NOTES ON PORTFOLIO OPTIMIZATION

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1. A Simple DP Formulation for one risky and one riskless asset

Assume we have T discrete time steps labeled as $t = 0, 1, \dots, T$. Wealth at time step t is denoted as W_t. We start with wealth W_0. The utility of wealth consumption at final time T is given as $U(W_T) = -\frac{e^{-\gamma W_T}}{\gamma}$ for some $\gamma > 0$. Let the one-step rate of return of the riskless asset be r and let the one-step rate of return of the risky asset be the random variable $R \sim N(\mu, \sigma^2)$. Let the single-time-step discount factor for wealth be ρ.

State will be represented as (t, W_t). Assume our decision (action) at any time step t is given by the quantity of investment in the risky asset at time step t for all $0 <= t <= T-1$ and is denoted by x_t (hence, quantity of investment in the riskless asset at time t will be $W_t - x_t$). We denote the policy as π, so $\pi((t, W_t)) = x_t$.

$$W_{t+1} = x_t(1+R) + (W_t - x_t)(1+r)$$

Denote excess return of the risk asset (over riskless return) as $S = R - r$. So,

$$W_{t+1} = x_t S + W_t(1+r)$$

Value function for a given policy is denoted as $V^\pi(t, W_t) = E_\pi[\rho^{T-t} \cdot U(W_T) | (t, W_t)] = E_\pi[-\rho^{T-t} \cdot \frac{e^{-\gamma W_T}}{\gamma} | (t, W_t)]$. Optimal Value function will be denoted as $V^*(t, W_t) = \max_\pi V^\pi(t, W_t) = \max_\pi E_\pi[\rho^{T-t} \cdot U(W_T) | (t, W_t)] = \max_\pi E_\pi[-\rho^{T-t} \cdot \frac{e^{-\gamma W_T}}{\gamma} | (t, W_t)]$.

Assume $V^*(t, W_t)$ has the form $-a_t e^{-b_t W_t}$. Since $V^*(T, W_T) = -\frac{e^{-\gamma W_T}}{\gamma}, a_T = \frac{1}{\gamma}, b_T = \gamma$.

The Bellman optimality equation is:

$$\begin{aligned}
V^*(t, W_t) &= max_{x_t}(E_{R \sim N(\mu, \sigma^2)}[\rho \cdot V^*(t+1, W_{t+1})]) \\
&= max_{x_t}(E_{R \sim N(\mu, \sigma^2)}[-\rho \cdot a_{t+1} e^{-b_{t+1}(x_t S + W_t(1+r))}]) \\
&= max_{x_t}(-\rho \cdot a_{t+1} e^{-b_{t+1} W_t(1+r) - b_{t+1} x_t(\mu - r) + \frac{\sigma^2}{2} b_{t+1}^2 x_t^2})
\end{aligned}$$

$\frac{\partial V^*(t, W_t)}{\partial x_t} = 0$ yields:

$$-b_{t+1}(\mu - r) + \sigma^2 b_{t+1}^2 x_t^* = 0$$

$$x_t^* = \frac{\mu - r}{\sigma^2 b_{t+1}}$$

Plugging in x_t^* in the above equation for $V^*(t, W_t)$ gives:

$$V^*(t, W_t) = -\rho \cdot a_{t+1} e^{-b_{t+1} W_t (1+r) - \frac{(\mu-r)^2}{2\sigma^2}}$$

But since,

$$V^*(t, W_t) = -a_t e^{-b_t W_t}$$

, we can write the following recursive equations for a_t and b_t.

$$a_t = \rho \cdot a_{t+1} e^{-\frac{(\mu-r)^2}{2\sigma^2}}$$
$$b_t = b_{t+1}(1+r)$$

Since we know that $a_T = \frac{1}{\gamma}, b_T = \gamma$,

$$a_t = \frac{\rho^{T-t}}{\gamma} e^{-\frac{(\mu-r)^2 \cdot (T-t)}{2\sigma^2}}$$
$$b_t = \gamma \cdot (1+r)^{T-t}$$

Hence,

$$x_t^* = \frac{\mu - r}{\sigma^2 \gamma (1+r)^{T-t-1}}$$
$$V^*(t, W_t) = -\frac{\rho^{T-t}}{\gamma} e^{-\frac{(\mu-r)^2 \cdot (T-t)}{2\sigma^2}} \cdot e^{-\gamma \cdot (1+r)^{T-t} \cdot W_t}$$

As extensions of this problem, consider:

- Discrete Amounts of shares to hold and discrete quantities of trades
- Transaction costs
- Locked-out days for trading
- Non-stationary/arbitrary distributions
- Borrowing rate changing/uncertain future borrowing rate