OPTIMAL ASSET ALLOCATION IN DISCRETE TIME

STANFORD UNIVERSITY - CME 241 ASSIGNMENT PROBLEM

We are given wealth W_0 at time 0. At each of discrete time steps labeled $t = 0, 1, \ldots, T$, we are allowed to allocate the current wealth W_t in a risky asset and a riskless asset in an unconstrained, frictionless manner. The risky asset yields a random rate of return $\sim N(\mu, \sigma^2)$ over each single time step. The riskless asset yields a rate of return denoted by r over each single time step.

Our goal is to maximize the Utility of Wealth at the final time step $t = T$ by dynamically allocating x_t in the risky asset and the remaining $W_t - x_t$ in the riskless asset for each $t = 0, 1, \ldots, T-1$ (assume no transaction costs and no restrictions on going long or short in either asset). Assume the single-time-step discount factor is γ and the Utility of Wealth at the final time step $t = T$ is $U(W_T) = -\frac{e^{-aW_T}}{a}$ for some fixed $a > 0$.

- Formulate this problem as a *Continuous States*, *Continuous Actions* MDP by specifying it's *State Transitions*, *Rewards* and *Discount Factor*. The problem then is to find the Optimal Policy.
- As always, we strive to find the Optimal Value Function. The first step in determining the Optimal Value Function is to write the Bellman Optimality Equation.
- Assume the functional form for the Optimal Value Function is $-b_t e^{-c_t W_t}$ where b_t, c_t are unknowns functions of only t. Express the Bellman Optimality Equation using this functional form for the Optimal Value Function.
- Since the right-hand-side of the Bellman Optimality Equation involves a max over x_t, we can say that the partial derivative of the term inside the max with respect to x_t is 0. This enables us to write the Optimal Allocation x_t^* in terms of c_{t+1}.
- Substituting this maximizing x_t^* in the Bellman Optimality Equation enables us to express b_t and c_t as recursive equations in terms of b_{t+1} and c_{t+1} respectively.
- We know b_T and c_T from the knowledge of the MDP *Reward* at $t = T$ (Utility of Terminal Wealth), which enables us to unroll the above recursions for b_t and c_t.
- Solving b_t and c_t yields the Optimal Policy and the Optimal Value Function.