

1. (1) 由于

$${}^0J = \begin{bmatrix} {}^0R_n & 0 \\ 0 & {}^0R_n \end{bmatrix} {}^nJ$$

其中 0R_n 为正交矩阵, 故

$$\begin{aligned} {}^nJ &= \begin{bmatrix} {}^0R_n & 0 \\ 0 & {}^0R_n \end{bmatrix}^{-1} {}^0J \\ &= \begin{bmatrix} {}^0R_n^{-1} & 0 \\ 0 & {}^0R_n^{-1} \end{bmatrix} {}^0J = \begin{bmatrix} {}^0R_n^T & 0 \\ 0 & {}^0R_n^T \end{bmatrix} {}^0J \end{aligned}$$

下面求解基座雅可比矩阵 0J

基于 D-H 参数, 求得各关节齐次变换矩阵如下:

$$\begin{aligned} {}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^1T_2 &= \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 & a_2 \cos \theta_2 \\ \sin \theta_2 & \cos \theta_2 & 0 & a_2 \sin \theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ {}^2T_3 &= \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & a_3 \cos \theta_3 \\ \sin \theta_3 & \cos \theta_3 & 0 & a_3 \sin \theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

进一步可计算得：

$$\begin{aligned}
{}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_2 &= \begin{bmatrix} \cos \theta_1 \cos \theta_2 & -\cos \theta_1 \sin \theta_2 & -\sin \theta_1 & a_2 \cos \theta_1 \cos \theta_2 \\ \sin \theta_1 \cos \theta_2 & -\sin \theta_1 \sin \theta_2 & \cos \theta_1 & a_2 \sin \theta_1 \cos \theta_2 \\ -\sin \theta_2 & -\cos \theta_2 & 0 & d_1 - a_2 \sin \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0T_3 &= \begin{bmatrix} \cos \theta_1 \cos(\theta_2 + \theta_3) & -\cos \theta_1 \sin(\theta_2 + \theta_3) & -\sin \theta_1 & \cos \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ \sin \theta_1 \cos(\theta_2 + \theta_3) & -\sin \theta_1 \sin(\theta_2 + \theta_3) & \cos \theta_1 & \sin \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ -\sin(\theta_2 + \theta_3) & -\cos(\theta_2 + \theta_3) & 0 & d_1 - a_2 \sin \theta_2 - a_3 \sin(\theta_2 + \theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

故构造基座雅可比矩阵所需各参数如下：

$$\begin{aligned}
{}^0\xi_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & {}^0\xi_1 &= \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ 0 \end{bmatrix} & {}^0\xi_2 &= \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ 0 \end{bmatrix} \\
{}^0p_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & {}^0p_1 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_2 &= \begin{bmatrix} a_2 \cos \theta_1 \cos \theta_2 \\ a_2 \sin \theta_1 \cos \theta_2 \\ d_1 - a_2 \sin \theta_2 \end{bmatrix} \\
{}^0p_3 &= \begin{bmatrix} \cos \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ \sin \theta_1 (a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3)) \\ d_1 - a_2 \sin \theta_2 - a_3 \sin(\theta_2 + \theta_3) \end{bmatrix}
\end{aligned}$$

构造得基座雅可比矩阵各分块如下：

$${}^0J_{v1} = {}^0\xi_0 \times ({}^0p_3 - {}^0p_0) = \begin{bmatrix} -\sin\theta_1(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ \cos\theta_1(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \\ 0 \end{bmatrix}$$

$${}^0J_{v2} = {}^0\xi_1 \times ({}^0p_3 - {}^0p_1) = \begin{bmatrix} -\cos\theta_1(a_2\sin\theta_2 + a_3\sin(\theta_2 + \theta_3)) \\ -\sin\theta_1(a_2\sin\theta_2 + a_3\sin(\theta_2 + \theta_3)) \\ -(a_2\cos\theta_2 + a_3\cos(\theta_2 + \theta_3)) \end{bmatrix}$$

$${}^0J_{v3} = {}^0\xi_2 \times ({}^0p_3 - {}^0p_2) = \begin{bmatrix} -a_3\cos\theta_1\sin(\theta_2 + \theta_3) \\ -a_3\sin\theta_1\sin(\theta_2 + \theta_3) \\ -a_3\cos(\theta_2 + \theta_3) \end{bmatrix}$$

$${}^0J_{\omega1} = {}^0\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0J_{\omega2} = {}^0\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}$$

$${}^0J_{\omega3} = {}^0\xi_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}$$

为转换到末端雅可比矩阵，计算可得：

$${}^0R_n^\top {}^0J_{v1} = \begin{bmatrix} 0 \\ 0 \\ a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3) \end{bmatrix}$$

$${}^0R_n^\top {}^0J_{v2} = \begin{bmatrix} a_2 \sin \theta_3 \\ a_2 \cos \theta_3 + a_3 \\ 0 \end{bmatrix}$$

$${}^0R_n^\top {}^0J_{v3} = \begin{bmatrix} 0 \\ a_3 \\ 0 \end{bmatrix}$$

$${}^0R_n^\top {}^0J_{\omega 1} = \begin{bmatrix} -\sin(\theta_2 + \theta_3) \\ -\cos(\theta_2 + \theta_3) \\ 0 \end{bmatrix}$$

$${}^0R_n^\top {}^0J_{\omega 2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0R_n^\top {}^0J_{\omega 3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

最终可得末端雅可比矩阵为

$${}^nJ = \begin{bmatrix} 0 & a_2 \sin \theta_3 & 0 \\ 0 & a_2 \cos \theta_3 + a_3 & a_3 \\ a_2 \cos \theta_2 + a_3 \cos(\theta_2 + \theta_3) & 0 & 0 \\ -\sin(\theta_2 + \theta_3) & 0 & 0 \\ -\cos(\theta_2 + \theta_3) & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

2. (1) 基于 D-H 参数，求得各关节齐次变换矩阵如下：

$$\begin{aligned}
 {}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^1T_2 &= \begin{bmatrix} \cos(\theta_2 - 90^\circ) & 0 & -\sin(\theta_2 - 90^\circ) & 0 \\ \sin(\theta_2 - 90^\circ) & 0 & \cos(\theta_2 - 90^\circ) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} \sin \theta_2 & 0 & \cos \theta_2 & 0 \\ -\cos \theta_2 & 0 & \sin \theta_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^2T_3 &= \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

进一步可计算得：

$$\begin{aligned}
 {}^0T_1 &= \begin{bmatrix} \cos \theta_1 & 0 & -\sin \theta_1 & 0 \\ \sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^0T_2 &= \begin{bmatrix} \cos \theta_1 \sin \theta_2 & \sin \theta_1 & \cos \theta_1 \cos \theta_2 & 0 \\ \sin \theta_1 \sin \theta_2 & -\cos \theta_1 & \sin \theta_1 \cos \theta_2 & 0 \\ \cos \theta_2 & 0 & -\sin \theta_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 {}^0T_3 &= \begin{bmatrix} c\theta_1 s\theta_2 c\theta_3 + s\theta_1 s\theta_3 & -c\theta_1 s\theta_2 s\theta_3 + s\theta_1 c\theta_3 & c\theta_1 c\theta_2 & d_3 c\theta_1 c\theta_2 \\ s\theta_1 s\theta_2 c\theta_3 - c\theta_1 s\theta_3 & -s\theta_1 s\theta_2 s\theta_3 - c\theta_1 c\theta_3 & s\theta_1 c\theta_2 & d_3 s\theta_1 c\theta_2 \\ c\theta_2 c\theta_3 & -c\theta_2 s\theta_3 & -s\theta_2 & d_1 - d_3 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

故构造基座雅可比矩阵所需各参数如下：

$$\begin{aligned}
{}^0\xi_0 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & {}^0\xi_1 &= \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} & {}^0\xi_2 &= \begin{bmatrix} \cos\theta_1 \cos\theta_2 \\ \sin\theta_1 \cos\theta_2 \\ -\sin\theta_2 \end{bmatrix} \\
{}^0p_0 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & {}^0p_1 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_2 &= \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} & {}^0p_3 &= \begin{bmatrix} d_3 \cos\theta_1 \cos\theta_2 \\ d_3 \sin\theta_1 \cos\theta_2 \\ d_1 - d_3 \sin\theta_2 \end{bmatrix}
\end{aligned}$$

构造得基座雅可比矩阵各分块如下：

$$\begin{aligned}
{}^0J_{v1} &= {}^0\xi_0 \times ({}^0p_3 - {}^0p_0) = \begin{bmatrix} -d_3 \sin\theta_1 \cos\theta_2 \\ d_3 \cos\theta_1 \cos\theta_2 \\ 0 \end{bmatrix} \\
{}^0J_{v2} &= {}^0\xi_1 \times ({}^0p_3 - {}^0p_1) = \begin{bmatrix} -d_3 \cos\theta_1 \sin\theta_2 \\ -d_3 \sin\theta_1 \sin\theta_2 \\ -d_3 \cos\theta_2 \end{bmatrix} \\
{}^0J_{v3} &= {}^0\xi_2 \times ({}^0p_3 - {}^0p_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
{}^0J_{\omega1} &= {}^0\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
{}^0J_{\omega2} &= {}^0\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \\
{}^0J_{\omega3} &= {}^0\xi_2 = \begin{bmatrix} \cos\theta_1 \cos\theta_2 \\ \sin\theta_1 \cos\theta_2 \\ -\sin\theta_2 \end{bmatrix}
\end{aligned}$$

最终可得基座雅可比矩阵为

$${}^0J = \begin{bmatrix} -d_3 \sin \theta_1 \cos \theta_2 & -d_3 \cos \theta_1 \sin \theta_2 & 0 \\ d_3 \cos \theta_1 \cos \theta_2 & -d_3 \sin \theta_1 \sin \theta_2 & 0 \\ 0 & -d_3 \cos \theta_2 & 0 \\ 0 & -\sin \theta_1 & \cos \theta_1 \cos \theta_2 \\ 0 & \cos \theta_1 & \sin \theta_1 \cos \theta_2 \\ 1 & 0 & -\sin \theta_2 \end{bmatrix}$$