1.

$$\Psi = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

则有

$$\begin{aligned}
\omega &= \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} + R_z(\alpha) \begin{bmatrix} 0 \\ \dot{\beta} \\ 0 \end{bmatrix} + R_z(\alpha) R_y(\beta) \begin{bmatrix} \dot{\gamma} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\beta} \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \dot{\gamma} \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\alpha} + \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \dot{\beta} + \begin{bmatrix} \cos\alpha\cos\beta \\ \sin\alpha\cos\beta \\ -\sin\beta \end{bmatrix} \dot{\gamma} \\
&= \begin{bmatrix} 0 & -\sin\alpha & \cos\alpha\cos\beta \\ 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \end{bmatrix} \dot{\Psi}
\end{aligned}$$

当 $\beta \neq 90^\circ$ 时

$$\begin{aligned}
\Psi &= \begin{bmatrix} 0 & -\sin\alpha & \cos\alpha\cos\beta \\ 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \end{bmatrix}^{-1} \omega \\
&= \begin{bmatrix} \cos\alpha\tan\beta & \sin\alpha\tan\beta & 1 \\ -\sin\alpha & \cos\alpha & 0 \\ \frac{\cos\alpha}{\cos\beta} & \frac{\sin\alpha}{\cos\beta} & 0 \end{bmatrix} \omega
\end{aligned}$$

当 $\beta = 90^\circ$ 时，ZYX 欧拉角奇异。

2.

$$\omega = \begin{bmatrix}\dot\alpha\\0\\0\end{bmatrix} + R_x(20^\circ)\begin{bmatrix}0\\\dot\beta\\0\end{bmatrix} + R_x(20^\circ)R_y(-10^\circ)\begin{bmatrix}0\\0\\\dot\gamma\end{bmatrix}$$

$$= \begin{bmatrix}\dot\alpha\\0\\0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & \cos 20^\circ & -\sin 20^\circ\\0 & \sin 20^\circ & \cos 20^\circ\end{bmatrix}\begin{bmatrix}0\\\dot\beta\\0\end{bmatrix} + \begin{bmatrix}1 & 0 & 0\\0 & \cos 20^\circ & -\sin 20^\circ\\0 & \sin 20^\circ & \cos 20^\circ\end{bmatrix}\begin{bmatrix}\cos -10^\circ & 0 & \sin -10^\circ\\0 & 1 & 0\\-\sin -10^\circ & 0 & \cos -10^\circ\end{bmatrix}\begin{bmatrix}0\\0\\\dot\gamma\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix}\dot\alpha + \begin{bmatrix}0\\\cos 20^\circ\\\sin 20^\circ\end{bmatrix}\dot\beta + \begin{bmatrix}-\sin 10^\circ\\-\sin 20^\circ\cos 10^\circ\\\cos 20^\circ\cos 10^\circ\end{bmatrix}\dot\gamma$$

$$= \begin{bmatrix}1 & 0 & -\sin 10^\circ\\0 & \cos 20^\circ & -\sin 20^\circ\cos 10^\circ\\0 & \sin 20^\circ & \cos 20^\circ\cos 10^\circ\end{bmatrix}\dot\Psi$$

$$\Psi = \begin{bmatrix}1 & 0 & -\sin 10^\circ\\0 & \cos 20^\circ & -\sin 20^\circ\cos 10^\circ\\0 & \sin 20^\circ & \cos 20^\circ\cos 10^\circ\end{bmatrix}^{-1}\omega$$

$$= \begin{bmatrix}1 & -\sin 20^\circ\tan 10^\circ & \cos 20^\circ\tan 10^\circ\\0 & \cos 20^\circ & \sin 20^\circ\\0 & -\frac{\sin 20^\circ}{\cos 10^\circ} & \frac{\cos 20^\circ}{\cos 10^\circ}\end{bmatrix}\begin{bmatrix}3\\-5\\8\end{bmatrix}$$

$$= \begin{bmatrix}4.627^\circ/s\\-1.962^\circ/s\\9.4533^\circ/s\end{bmatrix}$$

3. (1) 由题 1 可知

$$J_{Euler,ZYX} = \begin{bmatrix} 0 & -\sin\alpha & \cos\alpha\cos\beta \\ 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \end{bmatrix}$$

$$\begin{aligned} e = J_{Euler,ZYX}\Delta\Psi &= \begin{bmatrix} 0 & -\sin\alpha_c & \cos\alpha_c\cos\beta_c \\ 0 & \cos\alpha_c & \sin\alpha_c\cos\beta_c \\ 1 & 0 & -\sin\beta_c \end{bmatrix} \begin{bmatrix} \alpha_d-\alpha_c \\ \beta_d-\beta_c \\ \gamma_d-\gamma_c \end{bmatrix} \\ &= \begin{bmatrix} -\sin\alpha_c(\beta_d-\beta_c)+\cos\alpha_c\cos\beta_c(\gamma_d-\gamma_c) \\ \cos\alpha_c(\beta_d-\beta_c)+\sin\alpha_c\cos\beta_c(\gamma_d-\gamma_c) \\ (\alpha_d-\alpha_c)-\sin\beta_c(\gamma_d-\gamma_c) \end{bmatrix} \end{aligned}$$

(2)

$$\omega = J_{Euler,ZYX}\dot{\Psi}$$

则有

$$\dot{e} = J_{Euler,ZYX}\Delta\dot{\Psi} = J_{Euler,ZYX}\Delta\omega$$

故控制方程为

$$\begin{aligned} \tau &= K_{pp}e + K_{pi}\int e\mathrm{d}t + K_{pd}\dot{e} \\ &= K_{pp}J_{Euler,ZYX}\Delta\Psi + K_{pi}J_{Euler,ZYX}\int\Delta\Psi\mathrm{d}t + K_{pd}J_{Euler,ZYX}\Delta\omega \end{aligned}$$