}(x, y, r, t)$:

$$\begin{aligned} P(0) &= E_Q[\int_0^T (c + \pi^{(Q)}) \cdot B(t) \cdot e^{-\int_0^u (rs+r(u))du} dt] \\ &= E_Q[\int_0^T (c + \pi^{(Q)}) \cdot e^{-\int_0^u (rs+r(u)+\pi^{(Q)})du} dt] \end{aligned} \tag{10}$$

Alternative approaches to Pricing/Sensitivity Computations

- Closed-form approximations for pricing and price sensitivities
- Pricing on a grid (Markovian)
- Pricing and Hedging based on “risk-neutral prepayments” and residual spread (instead of OAS).

Closed-Form Approximations for Price and Sensitivities

- Closed-form Solution for Expected Discounted Cashflow (EDC) formula when $\pi(r,t) = \pi_0(t) + r\pi_1$ and $c(r,t) = c_0(t) + r \cdot c_1(t)$
- Closed-form Solution for Pricing PDE when r follows an affine process and $\pi(r,t) = \pi_0(t) + r\pi_1$ and $c(r,t) = c_0(t) + r \cdot c_1(t)$
- Closed-form Solution for EDC Price formula when π is linear in r with hard max and min
- The above gives bad greeks due to piecewise prepayment function. But this serves quite useful as a control variate.
- We can do analytical partial derivatives of EDC Price formula w.r.t:
    - a time-parallel shift to $r(t)$ (Duration approximation)
    - coupon c
    - a time-parallel shift to $\pi(r,t)$ (Prepayment sensitivity)
    - a time-parallel shift to refi multiplier $x(t)$ or sale multiplier $y(t)$
    - residual spread rs
- **These closed-forms very useful to reason about pricing/sensitivities**

Markovian Pricing (on a Grid)

- Model Burnout by creating a few (2-3) cohorts in the pool, each of which is homogeneous in terms of refinancing efficiency
- **Each cohort's prepayment incentive is Markovian**
- For each cohort, at every grid node, probabilistically discount sum of:
  - Interest cash flow c
  - Principal cash flow π
  - $(1 - \pi) \cdot P$
- This gives a backward induction for P for each cohort
- MBS Price = sum of cohort Prices
- Alternatively, we can solve Pricing PDE with Crank-Nicholson
- Making π a function of P (instead of r) yields a clean model

Caveat: ARMs and some CMOs have non-Markovian cash flows.

Pricing and Hedging with “risk-neutral prepayments”

Trading with “risk-neutral prepayments” and residual spread *rs*

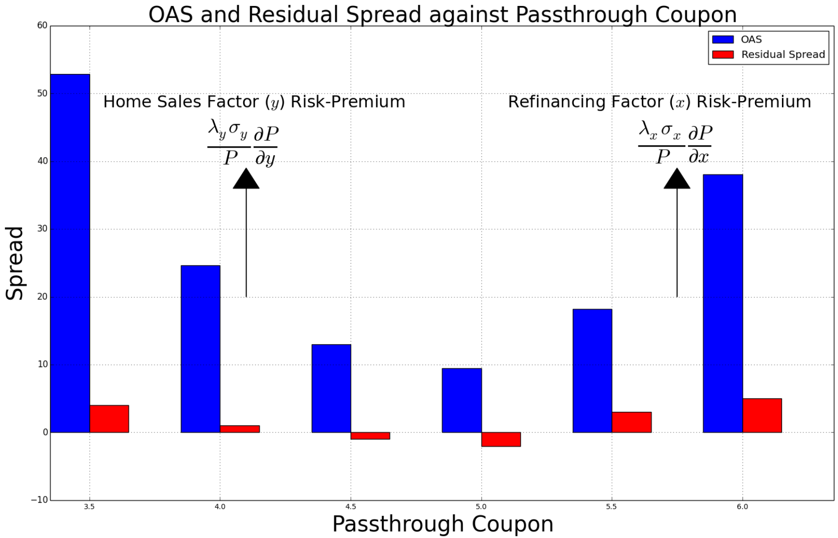
- For MBS with market prices, assess rich/cheap based on implied *rs*
- For illiquid MBS, Price using a prudent *rs* (easier than a OAS input)
- Price Sensitivities **using constant rs** (instead of constant OAS)

Practical considerations in designing the Pricer

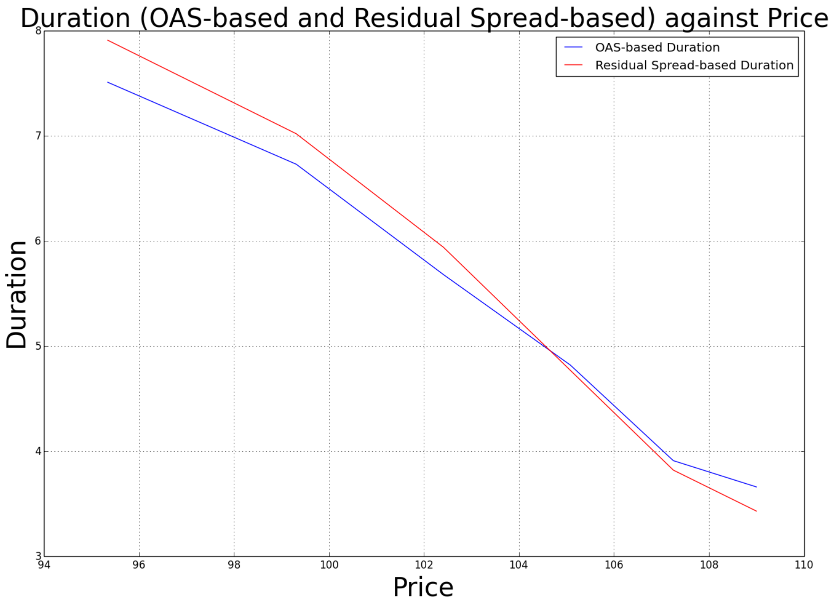
- I used TBA prices for calibration (liquid IOs/POs could also be used)
- I calibrated Price of Risk of uncertain initial values $x(0)$ and $y(0)$, along with Price of Risk of dz_x, dz_y
- I set $rs =$ Agency-Swap spread plus appropriate liquidity spread (we can do better by building a proper model for residual risk)
- **Small** $\frac{\partial^2 P}{\partial x^2}, \frac{\partial^2 P}{\partial y^2} \Rightarrow$ **we can avoid stochasticity of** x, y **in Pricer**
- So calibrated $\alpha_x^{(Q)}, \alpha_y^{(Q)}$ essentially give us $E_Q[x(t)], E_Q[y(t)]$
- This gives us a (deterministic) shifted prepayment function $\pi^{(Q)}(r, t)$
- **Practically, use same old Pricer, simply alter** $\pi(r, t)$ **to** $\pi^{(Q)}(r, t)$

Are we happy? Can we improve?

- "OAS Smile" captured (but sometimes small residual smile remains)
- Model Durations match Empirical ("OAS Directionality" captured)
- Model prices for IOs/POs are fairly close to their market prices
- IOs: *rs*-based Duration more negative than OAS-based Duration
- Superior hedge performance relative to OAS-based Duration
- Further work (will help in crisis periods) - make *rs* a function of:
    - General credit risk
    - Agency-specific credit risk
    - Supply/Demand-based liquidity risk
- Modeling x, y as a jump-diffusion would also be a good idea



OAS and Residual Spread against Passthrough Coupon
OAS
Residual Spread
Home Sales Factor (y) Risk-Premium
$\frac{\lambda_y \sigma_y}{P} \frac{\partial P}{\partial y}$
Refinancing Factor (x) Risk-Premium
$\frac{\lambda_x \sigma_x}{P} \frac{\partial P}{\partial x}$
Spread
60
50
40
30
20
10
0
−10
3.5
4.0
4.5
5.0
5.5
6.0
Passthrough Coupon


Duration (OAS-based and Residual Spread-based) against Price
OAS-based Duration
Residual Spread-based Duration
Duration
Price
3
4
5
6
7
8
94
96
98
100
102
104
106
108
110

Appendix: Sign of Price of Risk

For risk factor f, the vol of $\frac{dV}{V}$ w.r.t f is $\frac{\sigma_f}{P}\frac{\partial P}{\partial f}$.
So the risk-premium (spread) for the MBS due to the risk factor f is:

$$\frac{\lambda_f \sigma_f}{P}\frac{\partial P}{\partial f}$$

- We calibrate λ_x and λ_y to liquid passthrough market prices
- Premiums are mainly exposed to refinancing risk and $\frac{\partial P}{\partial x}$ is negative
- OAS smile $\Rightarrow \lambda_x$ (Price of Refinancing-Forecast Risk) is negative
- Discounts are mainly exposed to home sales risk and $\frac{\partial P}{\partial y}$ is positive
- OAS smile $\Rightarrow \lambda_y$ (Price of HomeSales-Forecast Risk) is positive

Appendix: Direction of risk-neutral adjustment of drift

For risk factor f, it's drift is subtracted by $\lambda_f \sigma_f$ to make the process for f risk-neutral. Risk-premium s due to the risk factor f is given by:

$$s = \frac{\lambda_f \sigma_f}{P} \frac{\partial P}{\partial f}$$

So, the drift of f is subtracted by:

$$\frac{sP}{(\frac{\partial P}{\partial f})}$$

- $s > 0 \Rightarrow$ drift is adjusted in a direction that worsens price
- So, adjusted Refinancing multiplier $E_Q[x(t)] > 1$
- And adjusted HomeSales multiplier $E_Q[y(t)] < 1$

Summary and Finishing Comments

- "Risk-Neutral Prepayments" have been discussed since the mid-90s.
- But practitioners shy away claiming "Pricing Theory is not for MBS"!
- I was first influenced to implement this by Alex Levin in 2006
- My motivations: A) Improved Rich/Cheap, & B) Hedge Performance
- Despite frailties, I see the following as a big upgrade for Trading: **Deep Learning models layered with Price of Risk calibration**
- Further work on residual Liquidity/Credit Risk (for times of crises!)
- Can we extend this idea to non-Agency (default-risky) MBS?