MONASH University

MONASH BUSINESS SCHOOL

Forecast Linear Augmented Projection (FLAP): A free lunch to reduce forecast error variance

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Forecast Linear Augmented Projection (FLAP)

A model-independent post-forecast adjustment method that can reduce forecast error variance.

- Averaging indirect forecasts from linear combinations (components)
- Projecting forecasts of augmented series
- Free lunch: no additional data or information needed

What to expect

- Intuition with data
- Literature
- Method formulation
- Properties
- Empirical applications and simulation

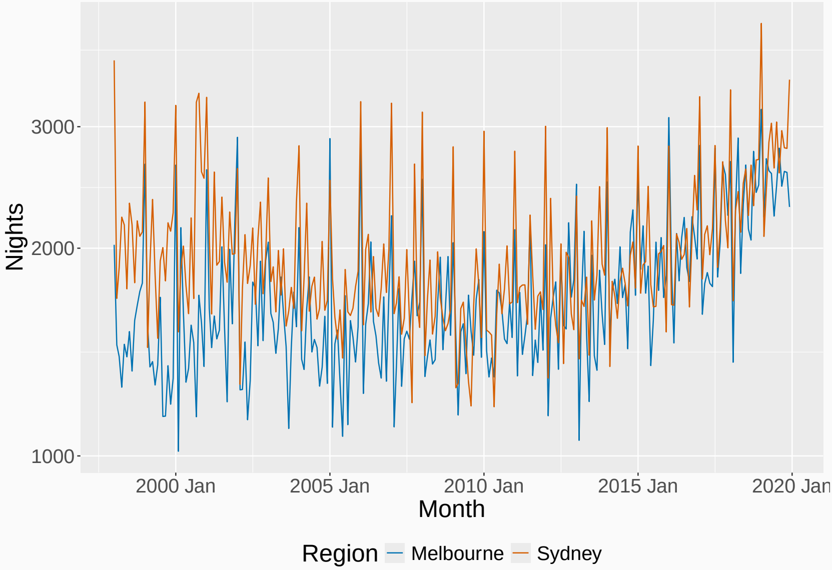
Australian tourism data

- The data include tourism information on seven states and territories which can be divided into 77 regions
  - For example, Melbourne, Sydney, East Coast

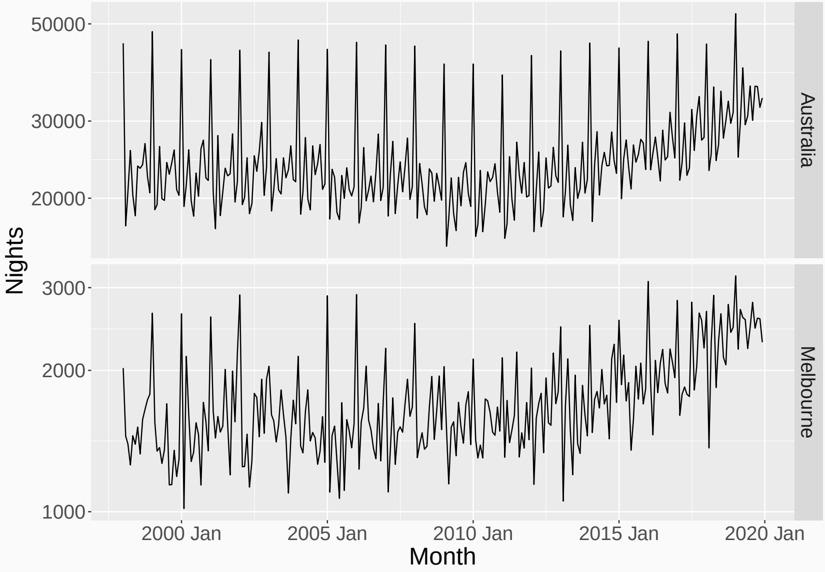
Visitor nights

The total number of nights spent by Australians away from home recorded monthly

Melbourne and Sydney



Total and Region



Intuition

Observation

1. Similar patterns are shared by different series.
2. Better signal-noise ratio in the linear combination.

Intuition

Observation

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One step further

Finding components that

1. are easy to forecast;
2. can capture the common signals;
3. can improve forecast of original series.

Literature

Forecast reconciliation

- Wickramasuriya, Athanasopoulos, and Hyndman (2019): Projecting forecasts to be consistent with the hierarchical structure

Forecast combination

- Combining forecasts of the target series
- Hollyman, Petropoulos, and Tipping (2021): Combining direct and indirect forecasts
- Petropoulos and Spiliotis (2021): Combining forecasts of selections and transformations of the target series (“wisdom of data”)

Literature

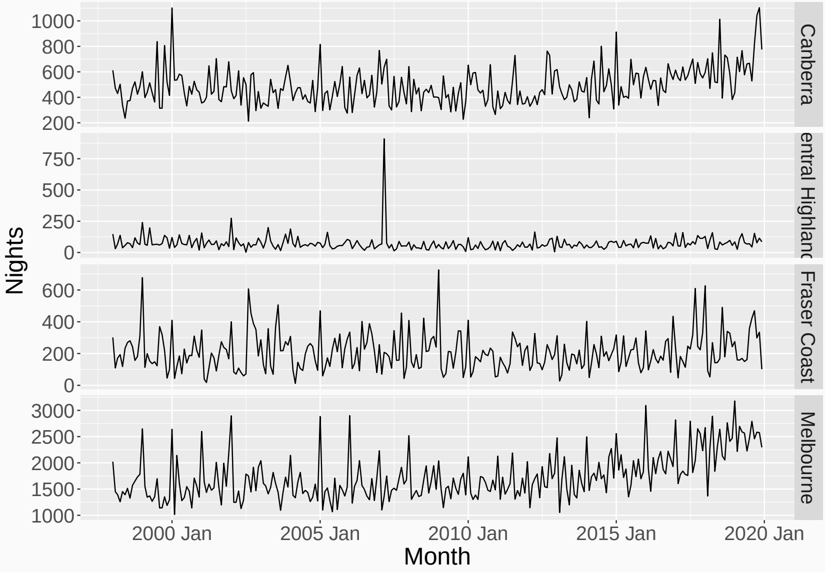
Bagging

- Bergmeir, Hyndman, and Benítez (2016): Bagging ETS models to forecast
- Petropoulos, Hyndman, and Bergmeir (2018): The benefits of bagging originate from the model uncertainty

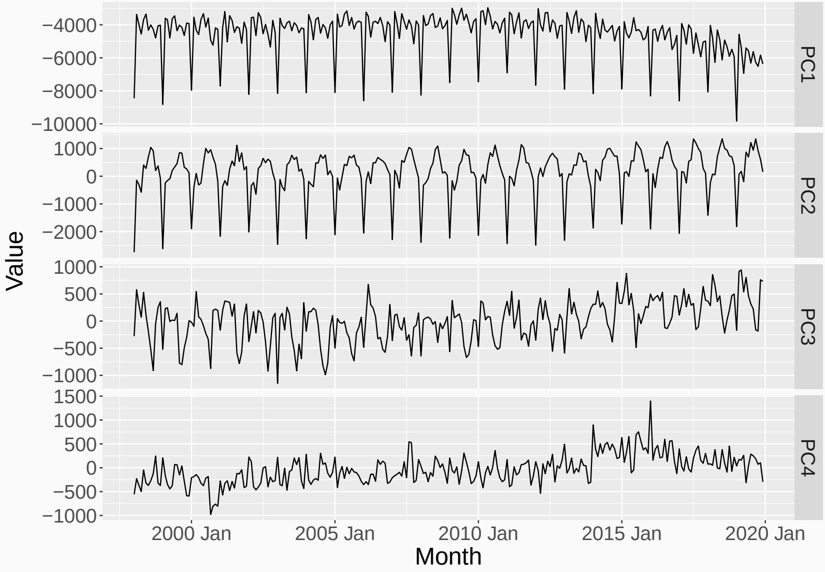
Dynamic factor model (DFM)

- Stock and Watson (2002a), Stock and Watson (2002b)
- De Stefani et al. (2019): Machine learning extension

Series $\boldsymbol{y}_t \in \mathbb{R}^m$



Components $\boldsymbol{c}_t = \Phi \boldsymbol{y}_t \in \mathbb{R}^p$



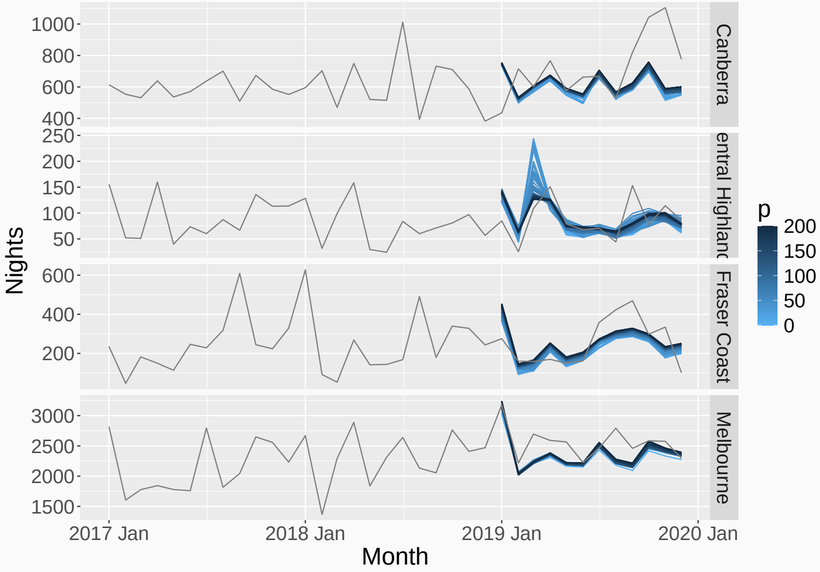
FLAP

$$\boldsymbol{z}_t = \begin{bmatrix} \boldsymbol{y}_t \\ \boldsymbol{c}_t \end{bmatrix} \qquad \tilde{\boldsymbol{z}}_{t+h} = \boldsymbol{M}\hat{\boldsymbol{z}}_{t+h}$$

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{J}\tilde{\boldsymbol{z}}_{t+h} = \boldsymbol{J}\boldsymbol{M}\hat{\boldsymbol{z}}_{t+h}$$

$$\begin{aligned}
\boldsymbol{M} &= \boldsymbol{I}_{m+p} - \boldsymbol{W}_h\boldsymbol{C}'\left(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}'\right)^{-1}\boldsymbol{C} \\
\boldsymbol{J} &= \boldsymbol{J}_{m,p} = \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{O}_{m\times p} \end{bmatrix} \\
\boldsymbol{C} &= \begin{bmatrix} -\Phi & \boldsymbol{I}_p \end{bmatrix} \\
\boldsymbol{W}_h &= \mathrm{Var}\left(\boldsymbol{z}_{t+h} - \hat{\boldsymbol{z}}_{t+h}\right)
\end{aligned}$$

Forecasts and FLAP of series



Nonnegative variance reduction

Under unbiasedness, the variance reduction is **positive semi-definite**:

$$\begin{aligned}\mathrm{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \mathrm{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}) \\ = \boldsymbol{J}\boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{J}'\end{aligned}$$

Projection matrix

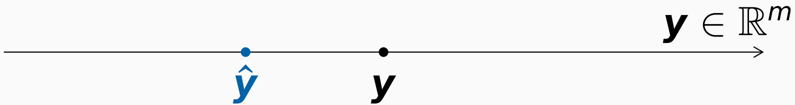
Projection matrix

The matrix $\boldsymbol{M}$ is a projection onto the space where the constraint $\boldsymbol{C}\boldsymbol{z}_t = \boldsymbol{0}$ is satisfied.

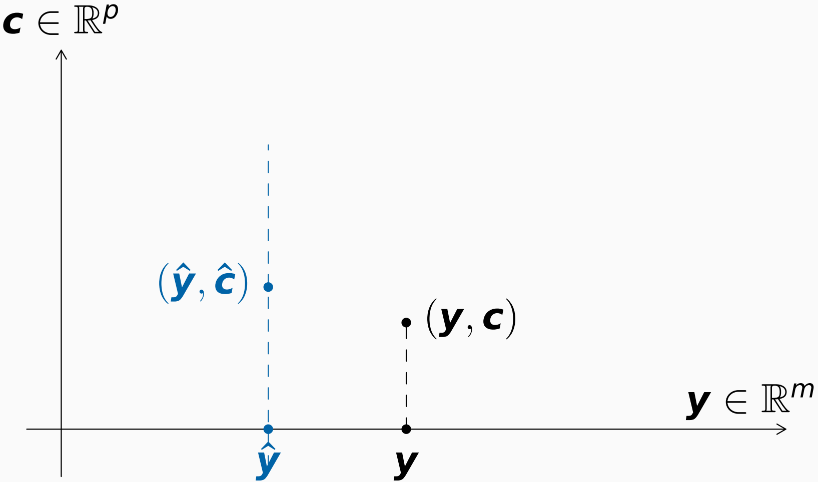
Properties

1. The projected forecast $\tilde{\boldsymbol{z}}_{t+h}$ satisfies the constraint $\boldsymbol{C}\tilde{\boldsymbol{z}}_{t+h} = \boldsymbol{0}$.
2. For $\boldsymbol{z}_{t+h}$ that already satisfies the constraint, the projection does not change its value: $\boldsymbol{M}\boldsymbol{z}_{t+h} = \boldsymbol{z}_{t+h}$
3. If the base forecasts are unbiased, then the FLAP forecasts are also unbiased.

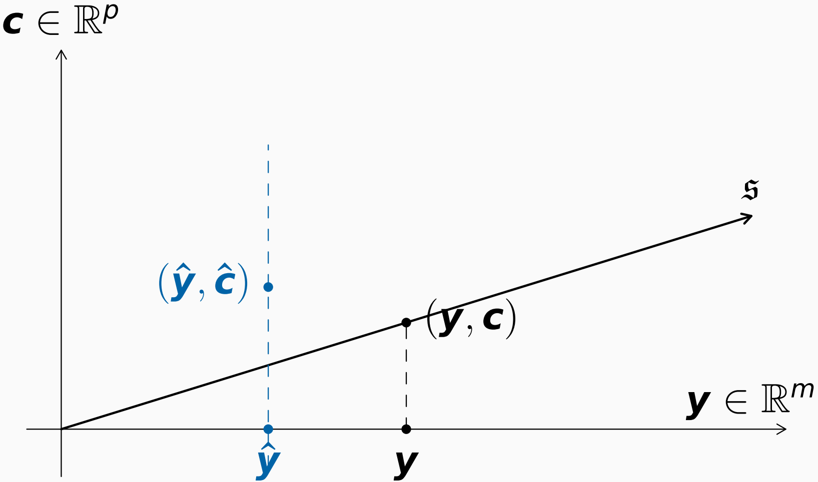
Geometry of FLAP



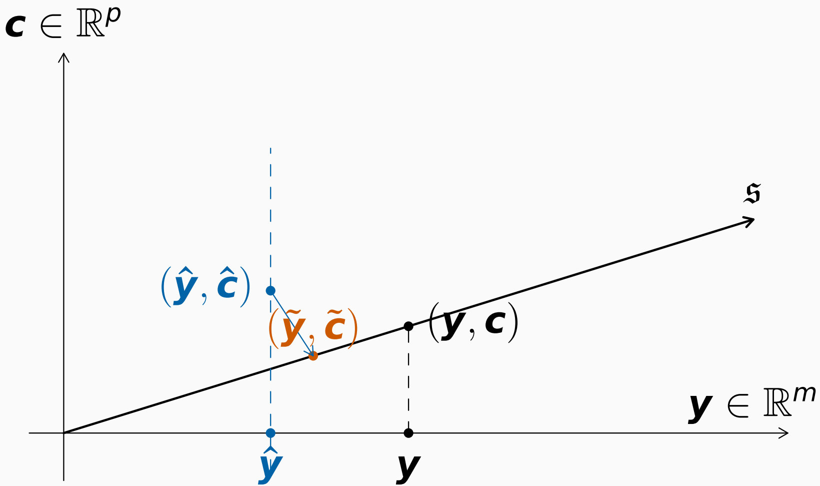
Geometry of FLAP



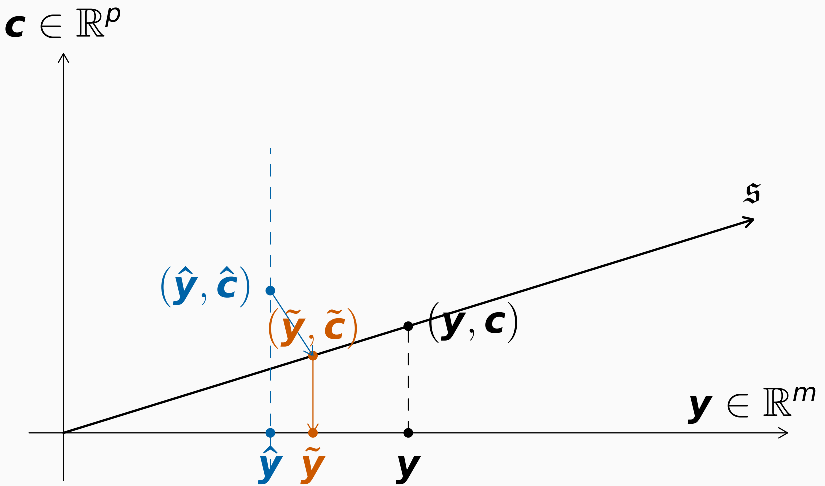
Geometry of FLAP



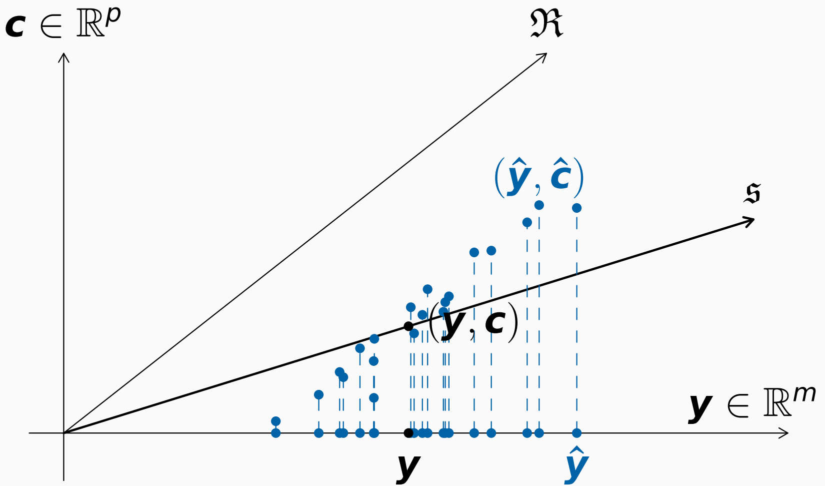
Geometry of FLAP



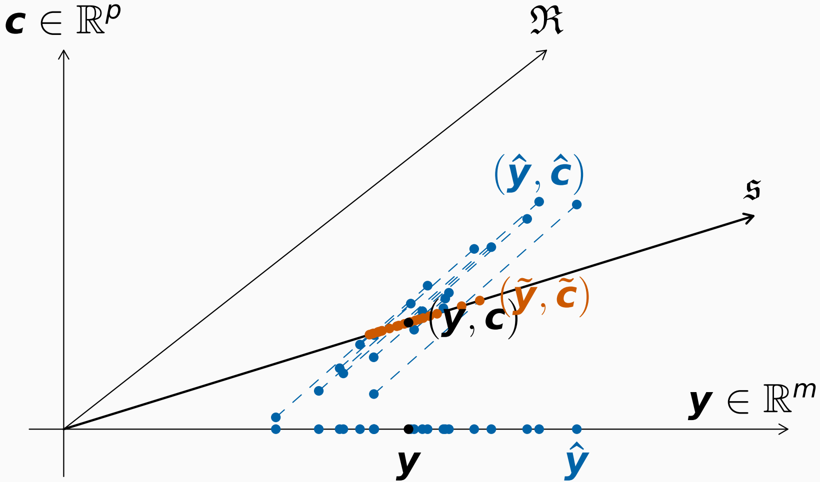
Geometry of FLAP



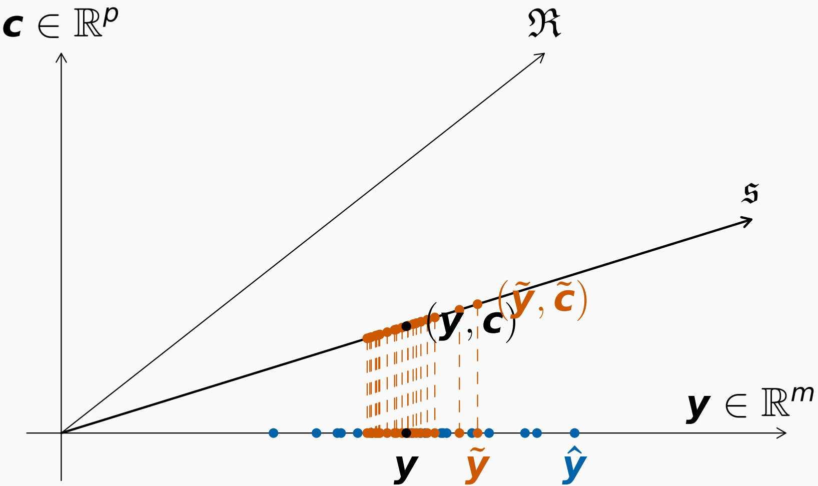
Geometry of FLAP



Geometry of FLAP



Geometry of FLAP



Example $\boldsymbol{W}_h = \boldsymbol{I}_{m+p}$

$$
\begin{aligned}
\mathrm{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \mathrm{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}) &= \boldsymbol{J}\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C}\boldsymbol{J}' \\
&= \boldsymbol{\Phi}'(\boldsymbol{\Phi}\boldsymbol{\Phi}' + \boldsymbol{I})^{-1}\boldsymbol{\Phi}
\end{aligned}
$$

Let $\boldsymbol{\Phi}$ consist of orthogonal unit vectors:

$$
\begin{aligned}
\boldsymbol{\Phi}\boldsymbol{\Phi}' &= \boldsymbol{I}_p \text{ when } p \leq m \\
\boldsymbol{\Phi}'\boldsymbol{\Phi} &= \boldsymbol{I}_m \text{ when } p = m.
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{tr}(\mathrm{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \mathrm{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h})) \\
= \frac{1}{2} tr(\boldsymbol{\Phi}'\boldsymbol{\Phi}) = \frac{1}{2} p
\end{aligned}
$$

Positive condition

For the first component to have a guaranteed reduction of forecast error variance, the following condition must be satisfied:

$$\phi_1 \boldsymbol{W}_{y,h} \neq \boldsymbol{w}_{c_1 y,h},$$

- ϕ_1 is the weight vector of the first component
- $\boldsymbol{W}_{y,h} = \mathrm{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})$
- $\boldsymbol{w}_{c_1 y,h}$ is the forecast error covariance between the first component and the original series.

Monotonicity

The forecast error variance reductions, i.e. the diagonal elements of

$$\begin{aligned}\mathrm{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \mathrm{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}) \\ = \boldsymbol{J}\boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{J}'\end{aligned}$$

is non-decreasing as p increases.

Minimum variance of individual series

The projection is equivalent to the mapping

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{G}\hat{\boldsymbol{z}}_{t+h},$$

where $\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_1 & \boldsymbol{g}_2 & \dots & \boldsymbol{g}_m \end{bmatrix}' \in \mathbb{R}^{m\times(m+p)}$ is the solution to

$$\arg\min_{\boldsymbol{G}} \boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}' \qquad \text{s.t. } \boldsymbol{G}\boldsymbol{S} = \boldsymbol{I}$$

or

$$\arg\min_{\boldsymbol{g}_i} \boldsymbol{g}_i'\boldsymbol{W}_h\boldsymbol{g}_i \qquad \text{s.t. } \boldsymbol{g}_i'\boldsymbol{s}_j = \boldsymbol{1}(i=j),$$

where $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{I}_m \\ \boldsymbol{\Phi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_1 \cdots \boldsymbol{s}_m \end{bmatrix}$.

Key results

1. The forecast error variance is **reduced** with FLAP
2. The forecast error variance **monotonically** decreases with increasing number of components
3. The forecast projection is **optimal** to achieve minimum forecast error variance of each series

In practice, we need to

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{J}\tilde{\boldsymbol{z}}_{t+h} = \boldsymbol{J}\boldsymbol{M}\hat{\boldsymbol{z}}_{t+h}$$

$$\boldsymbol{M} = \boldsymbol{I}_{m+p} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

$$\boldsymbol{W}_h = \mathrm{Var}(\boldsymbol{z}_{t+h} - \hat{\boldsymbol{z}}_{t+h})$$
$$\boldsymbol{C} = \begin{bmatrix} -\Phi & \boldsymbol{I}_p \end{bmatrix}$$

- Estimate $\boldsymbol{W}_h$
- Construct Φ

Estimation of W_h

Shrinking variance towards their median (Opgen-Rhein and Strimmer 2007) and **shrinking covariance** towards zero (Schäfer and Strimmer 2005).

The shrinkage estimator is

- Positive definite, and
- Numerically stable.

In empirical applications, we assume

$$\widehat{\boldsymbol{W}}_h^{shr} = \eta_h \widehat{\boldsymbol{W}}_1^{shr}.$$

Construction of Φ

Principal component analysis (PCA)

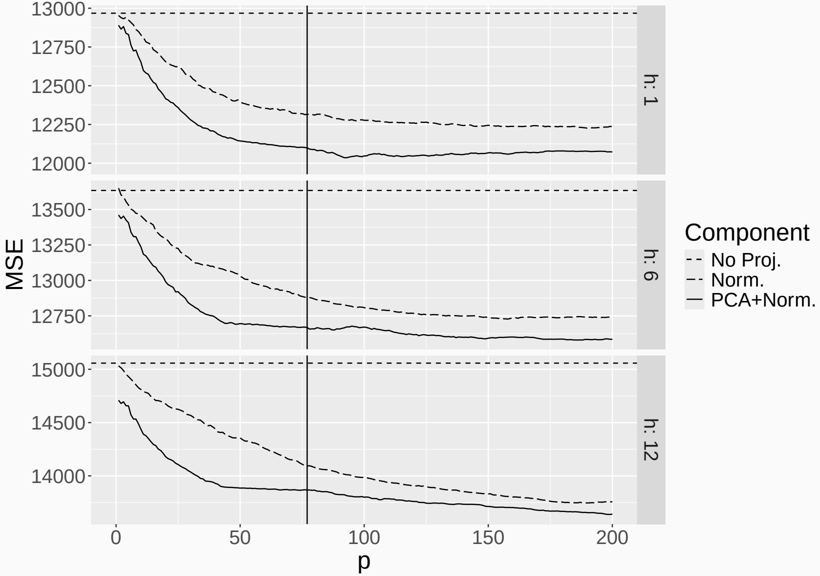
Finding the weights matrix so that the resulting components **maximise variance**

Simulation

Generating values from a random distribution and normalising them to unit vectors

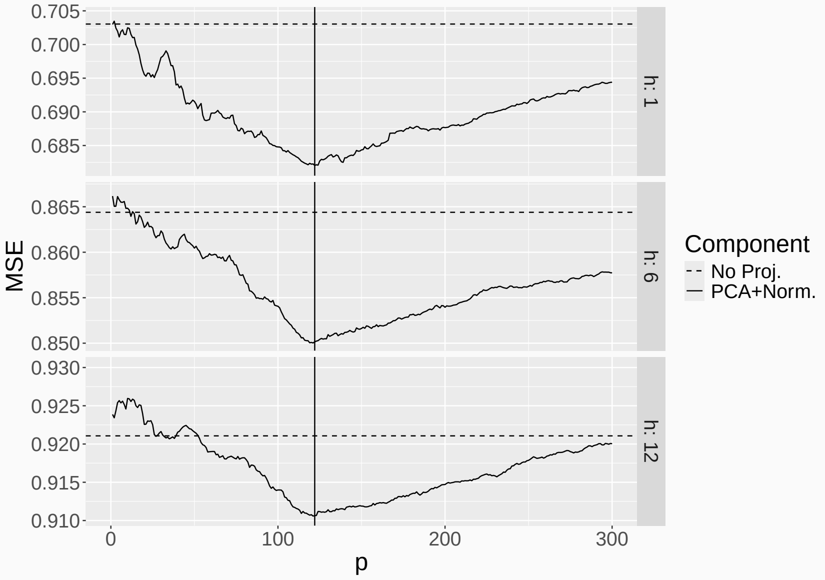
- Normal distribution
- Uniform distribution
- Orthonormal matrix (Borchers 2023)

Tourism (ETS)



13000
12750
12500
12250
12000
13500
13250
13000
12750
15000
14500
14000
MSE
h: 1
h: 6
h: 12
0
50
100
150
200
p
Component
No Proj.
Norm.
PCA+Norm.

FRED-MD (DFM)

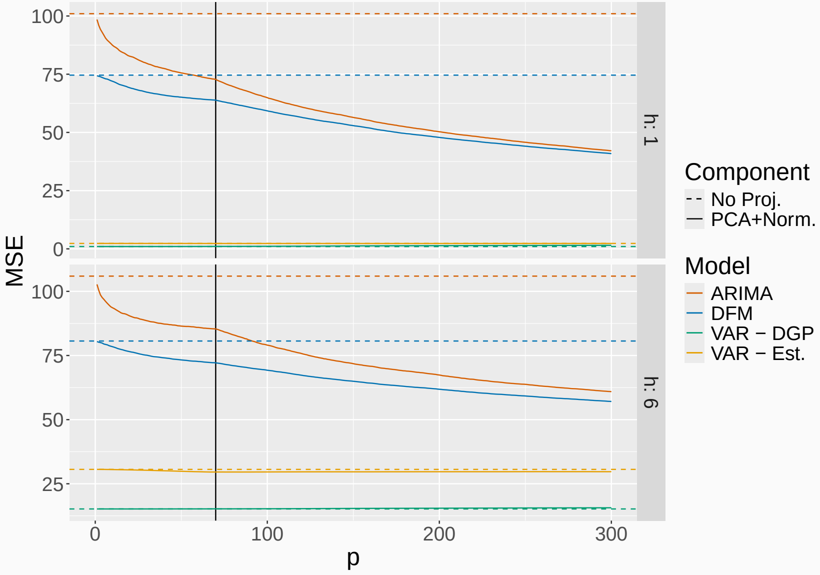


0.705
0.700
0.695
0.690
0.685
h: 1
0.865
0.860
0.855
0.850
h: 6
0.930
0.925
0.920
0.915
0.910
h: 12
MSE
0
100
200
300
p
Component
No Proj.
PCA+Norm.

Simulation

- Data generating process (DGP): VAR(3) with $m = 70$ variables
- Sample size: $T = 400$
- Number of repeated samples: 220
- Base model: ARIMA and DFM

Simulation



MSE
p
h: 1
h: 6
100
75
50
25
0
0
100
200
300
Component
No Proj.
PCA+Norm.
Model
ARIMA
DFM
VAR − DGP
VAR − Est.

R Package flap

You can install the stable version from CRAN

```r
## CRAN.R-project.org/package=flap
install.packages("flap")
```

or the development version from Github

```r
## github.com/FinYang/flap
# install.packages("remotes")
remotes::install_github("FinYang/flap")
```

Working paper and contact

- yangzhuoranyang.com/publication/flap/

- Email: `Fin.Yang@monash.edu`
- yangzhuoranyang.com

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