ANALYTIC FORMS FOR CLEARANCE PRICE OPTIMIZATION

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1. Introduction

This is a brief article to derive analytic forms for Clearance Price Optimization.

2. Bellman Equation

The *State* at time $t \in \{0, 1, \ldots, T\}$ is the Inventory $I_t \in \mathbb{Z}_{\geq 0}$. The *Action* at time t is the Price $p_t \in \mathbb{R}_{\geq 0}$ set at time t. Assume the probability mass function (PMF) of demand $k_t \in \mathbb{Z}_{\geq 0}$ at time t has mean $f(p_t)$ and variance $g(p_t)$ for some given functions $f, g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$. The *Reward* at time t is $p_t \cdot \min(k_t, I_t)$. Then, for all $t = 0, 1, \ldots, T-1$, we have the following Bellman Equation:

$$V_t^*(I_t) = \max_{p_t} \{ \sum_{k_t=0}^{\infty} Pr[k_t; f(p_t), g(p_t)] \cdot (p_t \cdot \min(k_t, I_t) + V_{t+1}^*(\max(I_t - k_t, 0))) \}$$

$$V_t^*(0) = 0, V_T^*(I_T) = 0$$

As an example, consider the Poisson distribution for the PMF of demand given by Poisson mean $f(p_t) = \alpha \cdot e^{-\beta \cdot p_t}$ for all $t = 0, 1, \ldots, T-1$. Then,

$$V_t^*(I_t) = \max_{p_t} \{ \sum_{k_t=0}^{I_t - 1} \frac{e^{-\alpha \cdot e^{-\beta \cdot p_t}} \cdot \alpha^{k_t} \cdot e^{-\beta \cdot k_t \cdot p_t}}{k_t!} \cdot (p_t \cdot k_t + V_{t+1}^*(I_t - k_t)) + \sum_{k_t = I_t}^{\infty} \frac{e^{-\alpha \cdot e^{-\beta \cdot p_t}} \cdot \alpha^{k_t} e^{-\beta \cdot k_t \cdot p_t}}{k_t!} \cdot p_t \cdot I_t \}$$

$$V_t^*(0) = 0, V_T^*(I_T) = 0$$

For $t = T-1$, we take the derivative of the right hand side with respect to p_{T-1} and set to 0. This enables us to solve for p_{T-1}^* and V_{T-1}^*. Then we do the same for $t = T-2$ and proceed back in time to obtain p_t^* and V_t^* for all $t = 0, 1, \ldots T-1$.

The above technique is easy for the case of deterministic demand.

3. Deterministic Case

Here we consider the case where demand is a deterministic function of price, denoted as $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ for each time step $t = 0, 1, \ldots T-1$. For the deterministic case, since demand at each time step t is a continuous variable ($\in \mathbb{R}_{\geq 0}$), the inventory I_t at time t is a continuous variable ($\in \mathbb{R}_{\geq 0}$). Then, for all $t = 0, 1, \ldots T-1$, we have the Bellman Equation:

$$V_t^*(I_t) = \max_{p_t} \{p_t \cdot \min(f(p_t), I_t) + V_{t+1}^*(\max(I_t - f(p_t), 0))\}$$
$$V_t^*(0) = 0, V_T^*(I_T) = 0$$

3.1. **Simplifying the Bellman Equation.** We will assume that f has an inverse $f^{-1} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and that f is a monotonically decreasing function. Then, we have the following Proposition.

Proposition 1. *Optimal Pricing will not permit the demand at time t to exceed the Inventory I_t at time t, for all $t = 0, 1, \ldots, T-1$, i.e., $f(p_t^*) \leq I_t$ or equivalently, $p_t^* \geq f^{-1}(I_t)$.*

Proof. Assume the contrary, that $f(p_t^*) > I_t$, or equivalently, $p_t^* < f^{-1}(I_t)$. Since demand $f(p_t^*)$ exceeds I_t, $I_{t+1} = 0$ and so, $V_{t+1}^*(I_{t+1}) = 0$. Therefore, $V_t^*(I_t) = p_t^* \cdot I_t < f^{-1}(I_t) \cdot I_t$. This says that a price of $f^{-1}(I_t)$ producing a demand of I_t attains a *Reward* of $f^{-1}(I_t) \cdot I_t$ at time step t that exceeds the assumed optimal value function $V_t^*(I_t) \Rightarrow$ Reductio Ad Absurdum. □

An important consequence of this Proposition is that we can simplify the Bellman Equation:

$$V_t^*(I_t) = \max_{p_t \geq f^{-1}(I_t)} \{p_t \cdot f(p_t) + V_{t+1}^*(I_t - f(p_t))\}$$
$$V_t^*(0) = 0, V_T^*(I_T) = 0$$

3.2. **Exponential Price Function.** Assume $f(p_t) = \alpha \cdot e^{-\beta \cdot p_t}$ for some given $\alpha, \beta \in \mathbb{R}_{>0}$. Then,

$$V_t^*(I_t) = \max_{p_t \geq \frac{1}{\beta} \log \frac{\alpha}{I_t}} \{p_t \cdot \alpha \cdot e^{-\beta \cdot p_t} + V_{t+1}^*(I_t - \alpha \cdot e^{-\beta \cdot p_t})\}$$
$$V_t^*(0) = 0, V_T^*(I_T) = 0$$

We note that as $I_t \rightarrow 0$, $p_t^* \geq \frac{1}{\beta} \log \frac{\alpha}{I_t} \rightarrow \infty$, Demand under Optimal Price $(= \alpha \cdot e^{-\beta \cdot p_t^*}) \rightarrow 0$, and *Reward* under Optimal Price $(= p_t^* \cdot \alpha \cdot e^{-\beta \cdot p_t^*}) \rightarrow 0$.

3.3. **Backward Recursion: Optimal Price/Value Function for $t = T - 1$.** We start with time $t = T - 1$ and solve the Bellman Equation by recursing back in time. The Bellman equation for time step $t = T - 1$ is:

$$V_{T-1}^*(I_{T-1}) = \max_{p_{T-1} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-1}}} \{\alpha \cdot p_{T-1} \cdot e^{-\beta \cdot p_{T-1}}\}$$

In order to perform the constrained maximization above, let us examine the function $h(x) = \alpha \cdot x \cdot e^{-\beta x}$ for all $x \in \mathbb{R}_{\geq 0}$.

- $h(0) = 0$
- $\lim_{x \rightarrow \infty} h(x) = 0$

- $h'(x) = \alpha \cdot e^{-\beta x} \cdot (1 - \beta x)$ and so, $h'(x) > 0$ for $x < \frac{1}{\beta}, h'(x) = 0$ for $x = \frac{1}{\beta}$, $h'(x) < 0$ for $x > \frac{1}{\beta}$

From the above analysis, we note that $h(x)$ has a single peak at $x = \frac{1}{\beta}$. If this peak occurs within the above-specified range for the price ($p_{T-1} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-1}}$), then the optimal price is where the peak occurs ($= \frac{1}{\beta}$). Otherwise, the optimal price is the lower bound of this price range ($= \frac{1}{\beta} \log \frac{\alpha}{I_{T-1}}$). We also note that the above condition ($\frac{1}{\beta} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-1}}$) can be succinctly expressed as: $I_t \geq \frac{\alpha}{e}$. Below, we summarize the Optimal Price and consequent Optimal Value Function for time step $t = T - 1$:

$$p^*_{T-1}(I_{T-1}) = \begin{cases} \frac{1}{\beta} & \text{if } I_{T-1} \geq \frac{\alpha}{e} \\ \frac{1}{\beta} \log \frac{\alpha}{I_{T-1}} & \text{if } I_{T-1} < \frac{\alpha}{e} \end{cases}$$

$$V^*_{T-1}(I_{T-1}) = \begin{cases} \frac{\alpha}{\beta \cdot e} & \text{if } I_{T-1} \geq \frac{\alpha}{e} \\ \frac{I_{T-1}}{\beta} \log \frac{\alpha}{I_{T-1}} & \text{if } I_{T-1} < \frac{\alpha}{e} \end{cases}$$

3.4. **Backward Recursion: Optimal Price/Value Function for $t = T - 2$.** Stepping back in time, the Bellman Equation for time step $t = T - 2$ is:

$$V^*_{T-2}(I_{T-2}) = \max_{p_{T-2} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-2}}} \{p_{T-2} \cdot \alpha \cdot e^{-\beta \cdot p_{T-2}} + V^*_{T-1}(I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}})\}$$

We start by making a couple of key observations:

- $V^*_{T-1}(I_{T-1})$ is a monotonically increasing function of I_{T-1} for $I_{T-1} < \frac{\alpha}{e}$ and is constant ($= \frac{1}{\beta}$) for $I_{T-1} \geq \frac{\alpha}{e}$
- $I_{T-1}(p_{T-2}) = I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}}$ is a monotonically increasing function of p_{T-2} for the entire range of I_{T-1} from 0 to I_{T-2} (see equations below):

$$I_{T-1}(p_{T-2} = \frac{1}{\beta} \log \frac{\alpha}{I_{T-2}}) = 0$$

$$I_{T-1}(p_{T-2} = \frac{1}{\beta} \log \frac{\alpha}{I_{T-2} - \frac{\alpha}{e}}) = \frac{\alpha}{e}$$

$$\lim_{p_{T-2} \to \infty} I_{T-1}(p_{T-2}) = I_{T-2}$$

Combining these two observations, we note that $V^*_{T-1}(I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}})$ is a monotonically increasing function of p_{T-2} for $\frac{1}{\beta} \log \frac{\alpha}{I_{T-2}} \leq p_{T-2} < \frac{1}{\beta} \log \frac{\alpha}{I_{T-2} - \frac{\alpha}{e}}$ and is constant for $p_{T-2} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-2} - \frac{\alpha}{e}}$. We also know that $p_{T-2} \cdot \alpha \cdot e^{-\beta \cdot p_{T-2}}$ has a single peak at $p_{T-2} = \frac{1}{\beta}$.

Therefore, if $\frac{1}{\beta} \geq \frac{1}{\beta} \log \frac{\alpha}{I_{T-2} - \frac{\alpha}{e}}$ (or equivalently, $I_{T-2} \geq \frac{2\alpha}{e}$), then we can assert that $p_{T-2} \cdot \alpha \cdot e^{-\beta \cdot p_{T-2}} + V^*_{T-1}(I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}})$ (right-hand side of Bellman Equation for $t = T - 2$) will have a single peak at $p_{T-2} = \frac{1}{\beta}$.

Summarizing the above arguments, we state that for $I_{T-2} \geq \frac{2\alpha}{e}$,

$$p^*_{T-2}(I_{T-2}) = \frac{1}{\beta}$$
$$V^*_{T-2}(I_{T-2}) = \frac{2\alpha}{\beta \cdot e}$$

Now we come to the remaining case of $I_{T-2} < \frac{2\alpha}{e}$. This case corresponds to: $\frac{1}{\beta} < \frac{1}{\beta} \log \frac{\alpha}{I_{T-2} - \frac{\alpha}{e}}$. This means $p_{T-2} \cdot \alpha \cdot e^{-\beta \cdot p_{T-2}} + V^*_{T-1}(I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}})$ (right-hand side of Bellman Equation for $t = T-2$) will peak for p_{T-2} within the range:

$$\frac{1}{\beta} \leq p_{T-2} \leq \frac{1}{\beta} \log \frac{\alpha}{\max(I_{T-2} - \frac{\alpha}{e}, 0)}$$

We also note that this range of p_{T-2} corresponds to $I_{T-1} < \frac{\alpha}{e}$, which in turn corresponds to $V^*_{T-1}(I_{T-1}) = \frac{I_{T-1}}{\beta} \log \frac{\alpha}{I_{T-1}}$. Hence, we can formally state the case of $I_{T-2} < \frac{2\alpha}{e}$ as maximization of:

$$s(p_{T-2}) = p_{T-2} \cdot \alpha \cdot e^{-\beta \cdot p_{T-2}} + \frac{I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}}}{\beta} \log \frac{\alpha}{I_{T-2} - \alpha \cdot e^{-\beta \cdot p_{T-2}}}$$

under the following range constraints for p_{T-2}:

$$\frac{1}{\beta} \leq p_{T-2} \leq \frac{1}{\beta} \log \frac{\alpha}{\max(I_{T-2} - \frac{\alpha}{e}, 0)}$$

Let us now analyze the function:

$$s(x) = x \cdot \alpha \cdot e^{-\beta x} + \frac{I - \alpha \cdot e^{-\beta x}}{\beta} \log \frac{\alpha}{I - \alpha \cdot e^{-\beta x}}$$

for all $x \in \mathbb{R}_{\geq 0}$.

- $s(0) = \frac{I-\alpha}{\beta} \log \frac{\alpha}{I-\alpha}$
- $\lim_{x \to \infty} s(x) = \frac{I}{\beta} \log \frac{\alpha}{I}$
- $s'(x) = \alpha \cdot e^{-\beta x} \cdot (\log \frac{\alpha}{I_{T-2} - \alpha \cdot e^{-\beta x}} - \beta x)$ and so, $h'(x) > 0$ for $x < \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$, $h'(x) = 0$ for $x = \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$, $h'(x) < 0$ for $x > \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$

From the above analysis, we note that $s(x)$ has a single peak at $x = \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$. Since we are considering the case of $I_{T-2} < \frac{2\alpha}{e}$, the Optimal Price $p^*_{T-2}(I_{T-2}) = \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$ satisfies the two constraints $p_{T-2} \geq \frac{1}{\beta}$ and $p_{T-2} \leq \frac{1}{\beta} \log \frac{\alpha}{\max(I_{T-2} - \frac{\alpha}{e}, 0)}$.

Summarizing the above arguments, we state that for $I_{T-2} < \frac{2\alpha}{e}$,

$$p^*_{T-2}(I_{T-2}) = \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}}$$
$$V^*_{T-2}(I_{T-2}) = \frac{I_{T-2}}{\beta} \log \frac{2\alpha}{I_{T-2}}$$

So now we are ready to summarize the Optimal Price and Optimal Value Function for $t = T - 2$:

$$p^*_{T-2}(I_{T-2}) = \begin{cases} \frac{1}{\beta} & \text{if } I_{T-2} \geq \frac{2\alpha}{e} \\ \frac{1}{\beta} \log \frac{2\alpha}{I_{T-2}} & \text{if } I_{T-2} < \frac{2\alpha}{e} \end{cases}$$

$$V^*_{T-2}(I_{T-2}) = \begin{cases} \frac{2\alpha}{\beta \cdot e} & \text{if } I_{T-2} \geq \frac{2\alpha}{e} \\ \frac{I_{T-2}}{\beta} \log \frac{2\alpha}{I_{T-2}} & \text{if } I_{T-2} < \frac{2\alpha}{e} \end{cases}$$

3.5. **Backward Recursion: Optimal Price/Value Function for all $t = 0, 1, \ldots T - 1$.** Stepping back in a likewise manner gives us the following Optimal Price and Optimal Value Function for all $t = 0, 1, \ldots, T - 1$.

$$p^*_t(I_t) = \begin{cases} \frac{1}{\beta} & \text{if } I_t \geq \frac{(T-t)\alpha}{e} \\ \frac{1}{\beta} \log \frac{(T-t)\alpha}{I_t} & \text{if } I_t < \frac{(T-t)\alpha}{e} \end{cases}$$

$$V^*_t(I_t) = \begin{cases} \frac{(T-t)\alpha}{\beta \cdot e} & \text{if } I_t \geq \frac{(T-t)\alpha}{e} \\ \frac{I_t}{\beta} \log \frac{(T-t)\alpha}{I_t} & \text{if } I_t < \frac{(T-t)\alpha}{e} \end{cases}$$

The interesting aspect of this result is that **there are only two possible optimal prices for the deterministic case** (separated by the line $I_t = \frac{(T-t)\alpha}{e}$ which we will refer to as the "Optimal Pricing Frontier").

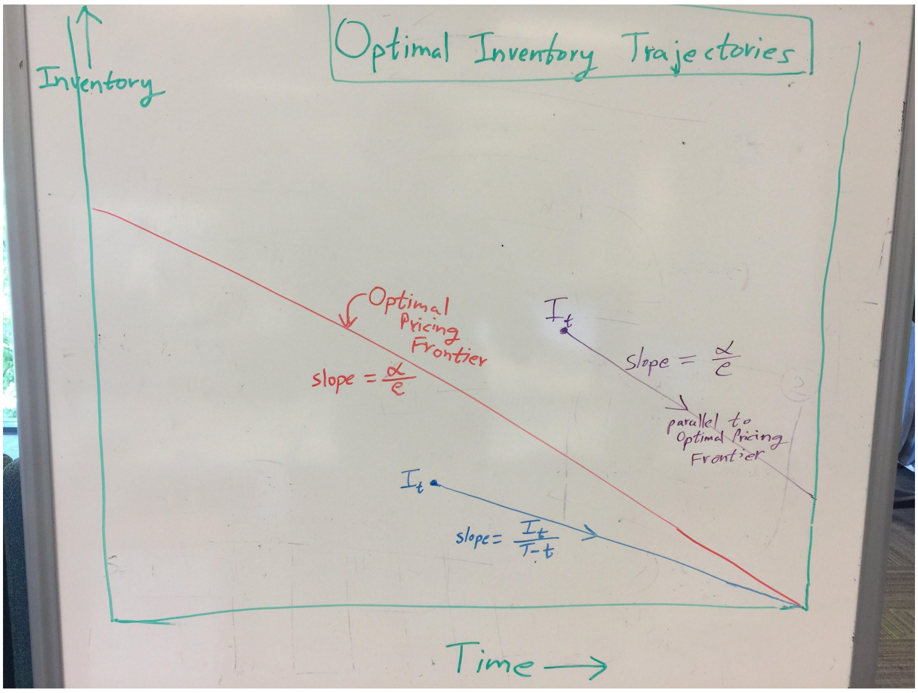

FIGURE 1. Optimal Pricing Frontier

We note that:

- Above the "Optimal Pricing Frontier", Demand (under Optimal Pricing) occurs at the rate of $\frac{\alpha}{e}$ per time step (i.e., inventory trajectory is a straight line parallel to the "Optimal Pricing Frontier")
- Below the "Optimal Pricing Frontier", Demand (under Optimal Pricing) occurs at the rate of $\frac{I_t}{T-t}$ per time step (i.e., inventory trajectory is a straight line taking the inventory to exactly 0 at $t = T$).

Hence, the demand rate (under Optimal Pricing) is higher above the "Optimal Pricing Frontier", and correspondingly Optimal Price is lower above the "Optimal Pricing Frontier" (as expected).

4. Calibration to "Demand Lift"

Let us assume that the "base price" is 1, for which the demand is D_0. Let us assume that the demand for "half price off" (i.e., for price of 0.5) is $D_{0.5} = (1 + L) \cdot D_0$ (we will refer to L as the "Demand Lift"). Let us calibrate the function $f(p) = \alpha \cdot e^{-\beta \cdot p}$ to these values:

$$D_0 = \alpha \cdot e^{-\beta}, D_{0.5} = \alpha \cdot e^{-\frac{\beta}{2}}$$

Solving for α and β:

$$\alpha = \frac{D_{0.5}^2}{D_0} = D_0 \cdot (1 + L)^2$$

$$\beta = 2 \log \left(\frac{D_{0.5}}{D_0}\right) = 2 \log (1 + L)$$

$$f(p) = D_{0.5}^{2(1-p)} \cdot D_0^{2p-1} = D_0 \cdot (1 + L)^{2(1-p)}$$

So, the "Optimal Price Frontier" (for the case of Deterministic Demand) is given by the line:

$$I_t = \frac{(T - t) \cdot D_0 \cdot (1 + L)^2}{e}$$

Above the "Optimal Price Frontier", we have:

$$p_t^*(I_t) = \frac{1}{2 \log (1 + L)}$$

$$V_t^*(I_t) = \frac{(T - t) \cdot D_0 \cdot (1 + L)^2}{2 \cdot e \cdot \log (1 + L)}$$

Below the "Optimal Price Frontier", we have:

$$p_t^*(I_t) = 1 + \frac{\log \frac{(T-t)D_0}{I_t}}{2 \log(1 + L)}$$

$$V_t^*(I_t) = I_t(1 + \frac{\log \frac{(T-t)D_0}{I_t}}{2 \log(1 + L)})$$