

# A Simple and Intuitive Coverage of The Fundamental Theorems of Asset Pricing

Ashwin Rao

ICME, Stanford University

August 28, 2019

# Overview

- 1 Simple Setting for Intuitive Understanding
- 2 Portfolios, Arbitrage and Risk-Neutral Measure
- 3 First Fundamental Theorem of Asset Pricing
- 4 Derivatives, Replicating Portfolios and Hedges
- 5 Second Fundamental Theorem of Asset Pricing
- 6 Derivatives Pricing
- 7 Examples
- 8 Summary and General Theory

# Simple Setting for Intuitive Understanding

- Single-period setting (two time points  $t = 0$  and  $t = 1$ )
- $t = 0$  has a single state (we'll call it "Spot" state)
- $t = 1$  has  $n$  random states represented by  $\Omega = \{\omega_1, \dots, \omega_n\}$
- With probability distribution  $\mu : \Omega \rightarrow [0, 1]$ , i.e,  $\sum_{i=1}^n \mu(\omega_i) = 1$
- $m + 1$  fundamental assets  $A_0, A_1, \dots, A_m$
- Spot Price (at  $t = 0$ ) of  $A_j$  denoted  $S_j^{(0)}$  for all  $j = 0, 1, \dots, m$
- Price of  $A_j$  in state  $\omega_i$  denoted  $S_j^{(i)}$  for all  $j = 0, \dots, m, i = 1, \dots, n$
- All asset prices are assumed to be positive real numbers, i.e. in  $\mathbb{R}^+$
- $A_0$  is a special asset known as risk-free asset with  $S_0^{(0)}$  normalized to 1
- $S_0^{(i)} = e^r$  for all  $i = 1, \dots, n$  where  $r$  is the constant risk-free rate
- $e^{-r}$  is the risk-free discount factor to represent "time value of money"

- A portfolio is a vector  $\theta = (\theta_0, \theta_1, \dots, \theta_m) \in \mathbb{R}^{m+1}$
- $\theta_j$  is the number of units held in asset  $A_j$  for all  $j = 0, 1, \dots, m$
- Spot Value (at  $t = 0$ ) of portfolio  $\theta$  denoted  $V_\theta^{(0)}$  is:

$$V_\theta^{(0)} = \sum_{j=0}^m \theta_j \cdot S_j^{(0)}$$

- Value of portfolio  $\theta$  in state  $\omega_i$  (at  $t = 1$ ) denoted  $V_\theta^{(i)}$  is:

$$V_\theta^{(i)} = \sum_{j=0}^m \theta_j \cdot S_j^{(i)} \text{ for all } i = 1, \dots, n$$

# Arbitrage Portfolio

- An Arbitrage Portfolio  $\theta$  is one that “makes money from nothing”
- Formally, a portfolio  $\theta$  such that:
  - $V_{\theta}^{(0)} \leq 0$
  - $V_{\theta}^{(i)} \geq 0$  for all  $i = 1, \dots, n$
  - $\exists i$  in  $1, \dots, n$  such that  $\mu(\omega_i) > 0$  and  $V_{\theta}^{(i)} > 0$
- So we never end with less value than what we start with and we end with expected value greater than what we start with
- Arbitrage allows market participants to make infinite returns
- In an efficient market, arbitrage disappears as participants exploit it
- Hence, Finance Theory typically assumes “arbitrage-free” markets

# Risk-Neutral Probability Measure

- Consider a Probability Distribution  $\pi : \Omega \rightarrow [0, 1]$  such that

$$\pi(\omega_i) = 0 \text{ if and only if } \mu(\omega_i) = 0 \text{ for all } i = 1, \dots, n$$

- Then,  $\pi$  is a Risk-Neutral Probability Measure if:

$$S_j^{(0)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot S_j^{(i)} \text{ for all } j = 0, 1, \dots, m \quad (1)$$

- So for each of the  $m + 1$  assets, the asset spot price (at  $t = 0$ ) is the discounted expectation (under  $\pi$ ) of the asset price at  $t = 1$
- $\pi$  is an artificial construct to connect expectation of asset prices at  $t = 1$  to their spot prices by the risk-free discount factor  $e^{-r}$

# 1st Fundamental Theorem of Asset Pricing (1st FTAP)

## Theorem

*1st FTAP: Our simple setting will not admit arbitrage portfolios if and only if there exists a Risk-Neutral Probability Measure.*

- First we prove the easy implication:  
Existence of Risk-Neutral Measure  $\Rightarrow$  Arbitrage-free
- Assume there is a risk-neutral measure  $\pi$
- Then, for each portfolio  $\theta = (\theta_0, \theta_1, \dots, \theta_m)$ ,

$$\begin{aligned} V_{\theta}^{(0)} &= \sum_{j=0}^m \theta_j \cdot S_j^{(0)} = \sum_{j=0}^m \theta_j \cdot e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot S_j^{(i)} \\ &= e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot \sum_{j=0}^m \theta_j \cdot S_j^{(i)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot V_{\theta}^{(i)} \end{aligned}$$

# 1st Fundamental Theorem of Asset Pricing (1st FTAP)

- So the portfolio spot value is the discounted expectation (under  $\pi$ ) of the portfolio value at  $t = 1$
- For any portfolio  $\theta$ , if the following two conditions are satisfied:
  - $V_{\theta}^{(i)} \geq 0$  for all  $i = 1, \dots, n$
  - $\exists i$  in  $1, \dots, n$  such that  $\mu(\omega_i) > 0 (\Rightarrow \pi(\omega_i) > 0)$  and  $V_{\theta}^{(i)} > 0$

Then,

$$V_{\theta}^{(0)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot V_{\theta}^{(i)} > 0$$

- This eliminates the possibility of arbitrage for any portfolio  $\theta$
- The other implication (Arbitrage-free  $\Rightarrow$  Existence of Risk-Neutral Measure) is harder to prove and covered in Appendix 1



# Derivatives, Replicating Portfolios and Hedges

- A Derivative  $D$  (in this simple setting) is a vector payoff at  $t = 1$ :

$$(V_D^{(1)}, V_D^{(2)}, \dots, V_D^{(n)})$$

where  $V_D^{(i)}$  is the payoff of the derivative in state  $\omega_i$  for all  $i = 1, \dots, n$

- Portfolio  $\theta \in \mathbb{R}^{m+1}$  is a *Replicating Portfolio* for derivative  $D$  if:

$$V_D^{(i)} = \sum_{j=0}^m \theta_j \cdot S_j^{(i)} \text{ for all } i = 1, \dots, n \quad (2)$$

- The components  $(\theta_0, \theta_1, \dots, \theta_m)$  are known as the *hedges* for  $D$  since they can be used to offset the risk in the payoff of  $D$  at  $t = 1$

# 2nd Fundamental Theorem of Asset Pricing (2nd FTAP)

An arbitrage-free market is said to be *Complete* if every derivative in the market has a replicating portfolio.

## Theorem

*2nd FTAP: A market is Complete in our simple setting if and only if there is a unique risk-neutral probability measure.*

Proof in Appendix 2. Together, the FTAPs classify markets into:

- ① Complete (arbitrage-free) market  $\Leftrightarrow$  Unique risk-neutral measure
- ② Market with arbitrage  $\Leftrightarrow$  No risk-neutral measure
- ③ Incomplete (arbitrage-free) market  $\Leftrightarrow$  Multiple risk-neutral measures

The next topic is derivatives pricing that is based on the concepts of *replication of derivatives* and *risk-neutral measures*, and so is tied to the concepts of *arbitrage* and *completeness*.

# Positions involving a Derivative

- Before getting into Derivatives Pricing, we need to define a *Position*
- We define a *Position* involving a derivative  $D$  as the combination of holding some units in  $D$  and some units in  $A_0, A_1, \dots, A_m$
- *Position* is an extension of the Portfolio concept including a derivative
- Formally denoted as  $\gamma_D = (\alpha, \theta_0, \theta_1, \dots, \theta_m) \in \mathbb{R}^{m+2}$
- $\alpha$  is the units held in derivative  $D$
- $\theta_j$  is the units held in  $A_j$  for all  $j = 0, 1, \dots, m$
- Extend the definition of Portfolio Value to Position Value
- Extend the definition of Arbitrage Portfolio to Arbitrage Position

# Derivatives Pricing: Elimination of candidate prices

- We will consider candidate prices (at  $t = 0$ ) for a derivative  $D$
- Let  $\theta = (\theta_0, \theta_1, \dots, \theta_m)$  be a replicating portfolio for  $D$
- Consider the candidate price  $\sum_{j=0}^m \theta_j \cdot S_j^{(0)} - x$  for  $D$  for any  $x > 0$
- Position  $(1, -\theta_0 + x, -\theta_1, \dots, -\theta_m)$  has value  $x \cdot e^r > 0$  in each of the states at  $t = 1$
- But this position has spot ( $t = 0$ ) value of 0, which means this is an Arbitrage Position, rendering this candidate price invalid
- Consider the candidate price  $\sum_{j=0}^m \theta_j \cdot S_j^{(0)} + x$  for  $D$  for any  $x > 0$
- Position  $(-1, \theta_0 + x, \theta_1, \dots, \theta_m)$  has value  $x \cdot e^r > 0$  in each of the states at  $t = 1$
- But this position has spot ( $t = 0$ ) value of 0, which means this is an Arbitrage Position, rendering this candidate price invalid
- So every candidate price for  $D$  other than  $\sum_{j=0}^m \theta_j \cdot S_j^{(0)}$  is invalid

# Derivatives Pricing: Remaining candidate price

- Having eliminated various candidate prices for  $D$ , we now aim to *establish* the remaining candidate price:

$$V_D^{(0)} = \sum_{j=0}^m \theta_j \cdot S_j^{(0)} \quad (3)$$

where  $\theta = (\theta_0, \theta_1, \dots, \theta_m)$  is a replicating portfolio for  $D$

- To eliminate prices, our only assumption was that  $D$  can be replicated
- This can happen in a complete market or in an arbitrage market
- To establish remaining candidate price  $V_D^{(0)}$ , we need to assume market is complete, i.e., there is a unique risk-neutral measure  $\pi$
- Candidate price  $V_D^{(0)}$  can be expressed as the discounted expectation (under  $\pi$ ) of the payoff of  $D$  at  $t = 1$ , i.e.,

$$V_D^{(0)} = \sum_{j=0}^m \theta_j \cdot e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot S_j^{(i)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot V_D^{(i)} \quad (4)$$

# Derivatives Pricing: Establishing remaining candidate price

- Now consider an *arbitrary portfolio*  $\beta = (\beta_0, \beta_1, \dots, \beta_m)$
- Define a position  $\gamma_D = (\alpha, \beta_0, \beta_1, \dots, \beta_m)$
- Spot Value (at  $t = 0$ ) of position  $\gamma_D$  denoted  $V_{\gamma_D}^{(0)}$  is:

$$V_{\gamma_D}^{(0)} = \alpha \cdot V_D^{(0)} + \sum_{j=0}^m \beta_j \cdot S_j^{(0)} \quad (5)$$

where  $V_D^{(0)}$  is the remaining candidate price

- Value of position  $\gamma_D$  in state  $\omega_i$  (at  $t = 1$ ), denoted  $V_{\gamma_D}^{(i)}$ , is:

$$V_{\gamma_D}^{(i)} = \alpha \cdot V_D^{(i)} + \sum_{j=0}^m \beta_j \cdot S_j^{(i)} \text{ for all } i = 1, \dots, n \quad (6)$$

- Combining the linearity in equations (1), (4), (5), (6), we get:

$$V_{\gamma_D}^{(0)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot V_{\gamma_D}^{(i)} \quad (7)$$

# Derivatives Pricing: Establishing remaining candidate price

- So the position spot value is the discounted expectation (under  $\pi$ ) of the position value at  $t = 1$
- For any  $\gamma_D$  (containing any arbitrary portfolio  $\beta$ ) and with  $V_D^{(0)}$  as the candidate price for  $D$ , if the following two conditions are satisfied:
  - $V_{\gamma_D}^{(i)} \geq 0$  for all  $i = 1, \dots, n$
  - $\exists i$  in  $1, \dots, n$  such that  $\mu(\omega_i) > 0 (\Rightarrow \pi(\omega_i) > 0)$  and  $V_{\gamma_D}^{(i)} > 0$

Then,

$$V_{\gamma_D}^{(0)} = e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot V_{\gamma_D}^{(i)} > 0$$

- This eliminates arbitrage possibility for remaining candidate price  $V_D^{(0)}$
- So we have eliminated all prices other than  $V_D^{(0)}$ , and we have established the price  $V_D^{(0)}$ , proving that it should be *the* price of  $D$
- The above arguments assumed a complete market, but what about an incomplete market or a market with arbitrage?

# Incomplete Market (Multiple Risk-Neutral Measures)

- Recall: Incomplete market means some derivatives can't be replicated
- Absence of replicating portfolio precludes usual arbitrage arguments
- 2nd FTAP says there are multiple risk-neutral measures
- So, multiple derivative prices (each consistent with no-arbitrage)
- The market “chooses” a risk-neutral measure
- ... reflecting the risk preference of the market
- ... that determines the market price of the derivative
- So we calibrate a model of “market-implied” risk-neutral measure



# Multiple Replicating Portfolios (Arbitrage Market)

- Assume there are replicating portfolios  $\alpha$  and  $\beta$  for  $D$  with

$$\sum_{j=0}^m \alpha_j \cdot S_j^{(0)} - \sum_{j=0}^m \beta_j \cdot S_j^{(0)} = x > 0$$

- Consider portfolio  $\theta = (\beta_0 - \alpha_0 + x, \beta_1 - \alpha_1, \dots, \beta_m - \alpha_m)$

$$V_{\theta}^{(0)} = \sum_{j=0}^m (\beta_j - \alpha_j) \cdot S_j^{(0)} + x \cdot S_0^{(0)} = -x + x = 0$$

$$V_{\theta}^{(i)} = \sum_{j=0}^m (\beta_j - \alpha_j) \cdot S_j^{(i)} + x \cdot S_0^{(i)} = x \cdot e^r > 0 \text{ for all } i = 1, \dots, n$$

- So  $\theta$  is an Arbitrage Portfolio  $\Rightarrow$  market with no risk-neutral measure
- Also note from previous elimination argument that every candidate price other than  $\sum_{j=0}^m \alpha_j \cdot S_j^{(0)}$  is invalid and every candidate price other than  $\sum_{j=0}^m \beta_j \cdot S_j^{(0)}$  is invalid, so  $D$  has no valid price at all

# Market with 2 states and 1 Risky Asset

- Consider a market with  $m = 1$  and  $n = 2$
- Assume  $S_1^{(1)} < S_1^{(2)}$
- No-arbitrage requires  $S_1^{(1)} \leq S_1^{(0)} \cdot e^r \leq S_1^{(2)}$
- Assuming absence of arbitrage and invoking 1st FTAP, there exists a risk-neutral probability measure  $\pi$  such that:

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)})$$

$$\pi(\omega_1) + \pi(\omega_2) = 1$$

- This implies:

$$\pi(\omega_1) = \frac{S_1^{(2)} - S_1^{(0)} \cdot e^r}{S_1^{(2)} - S_1^{(1)}}$$

$$\pi(\omega_2) = \frac{S_1^{(0)} \cdot e^r - S_1^{(1)}}{S_1^{(2)} - S_1^{(1)}}$$

# Market with 2 states and 1 Risky Asset (continued)

- We can use these probabilities to price a derivative  $D$  as:

$$V_D^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot V_D^{(1)} + \pi(\omega_2) \cdot V_D^{(2)})$$

- Now let us try to form a replicating portfolio  $(\theta_0, \theta_1)$  for  $D$

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)}$$

$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)}$$

- Solving this yields hedges  $(\theta_0, \theta_1)$  as follows:

$$\theta_0 = e^{-r} \cdot \frac{V_D^{(1)} \cdot S_1^{(2)} - V_D^{(2)} \cdot S_1^{(1)}}{S_1^{(2)} - S_1^{(1)}} \text{ and } \theta_1 = \frac{V_D^{(2)} - V_D^{(1)}}{S_1^{(2)} - S_1^{(1)}}$$

- This means this is a Complete Market
- Note that the derivative price can also be expressed as:

$$V_D^{(0)} = \theta_0 + \theta_1 \cdot S_1^{(0)}$$

# Market with 3 states and 1 Risky Asset

- Consider a market with  $m = 1$  and  $n = 3$
- Assume  $S_1^{(1)} < S_1^{(2)} < S_1^{(3)}$
- No-arbitrage requires  $S_1^{(1)} \leq S_1^{(0)} \cdot e^r \leq S_1^{(3)}$
- Assuming absence of arbitrage and invoking 1st FTAP, there exists a risk-neutral probability measure  $\pi$  such that:

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)} + \pi(\omega_3) \cdot S_1^{(3)})$$

$$\pi(\omega_1) + \pi(\omega_2) + \pi(\omega_3) = 1$$

- 2 equations & 3 variables  $\Rightarrow$  multiple solutions for  $\pi$
- Each of these solutions for  $\pi$  provides a valid price for a derivative  $D$

$$V_D^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot V_D^{(1)} + \pi(\omega_2) \cdot V_D^{(2)} + \pi(\omega_3) \cdot V_D^{(3)})$$

# Market with 3 states and 1 Risky Asset (continued)

- Now let us try to form a replicating portfolio  $(\theta_0, \theta_1)$  for  $D$

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)}$$

$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)}$$

$$V_D^{(3)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(3)}$$

- 3 equations & 2 variables  $\Rightarrow$  no replication for *some*  $D$
- This means this is an Incomplete Market
- Don't forget that we have multiple risk-neutral probability measures
- Meaning we have multiple valid prices for derivatives

# Market with 2 states and 2 Risky Assets

- Consider a market with  $m = 2$  and  $n = 3$
- Assume  $S_1^{(1)} < S_1^{(2)}$  and  $S_2^{(1)} < S_2^{(2)}$
- Let us try to determine a risk-neutral probability measure  $\pi$ :

$$S_1^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_1^{(1)} + \pi(\omega_2) \cdot S_1^{(2)})$$

$$S_2^{(0)} = e^{-r} \cdot (\pi(\omega_1) \cdot S_2^{(1)} + \pi(\omega_2) \cdot S_2^{(2)})$$

$$\pi(\omega_1) + \pi(\omega_2) = 1$$

- 3 equations & 2 variables  $\Rightarrow$  no risk-neutral measure  $\pi$
- Let's try to form a replicating portfolio  $(\theta_0, \theta_1, \theta_2)$  for a derivative  $D$

$$V_D^{(1)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(1)} + \theta_2 \cdot S_2^{(1)}$$

$$V_D^{(2)} = \theta_0 \cdot e^r + \theta_1 \cdot S_1^{(2)} + \theta_2 \cdot S_2^{(2)}$$

# Market with 2 states and 2 Risky Assets (continued)

- 2 equations & 3 variables  $\Rightarrow$  multiple replicating portfolios
- Each such replicating portfolio yields a price for  $D$  as:

$$V_D^{(0)} = \theta_0 + \theta_1 \cdot S_1^{(0)} + \theta_2 \cdot S_2^{(0)}$$

- Select two such replicating portfolios with different  $V_D^{(0)}$
- Combination of these replicating portfolios is an Arbitrage Portfolio
  - They cancel off each other's price in each  $t = 1$  states
  - They have a combined negative price at  $t = 0$
- So this is a market that admits arbitrage (no risk-neutral measure)

3 cases:

① *Complete market*

- Unique replicating portfolio for derivatives
- Unique risk-neutral measure, meaning we have unique derivatives prices

② *Arbitrage-free but incomplete market*

- Not all derivatives can be replicated
- Multiple risk-neutral measures, meaning we can have multiple valid prices for derivatives

③ *Market with Arbitrage*

- Derivatives have multiple replicating portfolios (that when combined causes arbitrage)
- No risk-neutral measure, meaning derivatives cannot be priced



# General Theory for Derivatives Pricing

- The theory for our simple setting extends nicely to the general setting
- Instead of  $t = 0, 1$ , we consider  $t = 0, 1, \dots, T$
- The model is a “recombining tree” of state transitions across time
- The idea of Arbitrage applies over multiple time periods
- Risk-neutral measure for each state at each time period
- Over multiple time periods, we need a *Dynamic Replicating Portfolio* to rebalance asset holdings (“self-financing trading strategy”)
- We obtain hedges and prices at each time period at each state
- By making time period smaller and smaller, the model turns into a stochastic process (in continuous time)
- Classical Financial Math theory based on stochastic calculus but has essentially the same ideas we developed for our simple setting

# Appendix 1: Arbitrage-free $\Rightarrow \exists$ a risk-neutral measure

- We will prove that if a risk-neutral probability measure doesn't exist, there exists an arbitrage portfolio
- Let  $\mathbb{V} \subset \mathbb{R}^m$  be the set of vectors  $(s_1, \dots, s_m)$  such that

$$s_j = e^{-r} \cdot \sum_{i=1}^n \mu(\omega_i) \cdot S_j^{(i)} \text{ for all } j = 1, \dots, m$$

spanning over all possible probability distributions  $\mu : \Omega \rightarrow [0, 1]$

- $\mathbb{V}$  is a bounded, closed, convex polytope in  $\mathbb{R}^m$
- If a risk-neutral measure doesn't exist,  $(S_1^{(0)}, \dots, S_m^{(0)}) \notin \mathbb{V}$
- Hyperplane Separation Theorem implies that there exists a non-zero vector  $(\theta_1, \dots, \theta_m)$  such that for any  $v = (v_1, \dots, v_m) \in \mathbb{V}$ ,

$$\sum_{j=1}^m \theta_j \cdot v_j > \sum_{j=1}^m \theta_j \cdot S_j^{(0)}$$

## Appendix 1: Arbitrage-free $\Rightarrow \exists$ a risk-neutral measure

- In particular, consider vectors  $v$  corresponding to the corners of  $\mathbb{V}$ , those for which the full probability mass is on a particular  $\omega_i \in \Omega$ , i.e.,

$$\sum_{j=1}^m \theta_j \cdot (e^{-r} \cdot S_j^{(i)}) > \sum_{j=1}^m \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \dots, n$$

- Choose a  $\theta_0 \in \mathbb{R}$  such that:

$$\sum_{j=1}^m \theta_j \cdot (e^{-r} \cdot S_j^{(i)}) > -\theta_0 > \sum_{j=1}^m \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \dots, n$$

- Therefore,

$$e^{-r} \cdot \sum_{j=0}^m \theta_j \cdot S_j^{(i)} > 0 > \sum_{j=0}^m \theta_j \cdot S_j^{(0)} \text{ for all } i = 1, \dots, n$$

- This means  $(\theta_0, \theta_1, \dots, \theta_m)$  is an arbitrage portfolio



## Appendix 2: Proof of 2nd FTAP

- We will first prove that in an arbitrage-free market, if every derivative has a replicating portfolio, there is a unique risk-neutral measure  $\pi$
- We define  $n$  special derivatives (known as *Arrow-Debreu securities*), one for each random state in  $\Omega$  at  $t = 1$
- We define the time  $t = 1$  payoff of *Arrow-Debreu security*  $D_k$  (for each of  $k = 1, \dots, n$ ) in state  $\omega_i$  as  $\mathbb{I}_{i=k}$  for all  $i = 1, \dots, n$ .
- Since each derivative has a replicating portfolio, let  $\theta^{(k)} = (\theta_0^{(k)}, \theta_1^{(j)}, \dots, \theta_m^{(k)})$  be the replicating portfolio for  $D_k$ .
- With usual no-arbitrage argument, the price (at  $t = 0$ ) of  $D_k$  is

$$\sum_{j=0}^m \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \dots, n$$

## Appendix 2: Proof of 2nd FTAP

- Now let us try to solve for an unknown risk-neutral probability measure  $\pi : \Omega \rightarrow [0, 1]$ , given the above prices for  $D_k, k = 1, \dots, n$

$$e^{-r} \cdot \sum_{i=1}^n \pi(\omega_i) \cdot \mathbb{I}_{i=k} = e^{-r} \cdot \pi(\omega_k) = \sum_{j=0}^m \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \dots, n$$

$$\Rightarrow \pi(\omega_k) = e^r \cdot \sum_{j=0}^m \theta_j^{(k)} \cdot S_j^{(0)} \text{ for all } k = 1, \dots, n$$

- This yields a unique solution for the risk-neutral probability measure  $\pi$
- Next, we prove the other direction of the 2nd FTAP
- To prove: if there exists a risk-neutral measure  $\pi$  and if there exists a derivative  $D$  with no replicating portfolio, we can construct a risk-neutral measure different than  $\pi$

## Appendix 2: Proof of 2nd FTAP

- Consider the following vectors in the vector space  $\mathbb{R}^n$

$$v = (V_D^{(1)}, \dots, V_D^{(n)}) \text{ and } s_j = (S_j^{(1)}, \dots, S_j^{(n)}) \text{ for all } j = 0, 1, \dots, m$$

- Since  $D$  does not have a replicating portfolio,  $v$  is not in the span of  $s_0, s_1, \dots, s_m$ , which means  $s_0, s_1, \dots, s_m$  do not span  $\mathbb{R}^n$
- Hence  $\exists$  a non-zero vector  $u = (u_1, \dots, u_n) \in \mathbb{R}^n$  orthogonal to each of  $s_0, s_1, \dots, s_m$ , i.e.,

$$\sum_{i=1}^n u_i \cdot S_j^{(i)} = 0 \text{ for all } j = 0, 1, \dots, n \quad (8)$$

- Note that  $S_0^{(i)} = e^r$  for all  $i = 1, \dots, n$  and so,

$$\sum_{i=1}^n u_i = 0 \quad (9)$$

## Appendix 2: Proof of 2nd FTAP

- Define  $\pi' : \Omega \rightarrow \mathbb{R}$  as follows (for some  $\epsilon > 0 \in \mathbb{R}$ ):

$$\pi'(\omega_i) = \pi(\omega_i) + \epsilon \cdot u_i \text{ for all } i = 1, \dots, n \quad (10)$$

- To establish  $\pi'$  as a risk-neutral measure different than  $\pi$ , note:
  - Since  $\sum_{i=1}^n \pi(\omega_i) = 1$  and since  $\sum_{i=1}^n u_i = 0$ ,  $\sum_{i=1}^n \pi'(\omega_i) = 1$
  - Construct  $\pi'(\omega_i) > 0$  for each  $i$  where  $\pi(\omega_i) > 0$  by making  $\epsilon > 0$  sufficiently small, and set  $\pi'(\omega_i) = 0$  for each  $i$  where  $\pi(\omega_i) = 0$
  - From Eq (8) and Eq (10), we derive:

$$\sum_{i=1}^n \pi'(\omega_i) \cdot S_j^{(i)} = \sum_{i=1}^n \pi(\omega_i) \cdot S_j^{(i)} = e^r \cdot S_j^{(0)} \text{ for all } j = 0, 1, \dots, m$$

