Understanding Risk-Aversion through Utility Theory

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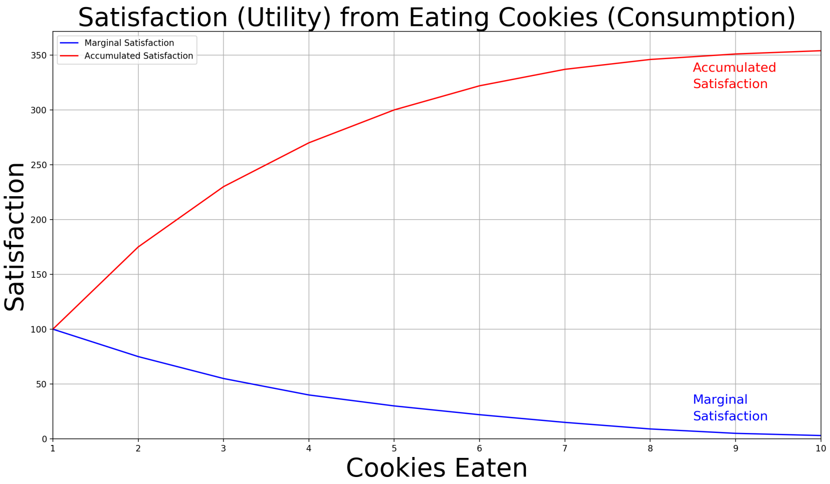
Intuition on Risk-Aversion and Risk-Premium

- Let's play a game where your payoff is based on outcome of a fair coin
- You get \$100 for HEAD and \$0 for TAIL
- How much would you pay to play this game?
- You immediately say: "Of course, \$50"
- Then you think a bit, and say: "A little less than \$50"
- Less because you want to "be compensated for taking the risk"
- The word *Risk* refers to the degree of variation of the outcome
- We call this risk-compensation as **Risk-Premium**
- Our *personality-based* degree of risk fear is known as **Risk-Aversion**
- So, we end up paying \$50 minus Risk-Premium to play the game
- **Risk-Premium grows with Outcome-Variance & Risk-Aversion**

Specifying Risk-Aversion through a Utility function

- We seek a "valuation formula" for the amount we'd pay that:
  - Increases one-to-one with the Mean of the outcome
  - Decreases as the Variance of the outcome (i.e.. Risk) increases
  - Decreases as our Personal Risk-Aversion increases
- The last two properties above define the Risk-Premium
- But fundamentally why are we Risk-Averse?
- Why don't we just pay the mean of the random outcome?
- **Reason: Our satisfaction to better outcomes grows non-linearly**
- We express this satisfaction non-linearity as a mathematical function
- Based on a core economic concept called **Utility of Consumption**
- We will illustrate this concept with a real-life example

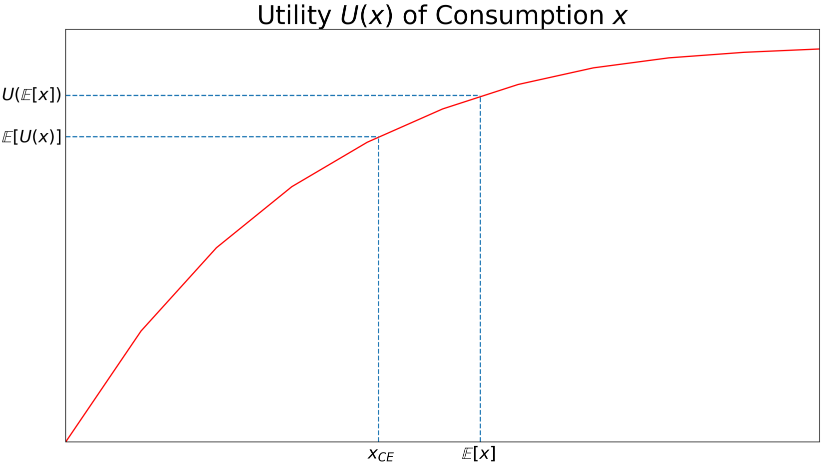
Law of Diminishing Marginal Utility



Utility of Consumption and Certainty-Equivalent Value

- Marginal Satisfaction of eating cookies is a diminishing function
- Hence, Accumulated Satisfaction is a concave function
- Accumulated Satisfaction represents Utility of Consumption $U(x)$
- Where x represents the uncertain outcome being consumed
- Degree of concavity represents extent of our Risk-Aversion
- Concave $U(\cdot)$ function $\Rightarrow \mathbb{E}[U(x)] < U(\mathbb{E}[x])$
- We define **Certainty-Equivalent Value** $x_{CE} = U^{-1}(\mathbb{E}[U(x)])$
- Denotes certain amount we'd pay to consume an uncertain outcome
- **Absolute Risk-Premium** $\pi_A = \mathbb{E}[x] - x_{CE}$
- **Relative Risk-Premium** $\pi_R = \frac{\pi_A}{\mathbb{E}[x]} = \frac{\mathbb{E}[x] - x_{CE}}{\mathbb{E}[x]} = 1 - \frac{x_{CE}}{\mathbb{E}[x]}$

Certainty-Equivalent Value



Calculating the Risk-Premium

- We develop mathematical formalism to calculate Risk-Premia π_A, π_R
- To lighten notation, we refer to $\mathbb{E}[x]$ as $\bar{x}$ and Variance of x as σ_x^2
- Taylor-expand $U(x)$ around $\bar{x}$, ignoring terms beyond quadratic

$$U(x) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x - \bar{x}) + \frac{1}{2} U''(\bar{x}) \cdot (x - \bar{x})^2$$

- Taylor-expand $U(x_{CE})$ around $\bar{x}$, ignoring terms beyond linear

$$U(x_{CE}) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x_{CE} - \bar{x})$$

- Taking the expectation of the $U(x)$ expansion, we get:

$$\mathbb{E}[U(x)] \approx U(\bar{x}) + \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2$$

- Since $\mathbb{E}[U(x)] = U(x_{CE})$, the above two expressions are $\approx$. Hence,

$$U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \approx \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2$$

Absolute & Relative Risk-Aversion

- From the last equation on the previous slide, Absolute Risk-Premium

$$\pi_A = \bar{x} - x_{CE} \approx -\frac{1}{2} \cdot \frac{U''(\bar{x})}{U'(\bar{x})} \cdot \sigma_x^2$$

- We refer to function $A(x) = -\frac{U''(x)}{U'(x)}$ as the **Absolute Risk-Aversion**

$$\pi_A \approx \frac{1}{2} \cdot A(\bar{x}) \cdot \sigma_x^2$$

- In multiplicative uncertainty settings, we focus on variance $\sigma^2_{\frac{x}{\bar{x}}}$ of $\frac{x}{\bar{x}}$
- In multiplicative settings, we also focus on Relative Risk-Premium π_R

$$\pi_R = \frac{\pi_A}{\bar{x}} \approx -\frac{1}{2} \cdot \frac{U''(\bar{x}) \cdot \bar{x}}{U'(\bar{x})} \cdot \frac{\sigma_x^2}{\bar{x}^2} = -\frac{1}{2} \cdot \frac{U''(\bar{x}) \cdot \bar{x}}{U'(\bar{x})} \cdot \sigma^2_{\frac{x}{\bar{x}}}$$

- We refer to function $R(x) = -\frac{U''(x) \cdot x}{U'(x)}$ as the **Relative Risk-Aversion**

$$\pi_R \approx \frac{1}{2} \cdot R(\bar{x}) \cdot \sigma^2_{\frac{x}{\bar{x}}}$$

Taking stock of what we're learning here

- We've shown that Risk-Premium can be expressed as the product of:
  - Extent of Risk-Aversion: either $A(\bar{x})$ or $R(\bar{x})$
  - Extent of uncertainty of outcome: either σ_x^2 or $\sigma_{\frac{x}{\bar{x}}}^2$
- We've expressed the extent of Risk-Aversion as the ratio of:
  - Concavity of the Utility function (at $\bar{x}$): $-U''(\bar{x})$
  - Slope of the Utility function (at $\bar{x}$): $U'(\bar{x})$
- For optimization problems, we ought to maximize $\mathbb{E}[U(x)]$ (not $\mathbb{E}[x]$)
- Linear Utility function $U(x) = a + b \cdot x$ implies *Risk-Neutrality*
- Now we look at typically-used Utility functions $U(\cdot)$ with:
  - Constant Absolute Risk-Aversion (CARA)
  - Constant Relative Risk-Aversion (CRRA)

Constant Absolute Risk-Aversion (CARA)

- Consider the Utility function $U(x) = \frac{-e^{-ax}}{a}$ for $a \neq 0$
- Absolute Risk-Aversion $A(x) = \frac{-U''(x)}{U'(x)} = a$
- a is called Coefficient of Constant Absolute Risk-Aversion (CARA)
- For $a = 0$, $U(x) = x$ (note: $A(x) = \frac{-U''(x)}{U'(x)} = 0$)
- If the random outcome $x \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}[U(x)] = \begin{cases} \frac{-e^{-a\mu + \frac{a^2\sigma^2}{2}}}{a} & \text{for } a \neq 0 \\ \mu & \text{for } a = 0 \end{cases}$$

$$x_{CE} = \mu - \frac{a\sigma^2}{2}$$

$$\text{Absolute Risk Premium } \pi_A = \mu - x_{CE} = \frac{a\sigma^2}{2}$$

- For optimization problems where σ^2 is a function of μ, we seek the distribution that maximizes $\mu - \frac{a\sigma^2}{2}$

A Portfolio Application of CARA

- We are given \$1 to invest and hold for a horizon of 1 year
- Investment choices are 1 risky asset and 1 riskless asset
- Risky Asset Annual Return $\sim \mathcal{N}(\mu, \sigma^2)$
- Riskless Asset Annual Return $= r$
- Determine unconstrained π to allocate to risky asset ($1 - \pi$ to riskless)
- Such that Portfolio has maximum Utility of Wealth in 1 year
- With CARA Utility $U(W) = \frac{-e^{-aW}}{a}$ for $a \neq 0$
- Portfolio Wealth $W \sim \mathcal{N}(1 + r + \pi(\mu - r), \pi^2\sigma^2)$
- From the section on CARA Utility, we know we need to maximize:

$$1 + r + \pi(\mu - r) - \frac{a\pi^2\sigma^2}{2}$$

- So optimal investment fraction in risky asset

$$\pi^* = \frac{\mu - r}{a\sigma^2}$$

Constant Relative Risk-Aversion (CRRA)

- Consider the Utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$
- Relative Risk-Aversion $R(x) = \frac{-U''(x) \cdot x}{U'(x)} = \gamma$
- γ is called Coefficient of Constant Relative Risk-Aversion (CRRA)
- For $\gamma = 1$, $U(x) = \log(x)$ (note: $R(x) = \frac{-U''(x) \cdot x}{U'(x)} = 1$)
- If the random outcome x is lognormal, with $\log(x) \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}[U(x)] = \begin{cases} \frac{e^{\mu(1-\gamma) + \frac{\sigma^2}{2}(1-\gamma)^2}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \mu & \text{for } \gamma = 1 \end{cases}$$

$$x_{CE} = e^{\mu + \frac{\sigma^2}{2}(1-\gamma)}$$

$$\text{Relative Risk Premium } \pi_R = 1 - \frac{x_{CE}}{\bar{x}} = 1 - e^{-\frac{\sigma^2 \gamma}{2}}$$

- For optimization problems where σ^2 is a function of μ, we seek the distribution that maximizes $\mu + \frac{\sigma^2}{2}(1-\gamma)$

A Portfolio Application of CRRA (Merton 1969)

- We work in the setting of Merton's 1969 Portfolio problem
- We only consider the single-period (static) problem with 1 risky asset
- Riskless asset: $dR_t = r \cdot R_t \cdot dt$
- Risky asset: $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$ (i.e. Geometric Brownian)
- We are given \$1 to invest, with continuous rebalancing for 1 year
- Determine constant fraction π of W_t to allocate to risky asset
- To maximize Expected Utility of Wealth $W = W_1$ (at time $t = 1$)
- Constraint: Portfolio is continuously rebalanced to maintain fraction π
- So, the process for wealth W_t is given by:

$$dW_t = (r + \pi(\mu - r)) \cdot W_t \cdot dt + \pi \cdot \sigma \cdot W_t \cdot dz_t$$

- Assume CRRA Utility $U(W) = \frac{W^{1-\gamma}}{1-\gamma}, 0 < \gamma \neq 1$

Recovering Merton's solution (for this static case)

Applying Ito's Lemma on $\log W_t$ gives us:

$$\log W_t = \int_0^t (r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}) \cdot du + \int_0^t \pi \cdot \sigma \cdot dz_u$$

$$\Rightarrow \log W \sim \mathcal{N}(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}, \pi^2 \sigma^2)$$

From the section on CRRA Utility, we know we need to maximize:

$$r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2} + \frac{\pi^2 \sigma^2 (1 - \gamma)}{2}$$

$$= r + \pi(\mu - r) - \frac{\pi^2 \sigma^2 \gamma}{2}$$

So optimal investment fraction in risky asset

$$\pi^* = \frac{\mu - r}{\gamma \sigma^2}$$