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Dynamic pricing in airline revenue management

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ABSTRACT

In this study, an exact Dynamic Programming formulation is developed for price-based revenue management in the airline industry. Structural properties of the optimal pricing policy are presented first. Then, the optimal policy is investigated for four different price-demand relationships and the characteristic properties of the airline demand are discussed. Performance of the proposed Dynamic Programming formulation is numerically compared with an alternative approximate approach based on the repeated use of a Mixed Integer Programming model and also with intuitive methods of temporal price discrimination. The numerical results show that significant revenue improvements are obtained by the proposed dynamic pricing methodology.

1. Introduction

The practice of price discrimination is almost as old as commerce itself and its basis could be spatial differences (location of market), temporal differences (time of sales) or income differences (customer wealth). Temporal differences lead to dynamically setting the product price through a complex decision process or just seller's intuition. Markdowns in the sales of perishable commodities and fashion goods are common business practices where the seller (in most cases) instinctively controls the price to recover possible sunk costs. The strategy of lowering the price gradually is referred to as price skimming and its benefits and shortcomings have been investigated in the literature on inventory control.

Revenue Management (RM) principally exploits temporal differences in customer valuations to increase revenues and its focus is on industries where customers' willingness to pay tends to increase during the sales horizon. A major challenge of price skimming is to minimize deferred purchases from *strategic* customers who are willing to pay the current price but postpone the purchase in order to benefit from potentially lower future prices. RM extends price skimming by introducing the possibility of the tendency of price increase

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during the sales horizon. For instance, the general anticipation of price increase in airline RM resolves the complications due to strategic customer considerations to a great extent. In this case, the seller's tradeoff is between setting lower prices to benefit from earlier arrivals and sparing inventory for possible future customers who are willing to pay higher prices.

Considerable RM applications result from the price deregulation in the airline industry in USA in 1978. The initial customer segmentation applications at the end of 1960s for two fare classes are simple yet effective; together with full-fare tickets, discount tickets are sold with restrictions on *availability*. An important part of the research on airline RM and related industry applications focuses on the seat allocation problem. Principally, the available flight capacity is allocated to the predetermined fare classes -each with a given price- by setting protection levels or booking limits for the fare classes. For the implementation of a seat allocation control, the seller only needs to keep track of the remaining inventory during the sales horizon. Due to ease of implementation, the seat allocation practices are favored in the airline industry.

This paper is on dynamic pricing for airline RM. Despite the popularity of seat allocation, the primary motivation here to study dynamic pricing is that dynamic pricing outperforms seat allocation control regarding input requirements. For seat allocation, the number of fare classes and the price for each class need to be determined in advance whereas price is the decision variable in dynamic pricing. Yet, the dominance of seat allocation in airline RM is mostly due to the heavier computational burden of dynamic pricing and easier price follow-up with seat allocation. As a matter of fact, the ticket prices of hundreds or thousands of flights in the next 3 to 6 months period must be determined in a business environment. Thus, considering the operational limitations in the airline industry, feasibility of the proposed dynamic pricing method -regarding the solution time and computational requirements- is also questioned in this paper.

1.1. Related studies in the literature

Seat allocation practices have been dominant in the field of airline RM and dynamic pricing has been introduced to this field more recently. Yet, there are earlier studies on dynamic pricing applications in other fields. To the best of our knowledge, the study of Kincaid and Darling [6] appears to be the earliest work on dynamic pricing. In this seminal work, the authors investigate dynamic pricing policies for various problem settings: the cases of single-item versus multiple-item pricing and the cases of seller posting prices versus seller evaluating customer bids are discussed. Airline RM is an example of multiple-item pricing when the seller posts the prices.

Elmaghraby and Keskinocak [4] present a detailed review of the dynamic pricing studies in the literature on inventory control and also give a classification of the dynamic pricing problems. The possibility of inventory replenishment, demand behavior over time (whether the demand is time-variant or not) and type of customer behavior (whether the customers are *myopic* or *strategic*) constitute the basis of their classification. The (airline RM) problem studied herein lacks the possibility of inventory replenishment and undergoes time-variant demand. The customers are assumed myopic; they make their purchasing decision without considering future price anticipations.

Modeling the demand along the entire sales horizon appropriately is a fundamental issue in dynamic pricing problem. Both Kincaid and Darling [6] and Gallego and van Ryzin [5] use a time-homogeneous demand model; the customer arrival rate is assumed to be a time-independent function of price. However, due to the aforementioned characteristics of air travel demand, this demand setting is not particularly appropriate for airline RM. Based on the work due to McGill and van Ryzin [9], the arrival process of customers is assumed to be a Poisson Process in most of the RM literature and the time-dependence of arrivals is mostly handled by modeling the arrivals by a Nonhomogeneous Poisson Process with a rate varying in time.

In airline RM, the customer's willingness to pay changes together with the customer arrival frequency within the sales horizon. In accordance with this phenomenon, Zhao and Zheng [12] study a demand setting allowing both customer arrival rate and the reservation price distribution to be time-variant for perishable assets. They indicate the conditions under which the monotonicity results obtained by Gallego and van Ryzin [5] hold true for time-variant arrival rates and reservation prices. The customer arrival rate and reservation price distributions are assumed to be time-variant here in this study as well.

In the dynamic pricing problems, the control tool of the seller for managing the demand is the price and the objective is to determine the optimal price as a function of time and other relevant factors. Alternatively, Lin [8] works with the probability of selling one item to the current customer. Under the assumption that the reservation price distribution of the customer is known, the probability of sale can be represented as a function of the posted price. With this alternative approach, Lin [8] considers a Dynamic Programming formulation and derives structural properties of the optimal policy. Using supermodularity of the optimal expected revenue which is expressed as a function of seat capacity and random demand, Lin [8] gives a rigorous proof for the "expected" characteristics of optimal policy: the seller's incentive for selling a ticket gets larger due to decrease in the optimal price with more seat inventory.

Bitran and Mondschein [1] study a dynamic pricing problem in retail industry in which customers have time-variant willingness to pay for a product. We adopt the time-variant reservation price definition in this work. They present the necessary condition for the optimal pricing policy. Yet, they point out that the optimal policy is not applicable to retail industry since price varies after each sale and that destroys "the notion of value".

The data being used for understanding the customer's willingness to pay is not limited to time to departure in airline RM literature. Kramer et al. [7] introduce new forms of price discrimination (browsing-history based pricing, past-behavior-pricing, device-based-pricing, demography-based-pricing) and propose personalized dynamic pricing. Also, using alternative sources of data to predict customer segment (leisure or business) and pricing accordingly have recently attracted attention. Interested readers are referred to Wittman and Belobaba [11] and Delahaye et al. [3] for price customization based on customer segment estimation.

1.2. Contributions

The research question addressed in this paper is the applicability of the existing findings in the literature on sequential dynamic pricing models to the airline revenue management problem characterized by the temporal behavior of customer willingness to buy. Static demand models do not generalize well for abstracting the real-life sales processes that cover a long period of time, as in the case of airline demand over the sales horizon. Also, working with realistic temporal demand formulations allows investigating the impact of different demand patterns to be possibly observed on the optimal pricing policy. In this respect, the analytical results due to Kincaid and Darling [6] and Lin [8] provide a guideline for this study. Findings in the aforementioned studies are generalized in this paper for the considered setting which represent air travel demand more accurately. The structural results given here are mostly extensions of the work on general dynamic pricing literature.

The primary purpose of this study is to propose a comprehensive method for airline dynamic pricing as an alternative. Towards that end, a framework is proposed here for investigating the temporal air travel demand at the *individual* customer level and the optimal price structure is studied under four different demand settings (formulations).

- The main difference in the approach considered in this paper for the dynamic pricing problem is disaggregation of demand to individual customer level. This perspective of demand requires a mathematical formulation of sales incentive as a function of price and allows modeling without discrete demand classes. The advantage of disaggregation of demand to individual customer level over working with aggregate demand turns out as having a different reservation price distribution for each customer.

- In addition to adopting linear and isoelastic demand models from the economics literature and working with exponential formulation introduced previously in the literature, a logarithmic formulation is proposed in this study.

Hence, the contribution of this paper is an outline proposed for implementation of the dynamic pricing policy in real life applications.

The paper is organized as follows. In Section 2, an exact Dynamic Programming formulation is given for the single-leg airline RM problem. This section also includes analytical findings regarding the optimal price and the marginal value function. The demand setting used in the proposed model is detailed in Section 3 and the optimal solution is analyzed for four alternative demand formulations. Section 4 presents the numerical findings and a comparison with approximate dynamic pricing formulations. To conclude, the limitations and shortcomings of the proposed models and possible extensions are considered in Section 5.

2. Exact Dynamic Programming formulation

The Dynamic Programming formulation proposed in this study for single-leg airline RM problem aims to find the optimal ticket price at a given state directly rather than selecting the best price in a given discrete set. The monopolist and risk-neutral seller's objective is to maximize the total revenue over the sales horizon and the marginal service costs are ignored. Batch bookings, overbooking and booking cancellations are not allowed; that is, booking requests are processed individually until all seats are sold.

2.1. Demand

To obtain an optimality equation for the ticket price, the sales process is modeled in a recursive fashion and at most one individual sales transaction is considered at each stage of the recursive model. In this respect, the sales horizon is divided into small time intervals of unit length during which at most one customer may arrive. An arriving customer books ticket for the flight or not depending on the price posted by the seller. Hence, the aggregate demand is disintegrated into the following two components.

1. Customer Arrival Rate: The probability of a customer arrival in an arbitrarily small time interval, ϵ, is determined by the rate of the demand arrival process. Letting λ_t denote the time-variant customer arrival rate when time to departure is t, the probability of a single customer arrival in period t corresponding to time interval $[t, t-\epsilon]$ is approximated as $\rho_t = \epsilon\lambda_t$. Assuming (nonhomogeneous) Poisson customer arrivals and sufficiently small ϵ, the possibility of multiple arrivals in $[t, t-\epsilon]$ is eliminated. Note that the arrival probabilities and rates are assumed to be independent of the prices posted by the seller.
2. Reservation Price Distribution: Instead of the relation between price and aggregate demand, the focus in this study is on the demand-price relationship of each individual. Each customer arrival does not necessarily result in a booking. Whenever the posted price is above the maximum amount the customer is willing to pay, a sale does not take place. Thus, the customer reservation price, P_t, is considered as a time-variant (continuous) random variable to represent the randomness in customers' willingness to pay and the change in customer valuation along the sales horizon. That is, the probability of selling one seat in time period t is $\rho_t Pr(P_t \geq p)$ when p is the ticket price posted by the seller.

The critical observation that triggered the first RM applications in airline industry back in 1960's is that leisure customers (who are willing to pay less for the air travel) tend to make bookings earlier than the business customers. As a generalization, the low-before-high arrival pattern is commonly accepted in the seat allocation literature which refers to an aggregate demand scheme that assumes arrival of customers from consecutive fare segments over non-overlapping time periods. The generic model proposed in this study does not require any temporal restrictions and the results are generalized for any given time-variant reservation price distribution that is differentiable. Yet, the numerical experiments are obtained for reservation price distributions that get stochastically smaller as the time to departure increases $P_t \geq_{st} P_\tau$ for $\tau \geq t$. The notation $\geq_{st}$ is used for first order stochastic dominance.

2.2. Sales process

The backward recursion for the Dynamic Programming formulation proposed in this study to maximize revenues over the entire sales horizon is given in (1). The state variables are time to departure, t, and the seat inventory, s. The optimal value function, $v_t(s)$, gives the optimal expected sales revenue that can be generated with the remaining s seats when time to departure is t. The unit time intervals are assumed sufficiently small to be able to neglect multiple customer arrivals in a period. The expected revenue is evaluated with respective probabilities of selling a single ticket, $\rho_t Pr(P_t \geq p)$, and no sales, $(1-\rho_t) + \rho_t Pr(P_t < p)$.

$$v_t(s) = \max_p \left\{\rho_t Pr(P_t \geq p)(p + v_{t-1}(s-1)) + [(1-\rho_t) + \rho_t Pr(P_t < p)]v_{t-1}(s)\right\}. \tag{1}$$

In the first term on the right hand side of (1), the following case is considered: a customer arrives in the coming period, t, (with probability ρ_t) and this customer is willing to pay p for a ticket (with probability $Pr(P_t \geq p)$); that is, (with probability $\rho_t Pr(P_t \geq p)$) the expected revenue for the remaining t periods is $p + v_{t-1}(s-1)$, where p is the revenue obtained from the sale of a seat of s seats for the current customer in period t and $v_{t-1}(s-1)$ is the optimal expected revenue for the following $t-1$ periods with the remaining available $s-1$ seats. In the second term on the right hand side of (1), the following two cases are considered: (1) a customer arrives in the coming period, t, (with probability ρ_t) and this customer is not willing to pay p for a ticket (with probability $Pr(P_t < p)$), (2) nobody arrives in the coming period (with probability $1-\rho_t$); that is, (with probability $[(1-\rho_t) + \rho_t Pr(P_t < p)]$) there is no sale in period t and $v_{t-1}(s)$ is the remaining optimal expected revenue for the following $t-1$ periods with the available s seats.

The terms in (1) are rearranged below by adding and subtracting $v_{t-1}(s)$. The second equation results from $[\rho_t Pr(P_t \geq p) + (1-\rho_t) + \rho_t Pr(P_t < p)] = 1$. In the third line, the second term of the maximization vanishes.

$$\begin{aligned}
v_t(s) &= \max_p \left\{\rho_t Pr(P_t \geq p)(p + v_{t-1}(s-1)) + [(1-\rho_t) + \rho_t Pr(P_t < p)]v_{t-1}(s)\right\} + v_{t-1}(s) - v_{t-1}(s) \\
&= \max_p \{\rho_t Pr(P_t \geq p)(p + v_{t-1}(s-1)) + [(1-\rho_t) + \rho_t Pr(P_t < p)]v_{t-1}(s) \\
&\qquad\qquad - [\rho_t Pr(P_t \geq p) + (1-\rho_t) + \rho_t Pr(P_t < p)]v_{t-1}(s)\} + v_{t-1}(s) \\
&= \max_p \big\{\rho_t Pr(P_t \geq p)(p - v_{t-1}(s) + v_{t-1}(s-1)) \\
&\qquad\qquad + [(1-\rho_t) + \rho_t Pr(P_t < p)](v_{t-1}(s) - v_{t-1}(s))\big\} + v_{t-1}(s) \\
&= \max_p \left\{\rho_t Pr(P_t \geq p)(p - [v_{t-1}(s) - v_{t-1}(s-1)])\right\} + v_{t-1}(s).
\end{aligned}$$

Letting $\Delta v_{t-1}(s) = v_{t-1}(s) - v_{t-1}(s-1)$ in the last equation above, the recursion in (1) is written as in (2). $\Delta v_{t-1}(s)$ is the opportunity cost of the sth available seat in period t; that is, it is the marginal value of capacity. Then, the optimal price at a given state (s, t) is the value of p that maximizes the expected net sales gain, $Pr(P_t \geq p)(p - \Delta v_{t-1}(s))$. The rearranged form of the recursion in (2) does not only present a better interpretation of the problem but also gives a lower bound on the ticket price; it is obvious that $p < \Delta v_{t-1}(s)$ cannot be optimal. Further analysis on the structural properties of the optimal price and the marginal value of capacity is in Section 2.3.

Thus, the Dynamic Pricing (DP) model for the single-leg airline RM problem under the aforementioned assumptions is given below. S denotes the capacity of the aircraft and T denotes the length of the sales horizon. The boundary conditions are given for the following two cases: (i) when the seat inventory depletes, revenue-to-go equals 0, (ii) when the flight departs with empty seats, the unsold inventory has no contribution to revenue-to-go.

$$\begin{aligned}
\text{DP:} \qquad & \text{Max} \qquad v_T(S) \\
& \text{subject to} \qquad v_t(s) = \rho_t \max_p \{Pr(P_t \geq p)(p - \Delta v_{t-1}(s))\} + v_{t-1}(s) \qquad \forall s \in [1, S], \forall t \in (0, T], \\
& \qquad v_0(s) = 0 \qquad \forall s \in [0, S], \\
& \qquad v_t(0) = 0 \qquad \forall t \in (0, T].
\end{aligned} \tag{2}$$

Then, the maximizer below would be the optimal ticket price for state (s, t) when the problem above is well-defined.

$$p^*_{st} = \underset{p}{\operatorname{argmax}}\{Pr(P_t \geq p)(p - \Delta v_{t-1}(s))\}.$$

2.3. Structural properties

In this section, structural properties of the optimal policy obtained by the model proposed in Section 2.2 are analyzed. The contribution that results from this analysis is twofold. Firstly, this analysis is critical for analytical validation of the developed model since the coherence of anticipated and analytically observed behavior is an indicator of the model validity. Besides, both marginal value of capacity and optimal price are common in different RM models and, therefore, this analysis provides a basis for comparing the model proposed in this study with the existing models in the literature.

The rearranged form of the optimality equation in (2) provides a direct lower bound on the optimal price: p^*_{st} cannot be less than the marginal value of capacity, $\Delta v_{t-1}(s)$. Lin [8] and Talluri and van Ryzin [10] give similar results regarding the optimal price. In both studies, *marginal revenue* is defined as the time-independent expected gain from one customer -as a counterpart of $Pr(P_t \geq p)p$ in our formulation- and a constraint on the optimal ticket price is obtained considering the behavior of the marginal revenue function. The generalization of this result in Lemma 1 below is for time-variant marginal revenue function. The proofs that are skipped in this section are given in Appendix A.

Lemma 1. *$Pr(P_t \geq p)p$ is a nonincreasing function of price at $p = p^*_{st}$.*

In Fig. 1, we consider an example reservation price distribution on a bounded support. For that particular choice, the marginal revenue is increasing on the intervals $[P(min), P(1))$ and $(P(2), P(3)]$. Thus, the prices in these two subintervals are not optimal due to Lemma 1. This restriction on optimal price can be used in preprocessing for elimination of the price values that are not optimal while trying to solve $\max_p \{Pr(P_t \geq p)(p - \Delta v_{t-1}(s))\}$ in the optimality equation.

The dynamic pricing problem studied by Lin [8] disregards the heterogeneity in customer valuations along the time horizon. Accordingly, the time dimensionality of the problem is eliminated and the challenge reduces to sequential pricing for customers waiting in a queue. He gives a similar recursive formulation in terms of the number of items in the inventory and the number of customers whose probability mass function is known. He proves the following.

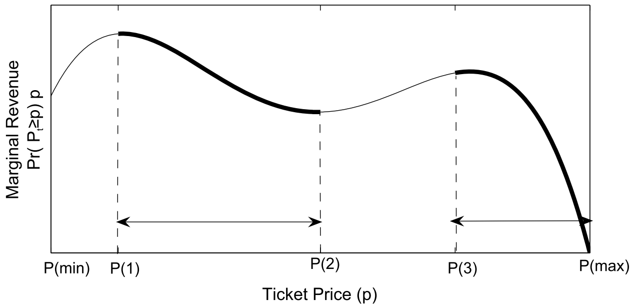

Fig. 1. Potential optimal prices due to Lemma 1.

- The marginal contribution of an additional unit inventory to the optimal expected revenue -counterpart of $\Delta v_t(s)$ in the proposed DP model in Section 2.2- and the optimal price are both decreasing in the number of items in the inventory for a given probability mass function of the number of customers.
- At a given inventory level, the marginal contribution of an additional unit inventory to the optimal expected revenue and the optimal ticket price both get larger if the number of customers gets (stochastically) larger.

Talluri and van Ryzin [10] also study a dynamic pricing example and investigate the monotonicity of marginal value of capacity. They prove that the marginal value of capacity is nondecreasing in time to departure and nonincreasing in remaining capacity under the assumption that the marginal revenue, $Pr(P_t \geq p)p$, is a concave function of demand. This assumption explicitly dictates a restriction on the form of the reservation price distribution. By eliminating this restriction, an extension of their findings is given below in this article. Lemma 2 below shows the concurrence of these monotonicity results; stating that the marginal value of capacity is nondecreasing in t if and only if it is nonincreasing in s.

Lemma 2. *$\Delta v_t(s)$ is a nondecreasing function of time to departure, t, if and only if it is a nonincreasing function of seat inventory, s.*

Proof. In the first part of the proof, it is assumed that $\Delta v_t(s)$ is nondecreasing in t and this implies that $\Delta v_t(s)$ is nonincreasing in s. From (2),

$$v_t(s+1) - v_{t-1}(s+1) = \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta v_{t-1}(s+1)]\},$$
$$v_t(s) - v_{t-1}(s) = \rho_t \max_p \{Pr(P_t \geq p)[p - \Delta v_{t-1}(s)]\}.$$

Taking the difference of the two equations above,

$$\Delta v_t(s+1) - \Delta v_{t-1}(s+1) = \rho_t \Big(\max_p \{Pr(P_t \geq p)[p - \Delta v_{t-1}(s+1)]\} - \max_p \{Pr(P_t \geq p)[p - \Delta v_{t-1}(s)]\} \Big). \quad (3)$$

Since $p^*_{s+1,t}$ is optimal for the first and suboptimal for the latter maximization in (3), the following inequality is obtained by substituting $p^*_{s+1,t}$ in both terms to be maximized in p:

$$\Delta v_t(s+1) - \Delta v_{t-1}(s+1) \leq \rho_t Pr(P_t \geq p^*_{s+1,t})[\Delta v_{t-1}(s) - \Delta v_{t-1}(s+1)].$$

Since $\Delta v_t(s)$ is nondecreasing in t, the left hand side of the inequality above is nonnegative. Therefore, $\Delta v_{t-1}(s) - \Delta v_{t-1}(s+1) \geq 0$, meaning that $\Delta v_t(s)$ is nonincreasing in s.

In the second part of the proof, it is assumed that $\Delta v_t(s)$ is nonincreasing in s and this implies that $\Delta v_t(s)$ is nondecreasing in t. Recall (3) we have obtained in the first part of the proof. By substituting p^*_{st} in both terms to be maximized in p, on the right hand side of (3), the following inequality is obtained:

$$\Delta v_t(s+1) - \Delta v_{t-1}(s+1) \geq \rho_t Pr(P_t \geq p^*_{st})[\Delta v_{t-1}(s) - \Delta v_{t-1}(s+1)].$$

Since $\Delta v_t(s)$ is nonincreasing in s, the right hand side of the equation is nonnegative. Therefore, $\Delta v_t(s+1) - \Delta v_{t-1}(s+1) \geq 0$, meaning that $\Delta v_t(s)$ is nondecreasing in t. □

Lemma 2 shows that the aforementioned monotonicity characteristics imply each other. Lemma 3 shows that $\Delta v_t(s)$ is nonincreasing in s. The proof of Lemma 3 is structurally the same as the proof given by Talluri and van Ryzin [10]. The joint conclusion drawn from Lemma 2 and Lemma 3 is that the marginal value of capacity is nonincreasing in seat inventory and nondecreasing in time to departure without making any assumptions on the distribution of P_t. Note that the concave marginal revenue requirement in Talluri and van Ryzin [10] is by-passed.

Lemma 3. $\Delta v_t(s)$ *is nonincreasing in* s *for any fixed* t.

The structural characteristics of $\Delta v_t(s)$ shed insight into the behavior of the optimal value function to make inferences about the optimal price, p^*_{st}. The following corollaries of Lemma 3 present behavioral facts about the optimal price.

Corollary 1. p^*_{st} *is nonincreasing in* s *for any fixed* t.

Corollary 1 shows that the optimal ticket price is nonincreasing in the remaining seat inventory s. Similarly, Corollary 2 shows that p^*_{st} is nondecreasing in time to departure, t, for a given seat inventory s if the distribution of the reservation price P_t is time-invariant. As mentioned previously, in airline RM the reservation prices tend to increase as time to departure decreases. Yet, if the reservation price distribution remains unchanged over a certain period in the sales horizon, the result in Corollary 2 will still be valid in this period (see Remark 3).

Corollary 2. *Let the probability distribution of* P_t *be time-invariant; that is, for a given value of* p, $Pr(P_t \geq p)$ *is constant for every* t*. Then,* p^*_{st} *is nondecreasing in* t *for fixed* s.

Regarding the case of time-variant reservation price distribution, a counterexample is presented below for monotonicity in t. It is shown that the optimal ticket price may get larger for constant s as the time to departure decreases if the reservation prices tend to get stochastically larger; that is, $p^*_{st} \geq p^*_{s,t+1}$ could be seen when $P_t \geq_{st} P_{t+1}$.

Example 1. For $s = 1$, the values of p^*_{st} at $t = 1$ and $t = 2$ are compared letting $\epsilon = 1$. Consider the following uniform distributions of the reservation prices: $P_1 \sim U(110, 130)$ and $P_2 \sim U(100, 120)$. In this case, $Pr(P_1 \leq p) \leq Pr(P_2 \leq p)$ for any ticket price p; hence, $P_2 \leq_{st} P_1$. The optimality equation in (2) for $v_1(1)$ is

$$v_1(1) = \rho_1 \max_p \{Pr(P_1 \geq p)[p - \Delta v_0(1)]\} + v_0(1),$$

where the terms $\Delta v_0(1)$ and $v_0(1)$ are both zero due the boundary conditions. The maximization for price p is on the interval $[110, 130]$ by the distribution of P_1; note that it does not make sense to set $p < 110$. Hence, the optimality equation for $t = 1$ is

$$v_1(1) = \rho_1 \max_{110 \le p \le 130} \left\{ \frac{(130-p)p}{20} \right\}. \tag{4}$$

Solving (4) for p, $p^*_{1,1} = 110$ and $v_1(1) = 110\rho_1$ are obtained.

Likewise, the optimality equation for $v_2(1)$ is

$$v_2(1) - 110\rho_1 = \rho_2 \max_{100 \le p \le 120} \left\{ \frac{(120-p)(p-110\rho_1)}{20} \right\}.$$

Then,

$$p^*_{1,2} = \begin{cases} 100 & \text{for } 0 \le \rho_1 \le \frac{40}{55}, \\ 60 + 55\rho_1 & \text{for } \frac{40}{55} < \rho_1 \le 1. \end{cases}$$

That is, $100 \le p^*_{1,2} < 110 = p^*_{1,1}$ for $0 \le \rho_1 < \frac{50}{55}$. Therefore, it is possible to observe $p^*_{s,t+1} < p^*_{st}$ when $P_t \ge_{st} P_{t+1}$. □

3. Demand-price relationship

The existing models in microeconomics primarily focus on understanding the relationship between the price of a commodity and the aggregate demand. When the supplier has a position in the market to set the price, the demand is controlled by the supplier's incentive for selling this commodity. In this respect, the analysis in this section for the demand-price relationship is considered at the individual customer level. In other words, instead of determining the quantity that could be sold to all prospective customers, the sales price announced at time t by the seller determines the sales probability for the customer arriving at time t.

The analysis in this section is for four different demand-price relationships: the logarithmic sales incentive presented in Section 3.1 and the exponential, linear and isoelastic sales incentives presented in Appendix C. In Section 3.2, the conditions under which a closed-form solution can be found for the continuous-time formulation are investigated. It is shown that time-invariant exponential reservation price $P_t = P$ with constant α yields a closed-form solution for the dynamic pricing problem. A summary of the findings for these four different demand-price relations are listed in Table 1.

Exponential sales incentive is formulated based on the demand formulations studied by Kincaid and Darling [6] and the extensions of their findings are analyzed in Section 3.2. The linear and isoelastic sales incentive formulations are derived from the aggregate demand counterparts in economics literature. Davis and Garces [2] give a detailed explanation regarding the specifications of linear and isoelastic demand functions. The other demand formulation in Table 1 called "logarithmic" is an alternative form proposed in this paper. For the logarithmic sales incentive, the optimality equation for the airline dynamic pricing problem is similar to that of the newsboy problem in the inventory theory as shown in Section 3.1. In this respect, this alternative formulation is interesting and deserves further investigation besides others aforementioned.

The parameters α, p_{up}, p_{low}, a and b in Table 1 are considered to be time-variant but to keep the notation simple the time index t is suppressed. f_{P_t} denotes probability density function of P_t. Price, p, values given in the last column are the solutions of the first order optimality conditions for given s and t. Note that the parameters of each demand setting should be defined as a function of time. For instance, if exponential demand setting is considered for the stochastic ordering between reservation prices, $P_t \ge_{st} P_{t+1}$ for all t, then the parameter α should be defined as a nondecreasing function of time to departure, t.

Remark 1. The transformation of linear and isoelastic demand functions into sales incentive functions can be constructed as a mapping from demanded quantity in $[0, q_{max}]$ to probability of sales in $[0, 1]$ over the

Table 1
Demand-sales incentive relationships.

Demand type	$Pr(P_t \geq p)$	Parameters	$f_{P_t}(p)$	Support	p
Exponential	$e^{-\alpha p}$	$\alpha > 0$	$\alpha e^{-\alpha p}$	$(0, \infty)$	$\Delta v_{t-1}(s) + \frac{1}{\alpha}$
Logarithmic	$\frac{\ln(p_{up}) - \ln(p)}{\ln(p_{up}) - \ln(p_{low})}$	$p_{up} > p_{low} > 0$	$\frac{1}{p\,(\ln(p_{up}) - \ln(p_{low}))}$	$[p_{low}, p_{up}]$	$\frac{\Delta v_{t-1}(s)}{1 - Pr(P_t \geq p)\ln(\frac{p_{up}}{p_{low}})}$
Linear	$a - bp$	$a > 1,\, b > 0$	b	$\left[\frac{a-1}{b}, \frac{a}{b}\right]$	$\frac{1}{2}\left(\Delta v_{t-1}(s) + \frac{a}{b}\right)$
Isoelastic	ap^{-b}	$a > 0,\, b > 1$	abp^{-b-1}	$[a^{1/b}, \infty)$	$\Delta v_{t-1}(s)\frac{b}{b-1}$

range of possible prices as follows: $z(p) = \frac{q(p)}{q_{max}} = \frac{q(p)}{q(p_{min})}$. q_{max} denotes the maximum quantity which is obtained at minimum price p_{min}. For the case of linear demand $q(p) = A - Bp$, the sales incentive is then given as $z(p) = a - bp = \dfrac{A - Bp}{A - Bp_{min}}$. That is, parameters of the sales incentive function are $a = \dfrac{A}{A - Bp_{min}}$ and $b = \dfrac{B}{A - Bp_{min}}$ in the linear case.

For the case of isoelastic demand function $q(p) = Ap^{-B}$, the sales incentive is given as $z(p) = a - p^{-b} = \dfrac{Ap^{-B}}{Ap_{min}^{-B}}$. That is, parameters of the sales incentive function are $a = \dfrac{1}{p_{min}^{-B}}$ and $b = B$.

Next, the optimality equation (2) in Section 2.2 is considered for the investigations outlined above. Since the reservation price is denoted by a continuous random variable, the sales incentive can be defined as the complementary cumulative distribution of the reservation price. Let $\Gamma(p)$ denote the term $Pr(P_t \geq p)(p - \Delta v_{t-1}(s))$ to be maximized in (2). Under the assumption that the (complementary) cumulative density function of reservation price is differentiable at any t, $\Gamma(p)$ is differentiable for any given p on the support defined for the cumulative density function. The first order optimality condition $\frac{d}{dp}\Gamma(p) = 0$ is

$$Pr(P_t \geq p) - f_{P_t}(p)(p - \Delta v_{t-1}(s)) = 0. \tag{5}$$

3.1. Logarithmic demand-price relationship

In this section, the following case is studied: the reservation price P_t has a probability density function f_{P_t} such that the probability density value $f_{P_t}(p)$ is inversely proportional to the value of p on a bounded interval. p_{up} and p_{low} denote the upper and lower bounds of the support, respectively. For the reservation price distribution

$$f_{P_t}(p) = \frac{1}{\kappa p} \quad \text{and} \quad Pr(P_t \leq p) = \frac{1}{\kappa}\ln(\frac{p}{p_{low}}) \qquad \text{for } \forall p \in [p_{low}, p_{up}], \tag{6}$$

the sales incentive as a function of price is $Pr(P_t \geq p) = \dfrac{1}{\kappa}\ln(\dfrac{p_{up}}{p})$ where $\kappa = \ln(\dfrac{p_{up}}{p_{low}})$. Proposition 1 and Proposition 2 present the observations regarding this particular demand-price relation between $Pr(P_t \geq p)$ and p. Fig. 2 exemplifies how the reservation price bounds evolve along the sales horizon and how the reservation price is distributed for given price bounds. Note that the example bounds given in Fig. 2 for the reservation price P_t increase as the time to departure, t, decreases; this behavior in time is in accordance with the real life practice in airline industry. As for the functional behavior of the sales incentive in Fig. 2, note that $Pr(P_t \geq p) = \dfrac{1}{\kappa}\ln(\dfrac{p_{up}}{p})$ decreases in p; also, $Pr(P_t \geq p)$ at $p = p_{low}$ is equal to $\dfrac{1}{\kappa}\ln(\dfrac{p_{up}}{p_{low}}) = 1$ due to the definition of κ and $Pr(P_t \geq p)$ at $p = p_{up}$ is equal to $\dfrac{1}{\kappa}\ln(\dfrac{p_{up}}{p_{up}}) = 0$.

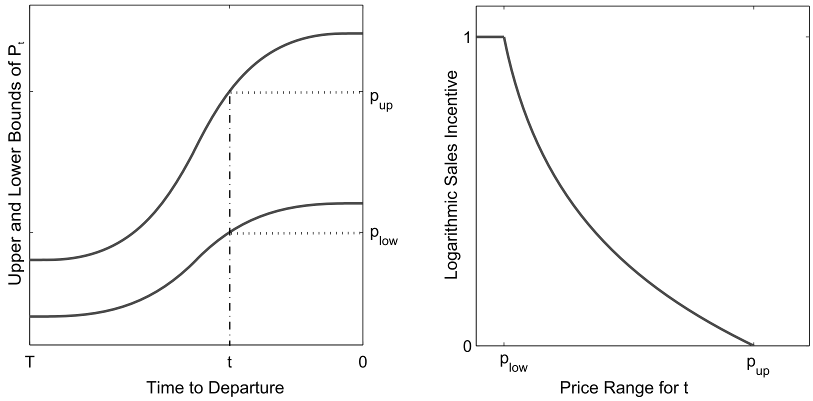

Fig. 2. Bounds for reservation price and sales incentive as a function of price posted at t.

Proposition 1. *Let* $Pr(P_t \geq p) = \dfrac{1}{\kappa}\ln(\dfrac{p_{up}}{p})$ *in* $\Gamma(p) = Pr(P_t \geq p)(p - \Delta v_{t-1}(s))$ *with* $\kappa = \ln(\dfrac{p_{up}}{p_{low}})$ *defined for arbitrary* $p > 0$*. Then, the first order condition is*

$$p = p\kappa Pr(P_t \geq p) + \Delta v_{t-1}(s) \tag{7}$$

and the p *value that satisfies (7) is the unique maximizer of* $\Gamma(p)$ *(if exists).*

Proof. Substituting $f_{P_t}(p) = \dfrac{1}{\kappa p}$ in (5), we obtain $Pr(P_t \geq p) - \dfrac{1}{\kappa p}(p - \Delta v_{t-1}(s)) = 0$. Rearranging the terms, we obtain the optimality equation given in (7).

Using $\dfrac{dPr(P_t \geq p)}{dp} = -\dfrac{1}{\kappa p}$ and $\dfrac{d^2 Pr(P_t \geq p)}{dp^2} = \dfrac{1}{\kappa p^2}$, the second derivative of Γ with respect to p is obtained below. This term is negative since κ is positive by definition, p is assumed to be positive and $\Delta v_{t-1}(s)$ is non-negative.

$$\frac{d^2\Gamma}{dp^2} = \frac{-1}{\kappa p}\left(1 + \frac{\Delta v_{t-1}(s)}{p}\right) < 0.$$

Hence, if there exists a solution to the first order optimality condition in (7), then that value of p is a local maximizer of $\Gamma(p)$.

Rearranging the terms, the first order optimality condition can be written as $p(1 - \kappa Pr(P_t \geq p)) = \Delta v_{t-1}(s)$ where $p(1 - \kappa Pr(P_t \geq p))$ is increasing in p and $\Delta v_{t-1}(s)$ is independent of the value of p for given s and t values. Having shown the monotonicity of $p(1 - \kappa Pr(P_t \geq p))$ to be compared with constant $\Delta v_{t-1}(s)$ over all p on the given support to solve the first order optimality condition, we conclude that there may exist at most one p that satisfies the first order optimality condition. □

Proposition 2. *For the reservation price distribution* $f_{P_t}(p) = \dfrac{1}{p(\ln(p_{up}) - \ln(p_{low}))}$ *defined on* $[p_{low}, p_{up}]$*, let* p *satisfy the condition in (7) in Proposition 1 for given* t *and* s *values. If* p *is in* $[max\left\{p_{low}, \frac{p_{up}}{e}\right\}, p_{up}]$*, then* $p^*_{st} = p$*. Otherwise,* p^*_{st} *is given by one of the endpoints of the closed interval* $[max\left\{p_{low}, \frac{p_{up}}{e}\right\}, p_{up}]$ *depending on which endpoint gives a higher* Γ *value.*

Proof. Lemma 1 states that for any reservation price distribution the marginal revenue, $Pr(P_t \geq p)p$, is nonincreasing in p at the optimal price. In the logarithmic demand case, $\frac{d}{dp}\left(Pr(P_t \geq p)p\right) = \frac{1}{\kappa}\left(\ln(\frac{p_{up}}{p}) - 1\right)$. For $\dfrac{p_{up}}{p} > e$, the marginal revenue function $Pr(P_t \geq p)p$ is increasing in p. That is, the price values less than $\frac{p_{up}}{e}$ are not optimal due to Lemma 1. Note that $\frac{d\Gamma}{dp} = \frac{1}{\kappa}\left(\ln(\frac{p_{up}}{p}) - 1\right) + \frac{\Delta v_{t-1}(s)}{\kappa p}$ is positive for $p < \frac{p_{up}}{e}$, meaning that Γ is increasing in p for $p < \frac{p_{up}}{e}$. Then, the largest price interval on which the optimal price could exist is $[\max\left\{p_{low}, \frac{p_{up}}{e}\right\}, p_{up}]$. As shown in the proof of Proposition 1, $\frac{d^2\Gamma}{dp^2} < 0$ leading to the concavity of Γ in p for $p > 0$, and thus the proof is complete. $\square$

Remark 2. Multiplication of both sides of the first order optimality condition in (7) by the factor $\dfrac{Pr(P_t \geq p)}{\kappa Pr(P_t \geq p)p + \Delta v_{t-1}(s)}$ gives

$$Pr(P_t \geq p) = \frac{Pr(P_t \geq p)p}{\kappa Pr(P_t \geq p)p + \Delta v_{t-1}(s)}. \tag{8}$$

For $\kappa = 1$, this solution is similar to the solution of the newsboy problem in the inventory theory as shown in (9). Note that restricting $\kappa = 1$ is equivalent to assuming that the relation $p_{up} = ep_{low}$ holds.

$$p = F_{P_t}^{-1}\left(\frac{\Delta v_{t-1}(s)}{Pr(P_t \geq p)p + \Delta v_{t-1}(s)}\right), \tag{9}$$

where F_{P_t} denotes the cumulative density function of P_t.

- In airline RM, for a given inventory s the objective is to determine the optimal price p^* under demand uncertainty formulated by the probability distribution function of the reservation price P_t. The classical newsboy problem presents a similar challenge: for a given sales price p the objective is to determine the optimal inventory level (order quantity) under uncertain demand.
- In the newsboy problem, the cumulative probability value of demand at the optimal inventory level is determined by the critical ratio, $\dfrac{c_u}{c_u + c_o}$, where c_u and c_o denote unit underage and overage costs, respectively. For the logarithmic demand-price relationship in RM studied in this section, the cumulative probability value of the reservation price at p that satisfies the first order optimality condition is also given by a similar ratio.

  In the newsboy problem, underage cost is the marginal revenue of an additional item that would have been sold to satisfy the excess demand, the difference between sales price and the purchase price. Similarly, in (9), $\Delta v_{t-1}(s)$ is defined as an approximation of the marginal revenue of an extra seat, $\Delta v_{t-1}(s) = v_{t-1}(s) - v_{t-1}(s-1)$. Overage cost, on the other hand, refers to the marginal loss due to each item that remains in the inventory after the sales, the difference between the purchase price and the salvage value. In the airline example, the inventory is fixed and an overage situation corresponds to having unsold seats at the time of departure. The marginal revenue, $Pr(P_t \geq p)p$, is the expected gain of selling one seat at the current time instant. Hence, it can be interpreted as an expected opportunity loss corresponding to not selling the seat immediately and is analogous to the overage cost in the newsboy problem in this respect.

3.2. Closed-form solution

A fundamental assumption about the model proposed in Section 2.2 is the restriction of at most one customer arrival in a period which is justified by using very short time periods in the model during the sales process. In the limiting case where the period lengths go to zero, one can switch to a continuous-time formulation similar to the model studied by Kincaid and Darling [6] using differential equations instead of difference equations. In their seminal work on dynamic pricing, the case of multiple items is also studied and the problem studied therein is an elementary form of the airline dynamic pricing problem that disregards the temporal behavior of demand. The significance of this continuous-time model is the following: for a particular reservation price distribution, Kincaid and Darling [6] derive a rigorous formula for the optimal price in terms of the state variables. Noticing the similarity between the model proposed in Section 2.2 and the one due to Kincaid and Darling [6], a generalization of the results for the latter is considered in this section for optimal pricing policies and the conditions under which the model in Section 2.2 promises a closed-form solution for the optimal price are presented.

Two contributions of Proposition 3 given below to the case studied by Kincaid and Darling [6] are the following.

- The closed-form solution is obtained for a general exponential reservation price distribution whereas the formulation due to Kincaid and Darling [6] is restricted in such a way that the mean of the exponential distribution is 1.
- A generalization is studied for the case the customer arrivals are not deterministic and the arrival rate is time-variant. Recall that the time-variant arrival probability, ρ_t, is used in the model proposed in Section 2.2 whereas Kincaid and Darling [6] consider deterministic customer arrivals by restricting the arrival probability in unit time interval to 1.

Proposition 3. *Let the distribution of $P_t = P$ be exponential with parameter α for all t. Then,*

a) *the continuous-time version of the recursion in (2) is $\dfrac{dv_t(s)}{dt} = \dfrac{\lambda_t}{\alpha e} e^{-\alpha \Delta v_t(s)}$ with boundary conditions $v_t(0) = 0$ for all t and $v_0(s) = 0$ for all s,*

b) *the closed-form solution of the model in part (a) is*

$$v_t(s) = \frac{1}{\alpha} \ln \left(\frac{(\Lambda_t)^s}{s! e^s} + \frac{(\Lambda_t)^{s-1}}{(s-1)! e^{s-1}} + \cdots \frac{\Lambda_t}{e} + 1 \right), \tag{10}$$

where $\Lambda_t = \int \lambda_t \mathrm{d}t$ assuming that Λ_t is well-defined for every t. This also provides a closed-form solution for the optimal price, $p^*_{st} = \Delta v_t(s) + 1/\alpha$.

Proof. a) Consider that the time period in which a single customer request can be handled is $[t, t-\epsilon]$ where $\epsilon > 0$ is arbitrarily small. Assuming the proportionality of the arrivals with the length of time period, the probability of a customer arrival equals $\epsilon\lambda_t$ and the optimal price is given in Table 1 as $p^*_{st} = \Delta v_{t-\epsilon}(s) + 1/\alpha$. Then, the optimality equation in (2) can be rewritten as

$$v_t(s) - v_{t-\epsilon}(s) = \epsilon \lambda_t Pr(P_t \geq p^*_{st}) \frac{1}{\alpha}.$$

The sales incentive term $Pr(P_t \geq p^*_{st})$ above in terms of the marginal value of capacity is $e^{-\alpha \Delta v_{t-\epsilon}(s) - 1}$. Rearranging the terms and taking one-sided limit $\epsilon \to 0^+$, the following equation is obtained:

$$\lim_{\epsilon \to 0^+} \frac{v_t(s) - v_{t-\epsilon}(s)}{\epsilon} = \frac{dv_t(s)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha \Delta v_t(s)}.$$

b) Proof is given in Appendix B. □

Remark 3. In airline RM, it is appropriate to use probabilistic customer arrivals with increasing frequency over the sales horizon due to the nature of air travel demand. Also, the maximum amount that customers are willing to pay for the flight also changes throughout the sales horizon and that does not comply with the time-invariant choice of the exponential distribution parameter, α, in Proposition 3. Accordingly, the formulation presented in Proposition 3, which is valid if the customer reservation price is exponentially distributed with constant mean, can be suitable only for modeling small portions of the sales horizon before the departure and should not be generalized to the entire horizon. The relevant models in the literature to be recalled at this point for a possible use or adaptation of the analytical results in Proposition 3 are the static models with the low-before-high demand arrival pattern and non-overlapping time periods for different fare segments.

While the continuous-time version of the DP model given in Proposition 3(a) offers a solution for a particular demand setting, the discrete-time DP model in Section 2.2 is more promising for analytical purposes. The optimal price at a given state, p^*_{st}, and the marginal value of capacity, $\Delta v_t(s)$, are two prominent figures which are important for understanding the behavior of the optimal value function. Recall the structural results presented in Section 2.3 for the discrete-time dynamic pricing DP model.

4. Numerical results

Using the recursive optimality equation given in (2), the optimal price can be found for any given state (s, t). The model requires the knowledge of the probability distribution of reservation price, P_t, at every t before the flight since the demand is not aggregated over time fragments but considered in terms of individual purchasing probabilities of the customers arriving at different time points.

- In order to assess numerical performance of the exact DP model proposed in Section 2.2, a *comparable* alternative but approximate dynamic pricing model is considered in Section 4.1 as an adaptation of the Mixed Integer Programming (MIP) model due to Talluri and van Ryzin [10]. This alternative model called MP-r is an MIP model obtained as a linear relaxation of an Integer Programming (IP) model called MP.
- A general trend of the optimal prices (with possible price markdowns towards the end of sales horizon) is numerically observed for the DP model. In order to avoid such markdowns, DP-m is proposed as a variation (a modified version) of DP.
- For the numerical experiments on the alternative pricing models, the setting and the parameters used are introduced in Section 4.2. The dynamic pricing policies obtained by the alternative models are compared in Section 4.3. Then, the performances of these pricing policies are tested interms of total revenue and load factor using a simulation model in Section 4.4. For the numerical results presented in this section, MATLAB is used with a computer having 3.0 GHz processor and 2 Gb RAM.

4.1. Approximate Mixed Integer Programming formulation

In the MP-r model, the sales horizon is partitioned into successive episodes of possibly unequal length. It is assumed that the price is constant during each episode. A set of alternative discrete prices are considered. Based on the knowledge of demand in each episode for each fare class and under the overall seat capacity constraint, the sequence of the optimal prices is obtained for the episodes in the sales horizon. Solving the model only once at the beginning of the sales horizon, a static price discrimination scheme is given: the nature of change in price is determined before the sales starts and is not affected from the actual demand realizations. Yet, a dynamic pricing framework can be developed by solving the MP-r model repeatedly along the sales horizon. A *daily price update scheme* is considered in Section 4.3 solving MP-r repeatedly and this scheme is compared to the one obtained by the recursive DP. That is, the MP-r model is solved at the end of each day to get a new sequence of prices for the rest of the sales horizon. An additional update scheme is considered in Section 4.4 to solve MP-r once in every 15 minutes.

t denotes time to departure; we have $t = T$ at the beginning of the sales horizon and $t = 0$ at the end of the sales horizon. When time to departure is equal to t_τ, there are τ more remaining episodes before departure and the ordered set $\{t_\tau, t_{\tau-1}, \dots, t_1, t_0\}$ with $t_\tau > t_{\tau-1} > \cdots > t_1 > t_0 = 0$ denotes the beginning and end points of these successive episodes in the MP-r model.

Episodes in the MP-r model allow us to classify customers into different groups depending on the customer arrival time. For the numerical studies in this section, the reservation prices $P_t = P$ of customers for $t \in [t_{i-1}, t_i)$ in episode i are considered to be independently and identically distributed (iid) random variables with cumulative distribution $F^{(i)}(p)$. In order to adopt the generally accepted low-fare before high-fare customer arrival pattern for consistency, it is assumed that the reservation price in episode $i-1$ is stochastically larger than the reservation price in the preceding episode i, i.e. $1 - F^{(i-1)}(p) \geq 1 - F^{(i)}(p)$ for every p.

Considering the dynamic nature of demand in airline RM, Nonhomogeneous Poisson Process (NHPP) is used in this section for modeling individual customer arrivals in the DP-based models with time-variant (daily) customer arrival rate λ_t. The alternative MP-r model adopts demand that is aggregated over episodes. Let parameter μ_{ij} denote the demand-price relationship in the alternative model, it denotes the expected number of seats that can be sold in episode i when the price posted in this episode is p_j, $j = 1, \dots, m$. In order to attain compatibility between two different problem settings, individual demands are aggregated as follows:

$$\mu_{ij} = (1 - F^{(i)}(p_j)) \int_{t_{i-1}}^{t_i} \lambda_t dt. \tag{11}$$

The decision variables used in the MP-r model are y_{ij} that denotes the number of seats that are sold in the ith episode at price level j and the binary variable x_{ij} that is defined below.

$$x_{ij} = \begin{cases} 1 & \text{if the price in the } i\text{th episode is } p_j, \\ 0 & \text{otherwise.} \end{cases}$$

At the beginning of the last τ episodes when the remaining number of available seats is equal to s, the MP-r model to be solved is given below. This model would give the solution for the whole sales horizon when $t_\tau = T$ and $s = S$.

$$\begin{aligned} \text{MP-r:} \qquad \text{Max} \quad & \sum_{i=1}^{\tau} \sum_{j=1}^{m} p_j y_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{\tau} \sum_{j=1}^{m} y_{ij} \leq s, \\ & \sum_{j=1}^{m} x_{ij} = 1 \qquad \forall i \in [1, \tau], \\ & 0 \leq y_{ij} \leq \mu_{ij} x_{ij} \qquad \forall i \in [1, \tau],\ j \in [1, m], \\ & y_{ij} \geq 0 \qquad \forall i \in [1, \tau],\ j \in [1, m], \\ & x_{ij} \in \{0, 1\} \qquad \forall i \in [1, \tau],\ j \in [1, m]. \end{aligned}$$

y_{ij}s are considered above as real decision variables, causing the MP-r model to be an MIP model. MP-r is a linear relaxation of the MP model with integer y_{ij}s. Note that y_{ij} denoting the number of seats should be defined as an integer decision variable for real life implementations. Based on the numerical test of MP given in Appendix D which is also referred to in Section 4.3, we prefer to proceed with MP-r giving a better benchmark than MP for comparison with DP.

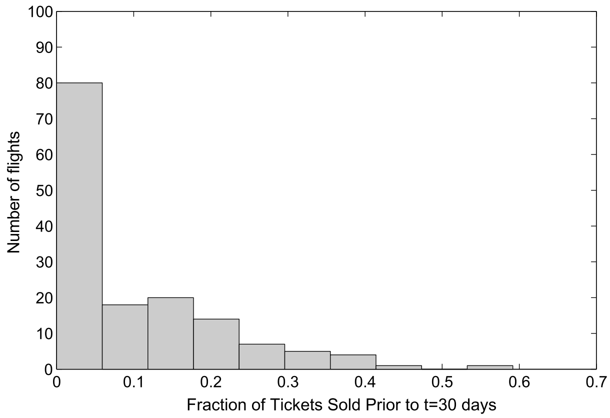

Fig. 3. Early booking ratio - percentage of tickets sold before the last 30 days.

Along with the ticket price in each episode, solutions of the MP-based models also provide the number of seats to be sold in each episode. Yet, the difference in the implementation of dynamic pricing from seat allocation practice is that the prices for the episodes are the only control variables that determine the seller's policy. Therefore, the dynamic pricing policy is characterized by only values of the binary variables x_{ij} representing the optimal price levels in the following episodes. Since a price update scheme is considered in this section to solve the model at the beginning of each episode, the only relevant decision variable value is that of $x_{\tau j}$ for the current episode τ.

4.2. Setting for the numerical experiments

The sales horizons in the airline industry commonly vary between 90 days and 360 days prior to flight; yet, the majority of the seats are sold in the last 30 days. In Fig. 3, a histogram plot is presented based on the sales realizations of 150 flights for a specific origin-destination pair of a mainstream airline company. Along the horizontal axis, the fraction of tickets sold before the last 30-days within overall sales for that flight is given. It is observed that, for majority of the flights, the early bookings (which are made before the last 30 days) constitute a very small ratio in total capacity sold. That is, pricing in the final month is critical. Thus, in the numerical experiments in Sections 4.3 and 4.4 the sales horizon for dynamic pricing is also considered as the last 30 days. Note that the demand data excludes the holidays with a different demand behavior than the general pattern.

In the DP-based models, the reservation price distribution is considered logarithmic with linearly increasing price upper and lower bounds along time. Daily customer arrival rate is convex increasing from 1 arrival/day to 25 arrivals/day $\lambda_t = 25^{1-\frac{t}{30}}$. In the MP-based models, the 30-days sales horizon is partitioned into 5 episodes. Lower and upper bounds of the reservation price are assumed fixed within each episode. For daily customer arrival rate, the average rate is used over each episode; that is, the rate function is piecewise constant. The continuous and piecewise constant versions of time-variant model parameters are depicted in Fig. 4 for the 30-days sales horizon. The values of these parameters are also given in Table 2.

As seen in Table 2, the average daily arrival rate is found as 2.48 for the first episode ($t \in [30, 15]$) and the expected number of total arrivals during this episode is $2.48 \times (30 - 15) = 37.2$. Within the first episode of the MP-based models, the reservation prices are assumed to follow logarithmic behavior with $p_{low} = 69$ and $p_{up} = 144$. If the ticket price is p_j, then the sales incentive for this fare class would be $1 - F^{(i)}(p_j) = \dfrac{\ln(p_{up}) - \ln(p_j)}{\ln(p_{up}) - \ln(p_{low})}$ and the aggregate expected number of seats that can be sold in the first episode at price p_j will be $37.2(1 - F^{(i)}(p_j))$ in the MP-based models.

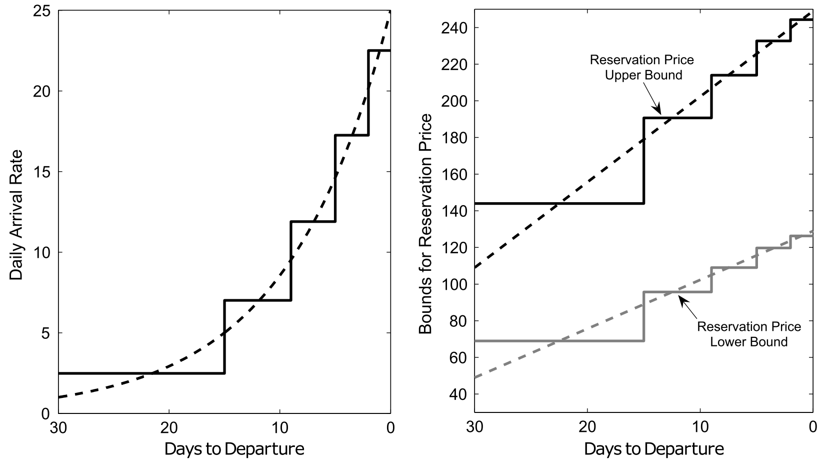

Fig. 4. Time-variant demand parameters - continuous (dash line) for DP-based models, piecewise constant (bold line) for MP-based models.

Table 2
Time-variant demand parameters.

Parameters	Continuous-time	Episode 1	Episode 2	Episode 3	Episode 4	Episode 5
Sales horizon (in days)	$[30, 0]$	[30,15]	(15,9]	(9,5]	(5,2]	(2,0]
(Daily) arrival rate	$[1, 25]$	2.48	7.01	11.89	17.25	22.5
P_t: lower bound (p_{low})	$[49, 129]$	69	95.7	109	119.7	126.3
P_t: upper bound (p_{up})	$[109, 249]$	144	190.7	214	232.7	244.3

During the periodical reruns of the MP-based models, the remaining length of the current episode gets shorter at each run and the respective demand parameter gets updated. Hence, more frequent model reruns result in a better approximation of the smoothly increasing demand along the time horizon.

4.3. Comparison of the alternative models

As previously noted, solving MP-r only once at the beginning of the sales horizon would give a static temporal price discrimination scheme such that the price levels of each episode would be fixed. In order to use MP-r as a tool for dynamic pricing, a daily price update scheme is considered in this section by solving MP-r 30 times. That is, MP-r is solved at the beginning of each day with updated remaining capacity and demand parameters.

For DP, the length of a unit time interval, ϵ, is chosen as 30-seconds. DP gives the optimal price for all time intervals and every possible value of seat inventory as p_{st}. The MP-r model, on the other hand, works once for a particular (s, t) pair. For the numerical analysis here, MP-r is solved repeatedly by fixing s or t to understand the evolution of optimal price as a function of time to departure or remaining seat inventory. The exemplary results are depicted in Figs. 5 and 6 in order to see the behavior of the optimal price in t and s by fixing one of the state variables.

In Fig. 5, the pricing schemes obtained by DP and MP-r are seen for the 30-days sales horizon. The initial increasing trend is due to the increase in customer reservation prices along the sales horizon. The decrease towards the end of the sales horizon can be due to an increase in expected revenue to be obtained by selling more of the remaining seats at good prices without lowering the prices much. The most significant difference between the findings for DP and MP-r is due to the difference between the definition of reservation price bounds. Recall that p_{up} and p_{low} are piecewise constant for MP-r whereas the bounds are linearly increasing for DP. In Fig. 6, the optimal prices obtained by DP are decreasing in the seat inventory level as expected.

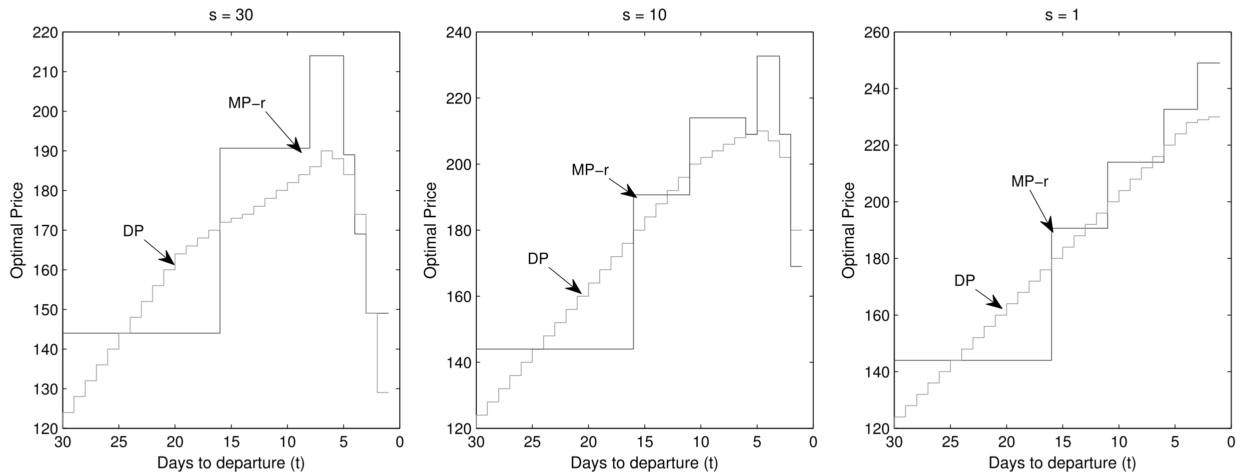

Fig. 5. Optimal prices for seat inventory levels $s = 1$, $s = 10$ and $s = 30$.

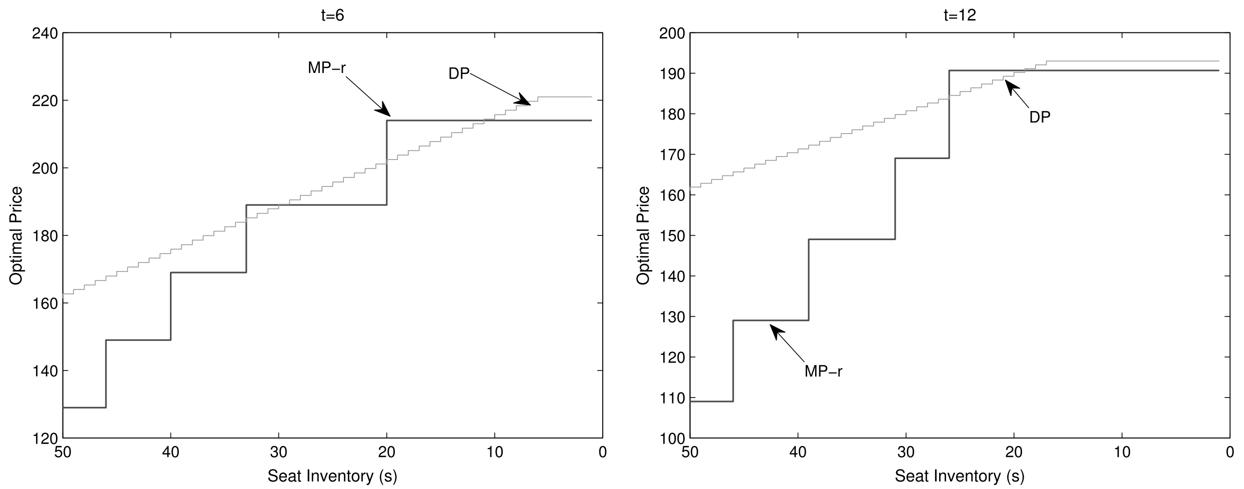

Fig. 6. Optimal prices for $t = 6$ and $t = 12$ days to departure.

We wrap up the numerical comparison of DP and MP-r in this section as follows.

- Time Resolution: In DP, continuous demand parameters are approximated with discrete versions on much finer time intervals as compared to MP-r. That is, DP could represent the continuous change in reservation price, P_t, and customer arrival probability, ρ_t, better while MP-r is restricted to a piecewise constant approximation to these time-variant parameters.
- Computational Time: For the small-scale sample problem under consideration, CPU time to solve DP is less than a minute and it is sufficient to solve it once at the beginning of the sales horizon. Solving MP-r takes less than a second. Note that MP-r is solved over and over again (every day in the example problem) throughout the sales horizon unlike DP.

The aforementioned anticipated characteristics indicate the plausibility of DP. However, price markdowns that may be observed in the results of DP towards the end of sales horizon are avoided in real life applications in airline RM in order not to promote the *strategic* customer behavior. (Elmaghraby and Keskinocak, [4], notice that the myopic customer behavior assumption is valid provided that the customers do not expect a price advantage from delayed purchases.) To overcome this disadvantage of DP, we modify the pricing policy obtained by DP as *offering maximum of the calculated optimal price*, p^*_{st}, and the price offered at

Table 3
Simulation results for pricing policies based on DP and MP.

Method	Revenue			Load factor	
	Initial projection	Sample mean	95% CI	Sample mean	95% CI
MP (1 day)	15,388	14,661	[13,407 − 15,915]	0.94	[0.85, 1.00]
MP-r (1 day)	15,609	15,277	[14,441 − 16,113]	0.95	[0.87, 1.00]
MP-r (15 min.)	16,311	16,125	[15,302 − 16,947]	0.95	[0.89, 1.00]
DP	18,002	18,069	[16,763 − 19,374]	0.99	[0.96, 1.00]
DP-m	18,011	17,243	[15,591 − 18,895]	0.91	[0.81, 1.00]

the beginning of the previous time interval $(t + 1)$. The acronym used for this modified DP-based policy is DP-m. CPU time to solve DP-m is comparable with that of DP.

In Figs. 8 and 9 in Appendix D, the solution of MP is given as compared to the solution of DP. The pricing policy obtained by MP is not implementable in real life. It is observed that the MP model could give pricing schemes that are not in accordance with the expected monotonicity results. Moreover, the MP model has limitations due to considerably long computation time. The related observations presented in Appendix D justify the use of MP-r giving a better benchmark with less computational burden than MP.

In the next section, total revenue and load factor performances of the alternative pricing policies are investigated using a simulation model. In addition to the alternative approximate models studied so far, rather simple elementary intuitive pricing models are introduced and tested.

4.4. Simulation results

Here, we simulate the sales process since analytical comparison (in terms of revenue and load factor) between dynamic pricing methods is quite difficult if not impossible. In Table 3, the simulation results are summarized for 500 replications. In addition to the daily update scenarios of the MP-based models, an update version with a higher frequency is simulated for MP-r. With a finer approximation to the continuous evolution of demand parameters, MP-r re-runs every 15 minutes during the sales horizon in this additional update version. For both total revenue and load factor, estimates for the expected values (sample means) and 95% confidence interval limits (CI) are presented. The initial projection column represents average of the optimal objective function values obtained for different replications by the optimization models -maximizing the expected revenue- at the beginning of the sales horizon.

- The revenue generation comparison also favors the DP-based models over the MP-based models.
- The load factor results of DP are rather high. This is mostly due to the assumption regarding future demand knowledge in the construction of DP formulation: it is assumed that arrival rates and customer reservation price distributions are known to the seller at every future instant. Real life sales agents have more limited knowledge regarding future demand.
- Another important issue regarding the high load factor of the DP policy is that the seller is allowed to decrease the price as the sales process continues. In our simulations, when the seat depletion rate is slow the seller tends to reduce the price to maximize the revenue of flight and, therefore, the load factors turn out to be higher than they are in real life implementations. In this respect, we observe DP-m as a more realistic pricing policy.
- For MP, MP-r and DP-m, the initial projection is slightly above the sample mean. We believe that this is due to the fact that the initial projection distributes the entire capacity over the sales horizon (with a load factor of 100%) while the simulation runs may result in lower load factors.
- MP-r with 15 min. updates shows a better revenue performance in comparison to MP-r with daily update, yet the improvement with higher frequency is limited in comparison to DP and DP-m. The

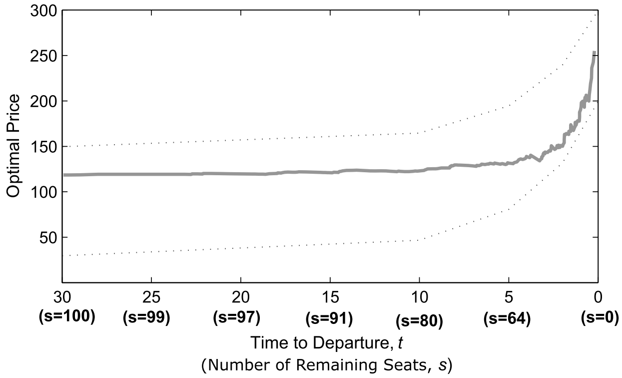

Fig. 7. Optimal prices posted by the seller and time-variant reservation price bounds.

primary reason for the difference between these revenue performances is that, even with a higher price update frequency, MP-r operates with a more limited ability to discretize the ranges of sales price and time to departure in comparison to the DP-based models.

Remark 4. Figs. 5 and 6 are given for understanding the marginal effect of each state variable by fixing the other variable to a constant value. However, these figures do not represent the change of price along the sales horizon since both s and t vary along sales horizon. In order to provide an insight for the DP pricing policy over the sales horizon, the sales process is simulated and evolution of the optimal price along sales horizon is presented in Fig. 7 for a single simulation run.

The number of days, t, before departure is shown along the horizontal axis; $t = 0$ is the time of departure. For each of these t values shown in the figure, the corresponding remaining number of seats is given in parentheses. E.g., the remaining number of seats is 97 for $t = 20$ and the seller is left with 64 seats when time to departure is $t = 5$ days. That is, the optimal prices presented in Fig. 7 are the prices determined by DP for the states, (s, t) pairs, visited under the optimal policy during the sales horizon for the demand scenario considered in the -single- simulation run. In other words, (s, t) pairs along the horizontal axis are the observed -realized- states over the sales horizon in the simulation run.

As seen in Fig. 7, the prices posted by the seller increases along the sales horizon with an increased rate at the last days of the sales horizon. This numerical observation is in accordance with what is mostly seen in real life airline RM practices. (Note that s may turn out to be greater than zero at $t = 0$; in the case presented in Fig. 7, a sold-out case is observed with $s = 0$ at $t = 0$.)

In order to test whether using DP is effective in maximizing revenue, we also simulate another demand scenario. We again consider a piecewise constant reservation price behavior. Given the upper and lower bounds of the reservation price, the following alternative probability distributions are considered for P_t: the logarithmic distribution as defined in Table 1 and the uniform distribution, which corresponds to the linear demand function in Table 1. The customer arrival frequency is assumed to be convex increasing towards the time of departure and the seat inventory is chosen so that the expected number of arrivals is almost twice the available inventory.

For both logarithmic and uniform distributions, alternative intuitive pricing policies are defined here as a benchmark for evaluating the performance of DP. These alternative sales prices are elementary distribution statistics (mean, median and percentiles) of the time-variant logarithmic and uniform distributions. These benchmark policies take the temporal changes in the reservation price into account but disregard the sales realizations and remaining inventory. For both settings of the reservation price distribution, performances of the alternative pricing policies and the policy obtained by DP are compared in the simulation model for the same demand realization string. The numerical results for 500 replications are summarized below.

- For the time-variant logarithmic distribution of P_t, the pricing policies that are compared with DP are posting the *mean* $(p_{up} \times p_{low})^{0.5}$ and the *median* $(p_{up} + p_{low})/2$ price values. Mean price and median price policies give the average load factor values of 91.7% and 97.3%, respectively. On the other hand, DP policy gives an average load factor value of 99.4%. Besides, the DP policy outperforms these two policies in terms of revenue with 19.6% and 15.8% improvements as compared to mean price and median price policies, respectively.
- For the time-variant uniform distribution of P_t, the pricing policies that are compared with DP are posting the *mean* and the 25th-*percentile* price values. Mean price and 25th-percentile price policies give the average load factor values of 97.9% and 100%, respectively. Both values are close to the average load factor value 99.4% that is obtained by the DP policy. Yet, the DP policy outperforms these two policies in terms of revenue with 13.5% and 41.3% as compared to mean price and median price policies, respectively.

In the simulation experiments, policy prices are calculated with full-accurate information assuming that future customer arrival rates and reservation price bounds are known with certainty. This assumption does not constitute a basis for bias towards DP in this comparative analysis. Yet, if the demand information is imperfect, the load factors obtained by the DP policy would not be that high.

In order to understand the effect of imperfect information on the DP pricing policy, we test the policy for underestimating and overestimating the customer reservation price. Keeping the arriving customers' reservation price distributions the same as in the base scenario, we simulate the lower and upper bounds of the estimated reservation prices to be 10 and 25 percent above and below in the model.

- For the simulation scenarios such that the seller overestimates the reservation prices by 10% and 25%, the respective average revenues are 7.6% and 16.3% lower than that of the perfect information scenario. In case of overestimation, seats deplete slower than anticipated and even if the DP policy increases the sales incentive in response, the take up rate does not increase that much since even the ensured sales price, p_{low}, assumed by the model with a sales incentive of 1 is greater than the real p_{low} value for some of the arriving customers due to overestimation. That is, sale for such customers can not be guaranteed even for the smallest price assumed by the model.
- The revenue impact of underestimating the reservation price by 10% and 25% is decrease by 4.3% and 10.9%, respectively. Improved robustness of the DP policy in case of underestimating the reservation price as compared to the case of overestimation is due to the fact that underestimating customer willingness to buy results in faster depletion of the seat inventory which in turn pushes the prices upwards.

5. Conclusion

This study focuses on generalization of the results in dynamic pricing literature for airlines and presents a methodology that could be implemented for real-time dynamic pricing in the airline industry. The dynamic pricing formulation, DP, proposed in this article is investigated considering structural comparisons to prior studies in the dynamic pricing literature. The proposed DP model is also analyzed for different demand-price relationships and novel results are presented. Statistical analysis of data for the observed demand behavior to investigate which demand formulations these observations are consistent with would be a challenging future research topic.

Numerical experiments show that DP gives a general trend of the optimal prices with possible price markdowns towards the end of sales horizon. Based on these observations, DP-m is considered as a variation of DP. Also, DP is compared to the alternative MP model and its linear relaxation MP-r considered with piecewise constant approximations of the demand parameters. Simulation results for total revenue and load factor values obtained by these alternative pricing models show plausibility of DP and DP-m. DP is also compared with rather simple elementary intuitive pricing models. The simulation results show that even when the seller could achieve a high load factor with an elementary dynamic pricing policy, these models are not satisfactory in revenue maximization when compared to DP.

In the proposed DP formulation with optimal value function $v_t(s)$, the restriction is $s \geq 0$ and the possibility of overbooking is disregarded. However, as Talluri and van Ryzin [10] state, overbooking is not only the oldest practice in RM but it also is among the most successful implementations. Thus, generalizing the results of the dynamic pricing methodology introduced here to include the possibility of overbooking is considered as a future work.

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Appendix A. Proof of Lemma 1

Proof is by contradiction. Assume that $Pr(P_t \geq p)p$ is increasing at $p = p^*_{st}$. Then, there exists $\theta > 0$ such that

$$Pr(P_t \geq p^*_{st} + \theta)(p^*_{st} + \theta) \geq Pr(P_t \geq p^*_{st})p^*_{st}. \tag{12}$$

Also, by definition of the complementary cumulative distribution function of P_t,

$$Pr(P_t \geq p^*_{st} + \theta) \leq Pr(P_t \geq p^*_{st}).$$

Since $\Delta v_{t-1}(s)$ is a nonnegative quantity for any given (s, t),

$$Pr(P_t \geq p^*_{st} + \theta)\Delta v_{t-1}(s) \leq Pr(P_t \geq p^*_{st})\Delta v_{t-1}(s). \tag{13}$$

Then, the difference of (12) and (13) gives

$$Pr(P_t \geq p^*_{st} + \theta)(p^*_{st} + \theta - \Delta v_{t-1}(s)) \geq Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_{t-1}(s)),$$

which contradicts the assumption that p^*_{st} is the maximizer. By contradiction, $Pr(P_t \geq p)p$ is nonincreasing at $p = p^*_{st}$. □

Proof of Lemma 3. Referring to the proof given by Talluri and van Ryzin [10] for Proposition 5.2 in their book, the proof here is given by induction on t.

- $\Delta v_0(s + 1) \leq \Delta v_0(s)$ holds true for every $s \in [1, S - 1]$ since $v_0(s + 1) = v_0(s) = 0$ by the boundary condition.
- Assume $\Delta v_{t-1}(s + 1) \leq \Delta v_{t-1}(s)$ for every $s \in [1, S - 1]$ and $t > 1$.
  Denoting the sales probability term $Pr(P_t \geq p^*_{s+i,t})$ by $z_{s+i,t}$, the following equalities are obtained for $\Delta v_t(s + 2)$ and $\Delta v_t(s + 1)$.

$$\Delta v_t(s+2) = \rho_t \left(z_{s+2,t}[p^*_{s+2,t} - \Delta v_{t-1}(s+2)] - z_{s+1,t}[p^*_{s+1,t} - \Delta v_{t-1}(s+1)]\right) + \Delta v_{t-1}(s+2)$$
$$\Delta v_t(s+1) = \rho_t \left(z_{s+1,t}[p^*_{s+1,t} - \Delta v_{t-1}(s+1)] - z_{st}[p^*_{st} - \Delta v_{t-1}(s)]\right) + \Delta v_{t-1}(s+1).$$

Since $p^*_{s+1,t}$ is the maximizer for the function $Pr(P_t \geq p)[p - \Delta v_{t-1}(s+1)]$, replacing $z_{s+1,t}$ and $p^*_{s+1,t}$ above with $z_{s+2,t}$ and $p^*_{s+2,t}$ (z_{st} and p^*_{st}) on the right hand side of the first (second) equation above gives the following inequalities:

$$\begin{aligned}
\Delta v_t(s+2) &\leq \rho_t \left(z_{s+2,t}[p^*_{s+2,t} - \Delta v_{t-1}(s+2)] - z_{s+2,t}[p^*_{s+2,t} - \Delta v_{t-1}(s+1)] \right) + \Delta v_{t-1}(s+2) \\
&= \rho_t z_{s+2,t} \left[\Delta v_{t-1}(s+1) - \Delta v_{t-1}(s+2) \right] + \Delta v_{t-1}(s+2), \\
\Delta v_t(s+1) &\geq \rho_t \left(z_{st}[p^*_{st} - \Delta v_{t-1}(s+1)] - z_{st}[p^*_{st} - \Delta v_{t-1}(s)] \right) + \Delta v_{t-1}(s+1) \\
&= \rho_t z_{st} \left[\Delta v_{t-1}(s) - \Delta v_{t-1}(s+1) \right] + \Delta v_{t-1}(s+1).
\end{aligned}$$

Then, the following is obtained for the difference $\Delta v_t(s+2) - \Delta v_t(s+1)$.

$$\begin{aligned}
\Delta v_t(s+2) - \Delta v_t(s+1) \leq\ & (1 - \rho_t z_{s+2,t})\Delta v_{t-1}(s+2) + \rho_t z_{s+2,t}\Delta v_{t-1}(s+1) \\
& - (1 - \rho_t z_{st})\Delta v_{t-1}(s+1) - \rho_t z_{st}\Delta v_{t-1}(s).
\end{aligned}$$

Rearranging the terms on the right hand side above,

$$\begin{aligned}
\Delta v_t(s+2) - \Delta v_t(s+1) \leq\ & (1 - \rho_t z_{s+2,t}) \left[\Delta v_{t-1}(s+2) - \Delta v_{t-1}(s+1) \right] \\
& + \rho_t z_{st} \left[\Delta v_{t-1}(s+1) - \Delta v_{t-1}(s) \right]
\end{aligned}$$

where ρ_t, z_{st} and $z_{s+2,t}$ are all probability values. Then, the terms $1 - \rho_t z_{s+2,t}$ and $\rho_t z_{st}$ are nonnegative. Using the induction assumption, both $\Delta v_{t-1}(s+2) - \Delta v_{t-1}(s+1)$ and $\Delta v_{t-1}(s+1) - \Delta v_{t-1}(s)$ are less than or equal to zero. Thus, $\Delta v_t(s+2) - \Delta v_t(s+1) \leq 0$, meaning that $\Delta v_t(s)$ is nonincreasing in s. □

Proof of Corollary 1. Proof is given by contradiction. Assume that $p^*_{st} < p^*_{s+1,t}$. By definition of p^*_{st},

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_{t-1}(s)) \geq Pr(P_t \geq p^*_{s+1,t})(p^*_{s+1,t} - \Delta v_{t-1}(s)). \tag{14}$$

From Lemma 3, $\Delta v_{t-1}(s)$ is nonincreasing in s. Hence, one can define $\delta \geq 0$ such that $\Delta v_{t-1}(s+1) = \Delta v_{t-1}(s) - \delta$. Since $\delta \geq 0$ and $p^*_{st} < p^*_{s+1,t}$,

$$\delta Pr(P_t \geq p^*_{st}) \geq \delta Pr(P_t \geq p^*_{s+1,t}). \tag{15}$$

Addition of (14) and (15) gives

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_{t-1}(s+1)) \geq Pr(P_t \geq p^*_{s+1,t})(p^*_{s+1,t} - \Delta v_{t-1}(s+1)),$$

which contradicts the optimality of $p^*_{s+1,t}$ for $Pr(P_t \geq p)(p - \Delta v_{t-1}(s+1))$. Hence, $p^*_{st} \geq p^*_{s+1,t}$. □

Proof of Corollary 2. By optimality of p^*_{st},

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_{t-1}(s)) \geq Pr(P_t \geq \hat{p})(\hat{p} - \Delta v_{t-1}(s)) \quad \text{for all } \hat{p}. \tag{16}$$

Consider $\hat{p}$ such that $\hat{p} \leq p^*_{st}$ implying $Pr(P_t \geq p^*_{st}) \leq Pr(P_t \geq \hat{p})$. Since the marginal value of capacity is nondecreasing in t from Lemma 2 and Lemma 3, $\Delta v_t(s) - \Delta v_{t-1}(s) \geq 0$. Then, also

$$Pr(P_t \geq p^*_{st})(\Delta v_t(s) - \Delta v_{t-1}(s)) \leq Pr(P_t \geq \hat{p})(\Delta v_t(s) - \Delta v_{t-1}(s)). \tag{17}$$

The difference of (16) and (17) gives

$$Pr(P_t \geq p^*_{st})(p^*_{st} - \Delta v_t(s)) \geq Pr(P_t \geq \hat{p})(\hat{p} - \Delta v_t(s)).$$

Since the probability distribution of P_t is time-invariant ($P_t = P$ for all t), P_t in the inequality above can be replaced with P_{t+1}. Then, $\hat{p}$, which is less than p^*_{st}, cannot yield a better value for the function to be maximized at state $(s, t+1)$; $Pr(P_{t+1} \geq p)(p - \Delta v_t(s))$. Thus, $p^*_{s,t+1}$ cannot be less than p^*_{st}. □

Appendix B. Proof of Proposition 3(b)

Proof is given by induction on the number of available seats for sale, s, at a given time period t.

- Consider $s = 1$. Due to the boundary condition $v_t(0) = 0$, we have $\Delta v_t(1) = v_t(1)$ in the optimality equation:

$$\frac{dv_t(1)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha v_t(1)}.$$

  Rearranging the terms in this differential equation as $\alpha e^{\alpha v_t(1)} dv_t(1) = \frac{\lambda_t}{e} dt$ and integrating both sides, the following equation is obtained:

$$e^{\alpha v_t(1)} = \frac{\Lambda_t}{e} + c. \tag{18}$$

  Notice that (18) holds for every t. At $t = 0$, the sales horizon ends and no more customers arrive at this point, i.e., $\Lambda_0 = 0$. Using also the boundary condition $v_0(1) = 0$ gives $c = 1$. Then, $v_t(1) = \frac{1}{\alpha} \ln\left(\frac{\Lambda_t}{e} + 1\right)$.
- Assume that $v_t(s)$ is given by (10). The optimality equation for $s+1$ available seats at time t is

$$\frac{dv_t(s+1)}{dt} = \frac{\lambda_t}{\alpha e} e^{-\alpha \Delta v_t(s+1)}.$$

  Rearranging the terms in this differential equation as $\alpha e^{\alpha v_t(s+1)} dv_t(s+1) = \frac{\lambda_t}{e} e^{\alpha v_t(s)} dt$ and using the induction assumption for $v_t(s)$,

$$\alpha e^{\alpha v_t(s+1)} dv_t(s+1) = \lambda_t \left(\frac{(\Lambda_t)^s}{s! e^{s+1}} + \frac{(\Lambda_t)^{s-1}}{(s-1)! e^s} + \cdots + \frac{\Lambda_t}{e^2} + \frac{1}{e} \right) dt \tag{19}$$

  is obtained. Notice that $\int \lambda_t (\Lambda_t)^i \mathrm{d}t = \frac{(\Lambda_t)^{i+1}}{i+1}$ for every positive integer i; the derivative of the right hand side with respect to t gives $\rho_t(\Lambda_t)^i$. Hence, integration of both sides in (19) leads to the following equation:

$$e^{\alpha v_t(s+1)} = \frac{(\Lambda_t)^{s+1}}{(s+1)! e^{s+1}} + \frac{(\Lambda_t)^s}{s! e^s} + \cdots + \frac{(\Lambda_t)^2}{2! e^2} + \frac{\Lambda_t}{e} + c.$$

  Using the boundary condition $v_0(s+1) = 0$ in the equation above gives $c = 1$. Then,

$$v_t(s+1) = \frac{1}{\alpha} \ln\left(\frac{(\Lambda_t)^{s+1}}{(s+1)! e^{s+1}} + \frac{(\Lambda_t)^s}{s! e^s} + \cdots + \frac{(\Lambda_t)^2}{2! e^2} + \frac{\Lambda_t}{e} + 1 \right). \qquad \square$$

Appendix C. Exponential, linear and isoelastic demand-price models

- Exponential Relationship: $Pr(P_t \geq p) = e^{-\alpha p}$.
  For exponentially distributed reservation price, $Pr(P_t \geq p) = e^{-\alpha p}$. Then, the first order optimality condition is

$$\frac{d}{dp}\{e^{-\alpha p}(p - \Delta v_{t-1}(s))\} \;=\; e^{-\alpha p}(1 - \alpha p + \alpha \Delta v_{t-1}(s)) \;=\; 0,$$

  where $\Delta v_{t-1}(s)$ is not a function of the price posted at time t. The solution of this condition gives

$$p \;=\; \Delta v_{t-1}(s) + \frac{1}{\alpha},$$

  which is positive since $\Delta v_{t-1}(s) \geq 0$ and $\alpha > 0$. Moreover, this p value is greater than $\Delta v_{t-1}(s)$ as implied by (2). Note that $\frac{d}{dp}\{e^{-\alpha p}(p - \Delta v_{t-1}(s))\} \;=\; -\alpha(p - \left[\Delta v_{t-1}(s) + \frac{1}{\alpha}\right])e^{-\alpha p}$ is negative for $p > \Delta v_{t-1}(s) + \frac{1}{\alpha}$ and positive for $p < \Delta v_{t-1}(s) + \frac{1}{\alpha}$. Also, $\frac{d^2 e^{-\alpha p}(p - \Delta v_{t-1}(s))}{dp^2}|_{p=\Delta v_{t-1}(s)+\frac{1}{\alpha}} \;=\; -\alpha e^{-\alpha \Delta v_{t-1}(s) - 1} < 0$ meaning that the second order optimality condition is satisfied at $p = \Delta v_{t-1}(s) + \frac{1}{\alpha}$. Thus, the optimal price at state (s, t) is $p^*_{st} = \Delta v_{t-1}(s) + \frac{1}{\alpha}$.
- Linear Relationship: $Pr(P_t \geq p) = a - bp$.
  s not aggregated in terms of quantity in this article for airline RM but interpreted as the probability of selling the commodity to an individual, preserving the linear relationship the price interval is given as $p \in [\frac{a-1}{b}, \frac{a}{b}]$ so that $Pr(P_t \geq p) \in [0, 1]$. The linear demand-price relationship implies $P_t \sim U(\frac{a-1}{b}, \frac{a}{b})$ where U stands for uniform distribution. That is, the linear sales incentive formulation is suitable when the probability density of reservation price is constant on its support. In this case, $f_{P_t}(p) = b$ and substituting it in the first order optimality equation in (5), $p = \frac{1}{2}\left(\Delta v_{t-1}(s) + \frac{a}{b}\right)$ is obtained.
  Consider $\max_p \{(a - bp)(p - \Delta v_{t-1}(s))\}$ in (2) for the linear demand. Since the function $(a - bp)(p - \Delta v_{t-1}(s)) = -bp^2 + (a + b\Delta v_{t-1}(s))p - a\Delta v_{t-1}(s)$ to be maximized is concave in p, the value of p^*_{st} that satisfies the first order optimality condition is the unique maximizer given that p^*_{st} lies on the support. Otherwise, the maximum functional value would be achieved at one of the endpoints of the closed interval $[\frac{a-1}{b}, \frac{a}{b}]$ depending on the value of $\Delta v_{t-1}(s)$.
- Isoelastic Relationship: $Pr(P_t \geq p) = ap^{-b}$.
  Isoelastic demand represents demand curves for which the price elasticity of demand $(\frac{dp}{dq} \times \frac{q}{p})$ is constant. In order to guarantee that $Pr(P_t \geq p)$ takes values in $[0, 1]$, the support for the reservation price distribution is defined as $p \geq a^{1/b}$. The probability density function of reservation price in the isoelastic demand formulation is given as $f_{P_t}(p) = abp^{-b-1}$ for $a > 0$ and $b > 1$. Hence, the first order optimality condition in (5) gives $p = \Delta v_{t-1}(s)\frac{b}{b-1}$. Note that this p value is nonnegative since $\Delta v_{t-1}(s) \geq 0$ and $\frac{b}{b-1} > 0$.
  Consider the optimization $\max_p \{Pr(P_t \geq p)(p - \Delta v_{t-1}(s))\}$ for the isoelastic relationship. That is, the function to be maximized is $ap^{-b}(p - \Delta v_{t-1}(s))$. The first derivative with respect to p is $\frac{d}{dp}ap^{-b}(p - \Delta v_{t-1}(s)) \;=\; ap^{-b-1}(b\Delta v_{t-1}(s) - p(b-1))$ which can be rewritten as $a(b-1)p^{-b-1}(\Delta v_{t-1}(s)\frac{b}{b-1} - p)$. Then, the first derivative is negative for $p > \Delta v_{t-1}(s)\frac{b}{b-1}$ and positive for $p < \Delta v_{t-1}(s)\frac{b}{b-1}$. Also, $\frac{d^2}{dp^2}ap^{-b}(p - \Delta v_{t-1}(s))|_{p=\Delta v_{t-1}(s)\frac{b}{b-1}} \;=\; -ab\Delta v_{t-1}(s)(\Delta v_{t-1}(s)\frac{b}{b-1})^{-b-2} \leq 0$ meaning that the second order optimality condition is satisfied at $p = \Delta v_{t-1}(s)\frac{b}{b-1}$. Thus, $p^*_{st} = \Delta v_{t-1}(s)\frac{b}{b-1}$ is the unique maximizer provided that $\Delta v_{t-1}(s)\frac{b}{b-1} \geq a^{1/b}$. If $\Delta v_{t-1}(s)\frac{b}{b-1} < a^{1/b}$, then the function is decreasing

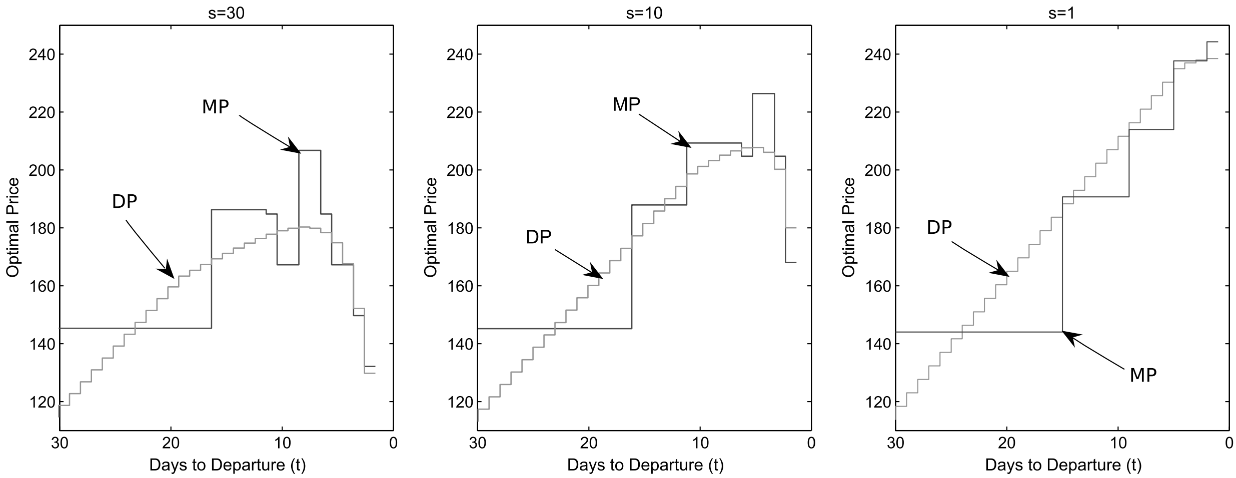

Fig. 8. Optimal prices for seat inventory levels $s = 1$, $s = 10$ and $s = 30$.

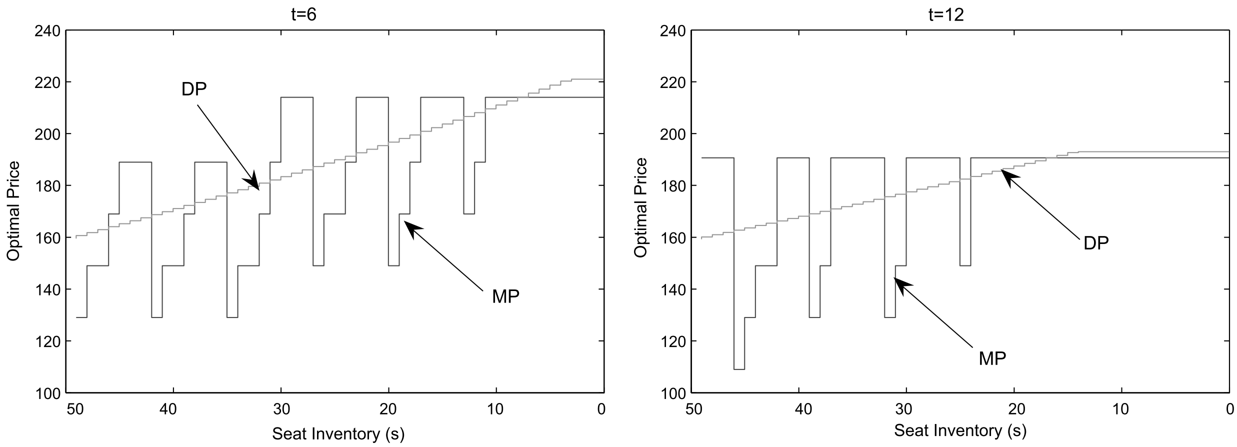

Fig. 9. Optimal prices for $t = 6$ and $t = 12$ days to departure.

on the support and the optimal price is $p^*_{st} = a^{1/b}$. Note that p^*_{st} is greater than $\Delta v_{t-1}(s)$ as implied by (2).

Appendix D. Numerical comparison of the DP and MP models

As seen in Figs. 8 and 9, the results are counterintuitive for the MP model: cyclical ups and downs are observed instead of a general trend. This phenomenon is due to the structure of the MP model to find an optimal seat allocation for the remaining episodes together with the optimal prices. In this respect, the optimal price for $(s+1, t)$ could be larger than that for (s, t) due to a major change in sales strategy with an additional available seat in period t. This could be noted as a shortcoming of the MP model. The behavior of optimal price obtained by DP is more parallel to our analytical findings and expectations.

Solution times for DP and MP-r are considerably lower than that of MP; solving MP for a given (s, t) pair typically lasts 60 to 90 seconds. Noticing the possibility of inconsistency in the probable pricing scenarios obtained by MP as seen in Figs. 8 and 9, we consider MP-r as the linear relaxation of MP by assuming that the number of seats to be sold at each episode, y_{ij}, is a real decision variable. MP-r gives a better benchmark than MP. CPU time to solve MP-r takes less than a second for the small-scale sample problem, allowing the decision maker to solve MP-r more frequently than MP with a considerable saving in the solution time.

Moreover, monotonicity of optimal price in time is preserved with MP-r until the last couple of days before the flight when the optimal price decreases rapidly if the remaining inventory is high. (Recall Figs. 5 and 6.)

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