In a decentralized market economy, all the agents make decisions and maximize their utilities independently. The resource allocation is all coordinated by market prices.

In a planned economy, resource allocation is through direct administrative orders and there is no voluntary transaction.

The second welfare theorem allows us to characterize the allocations in the decentralized general equilibrium by resorting to an artificial social planner problem.This social planner problem is a simple optimization problem which does not involve any prices and hence much easier to tackle.

Consider the following social planners problem

$$\max_{\{c_{A1},c_{A2},c_{B1},c_{B2}\}} \mu_A\left[\frac{1}{3}\ln(c_{A1})+\frac{2}{3}\ln(c_{A2})\right]+\mu_B\left[\frac{2}{3}\ln(c_{B1})+\frac{1}{3}\ln(c_{B2})\right]$$

subject to

$$c_{A1}+c_{B1}\leq 4,\ c_{A2}+c_{B2}\leq 4$$

We now set up the Lagrangian:

$$\begin{aligned}\mathcal{L}=\mu_A&\left[\frac{1}{3}\ln(c_{A1})+\frac{2}{3}\ln(c_{A2})\right]+\mu_B\left[\frac{2}{3}\ln(c_{B1})+\frac{1}{3}\ln(c_{B2})\right]\\&+\lambda_1(5-c_{A1}-c_{B1})+\lambda_2(3-c_{A2}-c_{B2})\end{aligned}$$

First order conditions(FOC):

$$\begin{cases}[c_{A1}]:\frac{\partial\mathcal{L}}{\partial c_{A1}}=0\Rightarrow\frac{1\mu_A}{3}\frac{1}{c_{A1}}=\lambda_1\Rightarrow c_{A1}=\frac{1\mu_A}{3\lambda_1} & (1)\\ [c_{A2}]:\frac{\partial\mathcal{L}}{\partial c_{A2}}=0\Rightarrow\frac{2\mu_A}{3}\frac{1}{c_{A2}}=\lambda_2\Rightarrow c_{A2}=\frac{2\mu_A}{3\lambda_2} & (2)\\ [c_{B1}]:\frac{\partial\mathcal{L}}{\partial c_{B1}}=0\Rightarrow\frac{2\mu_B}{3}\frac{1}{c_{B1}}=\lambda_1\Rightarrow c_{B1}=\frac{2\mu_B}{3\lambda_1} & (3)\\ [c_{B2}]:\frac{\partial\mathcal{L}}{\partial c_{B2}}=0\Rightarrow\frac{1\mu_B}{3}\frac{1}{c_{B2}}=\lambda_2\Rightarrow c_{B1}=\frac{1\mu_B}{3\lambda_2} & (4)\end{cases}$$

We take equal sign in the constraint since we wold not waste any resources,

$$\begin{cases}c_{A1}+c_{B1}=4 & (5)\\ c_{A2}+c_{B2}=4 & (6)\end{cases}$$

Notice that $\mu_A+\mu_B=1$, (1)(2)(3)(4) together with the resource constraint, we get

$$\begin{cases}\lambda_1=\frac{2-\mu_A}{12} & (7)\\ \lambda_2=\frac{1+\mu_A}{12} & (8)\end{cases}$$

In a competitive equilibrium households' choices are constrained by the budget constraint, while the planner is only concerned with resource balance. Hence, the last step is to ask which Pareto efficient allocations would be affordable for all households. We will introduce two methods.

Method 1: We can considerate the price in a decentralized market and write the budget constraint for A and B:

$$\begin{cases} \varphi_1 c_{A1} + \varphi_2 c_{A2} = 2\varphi_1 + 2\varphi_2 \Rightarrow \frac{\varphi 1}{\varphi 2} = -\frac{c_{A2}-2}{c_{A1}-2} & (9) \\ \varphi_1 c_{B1} + \varphi_2 c_{B2} = 2\varphi_1 + 2\varphi_2 \Rightarrow \frac{\varphi 1}{\varphi 2} = -\frac{c_{B2}-2}{c_{B1}-2} & (10) \end{cases}$$

We do not need to solve the general equilibrium as before, which is complex and not our purpose. But we do know that in Lecture 10, we used to learn equation(10.8) in page 151,

$$\frac{\pi_s u'(c_s)}{\pi_{s'} u'(c_{s'})} = \frac{\varphi_s}{\varphi_{s'}}$$

Using the equation, together with(9)(10), we get

$$\begin{cases} \frac{\varphi 1}{\varphi 2} = -\frac{c_{A2}-2}{c_{A1}-2} = \frac{\frac{1}{3}\frac{1}{c_{A1}}}{\frac{2}{3}\frac{1}{c_{A2}}} \Rightarrow (1 - \frac{2}{c_{A1}}) + 2(1 - \frac{2}{c_{A2}}) = 0 \\ \frac{\varphi 1}{\varphi 2} = -\frac{c_{B2}-2}{c_{B1}-2} = \frac{\frac{2}{3}\frac{1}{c_{B1}}}{\frac{1}{3}\frac{1}{c_{B2}}} \Rightarrow 2(1 - \frac{2}{c_{B1}}) + (1 - \frac{2}{c_{B2}}) = 0 \end{cases}$$

Substituting (1)(2)(3)(4) to the above equality, we obtain that

$$\begin{cases} 3 = 2\frac{1}{c_{A1}} + 4\frac{1}{c_{A2}} = 2\frac{3\lambda_1}{\mu_A} + 4\frac{3\lambda_2}{2\mu_A} = 0 \\ 3 = 4\frac{1}{c_{B1}} + 2\frac{1}{c_{B2}} = 4\frac{3\lambda_1}{\mu_B} + 2\frac{3\lambda_2}{2\mu_B} = 0 \end{cases}$$

Substituting (7)(8) to the above equality, we solve the pareto weight:

$$\mu_A = 0.5, \mu_B = 0.5$$

Method 2:Define the transfer functions $t_i(\mu)$ by

$$\begin{cases} t_A(\mu) = \lambda_1(c_{A1} - e_{A1}) + \lambda_2(c_{A2} - e_{A2}) = \lambda_1(c_{A1} - 2) + \lambda_2(c_{A2} - 2) \\ t_B(\mu) = \lambda_1(c_{B1} - e_{B1}) + \lambda_2(c_{B2} - e_{B2}) = \lambda_1(c_{B1} - 2) + \lambda_2(c_{B2} - 2) \end{cases}$$

The number t_i is the amount of the numeraire good that agent i would need as transfer in order to be able to afford the Pareto effcient allocation.

To find the competitive equilibrium allocation we now need to find the Pareto weight μ such that $t_1 = t_2 = 0$. That is to say,

$$\begin{cases} \lambda_1(c_{A1} - 2) + \lambda_2(c_{A2} - 2) = 0 \\ \lambda_1(c_{B1} - 2) + \lambda_2(c_{B2} - 2) = 0 \end{cases}$$

Substituting (1)(2)(3)(4)(7)(8) to the above equality, we can also solve the pareto weight:

$$\mu_A = 0.5, \mu_B = 0.5$$

As long as we solve find the pareto weight in the social planner problem, we can solve the problem easily.

$$\lambda_1 = \frac{1}{8}, \lambda_2 = \frac{1}{8}$$

$$c_{A1} = \frac{4}{3}, c_{A2} = \frac{8}{3}, \quad c_{AB1} = \frac{8}{3}, \quad c_{B2} = \frac{4}{3}$$

To summarize, to compute competitive equilibria using the social planner one does the following

1. Solve the social planners problem for Pareto efficient allocations indexed by Pareto weight μ.
2. Compute transfers, indexed by μ, necessary to make the efficient allocation affordable. As prices use Lagrange multipliers on the resource constraints in the planners' problem.
3. Find the Pareto weight(s) $\hat{\mu}$ that makes the transfer functions 0.
4. The Pareto e±cient allocations corresponding to $\hat{\mu}$ are equilibrium allocations; the supporting equilibrium prices are (multiples of) the Lagrange multipliers from the planning problem.