Understanding Risk-Aversion through Utility Theory

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Intuition on Risk-Aversion and Risk-Premium

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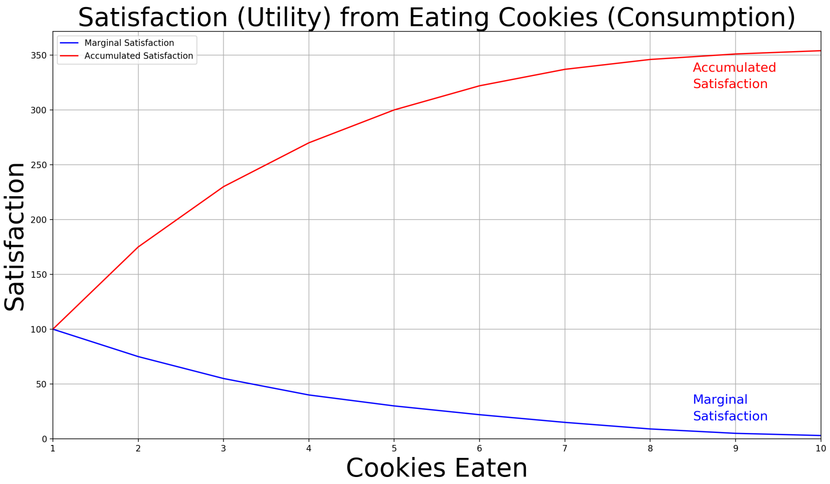
- We will illustrate this concept with a real-life example

Law of Diminishing Marginal Utility



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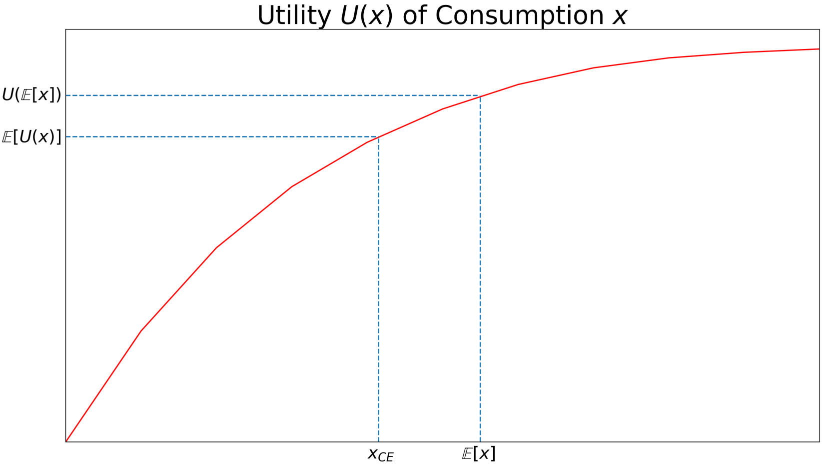

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- Taylor-expand $U(x)$ around $\bar{x}$, ignoring terms beyond quadratic

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- Since $\mathbb{E}[U(x)] = U(x_{CE})$, the above two expressions are $\approx$. Hence,

$$U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \approx \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2$$

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- a is called Coefficient of Constant Absolute Risk-Aversion (CARA)
- If the random outcome $x \sim \mathcal{N}(\mu, \sigma^2)$,

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- For optimization problems where σ^2 is a function of μ, we seek the distribution that (approximately) maximizes $\mu - \frac{a\sigma^2}{2}$

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$$\mathbb{E}[U(x)] = \begin{cases} \frac{e^{\mu(1-\gamma) + \frac{\sigma^2}{2}(1-\gamma)^2}}{1-\gamma} & \text{for } \gamma \neq 1 \\ \mu & \text{for } \gamma = 1 \end{cases}$$

$$x_{CE} = e^{\mu + \frac{\sigma^2}{2}(1-\gamma)}$$

$$\text{Relative Risk Premium } \pi_R = 1 - \frac{x_{CE}}{\bar{x}} = 1 - e^{-\frac{\sigma^2 \gamma}{2}}$$

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$$\Rightarrow \pi = \frac{\mu - r}{\sigma^2\gamma}$$