Case study - Scaling properties

1. Moving from distributional properties of time series to their dynamic properties over time

2. Comparison of different types of random walks

3. Multi-scaling and Generalized Hurst Exponents

4. Applications to data

Random walks

We call random walk a stochastic process of the following type

$$x(t+1) = x(t) + \eta(t) \qquad\qquad x(0) = \eta(0)$$

where $\eta(t)$ are i.i.d. (independent and identically distributed) random variables

Random walks

By summing up all terms we get

$$x(1) = x(0) + \eta(1) = \eta(0) + \eta(1)$$

$$x(2) = x(1) + \eta(2) = \eta(0) + \eta(1) + \eta(2)$$

$$\vdots$$

$$x(T) = \textstyle\sum_{t=0}^{T} \eta(t)$$

At each time step the value of a random walk is given by the sum of i.i.d. random variables

Scaling of Gaussian random walk

What happens to $x(T) = \sum_{t=0}^{T} \eta(t)$ when $T \to \infty$?

From the Central Limit Theorem we know that the addition of i.i.d. random variables with finite variance will lead to a Gaussian distribution.

Let us assume $\eta(t) \sim \mathcal{N}(0, \sigma^2)$ (a bad model of financial log-returns). Then, from the CLT we can conclude that for large times we will have

$$x(t) \sim \mathcal{N}(0, \sigma^2 t)$$

Scaling of Gaussian random walk

From the previous scaling result we have

$$p_t(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{x^2}{\sigma^2 t}} = \frac{1}{\sqrt{t}} p_1\left(\frac{x}{\sqrt{t}}\right)$$

Inverting the above we have

$$p_1(x) = \sqrt{t}\; p_t(x\sqrt{t})$$

So properly rescaling the distribution **at any time** will give **the same distribution**

Scaling of Gaussian random walk

The main take-home messages are

1. The **variance scales linearly** over time (standard deviation scales with square root of time)

2. The distribution is **invariant over time**! With a suitable rescaling, distributions at all times collapse on the same PDF

Lévy walks

What happens to $x(T) = \sum_{t=0}^{T} \eta(t)$ when $T \to \infty$ but the increments do not have finite variance?

In analogy with what we have seen before, we can invoke the generalisation of the Central Limit Theorem for Lévy stable distributions.

It follows from the Theorem that for increments distributed according to a Lévy stable distribution with tail exponent α

$$p_t(x) = \frac{1}{t^{1/\alpha}} p_1\left(\frac{x}{t^{1/\alpha}}\right)$$

This result recovers the Gaussian case for $\alpha = 2$, as it should

Scaling in the time domain:
Generalised Hurst Exponent

Tool to detect whether a time series has different scaling properties at different time scales

$$\frac{\mathbb{E}[|r(t+\tau)-r(t)|^q]}{\mathbb{E}[|r(t+1)-r(t)|^q]} \sim \tau^{qH(q)}$$

$0 < H(q) < 1/2$	Anti-persistent behaviour
$H(q) = 1/2$	Random walk
$1/2 < H(q) < 1$	Persistent behaviour

Generalized Hurst exponent for Lévy processes

$$\boxed{\alpha \leq 1} \longrightarrow \begin{cases} qH(q) = q & \text{for} \quad q < 1 \\ qH(q) = 1 & \text{for} \quad q \geq 1 \end{cases}$$

$$\boxed{1 < \alpha < 2} \longrightarrow \begin{cases} qH(q) = q/\alpha & \text{for} \quad q < \alpha \\ qH(q) = 1 & \text{for} \quad q \geq \alpha \end{cases}$$