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Pricing Structured Notes in YASN and SWPM

March 20, 2018

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Abstract

This document describes the pricing of structured notes using the Bloomberg Functions YASN and SWPM, as well as the important features of its underlying Linear Gaussian Markov (LGM) model.

Keywords. SWPM, YASN, CDSW, OAS, LGM, HW1F, Linear Gaussian Markov, Hull-White one-factor, Hybrid credit-rates.

Contents

Introduction

This document describes the pricing of interest rate structured notes in YASN and SWPM. Interest rate structured notes are instruments in which the buyer pays an up front principal amount and receives periodic coupon payments in addition to the return at maturity of the principal amount. These coupons are typically functions of market quoted interest rates, such as Libor and CMS rates. Often the notes are callable, in which case the seller has the option, exercisable at specific call dates, to redeem the note by paying back the principal (plus a potential "redemption fee").

The first chapter describes the *replication* and *convexity adjustment* pricing methodologies used in YASN for non-callable deals, and gives several examples illustrating these techniques.

The second chapter describes the underlying model used in SWPM and YASN, the Linear Gaussian Markov (LGM) model, which is a variant of the well-known Hull-White short rate model. Our presentation follows closely that given in the papers of Hagan.

Furthermore, the technique of *adjusters* is introduced to ensure consistent pricing of callable deals with their non-callable versions, and is explained in the final chapter. Additional topics and examples are provided in the Appendices.

1 Pricing of Non-callable Deals

In the absence of callability and path dependence, which is to say that the coupon payments do not depend on the values of previously paid coupons or prior interest rate events, it is often the case that the coupons can be valued, or "replicated", by representing them as a combination of vanilla instruments such as forwards, caps and floors. This forms the basis for the replication method, and is described briefly in §1.1.

In some deals, such as CMS based deals, there is a mismatch between the coupon payment times and the natural payment times of the replicating instruments. In these situations, an adjustment to the pricing, referred to as a *convexity correction*, must be applied, and is described in §1.2.

1.1 Replication Methodology

In SWPM and YASN, the pricing of non-callable products, such as digital options, CMS capped floaters, CMS spread deals, and range accruals, use the replication method to obtain prices from the vanilla market. This method, based on elementary calculus identities such as[1]

$$\frac{\partial}{\partial x}\left[(x-y)\cdot\mathbf{1}_{\{y<x\}}\right] = -\frac{\partial}{\partial y}\left[(x-y)\cdot\mathbf{1}_{\{y<x\}}\right] = \mathbf{1}_{\{y<x\}}, \tag{1.1a}$$

$$\int_a^b f''(x)(b-x)\,dx = f(b)-f(a)-f'(a)(b-a), \tag{1.1b}$$

deduces prices directly from the vanilla caps and swaptions markets without positing any model assumptions.

[1]The notation $\mathbf{1}_A$ denotes the indicator function of the set A, taking the value 1 for $x\in A$, and 0 for $x\notin A$. Identity (1.1b) is a form of the Lagrange Remainder $R_1(b)$.

Digital Option Replication

Recall that a Libor $L(t)$ floor payoff at strike K is given by:

$$F(L(t), K) = [K - L(t)]^+ .$$

Hence, a Libor digital $D(L(t), K) := \mathbf{1}_{L(t)\in[0,K]} \overset{(1.1a)}{=} \frac{\partial F(L(t),K)}{\partial K}$ can be replicated using floors[2]:

$$D(L(t), K) = \lim_{\eta\to 0} \frac{1}{\eta}\{F(L(t), K + \tfrac{1}{2}\eta) - F(L(t), K - \tfrac{1}{2}\eta)\}.$$

Range Accrual Replication

A coupon of a standard Libor range accrual is defined as

$$\frac{\alpha R_{fix}}{M} \cdot \mathbf{1}_{L(\tau_{st})\in[R_{min},R_{max}]} = \begin{cases} \alpha R_{fix}/M & \text{for } R_{min} \le L(\tau_{st}) \le R_{max}, \\ 0 & \text{otherwise.} \end{cases}$$

Here α is the daycount coverage of the coupon period, R_{fix} is the fixed coupon rate, M is the total number of days in the coupon period, $L(\tau_{st})$ is the Libor rate fixing at time τ_{st}, and finally $[R_{min}, R_{max}]$ is the accrual range. Since $\mathbf{1}_{[a,b]} = \mathbf{1}_{[0,b]} - \mathbf{1}_{[0,a)}$, range accruals defined in this way are long a digital floor at strike of R_{max}, and short a digital floor at strike of R_{min}, to which digital replication can be applied.

In range accruals, an additional complexity is introduced since the coupon is accrued every day depending on whether or not the rate is within range. Since the payment date is fixed, there is a payment date mismatch with the standard Libor floor option, and this mismatch can also be replicated in SWPM using the *replication convexity adjustment* described below. Further details describing this replication technique may be found in [Hag1].

1.2 Convexity Adjustment

CMS Rate Products

Constant Maturity Swaps (CMS) based products are very common among interest rate derivative deals. Exotics depending on CMS swaps include CMS capped floaters, and CMS range accruals. In these products the reset index is the swap rate, but accrual and payment use a standard coupon period, and therefore a convexity adjustment is required to correct the mismatch between receiving a single accrued coupon rather than a coupon stream associated with entering a swap. Note that SWPM and YASN provide three modeling methods for calculating a convexity adjustment; namely replication, lognormal-model, and normal-model based methods, the latter two being specific approximations to the general replication formula. Here we provide only the relevant results, detailed derivations of which can be found in [Hag2].

[2] In practice, one chooses $\eta = 10$ bps as a numerical approximation.

The CMS caplet fixed at time τ with swap schedule $\{\tau = t_0, t_1, \ldots, t_n\}$ and corresponding coverages $\{\alpha_1, \ldots, \alpha_n\}$ is defined as

$$\begin{aligned} V_{cap}^{CMS}(0) &= A(0)\mathbb{E}\left[\frac{(R_s(\tau)-K)^+ Z(\tau;t_p)}{A(\tau)}\,\middle|F_0\right] \\ &= D(t_p)\mathbb{E}\left[(R_s(\tau)-K)^+\,\middle|F_0\right] + D(t_p)\mathbb{E}\left[(R_s(\tau)-K)^+\left(\frac{G(R_s(\tau))}{G(R_s(0))}-1\right)\,\middle|F_0\right], \end{aligned} \tag{1.2}$$

where $\mathbb{E}\left[\cdot\,\middle|F_0\right]$ is the conditional expectation under the swap annuity measure, $D(t_p)$ is the discount factor at payment time t_p, $R_s(\tau)$ is the swap rate at the fixing time τ, and finally

$$\begin{aligned} G(R_s(\tau)) &:= Z(\tau;t_p)/A(\tau), \\ A(\tau) &:= \sum_{i=1}^{n} \alpha_i Z(\tau;t_i), \end{aligned}$$

where $Z(\tau;t_p)$ is the time τ price of the zero coupon bond maturing at t_p. These conventions, as well as the shorthand $R_s^0 := R_s(0)$, and $A_0 := A(0)$, will be assumed throughout this section.

1. **Replication convexity adjustment**: Using (1.1b), $V_{cap}^{CMS}(0)$ can be rewritten as

$$V_{cap}^{CMS}(0) = \frac{D(t_p)}{A_0}\left\{[1+f'(K)]C(K) + \int_K^{\infty} C(x)f''(x)\,dx\right\}, \tag{1.3}$$

where the value of an ordinary payer swaption with strike x is priced as:

$$C(x) = A_0\mathbb{E}\left[(R_s(\tau)-x)^+\,\middle|F_0\right],$$

and $f(x)$ is defined[3] by:

$$f(x) = [x-K]\left(\frac{G(x)}{G(R_s^0)}-1\right).$$

This formula replicates the value of a CMS caplet in terms of European swaptions $C(x)$, and without model assumptions on $R_s(t)$ can be calculated with the full market smile information. Using the same idea, one derives the CMS floorlet price from the ordinary receiver swaptions $P(x)$, from which an application of the put-call parity formula will give the CMS rate.

2. **Lognormal model convexity adjustment**: The integral appearing in the replication formula (1.3) will simplify if $f(x)$ can be approximated as a quadratic, which is to say $G(x)$ can be replaced with its linear approximation. Indeed, the function $G(x)$ is very smooth and slowly varying, the first few terms of its Taylor expansion about R_s^0 being:

$$G(x) \approx G(R_s^0) + G'(R_s^0)(x-R_s^0).$$

[3] In (1.1b) make the substitutions $a = K, b = R_s$ so $\int_K^{R_s} f''(x)(R_s-x)\,dx = f(R_s) - f'(K)(R_s-K)$ when $f(a)=0$. If one further replaces $(R_s-x), f(R_s), (R_s-K)$ with $(R_s-x)^+, f(R_s)^+, (R_s-K)^+$, respectively, then the identity persists and the upper-limit of integration R_s can be replaced with ∞. Apply $\mathbb{E}\left[\cdot\,\middle|F_0\right]$ to equate (1.2) and (1.3).

In practice, the slope $G'(R_s^0)$ is obtained from well known approximations to $L(\tau)$ and $Z(\tau, t_p)$ in terms of the swap rate R_s, such as the *bond math* formula derived in [Hag2, §A.3]:

$$G(R_s) \quad \approx \quad R_s w^{-\Delta}(1-w^{-n})^{-1}, \qquad w := (1+R_s/q), \quad \Delta := q(t_p-\tau),$$

where q is the frequency of payments. If one assumes the swap rate $R_s(t)$ follows a lognormal model with Black volatility σ_B, then[4]

$$R_s(t) = R_s^0 e^{-\frac{1}{2}\sigma_B^2 t + \sigma_B\sqrt{t}Z_t},$$

from which (1.3) (or (1.2) directly) implies an analytic formula for the CMS caplet price:

$$V_{cap}^{CMS}(0) = G'(R_s^0)A_0\Big[(R_s^0)^2 e^{\sigma_B^2\tau}\Phi(d_{3/2}) - R_s^0(R_s^0+K)\Phi(d_{1/2}) + R_s^0 K\Phi(d_{-1/2})\Big] + \frac{D(t_p)}{A_0}C(K), \tag{1.4}$$

where

$$d_\lambda = \frac{\log(R_s^0/K) + \lambda\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

3. **Normal model convexity adjustment**: In the case when $R_s(t)$ is assumed to follow a normal model with Normal volatility σ_N, one has

$$R_s(t) = R_s^0 + \sigma_N\sqrt{t}Z_t.$$

Using this distribution of $R_s(t)$, the pricing equation gives:

$$V_{cap}^{CMS}(0) \quad = \quad G'(R_s^0)A_0\sigma_N^2\tau\Phi\left(\frac{R_s^0-K}{\sigma_N\sqrt{\tau}}\right) + \frac{D(t_p)}{A_0}C(K). \tag{1.5}$$

Applying similar ideas to those employed for the Black model, one obtains the corresponding prices for floorlets and swaps.

CMS Spread Products

CMS spread products, such as CMS spread range accruals and capped floaters, are yet other types of pseudo-vanilla instruments which can be replicated. Consider the CMS spread defined as the spread $(S_1 - S_2)$ between the two CMS rates S_1, S_2. Once the convexity-adjusted rates of the CMS legs can be derived, the calculation of the CMS spread price is straightforward.

To price the spread option, such as a cap or floor, we need to model the dynamics of the CMS spread itself. Since the CMS spread market is not liquid, and there is little reliable market data, SWPM/YASN use historical information to estimate the spread Normal volatility. In SWPM, the historical spread volatility is directly estimated[5], while in YASN one uses the historical correlation ρ of the swap rates, together with the swaption Normal volatilities σ_1, σ_2 of each leg, to calculate the spread volatility σ_{spread}. These two methods give consistent pricing.

For callable and other exotic flavors of spread-based derivatives, in which each of the CMS legs is modeled with its own (correlated) Brownian motion, see §3 on the 2-factor version of LGM, as well as [Zha] where details of the pricing approximation are also presented.

[4] We use Z_t to denote the random variable drawn from the standard normal distribution, and $\Phi(\cdot)$ to denote the cumulative normal distribution.

[5] Historical spread volatilities from the last three years are used by default, but can be specified as a user setting.

2 The LGM Model

LGM dynamics, model parametrization, calibration and pricing techniques are briefly described below based on [Hag3], which should be consulted for more details.

2.1 LGM Dynamics

Under the risk neutral measure Q, the Hull-White one-factor model (HW1F) assumes that the short rate process r_t follows the following dynamics:

$$r_t = \theta_t + Y_t, \quad (2.1a)$$
$$dY_t = -\kappa Y_t \, dt + \sigma_t \, dW_t^Q, \quad (2.1b)$$

where κ is the mean-reversion constant, θ_t and σ_t are deterministic functions of t, and W_t^Q is a standard Brownian motion under Q.

The Linear Gauss Markov (LGM) model is adapted from the Hull-White model by applying the following transformation of parameters (see [Hag4, §B.2]):

$$h(t) := e^{-\kappa t}, \quad (2.2a)$$
$$\alpha_t := \tfrac{1}{h(t)}\sigma_t, \quad (2.2b)$$
$$X_t := \tfrac{1}{h(t)}Y_t. \quad (2.2c)$$

One then obtains the LGM dynamics written in terms of X_t:

$$r_t = \theta_t + h(t)X_t, \quad (2.3a)$$
$$dX_t = \alpha_t \, dW_t^Q, \quad (2.3b)$$

from which the solution $X_t = \int_0^t \alpha_s dW_s^Q$ is easily obtained.

Defining the additional quantities

$$\zeta(t) := \int_0^t \alpha^2(s) \, ds, \qquad H(t) := \int_0^t h(s) \, ds,$$

one sees that X_t is an $N(0, \zeta(t))$ Gaussian process, and that determination of any one of the (κ, σ_t), $(h(t), \alpha_t)$, or $(H(t), \zeta(t))$ parameter-pairs is equivalent[6] to the others.

In what follows, we take (2.3) as the LGM model equations, where we are given deterministic functions $\theta(t)$ and $h(t)$ obtained from the market's initial forward curve, and the mean-reversion constant κ, respectively. A piecewise constant term-structure will be assumed for the volatility function α_s, which is determined from calibrating $\zeta(t)$ to the vanilla swaptions market, as described below.

[6] For example, in terms of the original Hull-White parameters, Y_t is an $N(0, \int_0^t e^{-2\kappa(t-s)}\sigma_s^2 \, ds)$ Gaussian process.

2.2 LGM Calibration

The LGM model is calibrated to the vanilla swaption market. Under the simplified assumption of a single curve environment, we consider floating payments L_i that have no spread over the forward Libors F_i derived from the discount curve:

$$L_i = F_i := \frac{D_{i-1} - D_i}{\alpha_i D_i}.$$

Then, as shown in [Hag4, §5], a receiver swaption with strike K and expiry T will have value at time $t = 0$ given by

$$\begin{aligned} V_{rec}^{opt}(0) &= \sum_{i=1}^{n} K\alpha_i D_i \Phi\left(\frac{y^* + (H(t_i) - H(t_0))\,\zeta(T)}{\sqrt{\zeta(T)}}\right) \\ &+ D_n \Phi\left(\frac{y^* + (H(t_n) - H(t_0))\zeta(T)}{\sqrt{\zeta(T)}}\right) - D_0 \Phi\left(\frac{y^*}{\sqrt{\zeta(T)}}\right), \end{aligned} \tag{2.4}$$

where $\Phi(\cdot)$ is the cumulative normal distribution, $\{t_0, t_1, \ldots, t_n\}$ is the fixed leg schedule with corresponding coverages $\{\alpha_1, \ldots, \alpha_n\}$, and y^* is the unique solution of

$$\sum_{i=1}^{n} K\alpha_i D_i e^{-[H(t_i)-H(t_0)]y^* - \frac{1}{2}[H(t_i)-H(t_0)]^2\zeta(T)} + D_n e^{-[H(t_n)-H(t_0)]y^* - \frac{1}{2}[H(t_n)-H(t_0)]^2\zeta(T)} = D_0. \tag{2.5}$$

Therefore, LGM calibration involves solving for $\zeta(T)$ given market data $V_{rec}^{opt}(0)$ for specified strike K and maturity T.

Remark 2.1. In the presence of floating payment spreads s_i, as in a dual curve environment where the Libor spreads over OIS forwards are represented as $L_i = F_i + s_i$, the appearance of K in (2.4) and (2.5) must be modified, namely replaced by $(K - S_i)$ in accordance with [Hag4, 5.7c]. Alternatively, if we define

$$\widetilde{K} := K - \frac{\sum_{j=1}^{m} \widetilde{\alpha}_j s_j D(u_j)}{\sum_{i=1}^{n} \alpha_i D(t_i)},$$

where $\{u_j\}$ is the floating leg schedule with coverages $\{\widetilde{\alpha}_j\}$, then an effective approximation[7] to $V_{rec}^{opt}(0)$ is obtained by replacing K with $\widetilde{K}$ in both (2.4) and (2.5):

$$\begin{aligned} V_{rec}^{opt}(0) &\approx \widetilde{K} \sum_{i=1}^{n} \alpha_i D_i \Phi\left(\frac{y^* + (H(t_i) - H(t_0))\,\zeta(T)}{\sqrt{\zeta(T)}}\right) \\ &+ D_n \Phi\left(\frac{y^* + (H(t_n) - H(t_0))\zeta(T)}{\sqrt{\zeta(T)}}\right) - D_0 \Phi\left(\frac{y^*}{\sqrt{\zeta(T)}}\right). \end{aligned} \tag{2.6}$$

Remark 2.2. The default value for the mean-reversion constant κ is 3%.

[7]See §A.1 for details justifying the approximation (2.6). Note that the quantity $\left(\frac{\sum_{j=1}^{m} \widetilde{\alpha}_j s_j D(u_j)}{\sum_{i=1}^{n} \alpha_i D(t_i)}\right)$ is the spread between the par (Libor) swap rate $K_0 = \left(\frac{\sum_{j=1}^{m} \widetilde{\alpha}_j L_j D(u_j)}{\sum_{i=1}^{n} \alpha_i D(t_i)}\right)$ and the OIS swap rate $\widetilde{K}_0 = \left(\frac{D_0 - D_n}{\sum_{i=1}^{n} \alpha_i D(t_i)}\right)$.

Selection of Calibration Instruments

In order to accurately price a deal with options, the LGM model is calibrated to custom selected instruments from the swaptions market. Suppose a deal is callable on dates $\{T_j\}_{j=1}^J$, and matures on t^{mat}. For each exercisable date T_j, a swaption with expiry T_j and tenor $(t^{mat} - T_j)$, called a *diagonal* swaption[8], is used for calibration. Regarding the strike, there are two prevailing approaches:

1. **ATM Calibration**: The ATM swaptions are chosen for each applicable expiry and tenor.
2. **Diagonal-to-Moneyness Calibration**: The yield curve is parallel shifted by γ_j such that the j-th payoff becomes at-the-money, and the swaption strike is then set to $(\text{ATM} + \gamma_j)$.

While diagonal-to-moneyness calibration is preferred in general, it is not always desirable or feasible, and ATM calibration is often employed instead. For instance, ATM calibration is a natural choice for a callable floater. Moreover, an amortized cancellable swap might have no choice other than resorting to ATM calibration. As a general rule, callable fixed bonds should use diagonal-to-moneyness calibration, while floaters (with or without a cap or floor), perpetuals, and amortizing bonds should use ATM calibration. Examples showing calibration instruments are given in §A.2.

For callable deals which are not determined by a finite collection of exercise dates $\{T_j\}_{j=1}^J$, namely the American style options versus the Bermudan options, YASN applies Richardson extrapolation to an approximating Bermudan option.

Calibration Procedure

Given the J selected calibration instruments, one per exercise time, the LGM model is calibrated using the following procedure, starting with $j = 1$ and $\zeta(0) = 0$:

Step 1. For j-th swaption instrument, compute the initial guess of $\zeta(T_j)$

Step 2. Solve (2.4) for $\zeta(T_j)$

Step 3. Go to Step 1 for $(j + 1)$-th swaption, until $j = J$ and all J swaptions have been processed

A piecewise constant volatility model $(\sigma_t)_{t\geq 0}$ is assumed[9] in which

$$\sigma_t := \sigma_j, \quad t_{j-1} \leq t < t_j, \quad 1 \leq j \leq J,$$

and the J constants $\{\sigma_j\}$ determining the term-structure are given by the formula:

$$\sigma_j^2 = \frac{\zeta(t_j) - \zeta(t_{j-1})}{\int_{t_{j-1}}^{t_j} e^{2\kappa u} du}.$$

[8] A collection of diagonal swaptions of common maturity t_{mat} is sometimes referred to as being *coterminal*.

[9] Henceforth we assume $t_0 = 0$. Furthermore, piecewise constant σ_t implies $\int_{t_{j-1}}^{t_j} \alpha_t^2 \, dt \overset{(2.2b)}{=} \sigma_j^2 \int_{t_{j-1}}^{t_j} h^{-2}(t) \, dt$.

2.3 LGM Pricing

Given a calibrated LGM model, the value $V(t,x)$ of any deal without optionality can be determined from its value at any later time $T > t$ based on the expected value conditioned on the state (x,t):

$$V(t,x) \quad = \quad \frac{1}{\sqrt{2\pi\Delta\zeta(t,T)}} \int_{-\infty}^{\infty} e^{-(X-x)^2/2\Delta\zeta(t,T)} V(T,X) dX, \tag{2.7}$$

where $V(T,X)$ is the value of the payoff function at maturity, and

$$\Delta\zeta(t,T) = \zeta(T) - \zeta(t).$$

In the presence of optionality with the exercise dates $\{T_j\}$, a deal can be priced by utilizing (2.7) to compute the value of the option *if held* from T_j to T_{j+1}:

$$V_{hold}(T_j,x) \quad = \quad \frac{1}{\sqrt{2\pi\Delta\zeta(T_j,T_{j+1})}} \int_{-\infty}^{\infty} e^{-(X-x)^2/2\Delta\zeta(T_j,T_{j+1})} V(T_{j+1},X)\; dX. \tag{2.8}$$

The integration of (2.8) is performed numerically using a rollback table in which the state variable X is represented as a uniformly sampled grid centered on its mean $\sum_{i=0}^{j} H_{i+1}\Delta\zeta(T_i,T_{i+1})$, and which spans a range of five times its standard deviation $\sqrt{\Delta\zeta(T_j,T_{j+1})}$.

In the particular case of a European option, the option value $V(t=0, x=Spot)$ is obtained by applying (2.7) to the payoff function $V(T,X)$ at expiry, such as the caplet payoff

$$V(T,X) \quad = \quad \max(Libor(T,X) - K, 0). \tag{2.9}$$

When extended to the case of a Bermudan option with expiries at T_1 and T_2, one can equate the *hold value* of the option at states (T_1,X), namely $V_{hold}(T_1,X)$, with a European priced at T_1 with expiry at T_2 according to its payoff $V(T_2,X)$ (*e.g.* (2.9)). If we denote the *exercise value* of the option at T_1 as $V_{ex}(T_1,X)$, then the actual value at T_1 is the larger of the hold and exercise values:

$$V(T_1,X) \quad = \quad \max(V_{ex}(T_1,X), V_{hold}(T_1,X)). \tag{2.10}$$

Iterating the example of the European option, the option value $V(t=0, x=Spot)$ is obtained by applying (2.7) to the payoff function $V(T_1,X)$ at T_1 given by (2.10). These steps are easily generalized to Bermudan options with an arbitrary set of exercise dates $\{T_j\}$, and illustrate the *backward induction scheme* referred to above.

3 Adjusters for Callable Exotic Deals

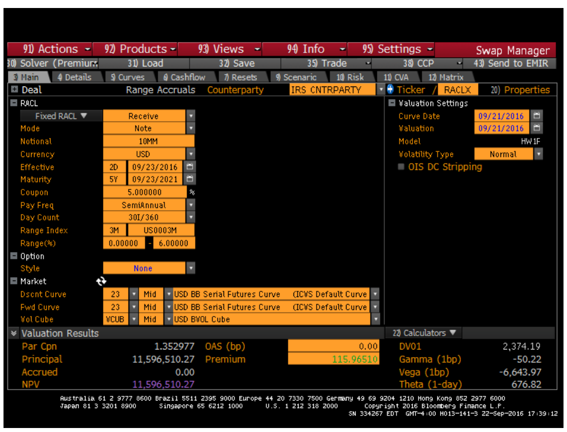
Callable exotic deals, such as callable range accruals, callable capped floaters, and callable CMS spreads, involve cash flows with embedded options. The low-dimensional LGM model cannot robustly price these deals due to its lack of parametric freedom (under-parameterized), and therefore adjuster methods are applied to improve the pricing.

For those callable exotics, we first price the non-callable deal using an applicable replication method described in §1. Next, the call option value is calculated based on the LGM model.

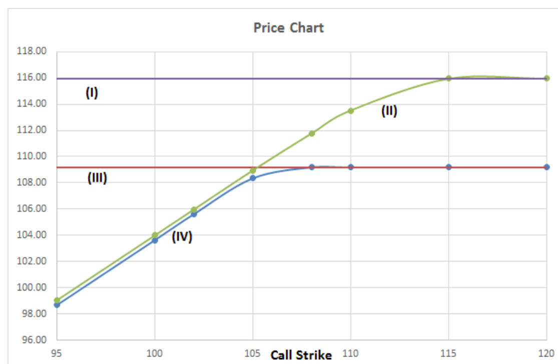
The LGM model is first calibrated to the vanilla swaptions with expiries aligned to the call schedule. Next, the calibrated LGM model is used to price the non-callable deal. In general, because the model is not calibrated to price the non-callable deal, this will not reproduce the non-callable price based on the replication method.

To improve this result, a simple adjuster method is used, whereby one scales the model price of the non-callable deal to the price based on the replication method used above, and then applies this constant scale to obtain the price of the callable option. The following example illustrates the results of applying this replication-with-adjuster methodology to pricing a callable range accrual. Figure 3.1(a) shows the non-callable range-accrual priced with the replication method, while Figure 3.1(b) shows the Bermudan callable range-accrual priced as a function of different call strikes. Here, the flat line (I) is the non-callable price based on the replication method. while curve (II) is the callable price using the adjuster method. Plots (III) and (IV) show the same deals priced using the pure LGM method, which is to say without adjusters or replication; the flat line (III) is the non-callable price, while curve (IV) is the callable price.

We can see from Figure 3.1 how, using this adjuster methodology, the callable deal price will converge to the replicated non-callable price in SWPM and YASN as the strike increases, namely, when the call option is far out of the money.



(a) Non-callable range-accrual priced by replication.



(b) Strike ladder of callable range-accrual prices.

Figure 3.1: Example of the adjuster method used in pricing callable range accruals.

Pricing Callable CMS Spreads with the LGM two-factor model

In general, callable CMS spread exotics cannot be modeled by a one factor term structure model. In these deals one must model both the spread options and call option, which have little correlation between them, and therefore SWPM and YASN use the LGM two-factor model to price these exotics. This two-factor model is calibrated using both the swaption leg volatilities and the volatility of the spread. The spread volatilities are estimated as above, and the swaption volatilities are obtained from market data. We use Fast Fourier Transform method to perform the pricing integration. A detailed description of the implementation can be found in [Zha].

A Appendices

A.1 Swaption Approximation Formula

In this section we derive the approximation formula for swaption pricing presented in (2.6), which is adapted from the exact formulas developed in [Hag4, §5.2]. We recall[10] the value of the receiver swap which receives the fixed coupon K is given by

$$V_{rec}(t) \quad = \quad K\sum_{i=1}^{n}\alpha_i P(t,t_i) + P(t,t_n) - P(t,t_0) - \sum_{j=1}^{m}\widetilde{\alpha}_j s_j P(t,u_j), \tag{A.1}$$

where $\{\alpha_i\}, \{\widetilde{\alpha}_j\}$ are coverages of the fixed and floating legs of schedules $\{t_i\}, \{u_j\}$ respectively, $P(t,T)$ is the time t valuation of the discount from t to T, and s_j is the spread of the floating payment L_j over the forward rate derived from the discounting curve:

$$L_j =: \frac{P(t,t_{j-1}) - P(t,t_j)}{\widetilde{\alpha}_j P(t,t_{j-1})} + s_j.$$

For $t = 0$, we can rewrite (A.1) in a form analogous to the zero-spread case:

$$V_{rec}(0) \quad = \quad \widetilde{K}\sum_{i=1}^{n}\alpha_i P(0,t_i) + P(0,t_n) - P(0,t_0),$$

where $\widetilde{K} := K - \frac{\sum_{j=1}^{m}\widetilde{\alpha}_j s_j P(0,u_j)}{\sum_{i=1}^{n}\alpha_i P(0,t_i)}$. We now make the observation, due to Rebonato[11] and described in [BM, §6.15], that the time t values of

$$\omega_j(t) := \frac{\widetilde{\alpha}_j P(t,u_j)}{\sum_{i=1}^{n}\alpha_i P(t,t_i)}$$

are very well approximated by the their time-0 values $\omega_j(0)$, which leads to the following useful approximation for $t > 0$:

$$V_{rec}(t) \quad \approx \quad \widetilde{K}\sum_{i=1}^{n}\alpha_i P(t,t_i) + P(t,t_n) - P(t,t_0). \tag{A.2}$$

The value of a receiver swaption at time zero with exercise time $t_{ex} \le t_0$ is

$$\begin{aligned} V_{rec}^{opt}(0) \quad &= \quad N_0 E^{Q_N}\left[\frac{\max\{V_{rec}(t_{ex}),0\}}{N_{t_{ex}}}\right] \\ &\approx \quad E^{Q_N}\left[\left(\widetilde{K}\sum_{i=1}^{n}\alpha_i \bar{P}(t_{ex},t_i;X_{t_{ex}}) + \bar{P}(t_{ex},t_n;X_{t_{ex}}) - \bar{P}(t_{ex},t_0;X_{t_{ex}})\right)^{+}\right], \end{aligned}$$

[10] Note that (A.1) differs slightly from [Hag4, (5.2c)] in that we do not adapt the floating leg spreads s_i to spreads S_i conforming to the fixed leg schedule as in [Hag4, (2.12b)].

[11] This technique of *freezing* the $\omega_j(t)$ to their initial values is sometimes referred to as *Rebonato's trick*.

where $X_{t_{ex}} \sim N(0, \zeta_{t_{ex}})$ under the martingale measure Q_N associated with numeraire N. By making the change of variable $y = x + H(t_0)\zeta_{t_{ex}}$, we have

$$
\begin{aligned}
V_{rec}^{opt}(0) &\approx \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\zeta_{t_{ex}}}} \Bigg(\widetilde{K} \sum_{i=1}^{n} \alpha_i P(0,t_i) \exp\left\{ -H(t_i)x - \frac{1}{2}H^2(t_i)\zeta_{t_{ex}} \right\} \\
&\quad + P(0,t_n) \exp\left\{ -H(t_n)x - \frac{1}{2}H^2(t_n)\zeta_{t_{ex}} \right\} - P(0,t_0) \exp\left\{ -H(t_0)x - \frac{1}{2}H^2(t_0)\zeta_{t_{ex}} \right\} \Bigg)^+ dx \\
&= \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\zeta_{t_{ex}}}} \Bigg(\widetilde{K} \sum_{i=1}^{n} \alpha_i D_i \exp\left\{ -(H_i - H_0)y - \frac{1}{2}(H_i - H_0)^2\zeta_{t_{ex}} \right\} \\
&\quad + D_n \exp\left\{ -(H_n - H_0)y - \frac{1}{2}(H_n - H_0)^2\zeta_{t_{ex}} \right\} - D_0 \Bigg)^+ dy, \qquad \text{(A.3)}
\end{aligned}
$$

where $H_i = H(t_i)$, $D_i = P(0,t_i)$ for $i = 0, 1, \ldots, n$. Following [Hag4, §5.2.2], we next assume without loss of generality that H is a strictly increasing function so that

$$
\exp\left\{ -[H(T) - H(t)]y - \frac{1}{2}[H(T) - H(t)]^2\zeta_{t_{ex}} \right\}, \; t_{ex} \le t \le T
$$

is a monotone decreasing function of y, with limit 0 as $y \to \infty$ and limit ∞ as $y \to -\infty$. Thus, there exists a unique break-even point y^* such that the term inside $(\cdots)^+$ appearing in (A.3) is

$$
\begin{cases} < 0 & \text{if } y > y^*, \\ = 0 & \text{if } y = y^*, \\ > 0 & \text{if } y < y^*. \end{cases}
$$

Consequently[12]

$$
\begin{aligned}
V_{rec}^{opt}(0) &\approx \frac{1}{\sqrt{2\pi\zeta_{t_{ex}}}} \int_{-\infty}^{y^*} e^{-\frac{y^2}{2\zeta_{t_{ex}}}} \left(\widetilde{K} \sum_{i=1}^{n} \alpha_i D_i e^{-(H_i-H_0)y - \frac{1}{2}(H_i-H_0)^2\zeta_{t_{ex}}} + D_n e^{-(H_n-H_0)y - \frac{1}{2}(H_n-H_0)^2\zeta_{t_{ex}}} - D_0 \right) dy \\
&= \widetilde{K} \sum_{i=1}^{n} \alpha_i D_i \Phi\left(\frac{y^* + (H_i - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}} \right) + D_n \Phi\left(\frac{y^* + (H_n - H_0)\zeta_{t_{ex}}}{\sqrt{\zeta_{t_{ex}}}} \right) - D_0 \Phi\left(\frac{y^*}{\sqrt{\zeta_{t_{ex}}}} \right), \qquad \text{(A.4)}
\end{aligned}
$$

where $\Phi(\cdot)$ is the c.d.f. of a standard normal distribution and y^* is the unique solution of

$$
\widetilde{K} \sum_{i=1}^{n} \alpha_i D_i e^{-[H(t_i)-H(t_0)]y^* - \frac{1}{2}[H(t_i)-H(t_0)]^2\zeta_{t_{ex}}} + D_n e^{-[H(t_n)-H(t_0)]y^* - \frac{1}{2}[H(t_n)-H(t_0)]^2\zeta_{t_{ex}}} = D_0.
$$

Note that in the case of a caplet where $n = 1$, $\widetilde{K} = K - s_1$ and (A.4) is exact.

[12] The technique of expressing a receiver (payer) swaption as a combination of ZCB calls (puts) is described in [BM, §3.11], and referred to as *Jamshidian's decomposition*.

A.2 Calibration Examples

In order to illustrate the instruments to which the LGM model may be calibrated, we consider the following examples using SWPM.

The first example is a 1Y×5Y Bermudan swaption with strike of 2% as shown in Figure A.1(a). To price this Bermudan option, the LGM model is calibrated using the diagonal-to-moneyness approach, where Figure A.1(b) shows the instruments selected for calibration. Note that the strikes of these calibration European swaptions are very close to the underlying strike of 2%.

The second example is a 5Y Bermudan callable capped floater as shown in Figure A.2(a). Diagonal-to-ATM calibration is used here, where the calibration instruments, which now have strikes very close to the forwards rates, are shown in Figure A.2(b).

(a) 1Y×5Y Bermudan swaption with 2% strike in SWPM

(b) Selected calibration instruments

Figure A.1: Diagonal-to-moneyness calibration example: 1Y×5Y Bermudan swaption

(a) Callable capped floater in SWPM

(b) Selected calibration instruments

Figure A.2: Diagonal-to-ATM calibration example: 5Y callable capped floater

A.3 Calculation of Greeks

At present, all Greeks calculations are based on the bump-and-reprice methodology, evaluated numerically using either one-sided or two-sided difference quotients.

Vega Calculation

Vega is calculated by taking the price difference between using bumped-up volatility and baseline volatility, and then scaling to the appropriate units. For Black volatility, a bump size of 0.1% is used, and the result is displayed to reflect the sensitivity to a 1% bump in volatility. For Normal volatility, a bump size of 0.1 bps is used, and the result is displayed to reflect the sensitivity to a 1 bp bump in volatility.

Figure A.3 illustrates this bump-and-reprice approach. Figure A.3(a) shows a fixed bond that is continuously callable three months before its maturity date, and priced using a LGM model calibrated to instruments in Figure A.3(b) showing Black volatilities slightly less than 32%. As an example demonstrating the reported Vega of (-0.0057) (see Figure A.3(c)), the deal is first priced with a flat volatility of 32% (Figure A.3(c)) to serve as the baseline price, and then repriced with a bumped volatility of 33% (Figure A.3(d)), resulting in the computed Vega value of (-0.00568).

Other Greeks Calculation

While Vega uses a one-sided difference quotient (indeed, a potentially negative volatility should be disallowed), all other first-order Greeks in YASN and SWPM use two-sided difference quotients. In Table A.1 we summarize the various Greeks calculated by YASN and SWPM, as well as their respective default bump-and-reprice settings currently used.

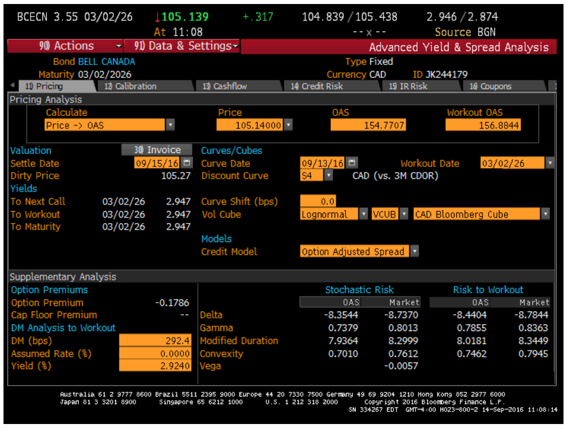
	SWPM		YASN	
Greek	Bump Size	Units	Bump Size	Units
Delta	-	-	10bp	%
Gamma	10bp	bps	10bp	%
-Duration	10bp	bps	10bp	bps
-DV01	10bp	bps	-	-
Convexity	-	-	10bp	%
KRR	10bp	%	10bp	%
Theta	1 day	1 day	1 day	1 day
Vega (Black)	10bp	%	10bp	%
Vega (Normal)	0.1bp	bps	0.1bp	bps

Table A.1: Summary of bump-and-reprice settings for Greek calculations

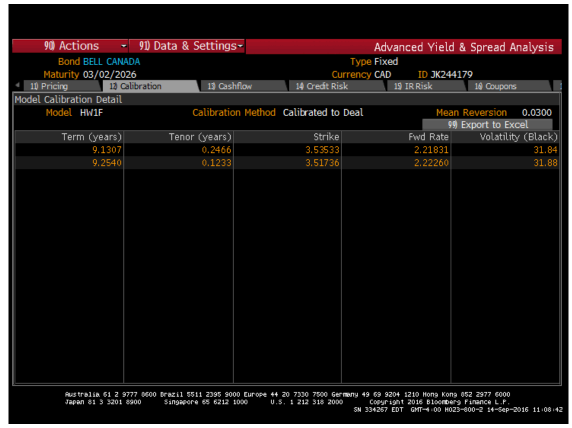
Several salient points are worth remarking:

- In Delta and its related calculations including KRR, bumping the interest rate curve can use one of four different methodologies, according to the user's DV01/KRR Curve Shift Options selection.[13] For example, the user can choose to shift the *forwards*, *par swap rates*, or *zero-rates* of the stripped curve, or simply shift the *market instruments* from which the interest rate curve is stripped. Note that this last *pre-strip bump* setting is equivalent to selecting the Shift checkbox in `SWDF` or `ICVS` and entering an appropriate (10 bps) shift-value.

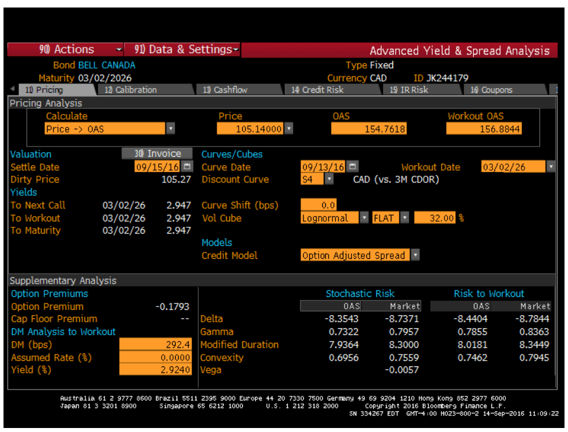
[13] See the `SWDF` terminal function, in particular `Swap Curve Default Settings`.



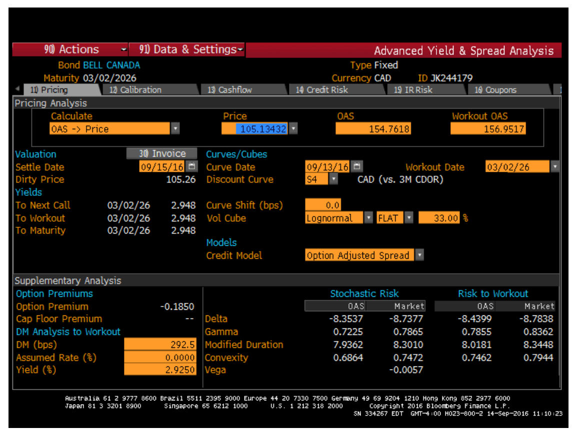
(a) A callable fixed bond in YASN



(b) Calibration instruments



(c) Base price at a flat vol of 32%



(d) Bumped price at a flat vol of 33%

Figure A.3: Example of Vega verification using CA07813ZBT09 (AKA JK244179)

- When Delta and its related calculations perform a bump on the interest rate curve, if the curve is constructed by adding a basis to another curve, for example obtaining *USD (vs. 6M Libor)* from a spread over *USD (vs. 3M Libor)*, then the basis curve is *not* independently bumped.
- When Delta and its related calculations perform a bump on the interest rate curve, not only are the forward rates shifted, but naturally the discounting curve is also regenerated[14].
- The DV01/KRR/Duration bump size in SWPM is settable in the Shift field of SWPM's Risk screen.
- Whereas the sign of some first order Greeks is subject to convention (DV01 vs Delta), the sign of second order Greeks (Gamma, Convexity) is unambiguous (*i.e.* invariant if left and right are reversed).
- The Theta calculation will include in its up-bump dirty pricing, using tomorrow's (or next business day's) market, any realized payments accrued from the time shift.

By searching on `Greek Methodologies` (Calculations→Risk Analytics→Greek Methodologies), more detail can be found on the `HELP SWPM` screen.

[14] An exception is the OAS Modified Duration discussed below, where *only* the discounting curve is regenerated.

DV01

One defines the risk factor "DV01"[15] to be the interest rate sensitivity of price $\Pi(r_t)$ given by

$$\text{DV01} := \frac{\Pi(r_t - 1\ bp) - \Pi(r_t + 1\ bp)}{2\ bps} \approx -\frac{\partial \Pi(r_t)}{\partial r_t}.$$

Thus, DV01 is a measure of price sensitivity to a downward shift in interest rates[16]. Assume the continuously compounded zero-rate curve is represented by the time-dependent function r_t, so that the discount factor at time t is given by

$$D(t) = e^{-r_t t}.$$

Furthermore, assume that bumps are applied to the continuously compounded zero-curve post-stripping, as described by the "Shift As Zeros" SWDF Default Setting. With these conventions, bumping the curve will translate into the straightforward substitution $r_t \to r_t + \delta$.

In the particular case of a *fixed coupon bond*, and more generally when coupon payments C_k have no dependence on future rates, the following calculation gives an explicit approximation for DV01:

$$\begin{aligned} \text{Dirty Price} \quad &:= \quad \sum_{k=1}^{N} C_k e^{-r_t t_k}, \\ \text{DV01} \quad &\approx \quad -\frac{d}{dr_t} \sum_{k=1}^{N} C_k e^{-r_t t_k} = \sum_{k=1}^{N} t_k\, C_k e^{-r_t t_k}. \end{aligned} \tag{A.5}$$

In the more general case when a change of cash flow may result from a change in future yields, we write $C_k = C_k(r_t)$. A particular case is provided by a *fixed coupon bond with embedded optionality*, in which case whenever r_t is sufficiently large or small, early exercise may occur implying $\{C_{ex} = N,\ C_{k>ex} = 0\}$. Thus, the sum $\sum_{k=1}^{N} t_k\, C_k(r_t) e^{-r_t t_k}$ will be truncated, and the DV01 will decrease. Another particular case is that of a *floating coupon bond with no spread*, whose DV01 is essentially[17] zero, since its present value is essentially just the notional. In these more general cases we can modify (A.5) accordingly:

$$\text{DV01} \quad \approx \quad \sum_{k=1}^{N} t_k\, C_k(r_t) e^{-r_t t_k} - \sum_{k=1}^{N} C_k'(r_t) e^{-r_t t_k}. \tag{A.6}$$

Note that this expression takes account of both the discounting sensitivity to rate change (the first term, leaving coupons unchanged), and also the coupon sensitivity (the second term, leaving discounts unchanged). In other words, the bump in rates affects both the *discounting curve* and

[15] DV01 is an acronym for "Dollar Value of a 01", *i.e.* one basis point change to yields at all tenors; alternate conventions, such as a post-stripping bump to continuously compounded zero rates, are equally valid.

[16] As bond prices increase with decreasing yields, receiving fixed cash flows is associated with positive DV01 (and negative Delta). Indeed, (A.5) implies $\text{DV01} = \Pi \cdot \sum_k t_k \omega_k$ is a weighted average ($1 = \sum_k \omega_k$) of positive times.

[17] One detail that must be accounted for is that the first coupon of the floater is actually a fixed coupon; thus, while bumping rates $r_t \to r_t + \delta$ will apply to discounting at all times, bumping is not applied to the first coupon which is tied to the historically fixed r_1. Indeed, the duration of a floater is that of a bullet bond maturing at t_1.

the *forward curve*. As already mentioned, these combined effects essentially cancel in the case of a floating coupon bond.

YASN also displays a Market Modified Duration defined as the ratio (DV01 ÷ Dirty Price). The motivation behind the OAS Modified Duration (versus the Market Modified Duration) in YASN is precisely to isolate the effect on the discounting alone. In other words, by bumping the OAS spread, one bumps the zero rates, and consequently the discount curve, according to

$$r_t + \text{oas-spread} \rightarrow r_t + \text{oas-spread} + \delta,$$

but leaves the forward rates unchanged. The bumping methodologies in the SWDF Defaults described above *do not apply* in this case, although the day count and discounting conventions discussed in §A.4 will apply. Needless to say, the OAS Duration of a floater may be non-negligible.

A.4 OAS Calculation Dependencies

Although not specific to the LGM implementation, it is worth commenting on the impact of day-count and compounding conventions to the OAS calculation. In particular, the OAS bump is, by definition, applied as a parallel shift to the zero-rates of the interest rate curve, and is therefore subject to the same dependencies on which the zero-rate calculation is performed. In other words, the *bumped* discounts which result from applying the additive bump δ to the zero-rates at different times T will be:

$$D_\delta(T) = f\left(\delta + f^{-1}(D(T))\right), \quad \text{where} \quad f^{-1}(D(T)) = Z(T) = \text{ Zero Rate at time } T.$$

If zero-rates are computed based on continuous compounding versus semi-annual compounding, then

$$f_{cc}(Z_{cc}, T) = e^{-TZ_{cc}}, \text{ versus } f_{sa}(Z_{sa}, T) = (1 + \tfrac{1}{2}Z_{sa})^{-2T}.$$

Thus, D_δ takes the especially simple form in the particular case of continuous compounding:

$$D_\delta(T) = e^{-\delta T} D(T).$$

Clearly, the non-linearity present in different $f(Z)$ implies that a parallel shift to the zero-rates in one convention will fail to be a parallel shift in another convention. Note that, while f and Z are distinguished with units specific to a compounding convention, the discount $D(T)$ is dimensionless.

Furthermore, when the above transformation is applied at some time T where a bond's cash flow occurs, then coverages (and possibly partial coverages in the case of stub-rates) will need to be computed according to a specific day-count convention. Needless to say, the application of different day-count conventions will impact pricing and OAS-evaluation.

YASN supports quarterly, semi-annual, annual, and continuous compounding as user choices in its OAS calculations.

A.5 Credit Models in YASN

YASN supports four pricing methodologies, as displayed in the *Credit Models* menu:

1. Option Adjusted Spread (OAS): The OAS for a certain market price is defined as the parallel shift of the zero rates of the discount curve at which the model price matches this given value.
2. Additive CDS Spread (ACDS): This model directly adds the spreads of the CDS curve to the par rates of the discount swap curve, while ignoring the CDS recovery rate value specified in `CDSW`. Note that, in those instances of a par rate maturity date for which no CDS spread quote is available, linear interpolation (of rate with respect to day-count) will be applied to the CDS spreads at their neighboring maturities.
3. Hybrid CDS/LGM (HCR): This model incorporates the Credit Default curve into the LGM model, under the assumptions that defaults are independent of interest rates, and that the hazard rates are deterministic. The pricing methodology incorporates not only the CDS recovery rate implicit in the risky discount curve, but also incorporates the bond recovery rate explicit in the HCR bond pricing formula. More details are presented below.
4. CDS Adjusted Par Curve (APC): In this methodology the (HCR) model is used to back out risky par coupons taking account of the bond recovery rate. These par coupons are then used, as with the benchmark coupons to which CDS spreads have been added in ACDS, to construct a risky default curve. More details are presented below.

For the last three models, OAS is calculated after taking into account their respective credit effects on pricing. The recovery rates for the CDS curves are standard Bloomberg default values, as seen in `CDSW`.

Hybrid Credit-Rates

The *Hybrid Credit-Rates* model (HCR) uses a methodology which more closely couples the credit and interest rate models. Specifically, by augmenting the LGM dynamics with an independent Poisson default process, where the interest rate state space is augmented with an absorbing default state, one directly incorporates credit risk into the interest rate model. Consequently, credit risk impact on exercise decisions of embedded options will be more accurately reflected than in other credit models. Additional details describing the Hybrid Credit-Rates model can be found in [Blo5].

Considering the valuation of a defaultable bond of maturity T, let $\tau > 0$ denote its random default time, and assume further that upon default the instrument recovers a fixed fraction R_B of par. By assumption, the default time τ is independent of the interest rate, and therefore the defaultable bond value $V^{HCR}(0)$ per unit notional of the defaultable bond is given simply as[18]:

$$V_{bond}^{HCR}(0) \quad = \quad V_{bond}(0; P_t^{risky}) + \mathbb{E}\left[R_B \frac{\mathbf{1}_{\{\tau \leq T\}}}{N(\tau)}\right], \tag{A.7}$$

where $V_{bond}(0; P_t^{risky})$ denotes the valuation of the bond using the risky discount curve

$$P_t^{risky} \quad := \quad \mathbb{E}\left[\frac{\mathbf{1}_{\{\tau > t\}}}{N(t)}\right] \overset{ind}{=} \mathbb{E}\left[N(t)^{-1}\right] \cdot \mathbb{E}\left[\mathbf{1}_{\{\tau > t\}}\right] = P_t \cdot (1 - Q_t), \tag{A.8}$$

[18] The description given here using (A.7) is essentially an explicit presentation of that found in [Blo5, §1]. Note that the numeraire $N(t)$ denotes the risk-free money market account, so $\mathbb{E}\left[N(t)^{-1}\right]$ is the riskless discount curve P_t.

obtained as the product of the riskless discounts P_t and survival probabilities $(1 - Q_t)$. Recall that Q_t is the risk-neutral default probability (cdf) of the issuer at time t, and is derived from the stripped $Q_{t_i}(C_k^{cds}, R_{cds})$ as in [Blo2]. Moreover, if τ is assumed to follow a Poisson process where

$$\mathbb{E}\left[\mathbf{1}_{\{\tau>t\}}\right] = e^{-\int_0^t \lambda(s)ds}, \tag{A.9}$$

then the hazard rates $\lambda(s)$ can be related directly to the risky discounts from (A.8):

$$P_t^{risky} = P_t \cdot e^{-\int_0^t \lambda(s)ds}. \tag{A.10}$$

By definition, $\mathbb{E}\left[\mathbf{1}_{\{t+dt>\tau>t\}}\right] = dQ_t$, which is to say $\mathbb{E}\left[f(\tau)\mathbf{1}_{\{T>\tau>0\}}\right] = \int_0^T f(t)dQ_t$, hence the expected recovery from default of the bond appearing in the second term of (A.7) can be written

$$\mathbb{E}\left[R_B \frac{\mathbf{1}_{\{\tau\leq T\}}}{N(\tau)}\right] \quad = \quad R_B \int_0^T P_t \, dQ_t. \tag{A.11}$$

In the case of a bond with fixed coupons $(C_{i<n} = \alpha_i C)$, and final repayment of unit notional $(C_n = 1 + \alpha_n C)$, the sum of the expected risky cash flows comprising the first term in (A.7) becomes

$$V_{fixed-bond}(0; P_t^{risky}) = \sum_{i=1}^{n} C_i P_{t_i}^{risky} \overset{(A.8)}{=} C \sum_{i=1}^{n} \alpha_i P_{t_i} \; (1 - Q_{t_i}) + P_T(1 - Q_T). \tag{A.12}$$

Approximating (A.11) by discretizing the integral over the partition $\{s_1, \ldots, s_m = T\}$, we may now write (A.7) for the fixed bond with notional N as

$$V_{fixed-bond}^{HCR}(0) \;\approx\; N\left[C \sum_{i=1}^{n} \alpha_i (1 - Q_{t_i}) P_{t_i} + (1 - Q_T)P_T + R_B \sum_{k=1}^{m} \left(Q_{s_{k+1}} - Q_{s_k}\right) P_{s_k}\right]. \tag{A.13}$$

Using (A.13), we see that, having constructed a default probability distribution (cdf) Q_t, one may directly evaluate a *risky bond price* of fixed coupons C, and bond recovery R_B.

In the general methodology for obtaining the recovery value in (A.11), one creates a discrete timeline $0 = t_0 < \ldots < t_n = T$, including the dates corresponding to events of interest such as coupon cash flows, curve tenors, exercises of embedded options, *etc.* A rollback from the terminal bond maturity date T, applying (A.7) to determine $V^{HCR}(t_i)$ at each time step t_i, eventually yields the defaultable present value $V^{HCR}(0)$ of the bond.

In the current implementation of this model, credit volatility is ignored, and Q_t is a deterministic distribution obtained from stripping the bond issuer's credit curve, while the bond valuation $V_{bond}(0; P_t^{risky}) = V_{bond}(0; P_t \cdot (1 - Q_t))$, is obtained from the one-factor LGM model.

CDS Adjusted Par Curve

As described above, the Hybrid Credit-Rates model evaluates a default-adjusted bond in two steps:

1. Strip the CDS curve to obtain a term structure of *hazard rates* λ_i, from which the *default probability* Q_t may be computed[19] for arbitrary future time t.

[19] Explicitly, $Q_t := 1 - S_t$ where $S_t := e^{-\int_0^t \lambda_t dt} = e^{-\lambda_{n+1}(t-t_n)} \Pi_{i=1}^n e^{-\lambda_i(t_i - t_{i-1})}$ is the *survival probability*.

2. Use these default probabilities to derive the bond's price from its recovery rate R_B, and its discounted risky cash flows.

Recalling (A.11) and (A.12), the model value $V_{fixed-bond}$ of the fixed bond of coupon rate C is

$$V_{fixed-bond} = N\left[R_B\int_0^T P_t\,dQ_t + C\sum_{i=1}^{n}\alpha_i P_{t_i}(1-Q_{t_i}) + (1-Q_T)P_T\right], \qquad \text{(A.14)}$$

where t_i are coupon payment times with $T = t_n$ denoting bond maturity, α_i is the year fraction accrued for the i^{th} coupon payment, P_t is the benchmark discount factor at time t, R_B is the recovery rate of the holder, and i ranges over the remaining n coupon cash flows of the bond.

The first term in formula (A.14) is the expected recovery value of the bond. The second term in the formula is the premium leg of the same CDS, valued assuming no premium accrual is paid upon default. The third term is the notional repayment if the bond did not default.

Consequently, the risky par coupon with maturity T_m is computed as that c_m satisfying

$$1 \;=\; R_B\int_0^{T_m} P_t\;dQ_t + c_m\sum_{i=1}^{m}\alpha_i(1-Q_{t_i})P_{t_i} + (1-Q_{t_m})P_{t_m}. \qquad \text{(A.15)}$$

In the *CDS Adjusted Par Curve* model (APV), (A.15) is applied to successive maturities T_m to bootstrap the risky par curve from the risky par coupons. Specifically, the Curve Stripping methodology, as configured in the `SWDF` Settings and described in [Blo4], is applied to the collection of risky par coupons c_m to obtain a risky discount curve. Additional details describing this methodology can be found in [Blo3].

A.6 Calibration in a Negative Rate Environment

In recent years, developed economies have shown signs of deflation, with negative interest rate scenarios becoming more plausible. When either the forward rate or the strike becomes negative, the standard Black implied volatility is no longer applicable. Recall that `VCUB` provides the market implied volatility of the vanilla swaption and cap/floor instruments. To support negative-rate market environments, `VCUB`'s default behavior is to rebase its quoting calculations on Normal volatilities, instead of Black volatilities. Note that, regardless of `VCUB`'s default behavior, it is always possible to manually change the quoting convention to use either Black or Normal volatilities. The significance of this remark is that the behavior of SWPM/YASN depends on the prevailing `VCUB` settings.

1. **Lognormal model CMS convexity adjustment is unselectable**: Whenever the `VCUB` behavior is set to Normal volatility, typically (though not exclusively) in a negative rate environment, the Lognormal model CMS convexity adjustment is invalid. In this case, only the Normal model and Replication methodologies are valid options.
2. **LGM model calibration not impacted**: The LGM model used to price callable deals will not be impacted, since the model is calibrated to *swaption premiums* rather than swaption volatilities, whether quoted as Black or Normal. As long as the option premiums based on the associated volatility quotes are not changed, the resulting calibrated model is unaffected.

A.7 FAQs

- When comparing swaption volatilities in the SWPM/YASN Calibration Tab with those displayed in `VCUB`, why does one find discrepencies? The volatilities for each (Expiry, Tenor, Strike)-triple in the SWPM/YASN Calibration Tab are displayed with the Expiry and Tenor in units of year fractions. Although one can coerce the Expiry and Tenor fields of `VCUB` to accept year fractions, the difference in the day count conventions used by YASN and VCUB, namely ACT/ACT and ACT/365 respectively, will result in a mismatch when comparing the two volatilities. The correct approach is to first convert YASN's year fractions to a number of days and thence a date, according to the ACT/ACT convention, and then use those dates directly in `VCUB`'s Expiry and Tenor fields. In practice, the formula floor(Expiry * 365.25 + 0.7) is an acceptable rule-of-thumb when converting to a number of days in the ACT/ACT convention. The distinguishing feature of the YASN methodology is that it allows for infinite time resolution by using a floating point year fraction representation, whereas `VCUB` employs daily resolution for the Expiry, and monthly resolution for the Tenor.

- What limitations apply when pricing American options? As mentioned in §2.2, American options are priced by applying Richardson extrapolation to a Bermudan option whose exercise dates, which can be viewed in SWPM's Calibration Tab, are determined internally. Before Richardson extrapolation can be applied to obtain the price of the American option, the approximating Bermudan option will be priced using the backward-induction method, which is described in §2.3. Due to performance considerations, SWPM will not exceed thirty exercise dates when constructing the approximating Bermudan. One implication of this constraint is that, while a long maturity deal may expect that the Bermudan sampling dates cover the entire American option period, a deal with amortization may require a denser coverage of dates in the early portion of the same option period, and consequently the long maturity coverage may be sacrificed. When such a situation arises, the recommended alternative is to directly price a Bermudan with a finely sampled granularity imposing, for example, a monthly exercise schedule. Note that this discussion of grid discretization in the time dimension, which is non-uniform and includes no more than thirty exercise sampling dates, is independent of grid discretization in the state-variable dimension, which is determined according to §2.3, and which relates to the numerical integration of (2.8).

- How are Perpetual Bonds priced with large negative OAS? Perpetual bonds are naturally modeled by pricing to an exceedingly distant date, say fifty years, and then applying an approximation to the present value of the *tail* cashflow. In the simplest example of a bond paying annual fixed coupons C, with a continuously compounded long-term interest rate[20] of r_∞, the tail payments beyond 50 years will contribute the following infinite sum to the price:

$$V_{tail}(0; r_\infty) \quad = \quad PV\left[N\sum_{t=50}^{\infty} Ce^{-r_\infty t}\right] = V_{50}\ (1 - e^{-r_\infty})^{-1}, \tag{A.16}$$

  where V_{50} is the present value of the coupon paid at 50 years. Clearly, the equality fails to hold if $r_\infty \leq 0$, in which case the summation is divergent. While we will assume any credit

[20] So $\frac{D(50+N)}{D(50)} \approx [\frac{D(51)}{D(50)}]^N = e^{-r_\infty N}$.

spread has been absorbed into r_∞, we will explicitly consider the effect of an OAS of δ, which is to say, of replacing r_∞ with $(r_\infty + \delta)$ in (A.16):

$$V_{tail}(0; r_\infty + \delta) = V_{50} e^{-50\delta} \cdot (1 - e^{-(r_\infty + \delta)})^{-1}. \tag{A.17}$$

The divergence of (A.17) can arise in any OAS→Price situation in which the user asserts a large negative OAS, or in the case of a rate sensitivity calculation (KRR) or other scenario analysis with relatively large (two-sided) bumps (Table A.1), especially in a negative rate environment. Indeed, in the absence of callability any OAS with $(r_\infty + \delta) < 0$ is disallowed (a "Calculation Error" is reported) for OAS→Price. On the other hand, when backing out OAS from Price→OAS, there will always exist $\delta > -r_\infty$ reproducing an arbitrarily large price.

The presence of callability fundamentally changes the above analysis. In particular, allowing for $(r_\infty + \delta) < 0$ when OAS falls below $-r_\infty$ will nonetheless produce finite exercise prices at each possible call date, and these valuations become less favorable to the issuer with increasing maturity. Therefore, a callable perpetual with $(r_\infty + \delta) < 0$, due to the almost certain eventuality of being called, is effectively equivalent to a non-perpetual whose cash flow is truncated at some future call date. In this sense, OAS→Price callable pricing is seen to be equivalent to that of a non-perpetual of some maturity (the more negative the OAS, the shorter the maturity), making (A.17) irrelevant[21].

Conversely, Price→OAS is problematic for callable deals. The limited range of OAS→Price restricts the feasible perpetual bond prices to which the Price→OAS solver can be applied.

Note that these remarks do not address the generic question of excessively negative OAS, where the numerical implications of short-term blow-up applies irrespective of callability or perpetuity. An obvious limitation for semi-annual OAS compounding, where the per-period discount will be $(1+0.5(LIBOR + OAS))^{-1}$, is that $OAS > -20,000$bps.

- The discussion in §1.2 on the replication method for CMS-related pricing in SWPM makes reference to the functional behavior of $G(x)$, to the call and put functions $C(K)$ and $P(K)$, and to the numerical evaluation of the improper integral in (1.3). How are each of these quantities obtained in practice? The functional form of $G(R_s(\tau))$ is explicitly presented in (2.13a) of [Hag2], and is obtained as an approximation to $Z(\tau; t_p)/L(\tau)$ via the "Bond Math" derivations in [Hag2, §A.3]. The evaluation of the Payer and Receiver Swaption prices, $C(K)$ and $P(K)$, are derived from Normal swaption volatility quotes obtained directly from the `VCUB` Terminal Function [Blo1].

- How does the discussion of DV01 in § A.3 explain a possible sign change in the Modified Duration in YASN in the presence of an OAS? Indeed, the presence of either a spread on the floating coupon or an OAS Spread on the discounting will impact the argument that DV01 (and hence Duration) is essentially zero for a pure floater. In particular, a positive coupon spread increases the price by a fixed bond which contributes a positive term to DV01, whereas positive OAS decreases the price while decreasing DV01 (since the offsetting effect of increased discount factors applied to decreased coupons is reduced[22]). The overall effect

[21] Although the callable perpetual can be priced with $(r_\infty + \delta) \leq 0$, the message "OAS unstable at given price. Non call oas-adjusted perpetual cashflows diverge." is displayed, and "discounted payments" in the Cashflow screen will be shown as "inf". Moreover, the Option Premium is undefined since there is no non-callable price for comparison.

[22] Which is to say, when discount factors are already at a higher-than-offsetting level due to the positive OAS.

of a Libor spread on the coupons (β) and an OAS spread on the zero-rates (δ) is ultimately determined by their combined effect in the expression

$$\text{DV01} \quad \approx \quad \sum_{k=1}^{N} t_k \left(C_k(r_t) + \beta\right) e^{-(r_t+\delta)t_k} - \sum_{k=1}^{N} C_k'(r_t) e^{-(r_t+\delta)t_k},$$

and hence a positive or a negative Modified Duration (DV01 ÷ Dirty Price) may result.

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