1. (1)

$$
\begin{aligned}
p_{ex} &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\
p_{ey} &= l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\
\psi_e &= \theta_1+\theta_2+\theta_3
\end{aligned}
$$

(2) 将 $\psi_e = \theta_1+\theta_2+\theta_3$ 代入 p_{ex}, p_{ey} 得

$$
\begin{aligned}
p_{ex} &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos\psi_e \\
p_{ey} &= l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin\psi_e
\end{aligned}
$$

适当移项后平方相加得

$$
(p_{ex} - l_3\cos\psi_e)^2 + (p_{ey} - l_3\sin\psi_e)^2 = l_1^2 + l_2^2 + 2l_1l_2\cos\theta_2
$$

解得

$$
\theta_2 = \pm\arccos\left(\frac{(p_{ex} - l_3\cos\psi_e)^2 + (p_{ey} - l_3\sin\psi_e)^2 - l_1^2 - l_2^2}{2l_1l_2}\right)
$$

将 θ_2 分别代入位置项有

$$
\begin{aligned}
p_{ex} - l_3\cos\psi_e &= (l_1 + l_2\cos\theta_2)\cos\theta_1 - l_2\sin\theta_2\sin\theta_1 \\
p_{ey} - l_3\sin\psi_e &= (l_1 + l_2\cos\theta_2)\sin\theta_1 + l_2\sin\theta_2\cos\theta_1
\end{aligned}
$$

令 $A = l_1 + l_2\cos\theta_2$，$B = l_2\sin\theta_2$，则上式可写为

$$
\begin{aligned}
p_{ex} - l_3\cos\psi_e &= A\cos\theta_1 - B\sin\theta_1 \\
p_{ey} - l_3\sin\psi_e &= A\sin\theta_1 + B\cos\theta_1
\end{aligned}
$$

这是标准的平面旋转形式，注意到这可以写成矩阵形式：

$$
\begin{bmatrix} p_{ex} - l_3\cos\psi_e \\ p_{ey} - l_3\sin\psi_e \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} \cos\theta_1 \\ \sin\theta_1 \end{bmatrix}
$$

其中 $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ 是旋转缩放矩阵。故其角度关系有

$$
\arctan 2(p_{ey} - l_3\sin\psi_e, p_{ex} - l_3\cos\psi_e) = \arctan 2(B, A) + \theta_1
$$

因此解得

$$\theta_1 = \arctan 2(p_{ey} - l_3 \sin\psi_e, p_{ex} - l_3 \cos\psi_e) - \arctan 2(l_2 \sin\theta_2, l_1 + l_2 \cos\theta_2)$$

最后易得

$$\theta_3 = \psi_e - \theta_1 - \theta_2$$

综上逆运动学求解结果为:

$$\theta_2 = \pm \arccos\left(\frac{(p_{ex} - l_3 \cos\psi_e)^2 + (p_{ey} - l_3 \sin\psi_e)^2 - l_1^2 - l_2^2}{2l_1 l_2}\right)$$

$$\theta_1 = \arctan 2(p_{ey} - l_3 \sin\psi_e, p_{ex} - l_3 \cos\psi_e) - \arctan 2(l_2 \sin\theta_2, l_1 + l_2 \cos\theta_2)$$

$$\theta_3 = \psi_e - \theta_1 - \theta_2$$

2. (1)

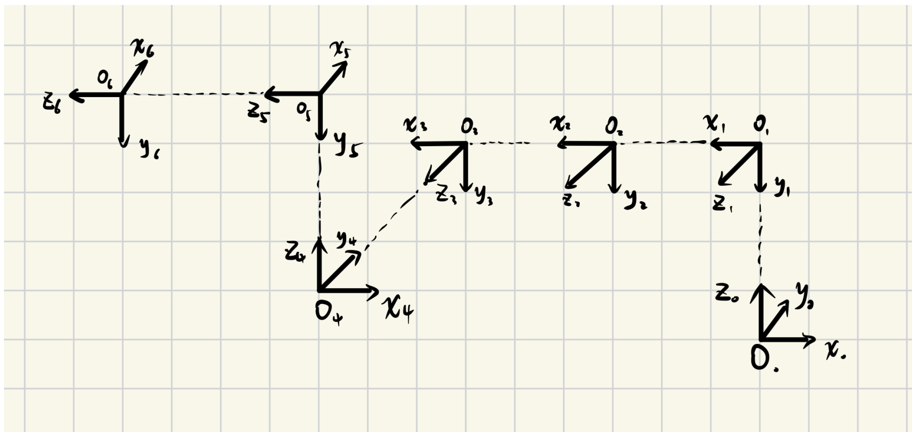

图 1: DH 坐标系

连杆 i	θ_i	d_i	a_i	α_i
1	$180°$	162.5	0	$-90°$
2	0	0	425	0
3	0	0	392.2	0
4	$180°$	133.3	0	$-90°$
5	$90°$	99.7	0	$-90°$
6	0	99.6	0	0

表 1: DH 参数表

(2)

$$
{}^{0}T_1 = \begin{bmatrix} \cos(\theta_1 + 180^\circ) & -\sin(\theta_1 + 180^\circ) & 0 & 0 \\ \sin(\theta_1 + 180^\circ) & \cos(\theta_1 + 180^\circ) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 162.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90^\circ) & -\sin(-90^\circ) & 0 \\ 0 & \sin(-90^\circ) & \cos(-90^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\cos\theta_1 & 0 & \sin\theta_1 & 0 \\ -\sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & -1 & 0 & 162.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 425 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 425\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & 425\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 392.2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 392.2\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & 392.2\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}T_{4} = \begin{bmatrix} \cos(\theta_4 + 180°) & -\sin(\theta_4 + 180°) & 0 & 0 \\ \sin(\theta_4 + 180°) & \cos(\theta_4 + 180°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 133.3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90°) & -\sin(-90°) & 0 \\ 0 & \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\cos\theta_4 & 0 & \sin\theta_4 & 0 \\ -\sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & -1 & 0 & 133.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}T_{5} = \begin{bmatrix} \cos(\theta_5 + 90°) & -\sin(\theta_5 + 90°) & 0 & 0 \\ \sin(\theta_5 + 90°) & \cos(\theta_5 + 90°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 99.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-90°) & -\sin(-90°) & 0 \\ 0 & \sin(-90°) & \cos(-90°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ \cos\theta_5 & 0 & -\sin\theta_5 & 0 \\ 0 & -1 & 0 & 99.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}T_{6} = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 99.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 99.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

故

$$
{}^0T_1 = \begin{bmatrix} -\cos\theta_1 & 0 & \sin\theta_1 & 0 \\ -\sin\theta_1 & 0 & -\cos\theta_1 & 0 \\ 0 & -1 & 0 & 162.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1T_4 = {}^1T_2 \cdot {}^2T_3 \cdot {}^3T_4 = \begin{bmatrix} -\cos(\theta_2+\theta_3+\theta_4) & 0 & \sin(\theta_2+\theta_3+\theta_4) & 425\cos\theta_2 + 392.2\cos(\theta_2+\theta_3) \\ -\sin(\theta_2+\theta_3+\theta_4) & 0 & -\cos(\theta_2+\theta_3+\theta_4) & 425\sin\theta_2 + 392.2\sin(\theta_2+\theta_3) \\ 0 & -1 & 0 & 133.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^4T_6 = {}^4T_5 \cdot {}^5T_6 = \begin{bmatrix} -\sin\theta_5\cos\theta_6 & \sin\theta_5\sin\theta_6 & -\cos\theta_5 & -99.6\cos\theta_5 \\ \cos\theta_5\cos\theta_6 & -\cos\theta_5\sin\theta_6 & -\sin\theta_5 & -99.6\sin\theta_5 \\ -\sin\theta_6 & -\cos\theta_6 & 0 & 99.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(3) 将 $[\theta_1\ \theta_2\ \theta_3\ \theta_4\ \theta_5\ \theta_6] = [16^\circ\ \ -124^\circ\ 63^\circ\ 152^\circ\ 88^\circ\ \ -166^\circ]$ 代入得

$$
\begin{aligned}
{}^0T_6 &= {}^0T_1 \cdot {}^1T_4 \cdot {}^4T_6 \\
&= \begin{bmatrix} -0.9612 & 0 & 0.2756 & 0 \\ -0.2756 & 0 & -0.9612 & 0 \\ 0 & -1 & 0 & 162.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.0174 & 0 & 0.9998 & -47.5146 \\ -0.9998 & 0 & 0.0174 & -695.3668 \\ 0 & -1 & 0 & 133.3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \begin{bmatrix} 0.9697 & -0.2417 & -0.0348 & -3.4759 \\ -0.0338 & 0.0084 & -0.9993 & -99.5393 \\ 0.2419 & 0.9703 & 0 & 99.7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -0.2397 & -0.9308 & 0.2760 & 14.0886 \\ -0.1038 & -0.2581 & -0.9605 & -238.1828 \\ 0.9653 & -0.2587 & -0.0349 & 852.6513 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

位置矢量

$$P = \begin{bmatrix} 14.0886 \\ -238.1828 \\ 852.6513 \end{bmatrix}$$

对于 XYZ 欧拉角有

$$\begin{aligned} R &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.2397 & -0.9308 & 0.2760 \\ -0.1038 & -0.2581 & -0.9605 \\ 0.9653 & -0.2587 & -0.0349 \end{bmatrix} \end{aligned}$$

解得

$$\beta = \arcsin(0.2760) = 16.02^\circ \text{ 或}\beta = 180^\circ - 16.02^\circ = 163.98^\circ$$

当 $\beta_1 = 16.02^\circ$ 时

$$\begin{aligned} \alpha_1 &= \arctan 2\left(-\frac{a_{23}}{\cos\beta_1}, \frac{a_{33}}{\cos\beta_1}\right) = 92.08^\circ \\ \gamma_1 &= \arctan 2\left(-\frac{a_{12}}{\cos\beta_1}, \frac{a_{11}}{\cos\beta_1}\right) = 104.45^\circ \end{aligned}$$

当 $\beta_2 = 163.98^\circ$ 时

$$\begin{aligned} \alpha_2 &= \arctan 2\left(-\frac{a_{23}}{\cos\beta_2}, \frac{a_{33}}{\cos\beta_2}\right) = -87.92^\circ \\ \gamma_2 &= \arctan 2\left(-\frac{a_{12}}{\cos\beta_2}, \frac{a_{11}}{\cos\beta_2}\right) = -75.55^\circ \end{aligned}$$

综上，XYZ 欧拉角解为

$$\Psi = \begin{bmatrix} 92.08^\circ \\ 16.02^\circ \\ 104.45^\circ \end{bmatrix} \text{ 或}\Psi = \begin{bmatrix} -87.92^\circ \\ 163.98^\circ \\ -75.55^\circ \end{bmatrix}$$