

1. (1) 由于

$${}^0J = \begin{bmatrix} {}^0R_n & 0 \\ 0 & {}^0R_n \end{bmatrix} {}^nJ$$

其中 0R_n 为正交矩阵, 故

$$\begin{aligned} {}^nJ &= \begin{bmatrix} {}^0R_n & 0 \\ 0 & {}^0R_n \end{bmatrix}^{-1} {}^0J \\ &= \begin{bmatrix} {}^0R_n^{-1} & 0 \\ 0 & {}^0R_n^{-1} \end{bmatrix} {}^0J = \begin{bmatrix} {}^0R_n^T & 0 \\ 0 & {}^0R_n^T \end{bmatrix} {}^0J \end{aligned}$$

下面求解基座雅可比矩阵 0J

基于 D-H 参数, 求得各关节齐次变换矩阵如下:

$$\begin{aligned} {}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^1T_2 &= \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & a_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & a_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^2T_3 &= \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & a_3 \cos \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & a_3 \sin \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

进一步可计算得：

$$\begin{aligned}
{}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_2 &= \begin{bmatrix} \cos \theta_1 \cos \theta_2 & -\cos \theta_1 \sin \theta_2 & -\sin \theta_1 & a_2 \cos \theta_1 \cos \theta_2 \\ \sin \theta_1 \cos \theta_2 & -\sin \theta_1 \sin \theta_2 & \cos \theta_1 & a_2 \sin \theta_1 \cos \theta_2 \\ -\sin \theta_2 & -\cos \theta_2 & 0 & d_1 - a_2 \sin \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_3 &= \begin{bmatrix} \cos \theta_1 \cos(\theta_2 + \theta_3) & -\cos \theta_1 \sin(\theta_2 + \theta_3) & -\sin \theta_1 & \cos \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ \sin \theta_1 \cos(\theta_2 + \theta_3) & -\sin \theta_1 \sin(\theta_2 + \theta_3) & \cos \theta_1 & \sin \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ -\sin(\theta_2 + \theta_3) & -\cos(\theta_2 + \theta_3) & 0 & d_1 - a_2 \sin \theta_2 - a_3 \sin(\theta_2 + \theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

故构造基座雅可比矩阵所需各参数如下：

$$\begin{aligned}
{}^0\xi_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & {}^0\xi_1 &= \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ 0 \end{bmatrix} & {}^0\xi_2 &= \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ 0 \end{bmatrix} \\
{}^0p_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & {}^0p_1 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_2 &= \begin{bmatrix} a_2 \cos \theta_1 \cos \theta_2 \\ a_2 \sin \theta_1 \cos \theta_2 \\ d_1 - a_2 \sin \theta_2 \end{bmatrix} \\
{}^0p_3 &= \begin{bmatrix} \cos \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ \sin \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ d_1 - a_2 \sin \theta_2 - a_3 \sin(\theta_2 + \theta_3) \end{bmatrix}
\end{aligned}$$

构造得基座雅可比矩阵各分块如下：

$$\begin{aligned}
 {}^0J_{v1} &= {}^0\xi_0 \times ({}^0p_3 - {}^0p_0) = \begin{bmatrix} -\sin\theta_1(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ \cos\theta_1(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ 0 \end{bmatrix} \\
 {}^0J_{v2} &= {}^0\xi_1 \times ({}^0p_3 - {}^0p_1) = \begin{bmatrix} -\cos\theta_1(a_2\sin\theta_2 + a_3\sin(\theta_2 + \theta_3)) \\ -\sin\theta_1(a_2\sin\theta_2 + a_3\sin(\theta_2 + \theta_3)) \\ -(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \end{bmatrix} \\
 {}^0J_{v3} &= {}^0\xi_2 \times ({}^0p_3 - {}^0p_2) = \begin{bmatrix} -a_3\cos\theta_1\sin(\theta_2 + \theta_3) \\ -a_3\sin\theta_1\sin(\theta_2 + \theta_3) \\ -a_3\cos(\theta_2 + \theta_3) \end{bmatrix} \\
 {}^0J_{\omega1} &= {}^0\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 {}^0J_{\omega2} &= {}^0\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \\
 {}^0J_{\omega3} &= {}^0\xi_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}
 \end{aligned}$$

为转换到末端雅可比矩阵，计算可得：

$$\begin{aligned}
 {}^0R_n^\top {}^0J_{v1} &= \begin{bmatrix} 0 \\ 0 \\ a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3) \end{bmatrix} \\
 {}^0R_n^\top {}^0J_{v2} &= \begin{bmatrix} a_2 \sin \theta_3 \\ a_2 \cos \theta_3 + a_3 \\ 0 \end{bmatrix} \\
 {}^0R_n^\top {}^0J_{v3} &= \begin{bmatrix} 0 \\ a_3 \\ 0 \end{bmatrix} \\
 {}^0R_n^\top {}^0J_{\omega 1} &= \begin{bmatrix} -\sin(\theta_2 + \theta_3) \\ -\cos(\theta_2 + \theta_3) \\ 0 \end{bmatrix} \\
 {}^0R_n^\top {}^0J_{\omega 2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 {}^0R_n^\top {}^0J_{\omega 3} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
 \end{aligned}$$

最终可得末端雅可比矩阵为

$${}^nJ = \begin{bmatrix} 0 & a_2 \sin \theta_3 & 0 \\ 0 & a_2 \cos \theta_3 + a_3 & a_3 \\ a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3) & 0 & 0 \\ -\sin(\theta_2 + \theta_3) & 0 & 0 \\ -\cos(\theta_2 + \theta_3) & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(2) 将 $\theta_1 = 30^\circ, \theta_2 = 50^\circ, \theta_3 = 20^\circ$ 代入雅可比矩阵表达式得:

$${}^n J_v = \begin{bmatrix} 0 & a_2 \sin(20^\circ) & 0 \\ 0 & a_2 \cos(20^\circ) + a_3 & a_3 \\ a_2 \cos(50^\circ) + a_3 \cos(70^\circ) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1710 & 0 \\ 0 & 0.9698 & 0.5 \\ 0.4924 & 0 & 0 \end{bmatrix}$$

对其进行 SVD 分解可得:

$${}^n J_v = U \Sigma V^\top, \quad \Sigma = \begin{bmatrix} 1.1016 & 0 & 0 \\ 0 & 0.4924 & 0 \\ 0 & 0 & 0.0776 \end{bmatrix}$$

由此可得:

最小奇异值 0.0776

条件数 $\frac{\sigma_1}{\sigma_3} = 14.19$

可操作度数 $\sigma_1 \sigma_2 \sigma_3 = 0.0421$

(3) 将 $\theta_1 = 5^\circ, \theta_2 = 30^\circ, \theta_3 = 40^\circ$ 代入雅可比矩阵表达式得:

$${}^n J_v = \begin{bmatrix} 0 & a_2 \sin(40^\circ) & 0 \\ 0 & a_2 \cos(40^\circ) + a_3 & a_3 \\ a_2 \cos(30^\circ) + a_3 \cos(70^\circ) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.3214 & 0 \\ 0 & 0.8830 & 0.5 \\ 0.6040 & 0 & 0 \end{bmatrix}$$

对其进行 SVD 分解可得:

$${}^n J_v = U \Sigma V^\top, \quad \Sigma = \begin{bmatrix} 1.0535 & 0 & 0 \\ 0 & 0.6040 & 0 \\ 0 & 0 & 0.1525 \end{bmatrix}$$

由此可得:

最小奇异值 0.1525

条件数 $\frac{\sigma_1}{\sigma_3} = 6.91$

可操作度数 $\sigma_1 \sigma_2 \sigma_3 = 0.0971$

2. (1) 基于 D-H 参数, 求得各关节齐次变换矩阵如下:

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1T_2 = \begin{bmatrix} \cos \theta_2 & 0 & -\sin \theta_2 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^2T_3 = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

进一步可计算得:

$${}^0T_1 = \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_2 = \begin{bmatrix} \cos \theta_1 \cos \theta_2 & \sin \theta_1 & -\cos \theta_1 \sin \theta_2 & 0 \\ \sin \theta_1 \cos \theta_2 & -\cos \theta_1 & -\sin \theta_1 \sin \theta_2 & 0 \\ -\sin \theta_2 & 0 & -\cos \theta_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0T_3 = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_3 + s\theta_1 s\theta_3 & -c\theta_1 c\theta_2 s\theta_3 + s\theta_1 c\theta_3 & -c\theta_1 s\theta_2 & -d_3 c\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 c\theta_3 - c\theta_1 s\theta_3 & -s\theta_1 c\theta_2 s\theta_3 - c\theta_1 c\theta_3 & -s\theta_1 s\theta_2 & -d_3 s\theta_1 s\theta_2 \\ -s\theta_2 c\theta_3 & s\theta_2 s\theta_3 & -c\theta_2 & d_1 - d_3 c\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

故构造基座雅可比矩阵所需各参数如下：

$$\begin{aligned}
 {}^0\xi_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & {}^0\xi_1 &= \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} & {}^0\xi_2 &= \begin{bmatrix} -\cos\theta_1 \sin\theta_2 \\ -\sin\theta_1 \sin\theta_2 \\ -\cos\theta_2 \end{bmatrix} \\
 {}^0p_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & {}^0p_1 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_2 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_3 &= \begin{bmatrix} -d_3 \cos\theta_1 \sin\theta_2 \\ -d_3 \sin\theta_1 \sin\theta_2 \\ d_1 - d_3 \cos\theta_2 \end{bmatrix}
 \end{aligned}$$

构造得基座雅可比矩阵各分块如下：

$$\begin{aligned}
 {}^0J_{v1} &= {}^0\xi_0 \times ({}^0p_3 - {}^0p_0) = \begin{bmatrix} d_3 \sin\theta_1 \sin\theta_2 \\ -d_3 \cos\theta_1 \sin\theta_2 \\ 0 \end{bmatrix} \\
 {}^0J_{v2} &= {}^0\xi_1 \times ({}^0p_3 - {}^0p_1) = \begin{bmatrix} -d_3 \cos\theta_1 \cos\theta_2 \\ -d_3 \sin\theta_1 \cos\theta_2 \\ d_3 \sin\theta_2 \end{bmatrix} \\
 {}^0J_{v3} &= {}^0\xi_2 \times ({}^0p_3 - {}^0p_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
 {}^0J_{\omega1} &= {}^0\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
 {}^0J_{\omega2} &= {}^0\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \\
 {}^0J_{\omega3} &= {}^0\xi_2 = \begin{bmatrix} -\cos\theta_1 \sin\theta_2 \\ -\sin\theta_1 \sin\theta_2 \\ -\cos\theta_2 \end{bmatrix}
 \end{aligned}$$

最终可得基座雅可比矩阵为

$${}^0J = \begin{bmatrix} d_3 \sin \theta_1 \sin \theta_2 & -d_3 \cos \theta_1 \cos \theta_2 & 0 \\ -d_3 \cos \theta_1 \sin \theta_2 & -d_3 \sin \theta_1 \cos \theta_2 & 0 \\ 0 & d_3 \sin \theta_2 & 0 \\ 0 & -\sin \theta_1 & -\cos \theta_1 \sin \theta_2 \\ 0 & \cos \theta_1 & -\sin \theta_1 \sin \theta_2 \\ 1 & 0 & -\cos \theta_2 \end{bmatrix}$$

(2) 将 $\theta_1 = 60^\circ, \theta_2 = 100^\circ, \theta_3 = 50^\circ$ 代入基座角速度雅可比分块表达式得:

$${}^0J_\omega = \begin{bmatrix} 0 & -\sin(60^\circ) & -\cos(60^\circ) \sin(100^\circ) \\ 0 & \cos(60^\circ) & -\sin(60^\circ) \sin(100^\circ) \\ 1 & 0 & -\cos(100^\circ) \end{bmatrix} = \begin{bmatrix} 0 & -0.8660 & -0.4924 \\ 0 & 0.5000 & -0.8529 \\ 1.0000 & 0 & 0.1736 \end{bmatrix}$$

对其进行 SVD 分解可得:

$${}^0J_\omega = U \Sigma V^\top, \quad \Sigma = \begin{bmatrix} 1.0834 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 0.9090 \end{bmatrix}$$

由此可得:

最小奇异值 0.9090

条件数 $\frac{\sigma_1}{\sigma_3} = 1.19$

可操作度数 $\sigma_1 \sigma_2 \sigma_3 = 0.9848$

(3) 将 $\theta_1 = 50^\circ, \theta_2 = 178^\circ, \theta_3 = 60^\circ$ 代入基座角速度雅可比分块表达式得：

$${}^0J_\omega = \begin{bmatrix} 0 & -\sin(50^\circ) & -\cos(50^\circ)\sin(178^\circ) \\ 0 & \cos(50^\circ) & -\sin(50^\circ)\sin(178^\circ) \\ 1 & 0 & -\cos(178^\circ) \end{bmatrix} = \begin{bmatrix} 0 & -0.7660 & -0.0224 \\ 0 & 0.6428 & -0.0267 \\ 1.0000 & 0 & 0.9994 \end{bmatrix}$$

对其进行 SVD 分解可得：

$${}^0J_\omega = U \Sigma V^\top, \quad \Sigma = \begin{bmatrix} 1.4140 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 0.0247 \end{bmatrix}$$

由此可得：

最小奇异值 0.0247

条件数 $\frac{\sigma_1}{\sigma_3} = 57.29$

可操作度数 $\sigma_1\sigma_2\sigma_3 = 0.0349$