Value Function Geometry and Gradient TD

Ashwin Rao

ICME, Stanford University

February 18, 2020

Overview

Motivation for understanding Value Function Geometry

- Helps us better understand transformations of Value Functions (VFs)
- Across the various DP and RL algorithms
- Particularly helps when VFs are approximated, esp. with linear approx
- Provides insights into stability and convergence
- Particularly when dealing with the "Deadly Triad"
- Deadly Triad := [Bootstrapping, Func Approx, Off-Policy]
- **Leads us to Gradient TD**

Notation

- Assume state space $\mathcal{S}$ consists of n states: $\{s_1, s_2, \ldots, s_n\}$
- Action space $\mathcal{A}$ consisting of finite number of actions
- This exposition extends easily to continuous state/action spaces too
- This exposition is for a fixed (often stochastic) policy denoted $\pi(a|s)$
- VF for a policy π is denoted as $\mathbf{v}_\pi : \mathcal{S} \to \mathbb{R}$
- m feature functions $\phi_1, \phi_2, \ldots, \phi_m : \mathcal{S} \to \mathbb{R}$
- Feature vector for a state $s \in \mathcal{S}$ denoted as $\phi(s) \in \mathbb{R}^m$
- For linear function approximation of VF with weights $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, VF $\mathbf{v}_\mathbf{w} : \mathcal{S} \to \mathbb{R}$ is defined as:

$$\mathbf{v}_\mathbf{w}(s) = \mathbf{w}^T \cdot \phi(s) = \sum_{j=1}^{m} w_j \cdot \phi_j(s) \text{ for any } s \in \mathcal{S}$$

- $\mu_\pi : \mathcal{S} \to [0, 1]$ denotes the states' probability distribution under π

VF Geometry and VF Linear Approximations

- Consider n-dim space $\mathbb{R}^n$, with each dim corresponding to a state in $\mathcal{S}$
- Think of a VF (typically denoted $\mathbf{v}$): $\mathcal{S} \rightarrow \mathbb{R}$ as a vector in this space
- Each dimension's coordinate is the VF for that dimension's state
- Coordinates of vector $\mathbf{v}_\pi$ for policy π are: $[\mathbf{v}_\pi(s_1), \mathbf{v}_\pi(s_2), \ldots, \mathbf{v}_\pi(s_n)]$
- Consider m vectors where j^{th} vector is: $[\phi_j(s_1), \phi_j(s_2), \ldots, \phi_j(s_n)]$
- These m vectors are the m columns of $n \times m$ matrix $\mathbf{\Phi} = [\phi_j(s_i)]$
- Their span represents an m-dim subspace within this n-dim space
- Spanned by the set of all $\mathbf{w} = [w_1, w_2, \ldots, w_m] \in \mathbb{R}^m$
- Vector $\mathbf{v_w} = \mathbf{\Phi} \cdot \mathbf{w}$ in this subspace has coordinates $[\mathbf{v_w}(s_1), \mathbf{v_w}(s_2), \ldots, \mathbf{v_w}(s_n)]$
- Vector $\mathbf{v_w}$ is fully specified by $\mathbf{w}$ (so we often say $\mathbf{w}$ to mean $\mathbf{v_w}$)

Some more notation

- Denote $r(s, a)$ as the Expected Reward upon action a in state s
- Denote $p(s, s', a)$ as the probability of transition $s \rightarrow s'$ upon action a
- Define

$$\mathbf{R}_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \cdot r(s, a)$$

$$\mathbf{P}_\pi(s, s') = \sum_{a \in \mathcal{A}} \pi(a|s) \cdot p(s, s', a)$$

- Denote $\mathbf{R}_\pi$ as the vector $[\mathbf{R}_\pi(s_1), \mathbf{R}_\pi(s_2), \ldots, \mathbf{R}_\pi(s_n)]$
- Denote $\mathbf{P}_\pi$ as the matrix $[\mathbf{P}_\pi(s_i, s_{i'})], 1 \leq i, i' \leq n$
- Denote γ as the MDP discount factor

Bellman operator $\mathbf{B}_\pi$

- Bellman operator $\mathbf{B}_\pi$ for policy π operating on VF vector $\mathbf{v}$ defined as:

$$\mathbf{B}_\pi \mathbf{v} = \mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{v}$$

- Note that $\mathbf{v}_\pi$ is the fixed point of operator $\mathbf{B}_\pi$ (meaning $\mathbf{B}_\pi \mathbf{v}_\pi = \mathbf{v}_\pi$)
- If we start with an arbitrary VF vector $\mathbf{v}$ and repeatedly apply $\mathbf{B}_\pi$, by Contraction Mapping Theorem, we will reach the fixed point $\mathbf{v}_\pi$
- This is the Dynamic Programming Policy Evaluation algorithm
- Monte Carlo without func approx also converges to $\mathbf{v}_\pi$ (albeit slowly)

Projection operator $\boldsymbol{\Pi_\Phi}$

- First we define "distance" $d(\mathbf{v_1}, \mathbf{v_2})$ between VF vectors $\mathbf{v_1}, \mathbf{v_2}$
- Weighted by μ_π across the n dimensions of $\mathbf{v_1}, \mathbf{v_2}$

$$d(\mathbf{v_1}, \mathbf{v_2}) = \sum_{i=1}^{n} \mu_\pi(s_i) \cdot (\mathbf{v_1}(s_i) - \mathbf{v_2}(s_i))^2 = (\mathbf{v_1} - \mathbf{v_2})^T \cdot \mathbf{D} \cdot (\mathbf{v_1} - \mathbf{v_2})$$

  where $\mathbf{D}$ is the square diagonal matrix consisting of $\mu_\pi(s_i), 1 \leq i \leq n$
- Projection operator for subspace spanned by $\boldsymbol{\Phi}$ is denoted as $\boldsymbol{\Pi_\Phi}$
- $\boldsymbol{\Pi_\Phi}$ performs an orthogonal projection of VF vector $\mathbf{v}$ on subspace $\boldsymbol{\Phi}$
- So, $\boldsymbol{\Pi_\Phi} \mathbf{v}$ is the VF in subspace $\boldsymbol{\Phi}$ defined by $\arg\min_{\mathbf{w}} d(\mathbf{v}, \mathbf{v_w})$
- This is a weighted least squares regression with solution:

$$\mathbf{w} = (\boldsymbol{\Phi}^T \cdot \mathbf{D} \cdot \boldsymbol{\Phi})^{-1} \cdot \boldsymbol{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{v}$$

- So, the Projection operator $\boldsymbol{\Pi_\Phi}$ can be written as:

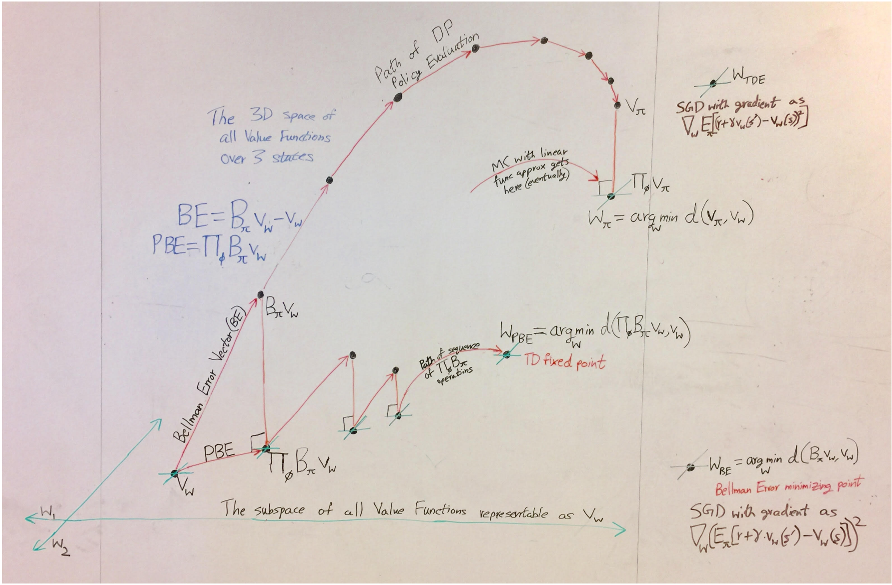
$$\boldsymbol{\Pi_\Phi} = \boldsymbol{\Phi} \cdot (\boldsymbol{\Phi}^T \cdot \mathbf{D} \cdot \boldsymbol{\Phi})^{-1} \cdot \boldsymbol{\Phi}^T \cdot \mathbf{D}$$

4 VF vectors of interest in the $\boldsymbol{\Phi}$ subspace

Note: We will refer to the $\boldsymbol{\Phi}$-subspace VF vectors by their weights $\mathbf{w}$

1. Projection $\mathbf{\Pi_\Phi v}_\pi$: $\mathbf{w}_\pi = \arg\min_{\mathbf{w}} d(\mathbf{v}_\pi, \mathbf{v_w})$
   - This is the VF we seek when doing linear function approximation
   - Because it is the VF vector "closest" to $\mathbf{v}_\pi$ in the $\boldsymbol{\Phi}$ subspace
   - Monte-Carlo with linear func approx will (slowly) converge to $\mathbf{w}_\pi$
2. Bellman Error (BE)-minimizing: $\mathbf{w}_{BE} = \arg\min_{\mathbf{w}} d(\mathbf{B}_\pi \mathbf{v_w}, \mathbf{v_w})$
   - This can be expressed as the solution of a linear system $\mathbf{Aw} = \mathbf{b}$
   - Matrix $\mathbf{A}$ and Vector $\mathbf{b}$ comprises of $\mathbf{P}_\pi, \mathbf{R}_\pi, \boldsymbol{\Phi}, \mu_\pi$
   - In model-free setting, $\mathbf{A}$ and $\mathbf{b}$ can be estimated with batch data
   - For non-linear approx or off-policy, Residual Gradient TD Algorithm
   - Based on observation: $\mathbf{w}_{BE} = \arg\min_{\mathbf{w}} (\mathbb{E}_\pi[\delta])^2$, where δ is TD Error
   - Cannot learn if we can only access features, and not underlying states
3. Temporal Difference Error (TDE)-minimizing:
   $\mathbf{w}_{TDE} = \arg\min_{\mathbf{w}} \mathbb{E}_\pi[\delta^2]$
   - Naive Residual Gradient TD Algorithm

4. Projected Bellman Error (PBE)-minimizing:
$\mathbf{w}_{PBE} = \arg\min_{\mathbf{w}} d((\mathbf{\Pi_\Phi} \cdot \mathbf{B}_\pi)\mathbf{v_w}, \mathbf{v_w})$
    - The minimum is 0, i.e., $\mathbf{\Phi} \cdot \mathbf{w}_{PBE}$ is the fixed point of operator $\mathbf{\Pi_\Phi} \cdot \mathbf{B}_\pi$
    - Starting with an arbitrary VF vector $\mathbf{v}$ and repeatedly applying $\mathbf{B}_\pi$ (taking it out of the subspace) followed by $\mathbf{\Pi_\Phi}$ (projecting it back to the subspace), we will reach the fixed point $\mathbf{\Phi} \cdot \mathbf{w}_{PBE}$
    - Also, $\mathbf{w}_{PBE}$ can be expressed as the solution of a linear system $\mathbf{Aw} = \mathbf{b}$
    - In model-free setting, $\mathbf{A}$ and $\mathbf{b}$ can be estimated with batch data
    - This yields the *Least Squares Temporal Difference (LSTD)* algorithm
    - For non-linear approx or off-policy, Gradient TD Algorithms



Path of DP Policy Evaluation
The 3D space of all Value Functions over 3 states
V_π
W_{TDE}
SGD with gradient as
$\nabla_w \mathbb{E}_\pi[(r+\gamma V_w(s')-V_w(s))^2]$
MC with linear func approx gets here (eventually)
$\Pi_\phi V_\pi$
$W_\pi = \arg\min_w d(V_\pi, V_w)$
$BE = B_\pi V_w - V_w$
$PBE = \Pi_\phi B_\pi V_w$
$B_\pi V_w$
Bellman Error Vector (BE)
$W_{PBE} = \arg\min_w d(\Pi_\phi B_\pi V_w, V_w)$
TD fixed point
Path of sequence of $\Pi_\phi B_\pi$ operations
PBE
$\Pi_\phi B_\pi V_w$
V_w
$W_{BE} = \arg\min_w d(B_\pi V_w, V_w)$
Bellman Error minimizing point
SGD with gradient as
$\nabla_w(\mathbb{E}_\pi[r+\gamma \cdot V_w(s')-V_w(s)])^2$
W_1
W_2
The subspace of all Value Functions representable as V_w

Solution of $\mathbf{w}_{BE}$ with a Linear System Formulation

$$\begin{aligned}
\mathbf{w}_{BE} &= \arg\min_{\mathbf{w}} d(\mathbf{v}_{\mathbf{w}}, \mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{v}_{\mathbf{w}}) \\
&= \arg\min_{\mathbf{w}} d(\mathbf{\Phi} \cdot \mathbf{w}, \mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w}) \\
&= \arg\min_{\mathbf{w}} d(\mathbf{\Phi} \cdot \mathbf{w} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w}, \mathbf{R}_\pi) \\
&= \arg\min_{\mathbf{w}} d((\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}) \cdot \mathbf{w}, \mathbf{R}_\pi)
\end{aligned}$$

This is a weighted least-squares linear regression of $\mathbf{R}_\pi$ versus $\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}$ with weights μ_π, whose solution is:

$$w_{BE} = ((\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}))^{-1} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi$$

Model-Free Learning of $\mathbf{w}_{BE}$

- Let us refer to $(\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})$ as $\mathbf{A}$
- Let us refer to $(\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi$ as $\mathbf{b}$
- So that $w_{BE} = \mathbf{A}^{-1} \cdot \mathbf{b}$
- Following policy π, each time we perform a model-free transition from s to s' getting reward r, we get a sample estimate of $\mathbf{A}$ and $\mathbf{b}$
- Estimate of $\mathbf{A}$ is the outer-product of vector $\phi(s) - \gamma \cdot \phi(s')$ with itself
- Estimate of $\mathbf{b}$ is scalar r times vector $\phi(s) - \gamma \cdot \phi(s')$
- Average these estimates across many such model-free transitions

Residual Gradient Algorithm to solve for $\mathbf{w}_{BE}$

- We defined $\mathbf{w}_{BE}$ as the vector in the $\mathbf{\Phi}$ subspace that minimizes BE
- But BE for a state is the expected TD error δ in that state when following policy π
- So we want to do SGD with gradient of square of expected TD error

$$
\begin{aligned}
\Delta w &= -\frac{1}{2}\alpha \cdot \nabla_w (\mathbb{E}_\pi[\delta])^2 \\
&= -\alpha \cdot \mathbb{E}_\pi[r + \gamma \cdot w^T \cdot \phi(s') - w^T \cdot \phi(s)] \cdot \nabla_w \mathbb{E}_\pi[\delta] \\
&= \alpha \cdot (\mathbb{E}_\pi[r + \gamma \cdot w^T \cdot \phi(s')] - w^T \cdot \phi(s)) \cdot (\phi(s) - \gamma \cdot \mathbb{E}_\pi[\phi(s')])
\end{aligned}
$$

- This is called the *Residual Gradient* algorithm
- Requires two independent samples of s' transitioning from s
- In that case, converges to $\mathbf{w}_{BE}$ robustly (even for non-linear approx)
- But it is slow, and doesn't converge to a desirable place
- Cannot learn if we can only access features, and not underlying states

Naive Residual Gradient Algorithm to solve for $\mathbf{w}_{TDE}$

- We defined $\mathbf{w}_{TDE}$ as the vector in the $\mathbf{\Phi}$ subspace that minimizes the expected square of the TD error δ when following policy π

$$\mathbf{w}_{TDE} = \arg\min_{\mathbf{w}} \sum_{s \in \mathcal{S}} \mu_\pi(s) \sum_{r,s'} prob_\pi(r, s'|s) \cdot (r + \gamma \cdot \mathbf{w}^T \cdot \phi(s') - \mathbf{w}^T \cdot \phi(s))^2$$

- To perform SGD, we have to estimate the gradient of the expected square of TD error by sampling
- The weight update for each sample in the SGD will be:

$$\begin{aligned}\Delta w &= -\frac{1}{2}\alpha \cdot \nabla_w (r + \gamma \cdot w^T \cdot \phi(s') - w^T \cdot \phi(s))^2 \\ &= \alpha \cdot (r + \gamma \cdot w^T \cdot \phi(s') - w^T \cdot \phi(s)) \cdot (\phi(s) - \gamma \cdot \phi(s'))\end{aligned}$$

- This algorithm (named *Naive Residual Gradient*) converges robustly, but not to a desirable place

Solution of $\mathbf{w}_{PBE}$ with a Linear System Formulation

$\mathbf{\Phi} \cdot \mathbf{w}_{PBE}$ is the fixed point of operator $\mathbf{\Pi_\Phi} \cdot \mathbf{B}_\pi$. We know:

$$\mathbf{\Pi_\Phi} = \mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D}$$

$$\mathbf{B}_\pi \mathbf{v} = \mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{v}$$

Therefore,

$$\mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w}_{PBE}) = \mathbf{\Phi} \cdot \mathbf{w}_{PBE}$$

Since columns of $\mathbf{\Phi}$ are assumed to be independent (full rank),

$$\begin{aligned}
(\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w}_{PBE}) &= \mathbf{w}_{PBE} \\
\mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w}_{PBE}) &= \mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi} \cdot \mathbf{w}_{PBE} \\
\mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}) \cdot \mathbf{w}_{PBE} &= \mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi
\end{aligned}$$

This is a square linear system of the form $\mathbf{A} \cdot \mathbf{w}_{PBE} = \mathbf{b}$ whose solution is:

$$\mathbf{w}_{PBE} = \mathbf{A}^{-1} \cdot \mathbf{b} = (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}))^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi$$

Model-Free Learning of $\mathbf{w}_{PBE}$

- How do we construct matrix $\mathbf{A} = \mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi})$ and vector $\mathbf{b} = \mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi$ without a model?
- Following policy π, each time we perform a model-free transition from s to s' getting reward r, we get a sample estimate of $\mathbf{A}$ and $\mathbf{b}$
- Estimate of $\mathbf{A}$ is the outer-product of vectors $\phi(s)$ and $\phi(s) - \gamma \cdot \phi(s')$
- Estimate of $\mathbf{b}$ is scalar r times vector $\phi(s)$
- Average these estimates across many such model-free transitions
- This algorithm is called Least Squares Temporal Difference (LSTD)
- Alternative: Our usual Semi-Gradient TD descent with updates:

$$\Delta w = \alpha \cdot (r + \gamma \cdot w^T \cdot \phi(s') - w^T \cdot \phi(s)) \cdot \phi(s)$$

- This converges to $\mathbf{w}_{PBE}$ because $\mathbb{E}_\pi[\Delta w] = 0$ yields

$$\mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{R}_\pi + \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi} \cdot \mathbf{w} - \mathbf{\Phi} \cdot \mathbf{w}) = 0$$

$$\Rightarrow \mathbf{\Phi}^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} - \gamma \mathbf{P}_\pi \cdot \mathbf{\Phi}) \cdot \mathbf{w} = \mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{R}_\pi$$

Gradient TD Algorithms to solve for $\mathbf{w}_{PBE}$

- For on-policy linear func approx, semi-gradient TD works
- For non-linear func approx or off-policy, we need Gradient TD
  - GTD: The original Gradient TD algorithm
  - GTD-2: Second-generation GTD
  - TDC: TD with Gradient correction
- We need to set up the loss function whose gradient will drive SGD
- $\mathbf{w}_{PBE} = \arg\min_{\mathbf{w}} d(\mathbf{\Pi_\Phi B}_\pi \mathbf{v_w}, \mathbf{v_w}) = \arg\min_{\mathbf{w}} d(\mathbf{\Pi_\Phi B}_\pi \mathbf{v_w}, \mathbf{\Pi_\Phi v_w})$
- So we define the loss function (denoting $\mathbf{B}_\pi \mathbf{v_w} - \mathbf{v_w}$ as $\delta_{\mathbf{w}}$) as:

$$\mathcal{L}(\mathbf{w}) = (\mathbf{\Pi_\Phi}\delta_{\mathbf{w}})^T \cdot \mathbf{D} \cdot (\mathbf{\Pi_\Phi}\delta_{\mathbf{w}}) = {\delta_{\mathbf{w}}}^T \cdot \mathbf{\Pi}_{\mathbf{\Phi}}^T \cdot \mathbf{D} \cdot \mathbf{\Pi_\Phi} \cdot \delta_{\mathbf{w}}$$

$$= {\delta_{\mathbf{w}}}^T \cdot (\mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D})^T \cdot \mathbf{D} \cdot (\mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D}) \cdot \delta_{\mathbf{w}}$$

$$= {\delta_{\mathbf{w}}}^T \cdot (\mathbf{D} \cdot \mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T) \cdot \mathbf{D} \cdot (\mathbf{\Phi} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot \mathbf{\Phi}^T \cdot \mathbf{D}) \cdot \delta_{\mathbf{w}}$$

$$= ({\delta_{\mathbf{w}}}^T \cdot \mathbf{D} \cdot \mathbf{\Phi}) \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi}) \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})$$

$$= (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})^T \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})$$

TDC Algorithm to solve for $\mathbf{w}_{PBE}$

We derive the TDC Algorithm based on $\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w})$

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = 2 \cdot (\nabla_{\mathbf{w}} (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})^T) \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi})^{-1} \cdot (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})$$

Now we express each of these 3 terms as expectations of model-free transitions $s \xrightarrow{\pi} (r, s')$, denoting $r + \gamma \cdot \mathbf{w}^T \cdot \phi(s') - \mathbf{w}^T \cdot \phi(s)$ as δ

- $\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}} = \mathbb{E}[\delta \cdot \phi(s)]$
- $\nabla_{\mathbf{w}} (\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \delta_{\mathbf{w}})^T = \nabla_{\mathbf{w}} (\mathbb{E}[\delta \cdot \phi(s)])^T = \mathbb{E}[(\nabla_{\mathbf{w}} \delta) \cdot \phi(s)^T] = \mathbb{E}[(\gamma \cdot \phi(s') - \phi(s)) \cdot \phi(s)^T]$
- $\mathbf{\Phi}^T \cdot \mathbf{D} \cdot \mathbf{\Phi} = \mathbb{E}[\phi(s) \cdot \phi(s)^T]$

Substituting, we get:

$$\nabla_{\mathbf{w}} \mathcal{L}(\mathbf{w}) = 2 \cdot \mathbb{E}[(\gamma \cdot \phi(s') - \phi(s)) \cdot \phi(s)^T] \cdot \mathbb{E}[\phi(s) \cdot \phi(s)^T]^{-1} \cdot \mathbb{E}[\delta \cdot \phi(s)]$$

Weight Updates of TDC Algorithm

$$\Delta w = -\frac{1}{2}\alpha \cdot \nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w})$$
$$= \alpha \cdot \mathbb{E}[(\phi(s) - \gamma \cdot \phi(s')) \cdot \phi(s)^T] \cdot \mathbb{E}[\phi(s) \cdot \phi(s)^T]^{-1} \cdot \mathbb{E}[\delta \cdot \phi(s)]$$
$$= \alpha \cdot (\mathbb{E}[\phi(s) \cdot \phi(s)^T] - \gamma \cdot \mathbb{E}[\phi(s') \cdot \phi(s)^T]) \cdot \mathbb{E}[\phi(s) \cdot \phi(s)^T]^{-1} \cdot \mathbb{E}[\delta \cdot \phi(s)]$$
$$= \alpha \cdot (\mathbb{E}[\delta \cdot \phi(s)] - \gamma \cdot \mathbb{E}[\phi(s') \cdot \phi(s)^T] \cdot \mathbb{E}[\phi(s) \cdot \phi(s)^T]^{-1} \cdot \mathbb{E}[\delta \cdot \phi(s)])$$
$$= \alpha \cdot (\mathbb{E}[\delta \cdot \phi(s)] - \gamma \cdot \mathbb{E}[\phi(s') \cdot \phi(s)^T] \cdot \theta)$$

where $\theta = \mathbb{E}[\phi(s) \cdot \phi(s)^T]^{-1} \cdot \mathbb{E}[\delta \cdot \phi(s)]$ is the solution to a weighted least-squares linear regression of $\mathbf{B}_\pi \mathbf{v} - \mathbf{v}$ against $\mathbf{\Phi}$, with weights as μ_π.

Cascade Learning: Update both w and θ (θ converging faster)

- $\Delta w = \alpha \cdot \delta \cdot \phi(s) - \alpha \cdot \gamma \cdot \phi(s') \cdot (\theta^T \cdot \phi(s))$
- $\Delta \theta = \beta \cdot (\delta - \theta^T \cdot \phi(s)) \cdot \phi(s)$

Note: $\theta^T \cdot \phi(s)$ operates as estimate of TD error δ for current state s