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Dimension Reduction in Stochastic Optimal Control

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Outline

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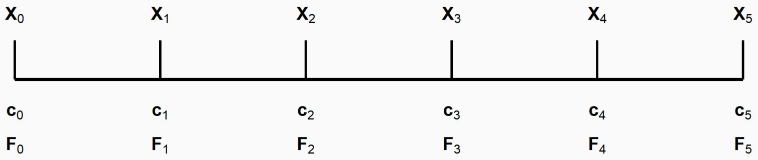
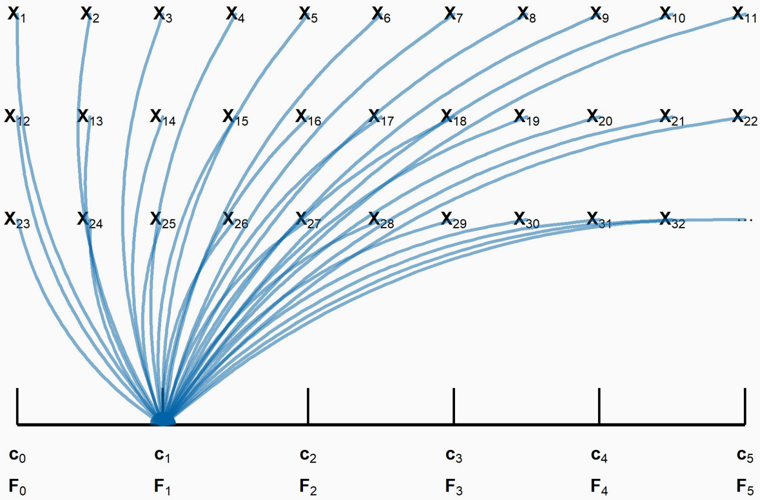
Our setting

In a **finite time horizon**, with available **assets** X_t and **information** F_t (e.g. past returns) at each time point, we try to optimise the **objective** function with a certain utility function with respect to the **control variable** c_t (e.g. consumption, proportion of money invested in each assets).



Our setting



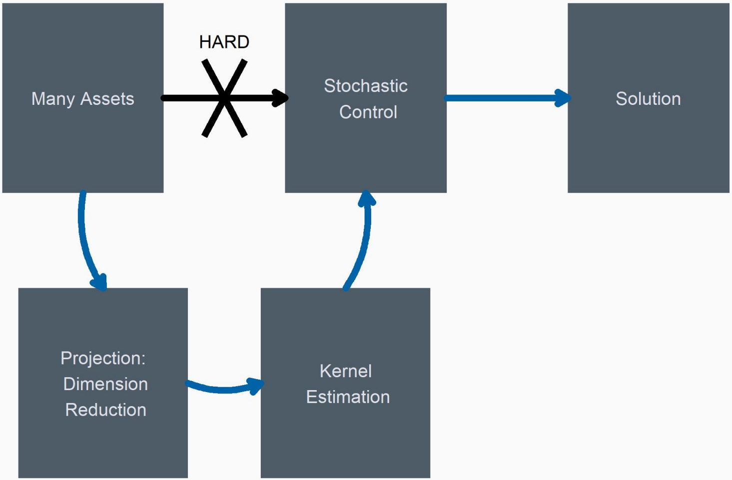
X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8 X_9 X_{10} X_{11}
X_{12} X_{13} X_{14} X_{15} X_{16} X_{17} X_{18} X_{19} X_{20} X_{21} X_{22}
X_{23} X_{24} X_{25} X_{26} X_{27} X_{28} X_{29} X_{30} X_{31} X_{32} $\cdots$
c_0 c_1 c_2 c_3 c_4 c_5
F_0 F_1 F_2 F_3 F_4 F_5

Goals

Goals

Developing an algorithm that achieves the **optimal portfolio selection** w.r.t. the objective utility function in an **optimal control setting** using **dimension reduction**, where the risky assets are projected onto one risky portfolio using linear regression.

The Big Picture



Significance

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- Overcoming the curse of dimensionality.
- Reducing the required computational power.
- Filling the gap to utilise the dynamics of portfolio selection in the stochastic control theory.

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- Reducing the required computational power.
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Significance

- We can do it when others cannot.
- We do it much faster than others.
- We can do it better.

Outline

Projection

The focus of portfolio selection has been on risk minimization.

$$\min_{\omega^T \mathbf{1}=1} Var(\omega^T \mathbf{R}) = \min_{\omega^T \mathbf{1}=1} \omega^T \mathbf{\Sigma} \omega$$

(Fan, Zhang, and Yu 2012)

where **R** is the return vector; **Σ** is its associated covariance matrix; ω is its portfolio allocation vector.

Projection

$$
\begin{aligned}
&\min_{\omega'\mathbf{1}=1} Var(\boldsymbol{\omega}'\mathbf{R}) \\
&= \min_{\boldsymbol{\omega}'\mathbf{1}=1,\xi} E(\boldsymbol{\omega}'\mathbf{R} - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^q,\xi} E((1 - \sum_{i=2}^q \omega_i)R_1 + \omega_2 R_2 + \cdots + \omega_q R_q - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^q,\xi} E(R_1 - \omega_2(R_1 - R_2) \cdots - \omega_q(R_1 - R_q) - \xi)^2 \\
&= \min_{\{\omega_i\}_{i=2}^q,\xi} E(R_1 - (\xi + \sum_{i=2}^q \omega_i(R_1 - R_i)))^2
\end{aligned}
$$

Projection

$$\min_{\boldsymbol{\omega}'\mathbf{1}=1} Var(\boldsymbol{\omega}'\boldsymbol{R}) = \min_{\{\omega_i\}_{i=2}^{q},\xi} E(R_1 - (\xi + \sum_{i=2}^{q} \omega_i(R_1 - R_i)))^2$$

Solving by OLS

$$R_t^r = \boldsymbol{\omega}_t'\mathbf{R}_t$$

Dimension reduced.

Single Index

Evolution of wealth

$$W_{t+1} = (W_t - \beta_t) \cdot R^f + \beta_t \cdot R^r_t - C_{t+1}$$

where W_t is the wealth at time t; R^f is the accumulation factor for risk free asset from time t to $t+1$; R^r_t is the accumulation factor for the risky portfolio from time t to $t+1$; β_t is the amount of wealth invested in the risky portfolio at time t; C_{t+1} is the consumption made at time $t+1$.

Single Index

The value function at time t is

$$f_t(W_t) = \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^{s-t} \cdot (C_s^2 - 2\lambda C_s) + \delta^{T-t} \cdot (W_T^2 - 2\lambda W_T) | \mathcal{F}_t]$$

with terminal condition

$$f_T(W_T) = W_T^2 - 2\lambda W_T$$

We have used a variance-mean utility function.

Single Index

Solving single index

1. Rewrite in the Bellman equation format, we have $V_t(W_t) = \inf_{\mathbf{c}_t, \beta_t} u(W_t, c_t, \beta_t) + \mathbf{E}[V_{t+1}(W_{t+1})]$
2. Mathematical induction:
   a. At time $T - 1$, solve the FOC of the Bellman equation
   b. Assume it is true at time $t + 1$, solve the FOC of the Bellman equation at time t

Single Index

Define

$$\widetilde{W_t} = W_t - \lambda \qquad c_t = C_t - \lambda$$

$$J_t = \begin{bmatrix} R_t^r - R^f \\ -1 \end{bmatrix} \qquad Z_t = \begin{bmatrix} \beta_t \\ c_{t+1} \end{bmatrix} \qquad \mathbb{I}^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Single Index

Define auxiliary variable

HDGF

$$
\begin{cases}
H_t = \mathbb{I}^{22} + D_{t+1} E[J_t J_t'] \\
D_t = \delta (R^f)^2 D_{t+1} \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t]) \\
G_t = (\delta R^f G_{t+1} + \delta R^f (R^f - 2) \lambda D_{t+1}) \\
\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t]) \\
F_t = (\delta (R^f - 2)^2 \lambda^2 D_{t+1} + 2\delta (R^f - 2) \lambda G_{t+1} + \delta F_{t+1}) \\
\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t])
\end{cases}
$$

Solution

Theorem

For $t = 0, 1, \cdots, T-1$, the optimal value for the aforementioned problem is

$$f_t(W_t) = \widetilde{W}_t^2 D_t + 2\widetilde{W}_t G_t + F_t - \lambda^2\left(\sum_{s=t}^{T-1} \delta^{s+1-t} + \delta^{T-t}\right)$$

and the corresponding optimal strategy is given by

$$Z_t = -H_t^{-1}(D_{t+1}(\widetilde{W}_t R^f + (R^f - 2)\lambda) + G_{t+1})E[J_t]$$

Solution

HDGF

$$
\begin{cases}
H_t = \mathbb{I}^{22} + D_{t+1} E[J_t J_t'] \\
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\qquad \cdot (1 - D_{t+1} E[J_t]' H_t^{-1} E[J_t])
\end{cases}
$$

Kernel Estimation

To find the empirical distribution $F_t^n(y; \beta_t)$, we employ the nonparametric kernel estimation.

Kernel estimator

$$\hat{f}_\eta(y) = \frac{1}{\eta n} \sum_{i=1}^{n} \kappa(\frac{y - y_i}{\eta})$$

where η is the smoothing parameter bandwidth and the kernel function κ the Gaussian kernel:

$$\kappa(u) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}u^2)$$

Lower Bound

As the kernel estimation and the projection approach have been employed, the set of control vector $\{c_i\}_{i=t}^{T-1}$ can be viewed as a sub-optimal to the problem. Given a sub-optimal $\{c_i^*\}_{i=t}^{T-1}$ obtained from the above steps, we can obtain a upper bound of the process

Upper bound

$$\overline{V}_t^*(x_t) = \mathbb{E}[\sum_{j=t}^{T-1} \delta^{j-t} u(X_j, c_j^*) + U(X_T)\delta^{T-t}]$$

Lower bound

We know that

$$\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T \geq \min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T$$

By Jensen's inequality, we can get

$$\inf_{\mathbf{c}_0} \mathbb{E}[\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] \geq \mathbb{E}[\min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0]$$

Lower bound

By definition

$$\inf_{\mathbf{c}} \mathbb{E}[\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] = V_0(x_0)$$

Lower bound

By definition

$$\inf_{\mathbf{c}} \mathbb{E}[\sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] = V_0(x_0)$$

Theorem

We have an lower bound of the value

$$\mathbb{E}[\min_{\mathbf{c}_0} \sum_{t=0}^{T-1} \delta^t u(X_t, c_t) + U(X_T)\delta^T | X_0] \leq V_0(x_0)$$

and furthermore when utility functions are linear, we have the equality.

Outline

Competing method: EM

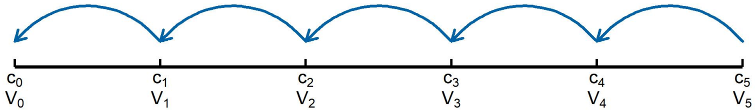
In each iteration

1. Simulate the return of the assets forward in time and calculates the corresponding wealth of the individual using the current parameters
2. Go backward in time to update the parameters of the control variables by optimizing the objective function

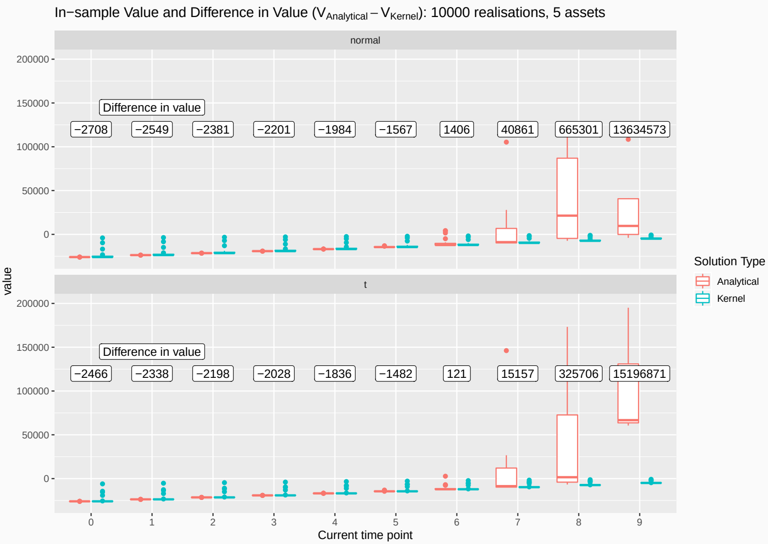
Competing method: EM

E step: Simulate forward in time

M step: Optimise backward in time



The accuracy of Kernel Estimator



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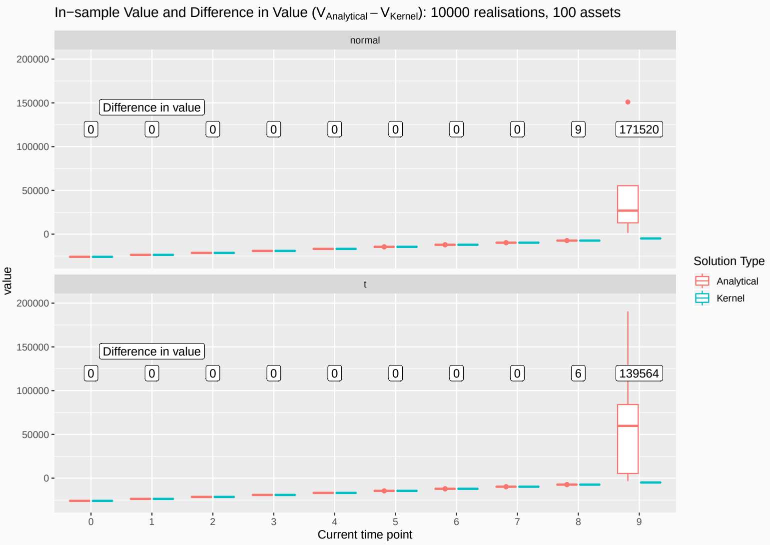
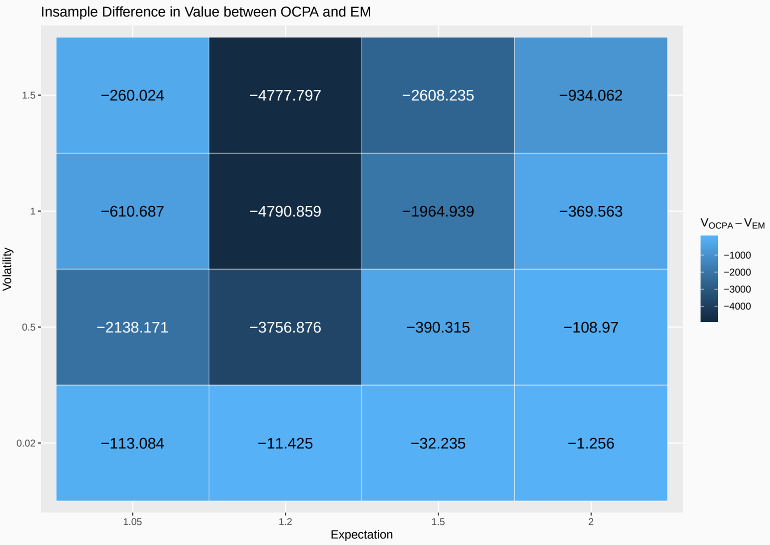


Comparison in Simulation



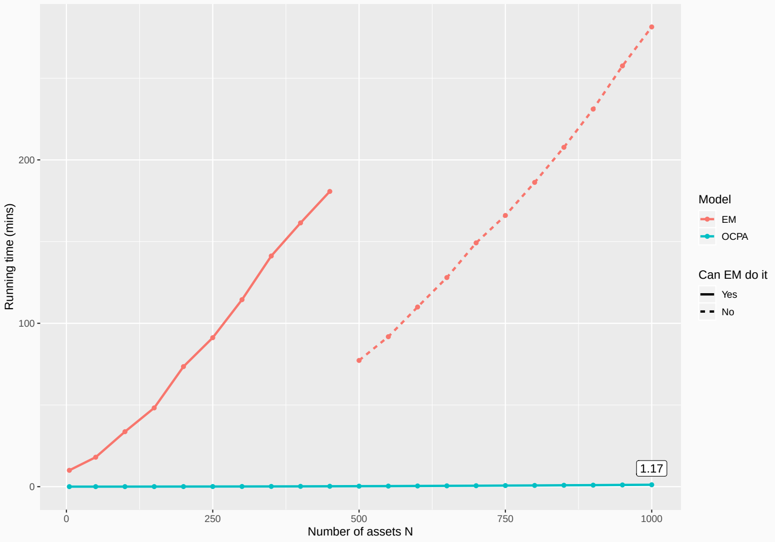
Lower bound

- Number of realisation = 5000
- Number of assets = 5

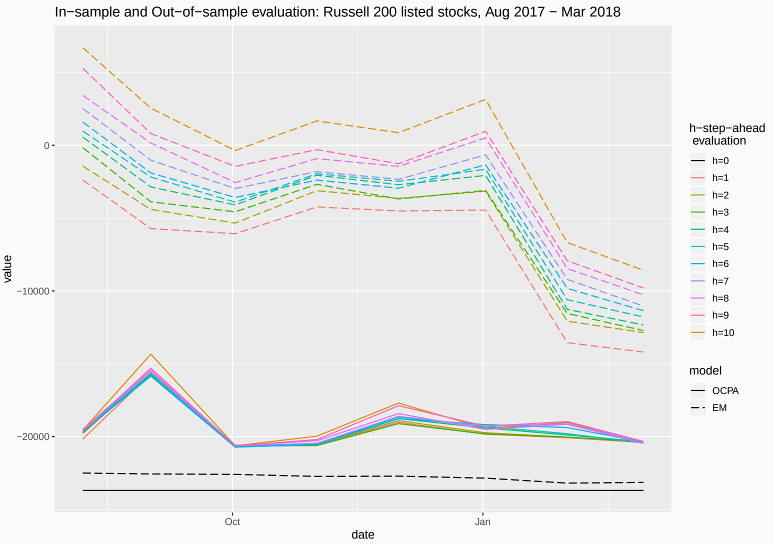
Table 1: Out-of-sample Evaluation

OCPA	EM	Lower.bound
55244.93	135794.4	-25694.6

Time difference



Real data and prediction



R package `stocon`

The package is in development. You can install the development version

```
devtools::install_github(FinYang/stocon)
```

The documentation can be found at

`https://pkg.yangzhuoranyang.com/stocon`

References i

Fan, Jianqing, Jingjin Zhang, and Ke Yu. 2012. "Vast Portfolio Selection with Gross-Exposure Constraints." *Journal of the American Statistical Association* 107 (498): 592–606. `https://doi.org/10.1080/01621459.2012.682825`.