A Quick/Terse Intro to Efficient Frontier Mathematics

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Overview

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2. Derivation of Efficient Frontier Curve
3. Interesting Efficient Portfolios
4. Linearity of Covariance Vector w.r.t. Mean Returns (a.k.a. CAPM)
5. Efficient Set with a Risk-Free Asset

Setting and Notation

- n assets in the economy with usual regularity/idealistic conditions
- Their mean returns denoted by column n-vector R
- Their covariance of returns denoted by V ($n \times n$ non-singular matrix)
- Column n-vector X_p denotes proportions of n assets in portfolio p
- Denote 1_n as a column n-vector of all 1's

$$X_p^T \cdot 1_n = 1$$

We drop subscript p whenever the reference to portfolio p is clear

Portfolio Returns

- A single portfolio's mean return is $X^T \cdot R$
- A single portfolio's variance of return is the quadratic form $X^T \cdot V \cdot X$
- Covariance between portfolios p and q is the bilinear form $X_p^T \cdot V \cdot X_q$
- Covariance of assets with a single portfolio is $V \cdot X$ (n-vector)

Derivation of Efficient Frontier Curve

- Efficient frontier is defined for a world with no risk-free assets
- It is the set of portfolios with minimum variance of return for each level of portfolio mean returns
- So, minimize portfolio variance $X^T \cdot V_p \cdot X$ subject to constraints:

$$X^T \cdot 1_n = 1$$

$$X^T \cdot R = r_p$$

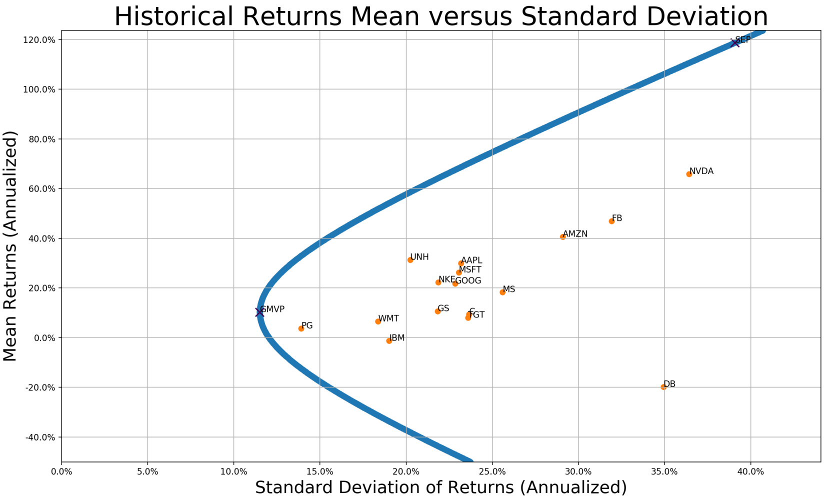
where r_p is the mean return for efficient portfolio p.

- Set up the Lagrangian and solve to express X in terms of R, V, r_p
- Substituting for X gives us the efficient frontier parabola:

$$\sigma_p^2 = \frac{a - 2br_p + cr_p^2}{ac - b^2} \text{ where}$$

$$a = R^T \cdot V^{-1} \cdot R, b = R^T \cdot V^{-1} \cdot 1_n, c = 1_n^T V^{-1} 1_n$$

The Efficient Frontier with 16 assets



Global Minimum Variance Portfolio (GMVP)

- Global minimum variance portfolio (GMVP) is the tip of the curve
- It has mean $r_0 = \frac{b}{c}$
- It has variance $\sigma_0^2 = \frac{1}{c}$
- It has investment proportions $X_0 = \frac{V^{-1} \cdot 1_n}{c}$
- GMVP is positively correlated with all portfolios and assets
- GMVP's covariance with all assets and all portfolios is a constant σ_0^2 (which is also equal to its own variance)

Orthogonal Efficient Portfolios

For every efficient portfolio p (other than GMVP), there exists a unique orthogonal efficient portfolio z (i.e. $Covariance(p, z) = 0$) with finite mean

$$r_z = \frac{a - br_p}{b - cr_p}$$

- z always lies on the opposite side of p on the efficient frontier
- In mean-variance space, the straight line from p to GMVP intersects the mean axis at r_z
- In mean-stdev space, the tangent to the efficient frontier at p intersects the mean axis at r_z
- All portfolios on one side of the efficient frontier are positively correlated with each other

Two-fund Theorem

- The X vector of any efficient portfolio is a linear combination of the X vectors of two other efficient portfolios
- Notationally, $X_p = \alpha X_{p_1} + (1-\alpha) X_{p_2}$ for some scalar α
- The range of α from $-\infty$ to $+\infty$ traces the efficient frontier
- So to construct all efficient portfolios, we just need to identify two canonical efficient portfolios
- One of them is GMVP
- The other is a portfolio we call Special Efficient Portfolio (SEP) with:
  - Mean $r_1 = \frac{a}{b}$
  - Variance $\sigma_1^2 = \frac{a}{b^2}$
  - Investment proportions $X_1 = \frac{V^{-1} \cdot R}{b}$
- The orthogonal portfolio to SEP has mean $r_z = \frac{a - b\frac{a}{b}}{b - c\frac{a}{b}} = 0$

Linearity of Covariance Vector w.r.t. Mean Returns

Important Theorem: The covariance vector of individual assets with a portfolio (= V X) can be expressed as an exact linear function of the individual mean returns vector iff the portfolio is efficient. If the efficient portfolio is p (and its orthogonal portfolio z), then:

$$R = r_z 1_n + \frac{r_p - r_z}{\sigma_p^2} CovarianceVector_p$$

$$= r_z 1_n + \frac{r_p - r_z}{\sigma_p^2}(V \cdot X_p) = r_z 1_n + (r_p - r_z)\beta_p$$

where $\beta_p = \frac{CovarianceVector_p}{\sigma_p^2}$ is the vector of slope coefficients of regressions where the explanatory variable is the portfolio return and the n dependent variables are the asset returns.
The linearity of βs w.r.t. mean returns is the (in)famous CAPM banner.

Useful Corollaries

- If p is SEP, $r_z = 0$ which would mean: $R = r_p \beta_p = \frac{r_p}{\sigma_p^2} V \cdot X_p$
- So, in this case, covariance vector and β_p are just scalar multiples of asset mean vector
- The investment proportion X in a given individual asset changes monotonically along the efficient frontier
- $Covariance = V \cdot X$ is also monotonic along the efficient frontier
- But β is not monotonic $\Rightarrow$ For every individual asset, there is a unique pair of efficient portfolios that result in max and min βs for that asset

Cross-Sectional Variance

- The cross-sectional variance in βs (variance in βs across assets for a fixed efficient portfolio) is zero when efficient portfolio is GMVP and when efficient portfolio has infinite mean
- The cross-sectional variance in βs is maximum for the two efficient portfolios with means: $r_0 \pm \sigma_0^2 \sqrt{|A|}$ where A is the 2×2 matrix consisting of a, b, b, c
- These two portfolios lie symmetrically on opposite sides of the efficient frontier (their βs are equal and of opposite signs), and are the only two orthogonal efficient portfolios with the same variance ($= 2\sigma_0^2$)

Efficient Set with a Risk-Free Asset

- If we have a risk-free asset with return r_F, V is singular
- First form the efficient frontier without the risk-free asset
- The efficient set (with a risk-free asset) is the tangent to the efficient frontier (without the risk-free asset) in mean-stdev space from $(0, r_F)$
- Let tangency point portfolio be T with return r_T
- If $r_F < r_0, r_T > r_F$
- If $r_F > r_0, r_T < r_F$
- All portfolios on this efficient set are perfectly correlated
- Homework: How is T related to SEP?