1. (1) 由于

$$^0J = \begin{bmatrix} ^0R_n & 0 \\ 0 & ^0R_n \end{bmatrix} {}^nJ$$

其中 ^0R_n 为正交矩阵，故

$$\begin{aligned} ^nJ &= \begin{bmatrix} ^0R_n & 0 \\ 0 & ^0R_n \end{bmatrix}^{-1} {}^0J \\ &= \begin{bmatrix} ^0R_n^{-1} & 0 \\ 0 & ^0R_n^{-1} \end{bmatrix} {}^0J \quad = \begin{bmatrix} ^0R_n^T & 0 \\ 0 & ^0R_n^T \end{bmatrix} {}^0J \end{aligned}$$

下面求解基座雅可比矩阵 ^0J

基于 D-H 参数，求得各关节齐次变换矩阵如下：

$$^0T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_3\cos\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & a_3\sin\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

进一步可计算得:

$$
{}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_2 = \begin{bmatrix} \cos\theta_1\cos\theta_2 & -\cos\theta_1\sin\theta_2 & -\sin\theta_1 & a_2\cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 & -\sin\theta_1\sin\theta_2 & \cos\theta_1 & a_2\sin\theta_1\cos\theta_2 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & d_1 - a_2\sin\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_3 = \begin{bmatrix} \cos\theta_1\cos(\theta_2+\theta_3) & -\cos\theta_1\sin(\theta_2+\theta_3) & -\sin\theta_1 & \cos\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ \sin\theta_1\cos(\theta_2+\theta_3) & -\sin\theta_1\sin(\theta_2+\theta_3) & \cos\theta_1 & \sin\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ -\sin(\theta_2+\theta_3) & -\cos(\theta_2+\theta_3) & 0 & d_1 - a_2\sin\theta_2 - a_3\sin(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

故构造基座雅可比矩阵所需各参数如下:

$$
{}^{0}\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad {}^{0}\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \quad {}^{0}\xi_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}
$$

$$
{}^{0}p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^{0}p_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \quad {}^{0}p_2 = \begin{bmatrix} a_2\cos\theta_1\cos\theta_2 \\ a_2\sin\theta_1\cos\theta_2 \\ d_1 - a_2\sin\theta_2 \end{bmatrix}
$$

$$
{}^{0}p_3 = \begin{bmatrix} \cos\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ \sin\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ d_1 - a_2\sin\theta_2 - a_3\sin(\theta_2+\theta_3) \end{bmatrix}
$$

构造得基座雅可比矩阵各分块如下：

$$
{}^0J_{v1} = {}^0\xi_0 \times ({}^0p_3 - {}^0p_0) = \begin{bmatrix} -\sin\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ \cos\theta_1\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \\ 0 \end{bmatrix}
$$

$$
{}^0J_{v2} = {}^0\xi_1 \times ({}^0p_3 - {}^0p_1) = \begin{bmatrix} -\cos\theta_1\big(a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3)\big) \\ -\sin\theta_1\big(a_2\sin\theta_2 + a_3\sin(\theta_2+\theta_3)\big) \\ -\big(a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3)\big) \end{bmatrix}
$$

$$
{}^0J_{v3} = {}^0\xi_2 \times ({}^0p_3 - {}^0p_2) = \begin{bmatrix} -a_3\cos\theta_1\sin(\theta_2+\theta_3) \\ -a_3\sin\theta_1\sin(\theta_2+\theta_3) \\ -a_3\cos(\theta_2+\theta_3) \end{bmatrix}
$$

$$
{}^0J_{\omega1} = {}^0\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
{}^0J_{\omega2} = {}^0\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}
$$

$$
{}^0J_{\omega3} = {}^0\xi_2 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}
$$

为转换到末端雅可比矩阵，计算可得：

$$
\begin{aligned}
{}^0R_n^\top\,{}^0J_{v1} &= \begin{bmatrix} 0 \\ 0 \\ a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) \end{bmatrix} \\
{}^0R_n^\top\,{}^0J_{v2} &= \begin{bmatrix} a_2\sin\theta_3 \\ a_2\cos\theta_3 + a_3 \\ 0 \end{bmatrix} \\
{}^0R_n^\top\,{}^0J_{v3} &= \begin{bmatrix} 0 \\ a_3 \\ 0 \end{bmatrix} \\
{}^0R_n^\top\,{}^0J_{\omega 1} &= \begin{bmatrix} -\sin(\theta_2+\theta_3) \\ -\cos(\theta_2+\theta_3) \\ 0 \end{bmatrix} \\
{}^0R_n^\top\,{}^0J_{\omega 2} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
{}^0R_n^\top\,{}^0J_{\omega 3} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}
$$

最终可得末端雅可比矩阵为

$$
{}^nJ = \begin{bmatrix}
0 & a_2\sin\theta_3 & 0 \\
0 & a_2\cos\theta_3 + a_3 & a_3 \\
a_2\cos\theta_2 + a_3\cos(\theta_2+\theta_3) & 0 & 0 \\
-\sin(\theta_2+\theta_3) & 0 & 0 \\
-\cos(\theta_2+\theta_3) & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}
$$

(2) 将 $\theta_1 = 30°, \theta_2 = 50°, \theta_3 = 20°$ 代入雅可比矩阵表达式得：

$$
{}^{n}J_v = \begin{bmatrix} 0 & a_2\sin(20°) & 0 \\ 0 & a_2\cos(20°) + a_3 & a_3 \\ a_2\cos(50°) + a_3\cos(70°) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.1710 & 0 \\ 0 & 0.9698 & 0.5 \\ 0.4924 & 0 & 0 \end{bmatrix}
$$

对其进行 SVD 分解可得：

$$
{}^{n}J_v = U\,\Sigma\,V^{\top}, \quad \Sigma = \begin{bmatrix} 1.1016 & 0 & 0 \\ 0 & 0.4924 & 0 \\ 0 & 0 & 0.0776 \end{bmatrix}
$$

由此可得：
最小奇异值 0.0776
条件数 $\frac{\sigma_1}{\sigma_3} = 14.19$
可操作度数 $\sigma_1\sigma_2\sigma_3 = 0.0421$

(3) 将 $\theta_1 = 5°, \theta_2 = 30°, \theta_3 = 40°$ 代入雅可比矩阵表达式得：

$$
{}^{n}J_v = \begin{bmatrix} 0 & a_2\sin(40°) & 0 \\ 0 & a_2\cos(40°) + a_3 & a_3 \\ a_2\cos(30°) + a_3\cos(70°) & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.3214 & 0 \\ 0 & 0.8830 & 0.5 \\ 0.6040 & 0 & 0 \end{bmatrix}
$$

对其进行 SVD 分解可得：

$$
{}^{n}J_v = U\,\Sigma\,V^{\top}, \quad \Sigma = \begin{bmatrix} 1.0535 & 0 & 0 \\ 0 & 0.6040 & 0 \\ 0 & 0 & 0.1525 \end{bmatrix}
$$

由此可得：
最小奇异值 0.1525
条件数 $\frac{\sigma_1}{\sigma_3} = 6.91$
可操作度数 $\sigma_1\sigma_2\sigma_3 = 0.0971$

2. (1) 基于 D-H 参数，求得各关节齐次变换矩阵如下:

$$
{}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^{1}T_2 &= \begin{bmatrix} \cos(\theta_2 - 90^\circ) & 0 & -\sin(\theta_2 - 90^\circ) & 0 \\ \sin(\theta_2 - 90^\circ) & 0 & \cos(\theta_2 - 90^\circ) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ -\cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
{}^{2}T_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

进一步可计算得:

$$
{}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_2 = \begin{bmatrix} \cos\theta_1\sin\theta_2 & \sin\theta_1 & \cos\theta_1\cos\theta_2 & 0 \\ \sin\theta_1\sin\theta_2 & -\cos\theta_1 & \sin\theta_1\cos\theta_2 & 0 \\ \cos\theta_2 & 0 & -\sin\theta_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}T_3 = \begin{bmatrix} c\theta_1\, s\theta_2\, c\theta_3 + s\theta_1\, s\theta_3 & -c\theta_1\, s\theta_2\, s\theta_3 + s\theta_1\, c\theta_3 & c\theta_1\, c\theta_2 & d_3\, c\theta_1\, c\theta_2 \\ s\theta_1\, s\theta_2\, c\theta_3 - c\theta_1\, s\theta_3 & -s\theta_1\, s\theta_2\, s\theta_3 - c\theta_1\, c\theta_3 & s\theta_1\, c\theta_2 & d_3\, s\theta_1\, c\theta_2 \\ c\theta_2\, c\theta_3 & -c\theta_2\, s\theta_3 & -s\theta_2 & d_1 - d_3\, s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

故构造基座雅可比矩阵所需各参数如下:

$$
{}^{0}\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad {}^{0}\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix} \quad {}^{0}\xi_2 = \begin{bmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ -\sin\theta_2 \end{bmatrix}
$$

$$
{}^{0}p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^{0}p_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \quad {}^{0}p_2 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \quad {}^{0}p_3 = \begin{bmatrix} d_3\cos\theta_1\cos\theta_2 \\ d_3\sin\theta_1\cos\theta_2 \\ d_1 - d_3\sin\theta_2 \end{bmatrix}
$$

构造得基座雅可比矩阵各分块如下:

$$
{}^{0}J_{v1} = {}^{0}\xi_0 \times ({}^{0}p_3 - {}^{0}p_0) = \begin{bmatrix} -d_3\sin\theta_1\cos\theta_2 \\ d_3\cos\theta_1\cos\theta_2 \\ 0 \end{bmatrix}
$$

$$
{}^{0}J_{v2} = {}^{0}\xi_1 \times ({}^{0}p_3 - {}^{0}p_1) = \begin{bmatrix} -d_3\cos\theta_1\sin\theta_2 \\ -d_3\sin\theta_1\sin\theta_2 \\ -d_3\cos\theta_2 \end{bmatrix}
$$

$$
{}^{0}J_{v3} = {}^{0}\xi_2 \times ({}^{0}p_3 - {}^{0}p_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
{}^{0}J_{\omega 1} = {}^{0}\xi_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
{}^{0}J_{\omega 2} = {}^{0}\xi_1 = \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{bmatrix}
$$

$$
{}^{0}J_{\omega 3} = {}^{0}\xi_2 = \begin{bmatrix} \cos\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 \\ -\sin\theta_2 \end{bmatrix}
$$

最终可得基座雅可比矩阵为

$$
{}^{0}J = \begin{bmatrix} -d_3 \sin\theta_1 \cos\theta_2 & -d_3 \cos\theta_1 \sin\theta_2 & 0 \\ d_3 \cos\theta_1 \cos\theta_2 & -d_3 \sin\theta_1 \sin\theta_2 & 0 \\ 0 & -d_3 \cos\theta_2 & 0 \\ 0 & -\sin\theta_1 & \cos\theta_1 \cos\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1 \cos\theta_2 \\ 1 & 0 & -\sin\theta_2 \end{bmatrix}
$$