1.

位置:

$$\text{B 相对于 \{A\} 为 } \begin{bmatrix} -100 & 200 & 120 \end{bmatrix}^\top,$$
$$\text{C 相对于 \{A\} 为 } \begin{bmatrix} 150 & 220 & -120 \end{bmatrix}^\top$$

姿态:

$${}^A R_B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$
$${}^A R_C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

方向余弦矩阵为对应旋转变换矩阵转置

$${}^A C_B = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$${}^A C_C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2.

$$^{a}R_b = R_z(-10^\circ)R_x(25^\circ)R_y(30^\circ)$$

$$^{b}R_c = R_z(5^\circ)R_x(15^\circ)R_y(20^\circ)$$

$$\begin{aligned}
^{a}R_c &= {^{a}R_b}\,{^{b}R_c} \\
&= R_z(-10^\circ)R_x(25^\circ)R_y(30^\circ)R_z(5^\circ)R_x(15^\circ)R_y(20^\circ) \\
&= \begin{bmatrix} \cos(-10^\circ) & -\sin(-10^\circ) & 0 \\ \sin(-10^\circ) & \cos(-10^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(25^\circ) & -\sin(25^\circ) \\ 0 & \sin(25^\circ) & \cos(25^\circ) \end{bmatrix} \begin{bmatrix} \cos(30^\circ) & 0 & \sin(30^\circ) \\ 0 & 1 & 0 \\ -\sin(30^\circ) & 0 & \cos(30^\circ) \end{bmatrix} \\
&\quad \begin{bmatrix} \cos(5^\circ) & -\sin(5^\circ) & 0 \\ \sin(5^\circ) & \cos(5^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(15^\circ) & -\sin(15^\circ) \\ 0 & \sin(15^\circ) & \cos(15^\circ) \end{bmatrix} \begin{bmatrix} \cos(20^\circ) & 0 & \sin(20^\circ) \\ 0 & 1 & 0 \\ -\sin(20^\circ) & 0 & \cos(20^\circ) \end{bmatrix} \\
&= \begin{bmatrix} 0.7110 & 0.1875 & 0.6778 \\ 0.3531 & 0.7382 & -0.5747 \\ -0.6081 & 0.6480 & 0.4586 \end{bmatrix}
\end{aligned}$$

对应 ZXY 欧拉角:

$$\begin{aligned}
R &= R_z(\alpha)R_x(\beta)R_y(\gamma) \\
&= \begin{bmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\sin\alpha\cos\beta & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{bmatrix}
\end{aligned}$$

解得:

$\beta = \arcsin(0.6480) = 40^\circ$ 或 $\beta = 180^\circ - 40^\circ = 140^\circ$

$\alpha = \arctan 2(-0.1875, 0.7382) = -14^\circ$ 或 $arctan2(0.1875, -0.7382) = 166^\circ$

$\gamma = \arctan 2(0.6081, 0.4586) = 53^\circ$ 或 $\arctan 2(-0.6081, -0.4586) = -127^\circ$

对应 XYX 欧拉角:

$$R = R_x(\alpha)R_y(\beta)R_x(\gamma)$$
$$= \begin{bmatrix} \cos\beta & \sin\beta\sin\gamma & \sin\beta\cos\gamma \\ \sin\alpha\sin\beta & \cos\alpha\cos\gamma - \sin\alpha\cos\beta\sin\gamma & -\cos\alpha\sin\gamma - \sin\alpha\cos\beta\cos\gamma \\ -\cos\alpha\sin\beta & \sin\alpha\cos\gamma + \cos\alpha\cos\beta\sin\gamma & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma \end{bmatrix}$$

解得:

$$\beta = \pm\arccos(0.7110) = 45^\circ$$
$$\alpha = \arctan 2(0.3531, 0.6081) = 30^\circ \text{ 或 } \arctan 2(-0.3531, -0.6081) = -150^\circ$$
$$\gamma = \arctan 2(0.1875, 0.6778) = 15^\circ \text{ 或 } \arctan 2(-0.1875, -0.6778) = -165^\circ$$

即 {c} 相对于 {a} 的 ZXY 欧拉角为

$$\begin{bmatrix} -14^\circ \\ 40^\circ \\ 53^\circ \end{bmatrix} \begin{bmatrix} 166^\circ \\ 140^\circ \\ -127^\circ \end{bmatrix}$$

XYX 欧拉角为

$$\begin{bmatrix} 30^\circ \\ 45^\circ \\ 15^\circ \end{bmatrix} \begin{bmatrix} -150^\circ \\ -45^\circ \\ -165^\circ \end{bmatrix}$$

3.

(1)

$$
\begin{aligned}
{}^{a}R_b &= R_y(20^\circ)R_x(-30^\circ)R_y(40^\circ) \\
&= \begin{bmatrix} \cos(20^\circ) & 0 & \sin(20^\circ) \\ 0 & 1 & 0 \\ -\sin(20^\circ) & 0 & \cos(20^\circ) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-30^\circ) & -\sin(-30^\circ) \\ 0 & \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} \begin{bmatrix} \cos(40^\circ) & 0 & \sin(40^\circ) \\ 0 & 1 & 0 \\ -\sin(40^\circ) & 0 & \cos(40^\circ) \end{bmatrix} \\
&= \begin{bmatrix} 0.5295 & -0.1710 & 0.8309 \\ -0.3214 & 0.8660 & 0.3830 \\ -0.7851 & -0.4698 & 0.4036 \end{bmatrix}
\end{aligned}
$$

$$
{}^{a}T_b = \begin{bmatrix} 0.5295 & -0.1710 & 0.8309 & -100 \\ -0.3214 & 0.8660 & 0.3830 & 400 \\ -0.7851 & -0.4698 & 0.4036 & 150 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(2)

$$
{}^{b}\overline{p}_b = \begin{bmatrix} -20 \\ 30 \\ -30 \\ 1 \end{bmatrix}
$$

$$
\begin{aligned}
{}^{a}\overline{p}_a &= {}^{a}T_b \, {}^{b}\overline{p}_b \\
&= \begin{bmatrix} 0.5295 & -0.1710 & 0.8309 & -100 \\ -0.3214 & 0.8660 & 0.3830 & 400 \\ -0.7851 & -0.4698 & 0.4036 & 150 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -20 \\ 30 \\ -30 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -140.65 \\ 420.92 \\ 139.50 \\ 1 \end{bmatrix}
\end{aligned}
$$

(3)

$$\omega = \begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix} + R_y(20^\circ) \begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix} + R_y(20^\circ) R_x(-30^\circ) \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix}$$

其中

$$\begin{aligned} R_y(20^\circ) &= \begin{bmatrix} \cos(20^\circ) & 0 & \sin(20^\circ) \\ 0 & 1 & 0 \\ -\sin(20^\circ) & 0 & \cos(20^\circ) \end{bmatrix} \\ &= \begin{bmatrix} 0.9397 & 0 & 0.3420 \\ 0 & 1 & 0 \\ -0.3420 & 0 & 0.9397 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} R_y(20^\circ) R_x(-30^\circ) &= \begin{bmatrix} \cos(20^\circ) & 0 & \sin(20^\circ) \\ 0 & 1 & 0 \\ -\sin(20^\circ) & 0 & \cos(20^\circ) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-30^\circ) & -\sin(-30^\circ) \\ 0 & \sin(-30^\circ) & \cos(-30^\circ) \end{bmatrix} \\ &= \begin{bmatrix} 0.9397 & -0.1710 & 0.2962 \\ 0 & 0.8660 & 0.5000 \\ -0.3420 & -0.4698 & 0.8138 \end{bmatrix} \end{aligned}$$

代入得

$$\begin{aligned} \omega &= \begin{bmatrix} 0 \\ \dot{\alpha} \\ 0 \end{bmatrix} + \begin{bmatrix} 0.9397 & 0 & 0.3420 \\ 0 & 1 & 0 \\ -0.3420 & 0 & 0.9397 \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.9397 & -0.1710 & 0.2962 \\ 0 & 0.8660 & 0.5000 \\ -0.3420 & -0.4698 & 0.8138 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\gamma} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0.9397\dot{\beta} - 0.1710\dot{\gamma} \\ \dot{\alpha} + 0.8660\dot{\gamma} \\ -0.3420\dot{\beta} - 0.4698\dot{\gamma} \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -8 \end{bmatrix} \end{aligned}$$

解得

$$\begin{cases} \dot{\alpha} = -0.0597 \\ \dot{\beta} = 7.4347 \\ \dot{\gamma} = 11.6163 \end{cases}$$

即

$$\dot{\Psi} = \begin{bmatrix} -0.0597 \\ 7.4347 \\ 11.6163 \end{bmatrix}$$

(4) $\beta = 0^\circ$ 或 $\beta = 180^\circ$ 时，YXY 欧拉角姿态奇异