

# Policy Gradient Algorithms

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# Overview

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# Why do we care about Policy Gradient (PG)?

- Let us review how we got here
- We started with Markov Decision Processes and Bellman Equations
- Next we studied several variants of DP and RL algorithms
- We noted that the idea of *Generalized Policy Iteration* (GPI) is key
- Policy Improvement step:  $\pi(a|s)$  derived from  $\operatorname{argmax}_a Q(s, a)$
- How do we do  $\operatorname{argmax}$  when action space is large or continuous?
- Idea: Do Policy Improvement step with a Gradient Ascent instead

# “Policy Improvement with a Gradient Ascent??”

- We want to find the Policy that fetches the “Best Expected Returns”
- Gradient Ascent on “Expected Returns” w.r.t params of Policy func
- So we need a func approx for (stochastic) Policy Func:  $\pi(s, a; \theta)$
- In addition to the usual func approx for Action Value Func:  $Q(s, a; w)$
- $\pi(s, a; \theta)$  func approx called *Actor*,  $Q(s, a; w)$  func approx called *Critic*
- Critic parameters  $w$  are optimized w.r.t  $Q(s, a; w)$  loss function min
- Actor parameters  $\theta$  are optimized w.r.t Expected Returns max
- We need to formally define “Expected Returns”
- But we already see that this idea is appealing for continuous actions
- GPI with Policy Improvement done as **Policy Gradient (Ascent)**

# Value Function-based and Policy-based RL

- Value Function-based
  - Learn Value Function (with a function approximation)
  - Policy is implicit - readily derived from Value Function (eg:  $\epsilon$ -greedy)
- Policy-based
  - Learn Policy (with a function approximation)
  - No need to learn a Value Function
- Actor-Critic
  - Learn Policy (Actor)
  - Learn Value Function (Critic)

# Advantages and Disadvantages of Policy Gradient approach

## Advantages:

- Finds the best *Stochastic* Policy (Optimal Deterministic Policy, produced by other RL algorithms, can be unsuitable for POMDPs)
- Naturally *explores* due to Stochastic Policy representation
- Effective in high-dimensional or continuous action spaces
- Small changes in  $\theta \Rightarrow$  small changes in  $\pi$ , and in state distribution
- This avoids the convergence issues seen in argmax-based algorithms

## Disadvantages:

- Typically converge to a local optimum rather than a global optimum
- Policy Evaluation is typically inefficient and has high variance
- Policy Improvement happens in small steps  $\Rightarrow$  slow convergence

- Discount Factor  $\gamma$
- Assume episodic with  $0 \leq \gamma \leq 1$  or non-episodic with  $0 \leq \gamma < 1$
- States  $s_t \in \mathcal{S}$ , Actions  $a_t \in \mathcal{A}$ , Rewards  $r_t \in \mathbb{R}$ ,  $\forall t \in \{0, 1, 2, \dots\}$
- State Transition Probabilities  $\mathcal{P}_{s,s'}^a = Pr(s_{t+1} = s' | s_t = s, a_t = a)$
- Expected Rewards  $\mathcal{R}_s^a = E[r_t | s_t = s, a_t = a]$
- Initial State Probability Distribution  $p_0 : \mathcal{S} \rightarrow [0, 1]$
- Policy Func Approx  $\pi(s, a; \theta) = Pr(a_t = a | s_t = s, \theta)$ ,  $\theta \in \mathbb{R}^k$

PG coverage will be quite similar for non-discounted non-episodic, by considering average-reward objective (so we won't cover it)

# “Expected Returns” Objective

Now we formalize the “Expected Returns” Objective  $J(\theta)$

$$J(\theta) = \mathbb{E}_{\pi} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \right]$$

Value Function  $V^{\pi}(s)$  and Action Value function  $Q^{\pi}(s, a)$  defined as:

$$V^{\pi}(s) = \mathbb{E}_{\pi} \left[ \sum_{k=t}^{\infty} \gamma^{k-t} r_k \mid s_t = s \right], \forall t \in \{0, 1, 2, \dots\}$$

$$Q^{\pi}(s, a) = \mathbb{E}_{\pi} \left[ \sum_{k=t}^{\infty} \gamma^{k-t} r_k \mid s_t = s, a_t = a \right], \forall t \in \{0, 1, 2, \dots\}$$

$$\text{Advantage Function } A^{\pi}(s, a) = Q^{\pi}(s, a) - V^{\pi}(s)$$

Also,  $p(s \rightarrow s', t, \pi)$  will be a key function for us - it denotes the probability of going from state  $s$  to  $s'$  in  $t$  steps by following policy  $\pi$



# Discounted-Aggregate State-Visitation Measure

$$\begin{aligned} J(\theta) &= \mathbb{E}_{\pi} \left[ \sum_{t=0}^{\infty} \gamma^t r_t \right] = \sum_{t=0}^{\infty} \gamma^t \mathbb{E}_{\pi} [r_t] \\ &= \sum_{t=0}^{\infty} \gamma^t \int_{\mathcal{S}} \left( \int_{\mathcal{S}} p_0(s_0) \cdot p(s_0 \rightarrow s, t, \pi) \cdot ds_0 \right) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot \mathcal{R}_s^a \cdot da \cdot ds \\ &= \int_{\mathcal{S}} \left( \int_{\mathcal{S}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(s_0) \cdot p(s_0 \rightarrow s, t, \pi) \cdot ds_0 \right) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot \mathcal{R}_s^a \cdot da \cdot ds \end{aligned}$$

## Definition

$$J(\theta) = \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot \mathcal{R}_s^a \cdot da \cdot ds$$

where  $\rho^{\pi}(s) = \int_{\mathcal{S}} \sum_{t=0}^{\infty} \gamma^t \cdot p_0(s_0) \cdot p(s_0 \rightarrow s, t, \pi) \cdot ds_0$  is the key function (for PG) we'll refer to as *Discounted-Aggregate State-Visitation Measure*.

# Policy Gradient Theorem (PGT)

## Theorem

$$\nabla_{\theta} J(\theta) = \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \nabla_{\theta} \pi(s, a; \theta) \cdot Q^{\pi}(s, a) \cdot da \cdot ds$$

- Note:  $\rho^{\pi}(s)$  depends on  $\theta$ , but there's no  $\nabla_{\theta} \rho^{\pi}(s)$  term in  $\nabla_{\theta} J(\theta)$
- So we can simply sample simulation paths, and at each time step, we calculate  $(\nabla_{\theta} \log \pi(s, a; \theta)) \cdot Q^{\pi}(s, a)$  (probabilities implicit in paths)
- Note:  $\nabla_{\theta} \log \pi(s, a; \theta)$  is Score function (Gradient of log-likelihood)
- We will estimate  $Q^{\pi}(s, a)$  with a function approximation  $Q(s, a; w)$
- We will later show how to avoid the estimate bias of  $Q(s, a; w)$
- This numerical estimate of  $\nabla_{\theta} J(\theta)$  enables **Policy Gradient Ascent**
- Let us look at the score function of some canonical  $\pi(s, a; \theta)$

# Canonical $\pi(s, a; \theta)$ for finite action spaces

- For finite action spaces, we often use Softmax Policy
- $\theta$  is an  $n$ -vector  $(\theta_1, \dots, \theta_n)$
- Features vector  $\phi(s, a) = (\phi_1(s, a), \dots, \phi_n(s, a))$  for all  $s \in \mathcal{S}, a \in \mathcal{A}$
- Weight actions using linear combinations of features:  $\theta^T \cdot \phi(s, a)$
- Action probabilities proportional to exponentiated weights:

$$\pi(s, a; \theta) = \frac{e^{\theta^T \cdot \phi(s, a)}}{\sum_b e^{\theta^T \cdot \phi(s, b)}} \text{ for all } s \in \mathcal{S}, a \in \mathcal{A}$$

- The score function is:

$$\nabla_{\theta} \log \pi(s, a; \theta) = \phi(s, a) - \sum_b \pi(s, b; \theta) \cdot \phi(s, b) = \phi(s, a) - \mathbb{E}_{\pi}[\phi(s, \cdot)]$$

# Canonical $\pi(s, a; \theta)$ for continuous action spaces

- For continuous action spaces, we often use Gaussian Policy
- $\theta$  is an  $n$ -vector  $(\theta_1, \dots, \theta_n)$
- State features vector  $\phi(s) = (\phi_1(s), \dots, \phi_n(s))$  for all  $s \in \mathcal{S}$
- Gaussian Mean is a linear combination of state features  $\theta^T \cdot \phi(s)$
- Variance may be fixed  $\sigma^2$ , or can also be parameterized
- Policy is Gaussian,  $a \sim \mathcal{N}(\theta^T \cdot \phi(s), \sigma^2)$  for all  $s \in \mathcal{S}$
- The score function is:

$$\nabla_{\theta} \log \pi(s, a; \theta) = \frac{(a - \theta^T \cdot \phi(s)) \cdot \phi(s)}{\sigma^2}$$

# Proof of Policy Gradient Theorem

We begin the proof by noting that:

$$J(\theta) = \int_{\mathcal{S}} p_0(s_0) \cdot V^\pi(s_0) \cdot ds_0 = \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot Q^\pi(s_0, a_0) \cdot da_0 \cdot ds_0$$

Calculate  $\nabla_\theta J(\theta)$  by parts  $\pi(s_0, a_0; \theta)$  and  $Q^\pi(s_0, a_0)$

$$\begin{aligned} \nabla_\theta J(\theta) &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \nabla_\theta \pi(s_0, a_0; \theta) \cdot Q^\pi(s_0, a_0) \cdot da_0 \cdot ds_0 \\ &\quad + \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot \nabla_\theta Q^\pi(s_0, a_0) \cdot da_0 \cdot ds_0 \end{aligned}$$

# Proof of Policy Gradient Theorem

Now expand  $Q^\pi(s_0, a_0)$  as  $\mathcal{R}_{s_0}^{a_0} + \int_{\mathcal{S}} \gamma \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot V^\pi(s_1) \cdot ds_1$  (Bellman)

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^\pi(s_0, a_0) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot \nabla_{\theta} (\mathcal{R}_{s_0}^{a_0} + \int_{\mathcal{S}} \gamma \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot V^\pi(s_1) \cdot ds_1) \cdot da_0 \cdot ds_0 \end{aligned}$$

Note:  $\nabla_{\theta} \mathcal{R}_{s_0}^{a_0} = 0$ , so remove that term

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^\pi(s_0, a) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot \nabla_{\theta} (\int_{\mathcal{S}} \gamma \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot V^\pi(s_1) \cdot ds_1) \cdot da_0 \cdot ds_0 \end{aligned}$$

# Proof of Policy Gradient Theorem

Now bring the  $\nabla_{\theta}$  inside the  $\int_{\mathcal{S}}$  to apply only on  $V^{\pi}(s_1)$

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^{\pi}(s_0, a) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \int_{\mathcal{S}} \gamma \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot \nabla_{\theta} V^{\pi}(s_1) \cdot ds_1 \cdot da_0 \cdot ds_0 \end{aligned}$$

Now bring the outside  $\int_{\mathcal{S}}$  and  $\int_{\mathcal{A}}$  inside the inner  $\int_{\mathcal{S}}$

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^{\pi}(s_0, a_0) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} \left( \int_{\mathcal{S}} \gamma \cdot p_0(s_0) \int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot da_0 \cdot ds_0 \right) \cdot \nabla_{\theta} V^{\pi}(s_1) \cdot ds_1 \end{aligned}$$

# Policy Gradient Theorem

Note that  $\int_{\mathcal{A}} \pi(s_0, a_0; \theta) \cdot \mathcal{P}_{s_0, s_1}^{a_0} \cdot da_0 = p(s_0 \rightarrow s_1, 1, \pi)$

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \cdot \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^{\pi}(s_0, a_0) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} \left( \int_{\mathcal{S}} \gamma \cdot p_0(s_0) \cdot p(s_0 \rightarrow s_1, 1, \pi) \cdot ds_0 \right) \cdot \nabla_{\theta} V^{\pi}(s_1) \cdot ds_1 \end{aligned}$$

Now expand  $V^{\pi}(s_1)$  to  $\int_{\mathcal{A}} \pi(s_1, a_1; \theta) \cdot Q^{\pi}(s_1, a_1) \cdot da_1$

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \cdot \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^{\pi}(s_0, a_0) \cdot da \cdot ds_0 \\ &+ \int_{\mathcal{S}} \left( \int_{\mathcal{S}} \gamma \cdot p_0(s_0) p(s_0 \rightarrow s_1, 1, \pi) ds_0 \right) \cdot \nabla_{\theta} \left( \int_{\mathcal{A}} \pi(s_1, a_1; \theta) Q^{\pi}(s_1, a_1) da_1 \right) ds_1 \end{aligned}$$



# Proof of Policy Gradient Theorem

We are now back to when we started calculating gradient of  $\int_{\mathcal{A}} \pi \cdot Q^{\pi} \cdot da$ . Follow the same process of splitting  $\pi \cdot Q^{\pi}$ , then Bellman-expanding  $Q^{\pi}$  (to calculate its gradient), and iterate.

$$\begin{aligned} &= \int_{\mathcal{S}} p_0(s_0) \int_{\mathcal{A}} \cdot \nabla_{\theta} \pi(s_0, a_0; \theta) \cdot Q^{\pi}(s_0, a_0) \cdot da_0 \cdot ds_0 \\ &+ \int_{\mathcal{S}} \int_{\mathcal{S}} \gamma p_0(s_0) p(s_0 \rightarrow s_1, 1, \pi) ds_0 \left( \int_{\mathcal{A}} \nabla_{\theta} \pi(s_1, a_1; \theta) Q^{\pi}(s_1, a_1) da_1 + \dots \right) ds_1 \end{aligned}$$

This iterative process leads us to:

$$= \sum_{t=0}^{\infty} \int_{\mathcal{S}} \int_{\mathcal{S}} \gamma^t \cdot p_0(s_0) \cdot p(s_0 \rightarrow s_t, t, \pi) \cdot ds_0 \int_{\mathcal{A}} \nabla_{\theta} \pi(s_t, a_t; \theta) \cdot Q^{\pi}(s_t, a_t) \cdot da_t \cdot ds_t$$

# Proof of Policy Gradient Theorem

Bring  $\sum_{t=0}^{\infty}$  inside the two  $\int_S$ , and note that  $\int_A \nabla_{\theta} \pi(s_t, a_t; \theta) \cdot Q^{\pi}(s_t, a_t) \cdot da_t$  is independent of  $t$ .

$$= \int_S \int_S \sum_{t=0}^{\infty} \gamma^t \cdot p_0(s_0) \cdot p(s_0 \rightarrow s, t, \pi) \cdot ds_0 \int_A \nabla_{\theta} \pi(s, a; \theta) \cdot Q^{\pi}(s, a) \cdot da \cdot ds$$

Reminder that  $\int_S \sum_{t=0}^{\infty} \gamma^t \cdot p_0(s_0) \cdot p(s_0 \rightarrow s, t, \pi) \cdot ds_0 \stackrel{\text{def}}{=} \rho^{\pi}(s)$ . So,

$$\nabla_{\theta} J(\theta) = \int_S \rho^{\pi}(s) \int_A \nabla_{\theta} \pi(s, a; \theta) \cdot Q^{\pi}(s, a) \cdot da \cdot ds$$

Q.E.D.

# Monte-Carlo Policy Gradient (REINFORCE Algorithm)

- Update  $\theta$  by stochastic gradient ascent using PGT
- Using  $G_t = \sum_{k=t}^T \gamma^{k-t} \cdot r_k$  as an unbiased sample of  $Q^\pi(s_t, a_t)$

$$\Delta\theta_t = \alpha \cdot \gamma^t \cdot \nabla_{\theta} \log \pi(s_t, a_t; \theta) \cdot G_t$$

## Algorithm 4.1: REINFORCE( $\cdot$ )

Initialize  $\theta$  arbitrarily

**for** each episode  $\{s_0, a_0, r_0, \dots, s_T, a_T, r_T\} \sim \pi(\cdot, \cdot; \theta)$

**do**  $\left\{ \begin{array}{l} \textbf{for } t \leftarrow 0 \textbf{ to } T \\ \textbf{do } \left\{ \begin{array}{l} G \leftarrow \sum_{k=t}^T \gamma^{k-t} \cdot r_k \\ \theta \leftarrow \theta + \alpha \cdot \gamma^t \cdot \nabla_{\theta} \log \pi(s_t, a_t; \theta) \cdot G \end{array} \right. \end{array} \right.$

**return** ( $\theta$ )

# Reducing Variance using a Critic

- Monte Carlo Policy Gradient has high variance
- We use a Critic  $Q(s, a; w)$  to estimate  $Q^\pi(s, a)$
- Actor-Critic algorithms maintain two sets of parameters:
  - Critic updates parameters  $w$  to approximate  $Q$ -function for policy  $\pi$
  - Critic could use any of the algorithms we learnt earlier:
    - Monte Carlo policy evaluation
    - Temporal-Difference Learning
    - $TD(\lambda)$  based on Eligibility Traces
    - Could even use LSTD (if critic function approximation is linear)
  - Actor updates policy parameters  $\theta$  in direction suggested by Critic
  - This is Approximate Policy Gradient due to *Bias* of Critic

$$\nabla_{\theta} J(\theta) \approx \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \nabla_{\theta} \pi(s, a; \theta) \cdot Q(s, a; w) \cdot da \cdot ds$$

# So what does the algorithm look like?

- Generate a sufficient set of simulation paths  $s_0, a_0, r_0, s_1, a_1, r_1, \dots$
- $s_0$  is sampled from the distribution  $p_0(\cdot)$
- $a_t$  is sampled from  $\pi(s_t, \cdot; \theta)$
- $s_{t+1}$  sampled from transition probs and  $r_{t+1}$  from reward func
- At each time step  $t$ , update  $w$  proportional to gradient of appropriate (MC or TD-based) loss function of  $Q(s, a; w)$
- Sum  $\gamma^t \cdot \nabla_{\theta} \log \pi(s_t, a_t; \theta) \cdot Q(s_t, a_t; w)$  over  $t$  and over paths
- Update  $\theta$  using this (biased) estimate of  $\nabla_{\theta} J(\theta)$
- Iterate with a new set of simulation paths ...

# Reducing Variance with a Baseline

- We can reduce variance by subtracting a baseline function  $B(s)$  from  $Q(s, a; w)$  in the Policy Gradient estimate
- This means at each time step, we replace  $\gamma^t \cdot \nabla_{\theta} \log \pi(s_t, a_t; \theta) \cdot Q(s_t, a_t; w)$  with  $\gamma^t \cdot \nabla_{\theta} \log \pi(s_t, a_t; \theta) \cdot (Q(s_t, a_t; w) - B(s))$
- Note that Baseline function  $B(s)$  is only a function of  $s$  (and not  $a$ )
- This ensures that subtracting Baseline  $B(s)$  does not add bias

$$\begin{aligned} & \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \nabla_{\theta} \pi(s, a; \theta) \cdot B(s) \cdot da \cdot ds \\ &= \int_{\mathcal{S}} \rho^{\pi}(s) \cdot B(s) \cdot \nabla_{\theta} \left( \int_{\mathcal{A}} \pi(s, a; \theta) \cdot da \right) \cdot ds = 0 \end{aligned}$$

# Using State Value function as Baseline

- A good baseline  $B(s)$  is state value function  $V(s; v)$
- Rewrite Policy Gradient algorithm using advantage function estimate

$$A(s, a; w, v) = Q(s, a; w) - V(s; v)$$

- Now the estimate of  $\nabla_{\theta} J(\theta)$  is given by:

$$\int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \nabla_{\theta} \pi(s, a; \theta) \cdot A(s, a; w, v) \cdot da \cdot ds$$

- At each time step, we update both sets of parameters  $w$  and  $v$

# TD Error as estimate of Advantage Function

- Consider TD error  $\delta^\pi$  for the *true* Value Function  $V^\pi(s)$

$$\delta^\pi = r + \gamma V^\pi(s') - V^\pi(s)$$

- $\delta^\pi$  is an unbiased estimate of Advantage function  $A^\pi(s, a)$

$$\mathbb{E}_\pi[\delta^\pi | s, a] = \mathbb{E}_\pi[r + \gamma V^\pi(s') | s, a] - V^\pi(s) = Q^\pi(s, a) - V^\pi(s) = A^\pi(s, a)$$

- So we can write Policy Gradient in terms of  $\mathbb{E}_\pi[\delta^\pi | s, a]$

$$\nabla_\theta J(\theta) = \int_{\mathcal{S}} \rho^\pi(s) \int_{\mathcal{A}} \nabla_\theta \pi(s, a; \theta) \cdot \mathbb{E}_\pi[\delta^\pi | s, a] \cdot da \cdot ds$$

- In practice, we can use func approx for TD error (and sample):

$$\delta(s, r, s'; v) = r + \gamma V(s'; v) - V(s; v)$$

- This approach requires only one set of critic parameters  $v$



# TD Error can be used by both Actor and Critic

## Algorithm 4.2: ACTOR-CRITIC-TD-ERROR( $\cdot$ )

Initialize Policy params  $\theta \in \mathbb{R}^m$  and State VF params  $v \in \mathbb{R}^n$  arbitrarily  
**for** each episode

**do** {  
    Initialize  $s$  (first state of episode)  
     $P \leftarrow 1$   
    **while**  $s$  is not terminal  
        {  
             $a \sim \pi(s, \cdot; \theta)$   
            Take action  $a$ , observe  $r, s'$   
             $\delta \leftarrow r + \gamma V(s'; v) - V(s; v)$   
            **do** {  
                 $v \leftarrow v + \alpha_v \cdot \delta \cdot \nabla_v V(s; v)$   
                 $\theta \leftarrow \theta + \alpha_\theta \cdot P \cdot \delta \cdot \nabla_\theta \log \pi(s, a; \theta)$   
                 $P \leftarrow \gamma P$   
                 $s \leftarrow s'$

# Using Eligibility Traces for both Actor and Critic

## Algorithm 4.3: ACTOR-CRITIC-ELIGIBILITY-TRACES( $\cdot$ )

Initialize Policy params  $\theta \in \mathbb{R}^m$  and State VF params  $v \in \mathbb{R}^n$  arbitrarily

**for** each episode

**do** {  
    Initialize  $s$  (first state of episode)  
     $z_\theta, z_v \leftarrow 0$  ( $m$  and  $n$  components eligibility trace vectors)  
     $P \leftarrow 1$   
    **while**  $s$  is not terminal  
        **do** {  
             $a \sim \pi(s, \cdot; \theta)$   
            Take action  $a$ , observe  $r, s'$   
             $\delta \leftarrow r + \gamma V(s'; v) - V(s; v)$   
             $z_v \leftarrow \gamma \cdot \lambda_v \cdot z_v + \nabla_v V(s; v)$   
             $z_\theta \leftarrow \gamma \cdot \lambda_\theta \cdot z_\theta + P \cdot \nabla_\theta \log \pi(s, a; \theta)$   
             $v \leftarrow v + \alpha_v \cdot \delta \cdot z_v$   
             $\theta \leftarrow \theta + \alpha_\theta \cdot \delta \cdot z_\theta$   
             $P \leftarrow \gamma P, s \leftarrow s'$   
        }

# Overcoming Bias

- We've learnt a few ways of how to reduce variance
- But we haven't discussed how to overcome bias
- All of the following substitutes for  $Q^\pi(s, a)$  in PG have bias:
  - $Q(s, a; w)$
  - $A(s, a; w, v)$
  - $\delta(s, s', r; v)$
- Turns out there is indeed a way to overcome bias
- It is called the *Compatible Function Approximation Theorem*

# Compatible Function Approximation Theorem

## Theorem

*If the following two conditions are satisfied:*

- 1 *Critic gradient is compatible with the Actor score function*

$$\nabla_w Q(s, a; w) = \nabla_\theta \log \pi(s, a; \theta)$$

- 2 *Critic parameters  $w$  minimize the following mean-squared error:*

$$\epsilon = \int_S \rho^\pi(s) \int_{\mathcal{A}} \pi(s, a; \theta) (Q^\pi(s, a) - Q(s, a; w))^2 \cdot da \cdot ds$$

*Then the Policy Gradient using critic  $Q(s, a; w)$  is exact:*

$$\nabla_\theta J(\theta) = \int_S \rho^\pi(s) \int_{\mathcal{A}} \nabla_\theta \pi(s, a; \theta) \cdot Q(s, a; w) \cdot da \cdot ds$$

# Proof of Compatible Function Approximation Theorem

For  $w$  that minimizes

$$\epsilon = \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot (Q^{\pi}(s, a) - Q(s, a; w))^2 \cdot da \cdot ds,$$

$$\int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot (Q^{\pi}(s, a) - Q(s, a; w)) \cdot \nabla_w Q(s, a; w) \cdot da \cdot ds = 0$$

But since  $\nabla_w Q(s, a; w) = \nabla_{\theta} \log \pi(s, a; \theta)$ , we have:

$$\int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot (Q^{\pi}(s, a) - Q(s, a; w)) \cdot \nabla_{\theta} \log \pi(s, a; \theta) \cdot da \cdot ds = 0$$

Therefore, 
$$\begin{aligned} & \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot Q^{\pi}(s, a) \cdot \nabla_{\theta} \log \pi(s, a; \theta) \cdot da \cdot ds \\ &= \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot Q(s, a; w) \cdot \nabla_{\theta} \log \pi(s, a; \theta) \cdot da \cdot ds \end{aligned}$$

# Proof of Compatible Function Approximation Theorem

$$\text{But } \nabla_{\theta} J(\theta) = \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot Q^{\pi}(s, a) \cdot \nabla_{\theta} \log \pi(s, a; \theta) \cdot da \cdot ds$$

$$\begin{aligned} \text{So, } \nabla_{\theta} J(\theta) &= \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot Q(s, a; w) \cdot \nabla_{\theta} \log \pi(s, a; \theta) \cdot da \cdot ds \\ &= \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \nabla_{\theta} \pi(s, a; \theta) \cdot Q(s, a; w) \cdot da \cdot ds \end{aligned}$$

Q.E.D.

**This means with conditions (1) and (2) of Compatible Function Approximation Theorem, we can use the critic func approx  $Q(s, a; w)$  and still have the exact Policy Gradient.**

# How to enable Compatible Function Approximation

A simple way to enable Compatible Function Approximation

$\frac{\partial Q(s, a; w)}{\partial w_i} = \frac{\partial \log \pi(s, a; \theta)}{\partial \theta_i}$ ,  $\forall i$  is to set  $Q(s, a; w)$  to be linear in its features.

$$Q(s, a; w) = \sum_{i=1}^n \phi_i(s, a) \cdot w_i = \sum_{i=1}^n \frac{\partial \log \pi(s, a; \theta)}{\partial \theta_i} \cdot w_i$$

We note below that a compatible  $Q(s, a; w)$  serves as an approximation of the advantage function.

$$\begin{aligned} \int_{\mathcal{A}} \pi(s, a; \theta) \cdot Q(s, a; w) \cdot da &= \int_{\mathcal{A}} \pi(s, a; \theta) \cdot \left( \sum_{i=1}^n \frac{\partial \log \pi(s, a; \theta)}{\partial \theta_i} \cdot w_i \right) \cdot da \\ &= \int_{\mathcal{A}} \left( \sum_{i=1}^n \frac{\partial \pi(s, a; \theta)}{\partial \theta_i} \cdot w_i \right) \cdot da = \sum_{i=1}^n \left( \int_{\mathcal{A}} \frac{\partial \pi(s, a; \theta)}{\partial \theta_i} \cdot da \right) \cdot w_i \\ &= \sum_{i=1}^n \frac{\partial}{\partial \theta_i} \left( \int_{\mathcal{A}} \pi(s, a; \theta) \cdot da \right) \cdot w_i = \sum_{i=1}^n \frac{\partial 1}{\partial \theta_i} \cdot w_i = 0 \end{aligned}$$

# Fisher Information Matrix

Denoting  $[\frac{\partial \log \pi(s, a; \theta)}{\partial \theta_i}]$ ,  $i = 1, \dots, n$  as the score column vector  $SC(s, a; \theta)$  and assuming compatible linear-approx critic:

$$\begin{aligned}\nabla_{\theta} J(\theta) &= \int_{\mathcal{S}} \rho^{\pi}(s) \int_{\mathcal{A}} \pi(s, a; \theta) \cdot (SC(s, a; \theta) \cdot SC(s, a; \theta)^T \cdot w) \cdot da \cdot ds \\ &= E_{s \sim \rho^{\pi}, a \sim \pi} [SC(s, a; \theta) \cdot SC(s, a; \theta)^T] \cdot w \\ &= FIM_{\rho^{\pi}, \pi}(\theta) \cdot w\end{aligned}$$

where  $FIM_{\rho^{\pi}, \pi}(\theta)$  is the Fisher Information Matrix w.r.t.  $s \sim \rho^{\pi}$ ,  $a \sim \pi$ .



# Natural Policy Gradient

- Recall the idea of Natural Gradient from Numerical Optimization
- Natural gradient  $\nabla_{\theta}^{nat} J(\theta)$  is the direction of optimal  $\theta$  movement
- In terms of the KL-divergence metric (versus plain Euclidean norm)
- Natural gradient yields better convergence (we won't cover proof)

Formally defined as:  $\nabla_{\theta} J(\theta) = FIM_{\rho_{\pi}, \pi}(\theta) \cdot \nabla_{\theta}^{nat} J(\theta)$

Therefore,  $\nabla_{\theta}^{nat} J(\theta) = w$

**This compact result is great for our algorithm:**

- Update Critic params  $w$  with the critic loss gradient (at step  $t$ ) as:

$$\gamma^t \cdot (r_t + \gamma \cdot SC(s_{t+1}, a_{t+1}, \theta) \cdot w - SC(s_t, a_t, \theta) \cdot w) \cdot SC(s_t, a_t, \theta)$$

- Update Actor params  $\theta$  in the direction equal to value of  $w$