Towards Improved Pricing and Hedging of Agency Mortgage-Backed Securities (MBS)

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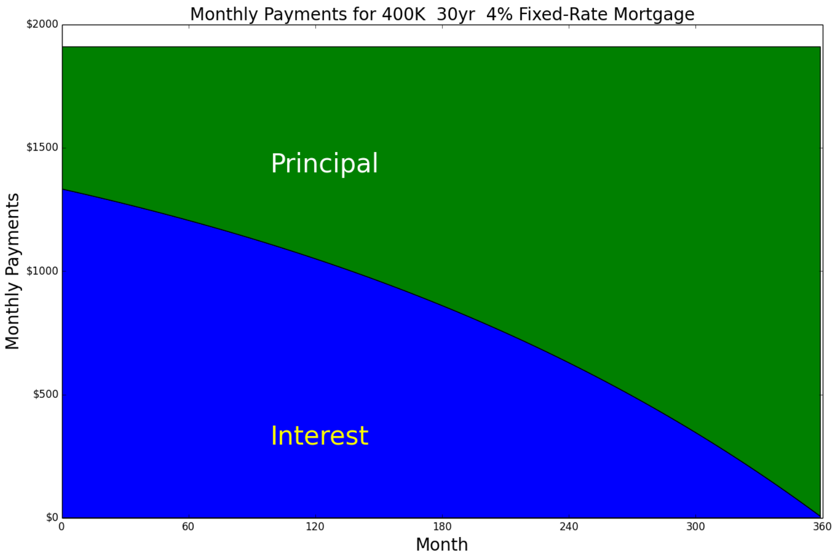
Overview

1. Mortgage Basics
2. Agency Securitization
3. Industry-Standard approach to Pricing & Hedging
   - Economic and Behavioral Intuition of Prepayment Models
   - Challenges with Trading based on "OAS"
4. Proposal: Pricing/Hedging with "risk-neutral prepayments"
   - Mathematical Formalism for Industry-Standard approach
   - Modeling of Prepayment Risks *other than Interest Rate Risk*
   - Trading based on calibrated Price of Prepayment Risk
   - Are we Happy? Can we do better?

Mortgage Basics: Borrower's Perspective

- A family wants to buy a house worth \$500K
- Can make down-payment of \$100K $\Rightarrow$ Loan to Value (LTV) of 80%
- Borrower has a good FICO score of 750
- Monthly family income is \$6000, Current monthly debt is only \$100
- So qualifies for a good mortgage rate (30 year, Fixed Rate) of 4%
- Monthly payment (Interest + Principal) amounts to about \$1900
- Debt to Income (DTI) is $\frac{1900+100}{6000} = 33\%$
- **Psychology less about debt mgmt, more about cash flows (DTI)**


Monthly Payments for 400K 30yr 4% Fixed-Rate Mortgage
Principal
Interest
Monthly Payments
$2000
$1500
$1000
$500
$0
0
60
120
180
240
300
360
Month

Mortgage Basics: Lender's Perspective

- The lender requires documentation on income, other assets/debts
- The lender is long a "Bond" receiving monthly Principal & Interest
- Lender's biggest risk is the borrower defaulting on monthly payments
- Think of this as the Borrower's (American) Put option on the House, with Strike = Remaining Principal (Default $\Rightarrow$ House taken away)
- **Lender will offer a rate commensurate with LTV, DTI, FICO**

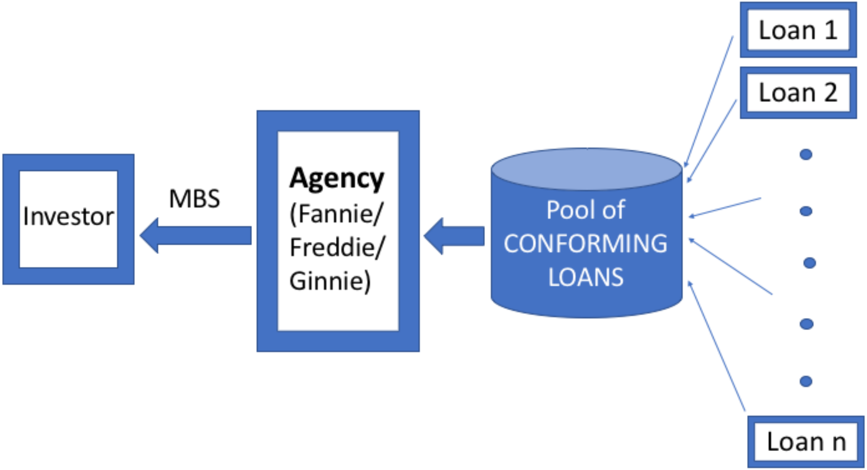
Lender also exposed to *Voluntary Prepayment* Risk

- *Prepayment*: Payment of Principal (full or partial) before schedule
- Borrower Defaulting is refered to as *Involuntary Prepayment*
- Voluntary Prepayment mainly due to Home Sale or Refinancing
- Think of this as the Borrower's (American) Call option on the Loan, with Strike = Remaining Principal
- Like any callable bond, Call Option is In-the-money when Price > Par
- In-the-money when offered rate is below rate on borrower's mortgage
- Refinancing: Borrowers often exercise deep in-the-money (suboptimal)
- HomeSale: Often happens when call option is out-of-the-money

Agency Securitization of a Pool of Mortgages

- "Agency" refers to the Government-sponsored entities (GSE)
- Pool of *Conforming* loans (conservative loan size, LTV, DTI, docs)
- Agency receives collective Principal & Interest from the pool
- In Exchange for cash extended to banks to originate mortgages
- Agency issues a Mortgage-Backed Security (backed by mortgage pool)
- Passthrough MBS simply forwards mortgage cash flows to investors
- Structured MBS: Pro Rata, Sequential, Planned Amortization Classes

Agency MBS securitized from a Conforming Pool of Loans



Agency MBS are essentially credit-risk-free

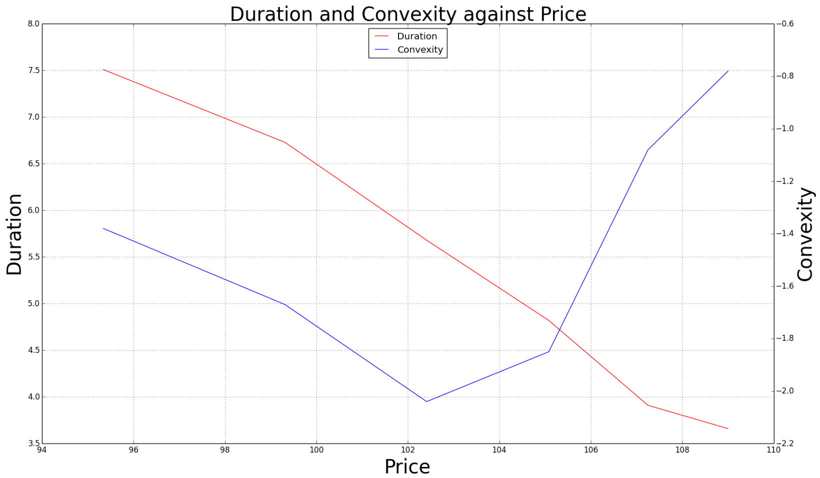
- Conforming loans $\Rightarrow$ risk of borrower delinquencies/default is low
- Agency guarantees investors principal even when a borrower defaults
- Defaulted loan is taken out of pool, investor **receives full principal**
- So Agency MBS investor has **no credit-risk** (government backing!)
- Agency receives a guarantee fee for this credit protection

Investor Perspective of Prepayment Risk for Passthroughs

- Prepayments $\Rightarrow$ Call option exercise on a fraction of the Pool
- So, Prepayments will move Price towards Par
  - Price for a Premium (Price $>$ Par) reduces when Prepayments rise
  - Price for a Discount (Price $<$ Par) increases when Prepayments rise
  - Price for the Par Passthrough is insensitive to Prepayments
- Prepayments for Premiums (Price $>$ Par) are mainly Refinancings
- Prepayments for Discounts (Price $<$ Par) are mainly HomeSales

Investor Perspective of Hedging Passthroughs

- Duration Definition: $-\frac{1}{P}\frac{\partial P}{\partial r}$
- Duration viewed as cashflows-PV-weighted average time of cash flows
- Duration is lower when Price is higher
- Duration reduces when Prepayments rise
- Short Prepayment Option $\Rightarrow$ Convexity and Vega are negative


Duration and Convexity against Price
Duration
Convexity
Price
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
−0.6
−0.8
−1.0
−1.2
−1.4
−1.6
−1.8
−2.0
−2.2
94
96
98
100
102
104
106
108
110

Industry-Standard approach to Pricing & Hedging

The key components of a typical Agency MBS Pricing/Hedging System:

- Stochastic interest rate model calibrated to liquid rate/vol instruments
- Monte Carlo simulation of interest rate model
- Econometric model of prepayments
- Model for mortgage rates (as a function of interest rates and vol)
- Cash flow generator for various MBS Structures
- Pricing/Sensitivities based on notion of "Option-Adjusted Spread"

Interest Rate Model

- Calibration to liquid rate/vol instruments
- Multi-factor model needed to capture non-parallel curve moves
- Capturing Vol Skew/Smile very important for MBS
- So need to also calibrate to OTM swaptions' market vol
- Model Choices: Local Vol, Stochastic Vol, CEV, Jump Diffusion
- Short rate or HJM models

Monte-Carlo Simulation

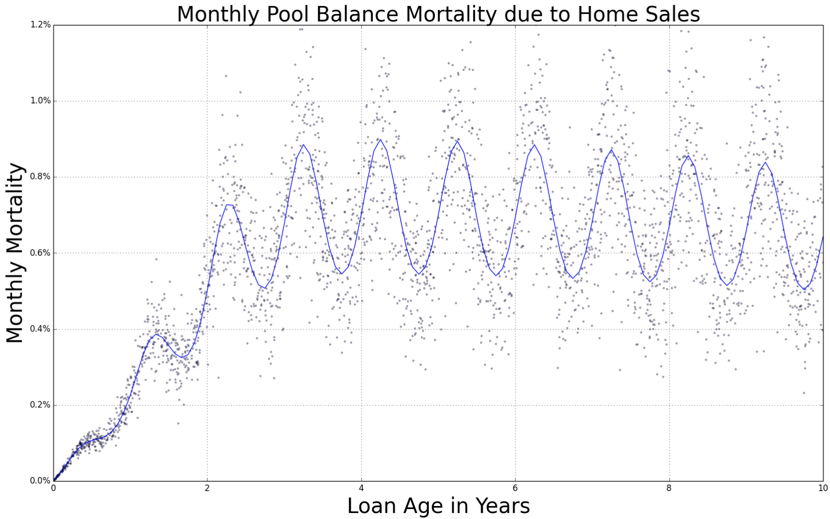
- Two reasons for needing to do non-Markovian Pricing of MBS:
  - Burnout and Media effects
  - Many CMOs have cash flows that depend on history of cash flows
- Sequence of draws of Brownian increments gives a path of states
- This provides discount factors at each time step
- For cash flows, need analytic/grid mapping from State to Rates/Vols
- Variance reduction:
  - Antithetic variables ($E[dz_t] = 0$ for all t)
  - Ensure $E[(dz_t)^2] = dt, E[(dz_s)(dz_t)] = 0$ using Orthonormalization
  - Control Variates, Quasi Random Numbers, Brownian Bridge

Prepayment Modeling

- Econometric forecast of fraction of pool terminating each month
- Explanatory variables built from historical data involving:
  - Loan Characteristics (Loan Size, Coupon, Term ...)
  - Borrower Information (Income, FICO, Other Debts ...)
  - Collateral Information (Single/Multi-Family, RE Taxes, ZIP ...)
  - Economic Conditions (Interest Rates, Unemployment, Home Prices ...)
- 4 sub-models, one for each of the following types of prepayments:
  - HomeSale (typically job-related or buying more expensive house)
  - Refinancing (to get a lower rate, or for cash-out refinance)
  - Default ($\Rightarrow$ loan paid off and removed from pool)
  - Curtailment (partial prepayment to curtail the life of the loan)
- Most prepayments for agency loans are home sales and refinancings

Economic and Behavioral Intuition of HomeSales Model

- HomeSales volume varies significantly by different seasons of the year
- Ramp up in home sales as a function of loan age
- Higher home prices leads to homeowners "trading up"
- Higher credit borrowers face less frictions
- "Lock-in" effect: Borrowers with low mortgage rate (relative to offered rate) reluctant to move as they like to hold on to their low rate

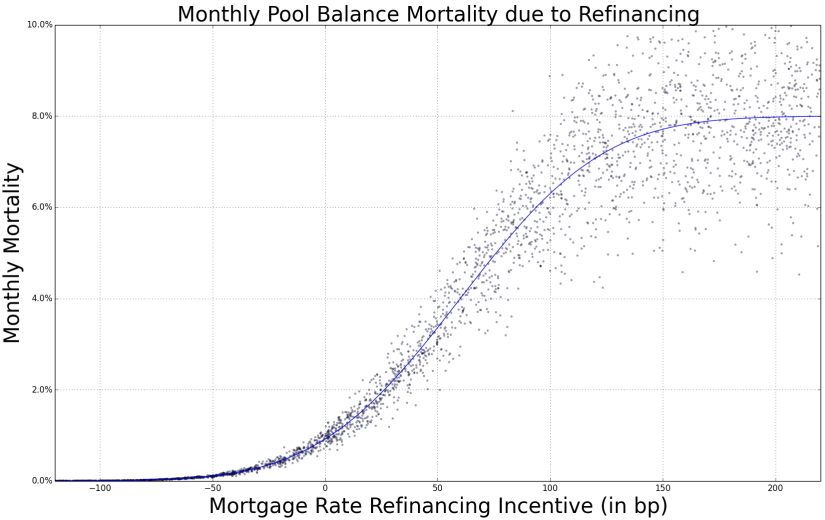


Monthly Pool Balance Mortality due to Home Sales
Monthly Mortality
Loan Age in Years
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0
2
4
6
8
10

Economic and Behavioral Intuition of Refinancing Model

- Incentive is typically the % improvement in monthly payments
- Mortgage Rate difference ("moneyness") is used as an approximation
- Lower credit quality diminishes the ability to refinance
- Credit quality indicators are SATO, FICO, LTV, DTI, Home Prices
- Higher home prices lead to cash-out refinancing
- Prepayment Model is a function of history of rates (non-Markovian)
    - Burnout Effect (Heterogeneity in borrower refinancing efficiency)
    - Media Effect (Spike in commercials when mortgage rates reach lows)

Topic for another day: Loan-level Deep Learning models


Monthly Pool Balance Mortality due to Refinancing
Monthly Mortality
Mortgage Rate Refinancing Incentive (in bp)
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
−100
−50
0
50
100
150
200

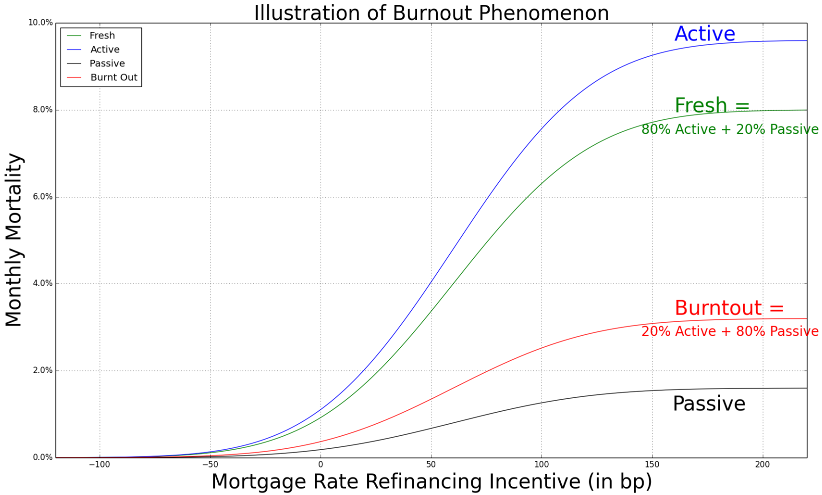


Illustration of Burnout Phenomenon
Fresh
Active
Passive
Burnt Out
Active
Fresh =
80% Active + 20% Passive
Burntout =
20% Active + 80% Passive
Passive
Monthly Mortality
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
−100
−50
0
50
100
150
200
Mortgage Rate Refinancing Incentive (in bp)

Regression model for mortgage rates

- Prepayment model typically based on mortgage rate "moneyness"
- Current Coupon (CC) model and Primary-Secondary Spread Model
- Historical CC regressed against historical rates and vol
- Instead, one can regress pricing-generated CC against rates and vol
- But then this CC model is input to pricing
- This fixed point is resolved by iterating to convergence
- Alternative approach: Price Moneyness and Backward Induction

Pricing/Sensitivities based on "Option-Adjusted Spread"

If T is maturity in months and N is number of Monte Carlo paths,

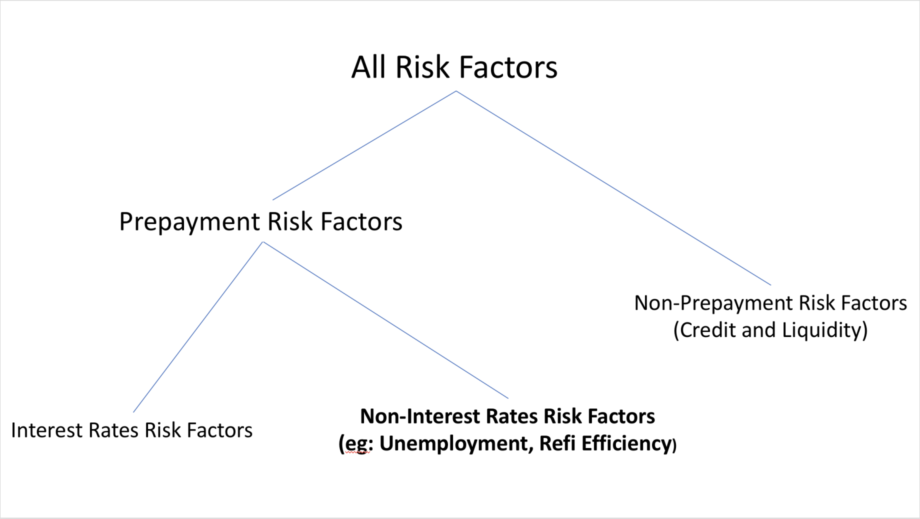
$$Price = \frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{T} (Principal_{i,t} + Interest_{i,t}) \cdot e^{-\sum_1^t (r_{i,u} + OAS)}$$

- OAS is a constant spread across time steps and MC paths
- **Risk-Premium for all risk factors other than interest rates**

Pricing/Hedging with this OAS-based (industry-standard) approach:

- For MBS with market prices, assess rich/cheap based on implied OAS
- For illiquid MBS, compute Price from trader-defined(!) OAS
- Price Sensitivities (duration, convexity, vega) with constant OAS

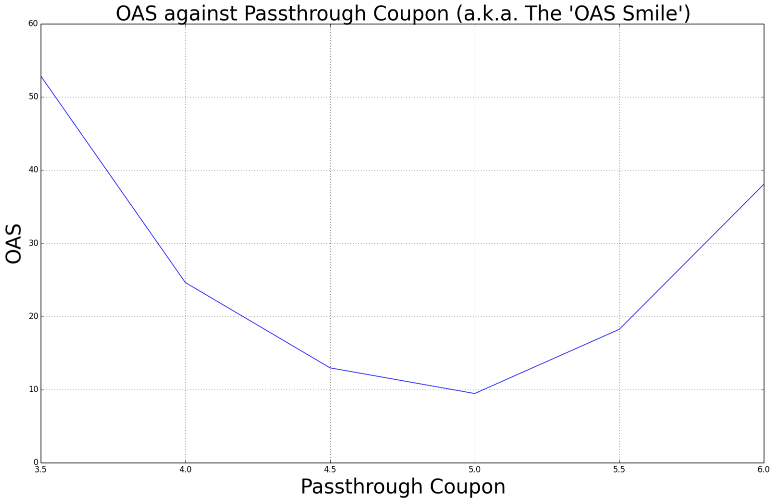
High-level Perspective of Agency MBS Risk Factors



Significant challenges in Trading based on OAS

Implicit assumption of constancy of OAS, **but market doesn't agree**

- Passthrough MBS exhibit an "OAS smile"
- Empirical Duration performs better than Model Duration
- Ill-formed theories on "OAS Directionality"
- IO/PO from same collateral have vastly different OAS
- IO Duration + PO Duration $\neq$ Underlying Pool Duration
- Unclear what OAS to set to price illiquid MBS


OAS against Passthrough Coupon (a.k.a. The 'OAS Smile')
OAS
Passthrough Coupon
0
10
20
30
40
50
60
3.5
4.0
4.5
5.0
5.5
6.0

Towards an improved approach for Pricing/Hedging

- First establish mathematical formalism for Industry-Standard approach
- Calibrate *Market Price of Prepayment Risk* (modulo Interest Rates)
- Trade based on constancy of calibrated Price of Risk (instead of OAS)

Notation for Continuous-Cashflow Pricing

$$dr(t) = \alpha_r(r,t) \cdot dt + \sigma_r(r,t) \cdot dz_r(t)$$

- $V(t)$ is *Value* at time t of (uncertain) cash flows after time t
- $B(t)$ is *Balance* = Remaining Principal at time t
- $P(t)$ is *Price* = "Normalized" *Value* $= \frac{Value}{Balance} = \frac{V(t)}{B(t)}$
- $c(r,t)$ is Coupon (interest paid per unit Balance per unit time)
- $\pi(r,t)$ is Principal paid per unit Balance per unit time

$$\pi(r,t) = sched(t) + sale(r,t) + refi(r,t) + curtail(r,t) + defaults(r,t)$$

Cashflow over time dt is $(c(r,t) + \pi(r,t)) \cdot B(t) \cdot dt$

$$dB(t) = -\pi(r,t) \cdot B(t) \cdot dt$$

Derivation of Continuous-Cashflow Pricing PDE

First assume MBS is a pure interest rate derivative, i.e., assume Prepayments are guaranteed to obey our model $\pi(r, t)$ function.

Follow a derivation similar to zero-coupon-bond-pricing PDE.

- Create portfolio of two MBS whose dV doesn't have dz term
- So portfolio *Value* grows at risk-free rate r
- The two MBS have the same *Market Price of Interest Rate Risk* λ_r

$$\frac{\partial P}{\partial t} + \alpha_r^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r) P$$

$$dr(t) = \alpha_r^{(Q)} dt + \sigma_r dz_r^{(Q)}(t) \text{ where } \alpha_r^{(Q)} = \alpha_r - \lambda_r \sigma_r$$

Pricing PDE for Industry-Standard (OAS-based) approach

- Now relax the assumption that MBS is a pure interest rate derivative
- Actual prepayments depend on stochastic (risk) factors other than r
- Each of them command a *Market Price of Risk* (i.e., a risk premium)
- Our Model Pricing is devoid of the risk premium of these other factors
- For Pricing to match Market, we need to adjust by this risk premium

$$\frac{\partial P}{\partial t} + \alpha_r^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r + s) P$$

where s is our "good-old" Option-Adjusted Spread (OAS)

$$P(0) = E_Q[\int_0^T (c + \pi) \cdot e^{-\int_0^u (\pi + r + s) du} dt]$$

Modeling Prepayment Risk Factors *modulo Interest Rates*

- To unravel s, model risk factors for prepayments *other than* r
- We won't model economic factors like employment, refinance frictions
- Instead we model processes for **forecast-errors** of prepayment model

$$refi'(x, r, t) = refi(r, t) \cdot x(t)$$

$$sale'(y, r, t) = sale(r, t) \cdot y(t)$$

$$dx(t) = \beta_x(1 - x)dt + \sigma_x dz_x(t) \text{ with } x(0) = 1$$

$$dy(t) = \beta_y(1 - y)dt + \sigma_y dz_y(t) \text{ with } y(0) = 1$$

Analogous to rate risk, shifting to risk-neutral measure, we have:

$$dx(t) = \alpha_x^{(Q)} dt + \sigma_x dz_x^{(Q)}(t) \text{ where } \alpha_x^{(Q)} = \beta_x(1 - x) - \lambda_x \sigma_x$$

$$dy(t) = \alpha_y^{(Q)} dt + \sigma_y dz_y^{(Q)}(t) \text{ where } \alpha_y^{(Q)} = \beta_y(1 - y) - \lambda_y \sigma_y$$

Calibrating the Market Price of Risk λ_x and λ_y

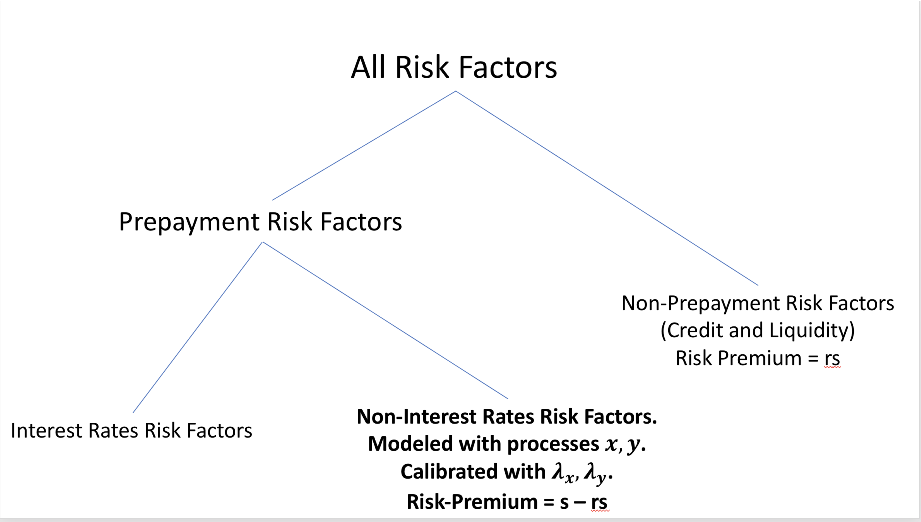
$$\frac{\partial P}{\partial t} + \alpha_r^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2 P}{\partial r^2} + \alpha_x^{(Q)} \frac{\partial P}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 P}{\partial x^2} + \alpha_y^{(Q)} \frac{\partial P}{\partial y} + \frac{\sigma_y^2}{2} \frac{\partial^2 P}{\partial y^2}$$

$$+c + \pi' = (\pi' + r + rs)P$$

- $\pi'(x, y, r, t) = sched(t) + refi(r, t) \cdot x(t) + sales(r, t) \cdot y(t) + \ldots$
- rs is the spread due to any residual risk such as liquidity or credit
- $\sigma_x, \sigma_y, \beta_x, \beta_y$ estimated from errors in prepayment forecasts
- λ_x, λ_y calibrated to liquid MBS market prices (shifting x, y drifts)

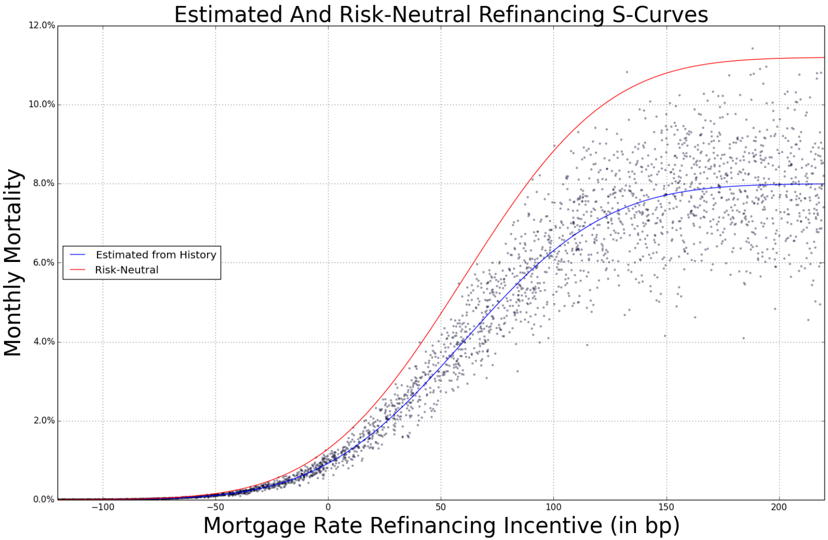
$$P(0) = E_Q[\int_0^T (c + \pi') \cdot e^{-\int_0^u (\pi' + r + rs) du} dt]$$

High-level Perspective of Agency MBS Risk Factors

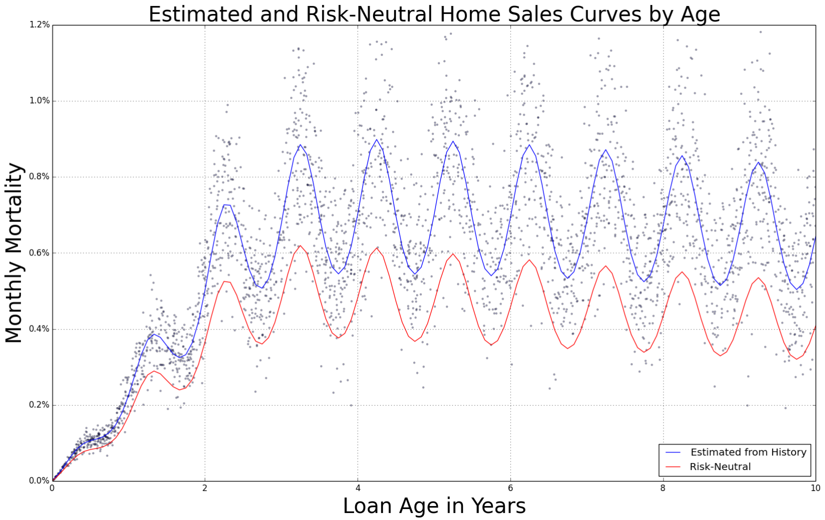


Pricing and Hedging with "risk-neutral prepayments"

- $\pi^{(Q)}(r,t) = \pi'(E_Q[x], E_Q[y], r, t)$ called "risk-neutral prepayments"
- λ_x, λ_y calibration $\Rightarrow E_Q[x] > 1, E_Q[y] < 1$,
  which dictates the alteration of $\pi(r,t)$ to $\pi^{(Q)}(r,t)$
- Trading with "risk-neutral prepayments" and residual spread *rs*
  - For MBS with market prices, assess rich/cheap based on implied *rs*
  - For illiquid MBS, Price using a prudent *rs* (easier than a OAS input)
  - Price Sensitivities **using constant rs** (instead of constant OAS)
- **Effectively, we trade based on constancy of λ_x, λ_y and *rs* (versus constancy of OAS)**



Estimated And Risk-Neutral Refinancing S-Curves
Monthly Mortality
Mortgage Rate Refinancing Incentive (in bp)
Estimated from History
Risk-Neutral
12.0%
10.0%
8.0%
6.0%
4.0%
2.0%
0.0%
−100
−50
0
50
100
150
200



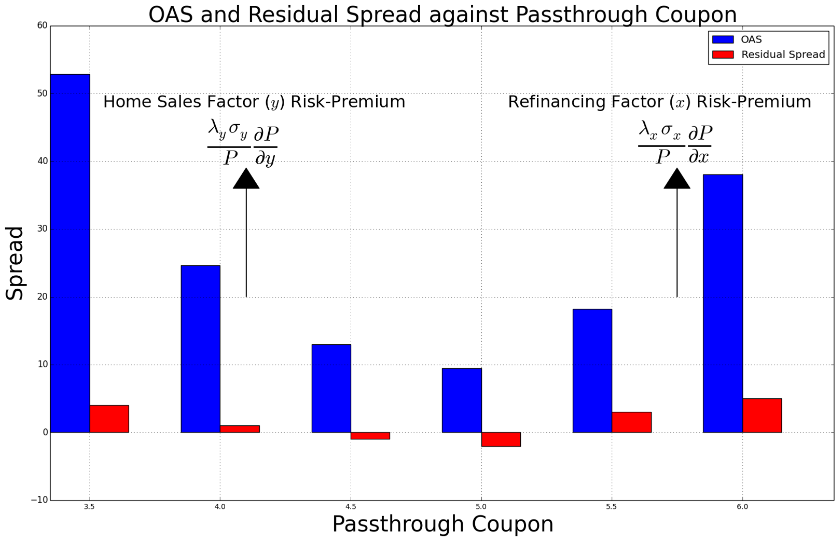
Estimated and Risk-Neutral Home Sales Curves by Age
Monthly Mortality
Loan Age in Years
1.2%
1.0%
0.8%
0.6%
0.4%
0.2%
0.0%
0
2
4
6
8
10
Estimated from History
Risk-Neutral

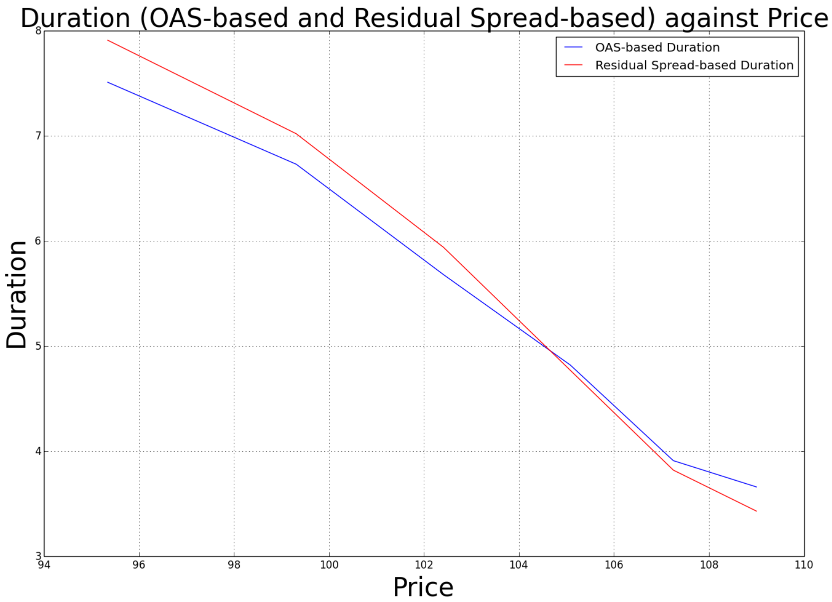
Practical considerations in designing the Pricer

- I used TBA prices for calibration (liquid IOs/POs could also be used)
- I calibrated Market Price of Risk of uncertain initial values $x(0)$ and $y(0)$, and Market Price of Risk of dz_x, dz_y
- I set $rs =$ Agency-Swap spread plus appropriate liquidity spread (we can do better by building a proper model for residual risk)
- **Small** $\frac{\partial^2 P}{\partial x^2}, \frac{\partial^2 P}{\partial y^2} \Rightarrow$ **we can avoid stochasticity of** x, y **in Pricer**
- So calibrated λ_x, λ_y essentially give us $E_Q[x(t)], E_Q[y(t)]$
- This gives us an altered prepayment function $\pi^{(Q)}(r, t)$
- **Practically, use same old Pricer, simply alter** $\pi(r, t)$ **to** $\pi^{(Q)}(r, t)$

Are we Happy? Can we do better?

- "OAS Smile" captured (but sometimes small residual smile remains)
- Model Durations match Empirical ("OAS Directionality" captured)
- Model prices for IOs/POs are fairly close to their market prices
- IOs: *rs*-based Duration more negative than OAS-based Duration
- Superior hedge performance relative to OAS-based Duration
- Further work (will help in crisis periods) - make *rs* a function of:
  - General credit risk
  - Agency-specific credit risk
  - Supply/Demand-based liquidity risk
- Modeling x, y as a jump-diffusion might also be a good idea


OAS and Residual Spread against Passthrough Coupon
OAS
Residual Spread
Home Sales Factor (y) Risk-Premium
$\frac{\lambda_y \sigma_y}{P} \frac{\partial P}{\partial y}$
Refinancing Factor (x) Risk-Premium
$\frac{\lambda_x \sigma_x}{P} \frac{\partial P}{\partial x}$
Spread
60
50
40
30
20
10
0
−10
3.5
4.0
4.5
5.0
5.5
6.0
Passthrough Coupon


Duration (OAS-based and Residual Spread-based) against Price
OAS-based Duration
Residual Spread-based Duration
Duration
Price
3
4
5
6
7
8
94
96
98
100
102
104
106
108
110

Summary and Finishing Comments

- “Risk-Neutral Prepayments” have been discussed since the mid-90s.
- But practitioners shy away claiming “Pricing Theory is not for MBS”!
- I was first influenced to implement this by Alex Levin in 2006
- My motivations: A) Improved Rich/Cheap, & B) Hedge Performance
- **My recommendation from a Trading perspective:**
    - Develop a loan-level econometric prepayment model
    - Capitalize on recent advances in Deep Learning techniques
    - Adjust this model with Market Price of Risk calibration to MBS Prices
- Further work on residual Liquidity/Credit Risk (for times of crises!)
- Can we extend this idea to non-Agency (default-risky) MBS?

Appendix 1: Derivation of Pricing PDE

$$dr(t) = \alpha(r,t) \cdot dt + \sigma(r,t) \cdot dz(t)$$

- $V(t)$ is *Value* at time t of (uncertain) cash flows after time t
- $B(t)$ is *Balance* = Remaining Principal at time t
- $P(t)$ is *Price* = "Normalized" *Value* = $\frac{Value}{Balance} = \frac{V(t)}{B(t)}$
- $c(r,t)$ is Coupon (interest paid per unit Balance per unit time)
- $\pi(r,t)$ is Principal paid per unit Balance per unit time

$$\pi(r,t) = sched(t) + sale(r,t) + refi(r,t) + curtail(r,t) + defaults(r,t)$$

Cashflow over time dt is $(c(r,t) + \pi(r,t)) \cdot B(t) \cdot dt$

$$dB(t) = -\pi(r,t) \cdot B(t) \cdot dt$$

Appendix 1: Pricing PDE (assuming only IR risk)

Assume MBS is a pure interest rate derivative, i.e., assume Prepayments are guaranteed to obey our model $\pi(r,t)$ function. Consider two different MBS with notation subscripted with "1" and "2".

$$dV_1 = B_1 dP_1 + P_1 dB_1$$

Ito's Lemma for dP_1 and substituting $dB_1 = -\pi_1 B_1 dt$ gives:

$$dV_1 = B_1(\frac{\partial P_1}{\partial t}dt + \frac{\partial P_1}{\partial r}dr + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2}dt) - P_1\pi_1 B_1 dt$$

$$= B_1(\frac{\partial P_1}{\partial t} + \alpha\frac{\partial P_1}{\partial r} - \pi P_1 + \frac{1}{2}\sigma^2\frac{\partial^2 P_1}{\partial r^2})dt + B_1\sigma\frac{\partial P_1}{\partial r}dz$$

$$\frac{dV_1}{V_1} = (\frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2})dt + (\frac{\sigma}{P_1}\frac{\partial P_1}{\partial r})dz \qquad (1)$$

Appendix 1: Pricing PDE (assuming only IR risk)

Denote the Ito drift and dispersion of $\frac{dV_1}{V_1}$ as α_1 and σ_1.

$$\frac{dV_1}{V_1} = \alpha_1 dt + \sigma_1 dz \tag{2}$$

$$\alpha_1 = \frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2} \tag{3}$$

$$\sigma_1 = \frac{\sigma}{P_1}\frac{\partial P_1}{\partial r} \tag{4}$$

Consider a portfolio $W = V_1 + V_2 = P_1 B_1 + P_2 B_2$ with values of B_1 and B_2 such that we can eliminate the dz term in the expression for dW

$$B_1 \frac{\partial P_1}{\partial r} = -B_2 \frac{\partial P_2}{\partial r} \tag{5}$$

Since we have eliminated the dz term, portfolio W together with the cashflow it generates should grow at the risk-free rate r. In other words,

$$dV_1 + dV_2 + B_1(c_1 + \pi_1)dt + B_2(c_2 + \pi_2)dt = r(V_1 + V_2)dt$$

Appendix 1: Pricing PDE (assuming only IR risk)

Expanding this out,

$$B_1(\frac{\partial P_1}{\partial t} + \alpha_1 \frac{\partial P_1}{\partial r} - \pi_1 P_1 + (c_1 + \pi_1) + \frac{\sigma^2}{2} \frac{\partial^2 P_1}{\partial r^2})dt +$$

$$B_2(\frac{\partial P_2}{\partial t} + \alpha_2 \frac{\partial P_2}{\partial r} - \pi_2 P_2 + (c_2 + \pi_2) + \frac{\sigma^2}{2} \frac{\partial^2 P_2}{\partial r^2})dt = r(B_1 P_1 + B_2 P_2)dt$$

Combining this with equation 5, we get:

$$\frac{\frac{\partial P_1}{\partial t} + \alpha_1 \frac{\partial P_1}{\partial r} - (\pi_1 + r)P_1 + (c_1 + \pi_1) + \frac{\sigma^2}{2} \frac{\partial^2 P_1}{\partial r^2}}{\frac{\partial P_1}{\partial r}} =$$

$$\frac{\frac{\partial P_2}{\partial t} + \alpha_2 \frac{\partial P_2}{\partial r} - (\pi_2 + r)P_2 + (c_2 + \pi_2) + \frac{\sigma^2}{2} \frac{\partial^2 P_2}{\partial r^2}}{\frac{\partial P_2}{\partial r}}$$

Appendix 1: Pricing PDE (assuming only IR risk)

Using equations 3 and 4, the above equation can be expressed as:

$$\frac{\alpha_1 + \frac{c_1+\pi_1}{P_1} - r}{\sigma_1} = \frac{\alpha_2 + \frac{c_2+\pi_2}{P_2} - r}{\sigma_2} \tag{6}$$

- Note the numerator of LHS of Eq 6 is the expected *excess return* per unit time of investing in MBS 1 (expected growth rate of process V_1 together with its P & I cash flows, less r).
- The denominator is the standard deviation of the return per unit time.
- Their ratio is the familiar *Market Price of Interest-Rate Risk* λ_r (which is the same for every security exposed to only interest-rate risk).
- This is what equation 6 is telling us, which can be re-expressed as:

$$\frac{\alpha_1 + \frac{c_1+\pi_1}{P_1} - r}{\sigma_1} = \frac{\alpha_2 + \frac{c_2+\pi_2}{P_2} - r}{\sigma_2} = \lambda_r$$

Appendix 1: Pricing PDE (assuming only IR risk)

Substituting in the above equation for α_1 from equation 3 and for σ_1 from equation 4, we get:

$$\frac{\frac{1}{P_1}\frac{\partial P_1}{\partial t} + \frac{\alpha}{P_1}\frac{\partial P_1}{\partial r} - \pi_1 + \frac{\sigma^2}{2P_1}\frac{\partial^2 P_1}{\partial r^2} + \frac{c_1+\pi_1}{P_1} - r}{\frac{\sigma}{P_1}\frac{\partial P_1}{\partial r}} = \lambda_r$$

Reorganize and drop the subscript 1 in P_1, c_1, π_1 to arrive at the PDE:

$$\frac{\partial P}{\partial t} + (\alpha - \lambda_r \sigma)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r)P$$

Appendix 1: Pricing PDE (assuming only IR risk)

If we shift to the risk-neutral measure (denoted as Q), V_1 together with the cash flows it generates should grow at risk-free rate r. Hence,

$$\frac{dV_1}{V_1} = (r - \frac{c_1 + \pi_1}{P_1})dt + \sigma_1 dz^{(Q)} = (\alpha_1 - \lambda_r \sigma_1)dt + \sigma_1 dz^{(Q)}$$

Since, $dz^{(Q)} = \lambda_r dt + dz$, the Ito process for the short-rate r in the risk-neutral measure can be written as:

$$dr = (\alpha - \lambda_r \sigma) \cdot dt + \sigma \cdot dz^{(Q)} = \alpha^{(Q)} \cdot dt + \sigma \cdot dz^{(Q)}$$

where $\alpha^{(Q)} = \alpha - \lambda_r \sigma$ is the risk-neutral drift for r.
So, the PDE can also be expressed in terms of the risk-neutral drift of r:

$$\frac{\partial P}{\partial t} + \alpha^{(Q)} \frac{\partial P}{\partial r} + \frac{\sigma^2}{2} \frac{\partial^2 P}{\partial r^2} + c + \pi = (\pi + r)P$$

Appendix 2: Martingale-based Pricing

Define the money-market process $M(t)$ as:

$$dM(t) = r(t) \cdot M(t) \cdot dt, M(0) = 1$$

Consider a process $\theta(t)$ derived from $V(t)$ and the cash flows, as follows:

$$d\theta(t) = dV(t) + (c(r,t) + \pi(r,t))B(t)dt$$

Assume MBS is a pure interest rate derivative, i.e., assume Prepayments are guaranteed to obey our model $\pi(r,t)$ function.

Then, $\frac{\theta(t)}{M(t)}$ is a martingale in risk-neutral measure Q. So, for any $T > 0$,

$$\frac{\theta(0)}{M(0)} = E_Q[\frac{\theta(T)}{M(T)}]$$

Since $\theta(0) = V(0), M(0) = 1$, and without loss of generality, $B(0) = 1$,

$$P(0) = \frac{\theta(0)}{M(0)} = E_Q[\frac{V(T) + \int_0^T (c(r,t) + \pi(r,t)) \cdot B(t) \cdot dt}{M(T)}]$$

Appendix 2: Martingale-based Pricing

If we set T to MBS maturity, $V(T) = B(T) = 0$. Then,

$$P(0) = E_Q[\int_0^T \frac{(c(r,t) + \pi(r,t)) \cdot B(t)}{M(t)} dt]$$
$$= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot B(t) \cdot e^{-\int_0^u r(u)du} dt] \quad (7)$$
$$= E_Q[\int_0^T (c(r,t) + \pi(r,t)) \cdot e^{-\int_0^u (r(u) + \pi(r,u))du} dt]$$

Conceptualize as cash flows discounted at rate $r + \pi$ in Q-measure.
We refer to this as the Expected Discounted Cashflow (EDC) formula.
Note: This formula holds only when assuming there is no risk factor other than interest rates. Otherwise, we require further (OAS) discounting.

Appendix 3: Closed-Form Approx. for Price/Sensitivities

- Closed-form Solution for Expected Discounted Cashflow (EDC) formula when $\pi(r,t) = \pi_0(t) + r\pi_1$ and $c(r,t) = c_0(t) + r \cdot c_1(t)$
- Closed-form Solution for Pricing PDE when r follows an affine process and $\pi(r,t) = \pi_0(t) + r\pi_1$ and $c(r,t) = c_0(t) + r \cdot c_1(t)$
- Closed-form Solution for EDC Price formula when π is linear in r with hard max and min
- The above gives bad greeks due to piecewise prepayment function. But this serves quite useful as a control variate.
- We can do analytical partial derivatives of EDC Price formula w.r.t:
    - a time-parallel shift to $r(t)$ (Duration approximation)
    - coupon c
    - a time-parallel shift to $\pi(r,t)$ (Prepayment sensitivity)
    - a time-parallel shift to refi multiplier $x(t)$ or sale multiplier $y(t)$
    - residual spread rs
- **These closed-forms useful to reason about pricing/sensitivities**

Appendix 4: Markovian Pricing (Alex Levin APD paper)

- Model Burnout by creating a few (2-3) cohorts in the pool, each of which is homogeneous in terms of refinancing efficiency
- **Each cohort's prepayment incentive is Markovian**
- For each cohort, at every grid node, probabilistically discount sum of:
    - Interest cash flow c
    - Principal cash flow π
    - $(1 - \pi) \cdot P$
- This gives a backward induction for P for each cohort
- MBS Price = sum of cohort Prices
- Alternatively, we can solve Pricing PDE with Crank-Nicholson
- Making π a function of P (instead of r) yields a clean model

Caveat: ARMs and some CMOs have non-Markovian cash flows.

Appendix 5: Sign of Price of Risk

For risk factor f, the vol of $\frac{dV}{V}$ w.r.t f is $\frac{\sigma_f}{P}\frac{\partial P}{\partial f}$.
So the risk-premium for the MBS due to the risk factor f is:

$$\frac{\lambda_f \sigma_f}{P}\frac{\partial P}{\partial f}$$

- We calibrate λ_x and λ_y to liquid passthrough market prices
- Premiums are mainly exposed to refinancing risk and $\frac{\partial P}{\partial x}$ is negative
- OAS smile $\Rightarrow \lambda_x$ (Price of Refinancing-Forecast Risk) is negative
- Discounts are mainly exposed to home sales risk and $\frac{\partial P}{\partial y}$ is positive
- OAS smile $\Rightarrow \lambda_y$ (Price of HomeSales-Forecast Risk) is positive

Appendix 6: Direction of risk-neutral adjustment of drift

For risk factor f, it's drift is subtracted by $\lambda_f \sigma_f$ to make the process for f risk-neutral. Risk-premium s due to the risk factor f is given by:

$$s = \frac{\lambda_f \sigma_f}{P} \frac{\partial P}{\partial f}$$

So, the drift of f is subtracted by:

$$\frac{sP}{(\frac{\partial P}{\partial f})}$$

- $s > 0 \Rightarrow$ drift is adjusted in a direction that worsens price
- So, adjusted Refinancing multiplier $E_Q[x(t)] > 1$
- And adjusted HomeSales multiplier $E_Q[y(t)] < 1$