HJB Equation and Merton's Portfolio Problem

Ashwin Rao

ICME, Stanford University

January 22, 2020

Overview

Informal Problem Statement

- Assume: Current wealth is $W_0 > 0$, and you'll live for T more years
- You can invest in (allocate to) n risky assets and a riskless asset
- Each risky asset has known normal distribution of returns
- Allowed to long or short any fractional quantities of assets
- Trading in continuous time $0 \leq t < T$, with no transaction costs
- You can consume any fractional amount of wealth at any time
- Dynamic Decision: Optimal Allocation and Consumption at each time
- To maximize lifetime-aggregated utility of consumption
- Consumption Utility assumed to have constant Relative Risk-Aversion

Problem Notation

For simplicity, we state and solve the problem for 1 risky asset but the solution generalizes easily to n risky assets.

- Riskless asset: $dR_t = r \cdot R_t \cdot dt$
- Risky asset: $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$ (i.e. Geometric Brownian)
- $\mu > r > 0, \sigma > 0$ (for n assets, we work with a covariance matrix)
- Wealth at time t is denoted by $W_t > 0$
- Fraction of wealth allocated to risky asset denoted by $\pi(t, W_t)$
- Fraction of wealth in riskless asset will then be $1 - \pi(t, W_t)$
- Wealth consumption per unit time denoted by $c(t, W_t) \geq 0$
- Utility of Consumption function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for $0 < \gamma \neq 1$
- Utility of Consumption function $U(x) = \log(x)$ for $\gamma = 1$
- $\gamma =$ (constant) Relative Risk-Aversion $\frac{-x \cdot U''(x)}{U'(x)}$

Problem Statement

- We write π_t, c_t instead of $\pi(t, W_t), c(t, W_t)$ to lighten notation
- Balance constraint implies the following process for Wealth W_t

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$$

- At any time t, determine optimal $[\pi(t, W_t), c(t, W_t)]$ to maximize:

$$E[\int_t^T \frac{e^{-\rho(s-t)} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds + \frac{e^{-\rho(T-t)} \cdot B(T) \cdot W_T^{1-\gamma}}{1-\gamma} \mid W_t]$$

- where $\rho \geq 0$ is the utility discount rate, $B(T)$ is the bequest function
- We can solve this problem for arbitrary bequest $B(T)$ but for simplicity, will consider $B(T) = \epsilon^\gamma$ where $0 < \epsilon \ll 1$, meaning "no bequest" (we need this ϵ-formulation for technical reasons).
- We will solve this problem for $\gamma \neq 1$ ($\gamma = 1$ is easier, hence omitted)

Continuous-Time Stochastic Control

- Think of this as a continuous-time Stochastic Control problem
- The *State* is (t, W_t)
- The *Action* is $[\pi_t, c_t]$
- The *Reward* per unit time is $U(c_t)$
- The *Return* is the usual accumulated discounted *Reward*
- Find *Policy* : $(t, W_t) \rightarrow [\pi_t, c_t]$ that maximizes the *Expected Return*
- Note: $c_t \geq 0$, but π_t is unconstrained

Optimal Discounted Value Function

- Instead of the usual Value Function (*Expected Return* from a given *State*), we consider the Discounted Value Function
- Discounted Value Function is simply the Value Function further discounted to time 0
- We focus on the Optimal Discounted Value Function $V^*(t, W_t)$

$$V^*(t, W_t) = \max_{\pi_t, c_t} E[\int_t^T \frac{e^{-\rho s} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds + \frac{e^{-\rho T} \cdot \epsilon^\gamma \cdot W_T^{1-\gamma}}{1-\gamma}]$$

- $V^*(t, W_t)$ satisfies a simple recursive formulation for $0 \leq t < t_1 < T$.

$$V^*(t, W_t) = \max_{\pi_t, c_t} E[V^*(t_1, W_{t_1}) + \int_t^{t_1} \frac{e^{-\rho s} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds]$$

HJB Equation for Optimal Discounted Value Function

Rewriting in stochastic differential form, we have the HJB formulation

$$\max_{\pi_t, c_t} E[dV^*(t, W_t) + \frac{e^{-\rho t} \cdot c_t^{1-\gamma}}{1-\gamma} \cdot dt] = 0$$

Use Ito's Lemma on dV^*, remove the dz_t term since it's a martingale, and divide throughout by dt to produce the HJB Equation in PDE form:

$$\max_{\pi_t, c_t} [\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}((\pi_t(\mu - r) + r)W_t - c_t) + \frac{\partial^2 V^*}{\partial W_t^2} \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \frac{e^{-\rho t} \cdot c_t^{1-\gamma}}{1-\gamma}] = 0$$

Let us write the above equation more succinctly as:

$$\max_{\pi_t, c_t} \Phi(t, W_t; \pi_t, c_t) = 0$$

Note: we are working with the constraints $W_t > 0, c_t \geq 0$ for $0 \leq t < T$

Optimal Allocation and Consumption

Find optimal π_t^*, c_t^* by taking partial derivatives of $\Phi(t, W_t; \pi_t, c_t)$ with respect to π_t and c_t, and equate to 0 (first-order conditions for Φ).

- Partial derivative of Φ with respect to π_t:

$$(\mu - r) \cdot \frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \cdot \pi_t \cdot \sigma^2 \cdot W_t = 0$$

$$\Rightarrow \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t} \cdot (\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \cdot \sigma^2 \cdot W_t}$$

- Partial derivative of Φ with respect to c_t:

$$-\frac{\partial V^*}{\partial W_t} + e^{-\rho t} \cdot (c_t^*)^{-\gamma} = 0$$

$$\Rightarrow c_t^* = (\frac{\partial V^*}{\partial W_t} \cdot e^{\rho t})^{\frac{-1}{\gamma}}$$

Optimal Discounted Value Function PDE

Now substitute π_t^* and c_t^* in $\Phi(t, W_t; \pi_t, c_t)$ and set it to 0, which gets us the Optimal Discounted Value Function PDE:

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{(\frac{\partial V^*}{\partial W_t})^2}{\frac{\partial^2 V^*}{\partial W_t^2}} + \frac{\partial V^*}{\partial W_t} \cdot r \cdot W_t + \frac{\gamma}{1-\gamma} \cdot e^{\frac{-\rho t}{\gamma}} \cdot (\frac{\partial V^*}{\partial W_t})^{\frac{\gamma - 1}{\gamma}} = 0$$

The boundary condition is:

$$V^*(T, W_T) = e^{-\rho T} \cdot \epsilon^\gamma \cdot \frac{W_T^{1-\gamma}}{1-\gamma}$$

The second-order conditions for Φ are satisfied **under the assumptions** $c_t^* > 0, W_t > 0, \frac{\partial^2 V^*}{\partial W_t^2} < 0$ for all $0 \leq t < T$ (we will later show that these are all satisfied in the solution we derive), and for concave $U(\cdot)$, i.e., $\gamma > 0$

Solving the PDE with a guess solution

We surmise with a guess solution

$$V^*(t, W_t) = f(t)^\gamma \cdot e^{-\rho t} \cdot \frac{W_t^{1-\gamma}}{1-\gamma}$$

Then,

$$\frac{\partial V^*}{\partial t} = (\gamma \cdot f(t)^{\gamma-1} \cdot f'(t) - \rho \cdot f(t)^\gamma) \cdot e^{-\rho t} \cdot \frac{W_t^{1-\gamma}}{1-\gamma}$$

$$\frac{\partial V^*}{\partial W_t} = f(t)^\gamma \cdot e^{-\rho t} \cdot W_t^{-\gamma}$$

$$\frac{\partial^2 V^*}{\partial W_t^2} = -f(t)^\gamma \cdot e^{-\rho t} \cdot \gamma \cdot W_t^{-\gamma-1}$$

PDE reduced to an ODE

Substituting the guess solution in the PDE, we get the simple ODE:

$$f'(t) = \nu \cdot f(t) - 1$$

where

$$\nu = \frac{\rho - (1-\gamma) \cdot (\frac{(\mu-r)^2}{2\sigma^2\gamma} + r)}{\gamma}$$

with boundary condition $f(T) = \epsilon$.
The solution to this ODE is:

$$f(t) = \begin{cases} \frac{1+(\nu\epsilon - 1) \cdot e^{-\nu(T-t)}}{\nu} & \text{for } \nu \neq 0 \\ T - t + \epsilon & \text{for } \nu = 0 \end{cases}$$

Optimal Allocation and Consumption

Putting it all together (substituting the solution for $f(t)$), we get:

$$\pi^*(t, W_t) = \frac{\mu - r}{\sigma^2 \gamma}$$

$$c^*(t, W_t) = \frac{W_t}{f(t)} = \begin{cases} \frac{\nu \cdot W_t}{1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)}} & \text{for } \nu \neq 0 \\ \frac{W_t}{T-t+\epsilon} & \text{for } \nu = 0 \end{cases}$$

$$V^*(t, W_t) = \begin{cases} e^{-\rho t} \cdot \frac{(1+(\nu\epsilon-1)\cdot e^{-\nu(T-t)})^\gamma}{\nu^\gamma} \cdot \frac{W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu \neq 0 \\ e^{-\rho t} \cdot \frac{(T-t+\epsilon)^\gamma \cdot W_t^{1-\gamma}}{1-\gamma} & \text{for } \nu = 0 \end{cases}$$

- $f(t) > 0$ for all $0 \leq t < T$ (for all ν) ensures $W_t, c_t^* > 0, \frac{\partial^2 V^*}{\partial W_t^2} < 0$. This ensures the constraints $W_t > 0$ and $c_t \geq 0$ are satisfied and the second-order conditions for Φ are also satisfied.
- The HJB Formulation was key and this solution approach provides a template for similar continuous-time stochastic control problems.

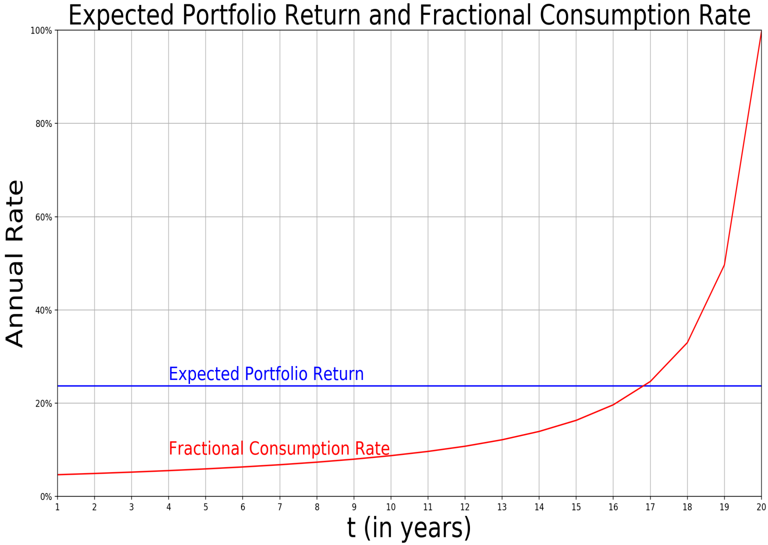
Gaining Insights into the Solution

- Optimal Allocation $\pi^*(t, W_t)$ is constant (independent of t and W_t)
- Optimal Fractional Consumption $\frac{c^*(t,W_t)}{W_t}$ depends only on t $(= \frac{1}{f(t)})$
- With Optimal Allocation & Consumption, the Wealth process is:

$$\frac{dW_t}{W_t} = (r + \frac{(\mu - r)^2}{\sigma^2 \gamma} - \frac{1}{f(t)}) \cdot dt + \frac{\mu - r}{\sigma \gamma} \cdot dz_t$$

- Expected Portfolio Return is constant over time $(= r + \frac{(\mu-r)^2}{\sigma^2\gamma})$
- Assuming $\epsilon < \frac{1}{\nu}$, Fractional Consumption $\frac{1}{f(t)}$ increases over time
- Expected Rate of Wealth Growth $r + \frac{(\mu-r)^2}{\sigma^2\gamma} - \frac{1}{f(t)}$ decreases over time
- If $r + \frac{(\mu-r)^2}{\sigma^2\gamma} > \frac{1}{f(0)}$, we start by Consuming $<$ Expected Portfolio Growth and over time, we Consume $>$ Expected Portfolio Growth
- Wealth Growth Volatility is constant $(= \frac{\mu-r}{\sigma\gamma})$

Portfolio Return versus Consumption Rate



Porting this to Real-World Portfolio Optimization

- Analytical tractability in Merton's formulation was due to:
  - Normal distribution of asset returns
  - Constant Relative Risk-Aversion
  - Frictionless, continuous trading
- However, real-world situation involves:
  - Discrete amounts of assets to hold and discrete quantities of trades
  - Transaction costs
  - Locked-out days for trading
  - Non-stationary/arbitrary/correlated processes of multiple assets
  - Changing/uncertain risk-free rate
  - Consumption constraints
  - Arbitrary Risk-Aversion/Utility specification
- $\Rightarrow$ Approximate Dynamic Programming or Reinforcement Learning
- Large Action Space points to Policy Gradient Algorithms