Building the Bloomberg Interest Rate Curve – Definitions and Methodology.

Abstract

The goal of this document is to describe the process of interest rate (IR) curve construction and stripping in the Bloomberg terminal. We first introduce various types of rates used during curve stripping, then discuss the type of instruments commonly used in building the IR curves (cash rates, futures and IR swaps). Attention is given to a functional form of the curve (a.k.a. interpolation methods) and algorithms for building the curve under these different interpolation methods (i.e. curve *stripping*). We also cover various types of IR curve stripping including both single-currency and cross-currency curve stripping.

1. Interest Rate Curve – Definition

The IR Curve is an object which allows one to calculate a discount factor for every date in the future, thus providing us with the risk-free present value (PV) of a unit of currency (say, $\$1$) paying on that particular future date. It is widely used to calculate present values of a known set of payments (i.e. cash flows) for certain IR instruments. While in some situations one can construct an IR curve which takes in an additional discount (i.e. spread over risk free curve) due to risk of default of the counterparty, this document leaves the discussion of default or credit risk out. For the sake of simplicity, we will assume that the IR curves described in this document produce risk-free present values.

A second use for IR curves is to calculate projected forward rates between two dates ($d_1 \rightarrow d_2$) in the future. A typical example is to construct the payments of a 'floating leg' of an IR swap which pays, say, quarterly an amount of interest equal to 3-month LIBOR rate on a given notional. While the actual payments that will be made in the future are not known until we reach that point in time when LIBOR is fixed, the PV of this stream of payments is correct if we apply current projections of forward rates based on the known curve.

2. Types of Interest Rate

The definition of a *simple spot rate* r_s is expressed as:

$$DF(d_0, d) = \frac{1}{1 + r_s(t) \cdot \tau} \qquad (1)$$

Here d_0 is the start date. Usually it is the settlement date of a financial instrument. d is some date in the future, $\tau = \tau(d_0, d)$ is the time interval between two dates (d_0, d) in years, and $DF(d_0, d)$ is the discount factor from the date d to start date d_0. The only undefined term in the above definition is the method to convert a pair of dates (d_0, d) into a time interval τ in years. This conversion method is formally called *Day Count Convention*. There is more than one way of doing this: e.g. both *ACT/360* and *ACT/365* are very widely accepted conventions in the financial markets. For *ACT/360*, we assume there are 360 days per year and $\tau(d_1, d_2) =$

$(d_2 - d_1)/360$. The operation $d_2 - d_1$ returns the actual number of calendar days between two dates. Similarity, for *ACT/365* $\tau(d_1, d_2) = (d_2 - d_1)/365$. There are other day count conventions such as ACT/ACT and 30U/360 which are outside the scope of this document.

The discount factor between two dates $DF(d_0, d)$ is fixed by the market, but day count convention is not, we can get different τ using different day count conventions, and therefore the corresponding rates will be different. Thus, when quoting the rate, it is important to note not only which rate it is (and frequency of compounding) but a day count convention as well. In this document every time we mention conversion of a pair of dates into a time interval t in years, it will be assumed that a day count convention is specified also.

A *simple forward rate* r_{sf} between two dates (d_1, d_2) in the future is defined as:

$$\frac{DF(d_0, d_2)}{DF(d_0, d_1)} = \frac{1}{1 + r_{sf}(t) \cdot \tau} \qquad (2)$$

where $\tau = \tau(d_1, d_2)$.

A *continuously compounded forward rate* $r_{cf}(\tau)$ between two dates (d_1, d_2) in the future is defined as:

$$DF(d_1, d_2) = exp\left[-\int_{T_1}^{T_2} r_{cf}(\tau) \cdot d\tau\right] \qquad (3)$$

here times T_1 and T_2 are equal to the time intervals between date d_0 now and dates d_1 and d_2 in future correspondingly: $T_1 = \tau(d_0, d_1)$ and $T_2 = \tau(d_0, d_2)$.

The *continuously compounded zero rate* $r_{cz}(t)$ is defined as:

$$DF(d_0, d) = exp[-r_{cz}(\tau) \cdot \tau] \qquad (4)$$

here time τ is the time interval between date d_0 now and date d in the future.

3. Building the Interest Rate Curve

The process of building of the IR curve (a.k.a. *curve stripping*) is a process of creating a curve object which would correctly price a set of N given instruments, e.g. produces correct discount factors and forward rates implied by these instruments.

Let us consider various types of IR instruments that can be used to build a typical vanilla swap curve (e.g. S23) in Bloomberg. They typically belong to 3 groups:

1) Cash or Deposit Rates in the short end
2) Interest Rate Futures or Forward Rate Agreements (FRAs) in the middle segment
3) Interest Rate Swaps in the long end

The user may choose the set of instruments to be applied in curve construction by typing **{SWDF <GO>}** in the Bloomberg terminal, choosing the right country/currency, selecting the source 8 (custom curve), saving their choice (**1 <GO>**) and clicking on the curve of interest. If the user knows the curve number, he can directly type **{SWDF ## <GO>}** in the terminal. A screen like the below should appear:

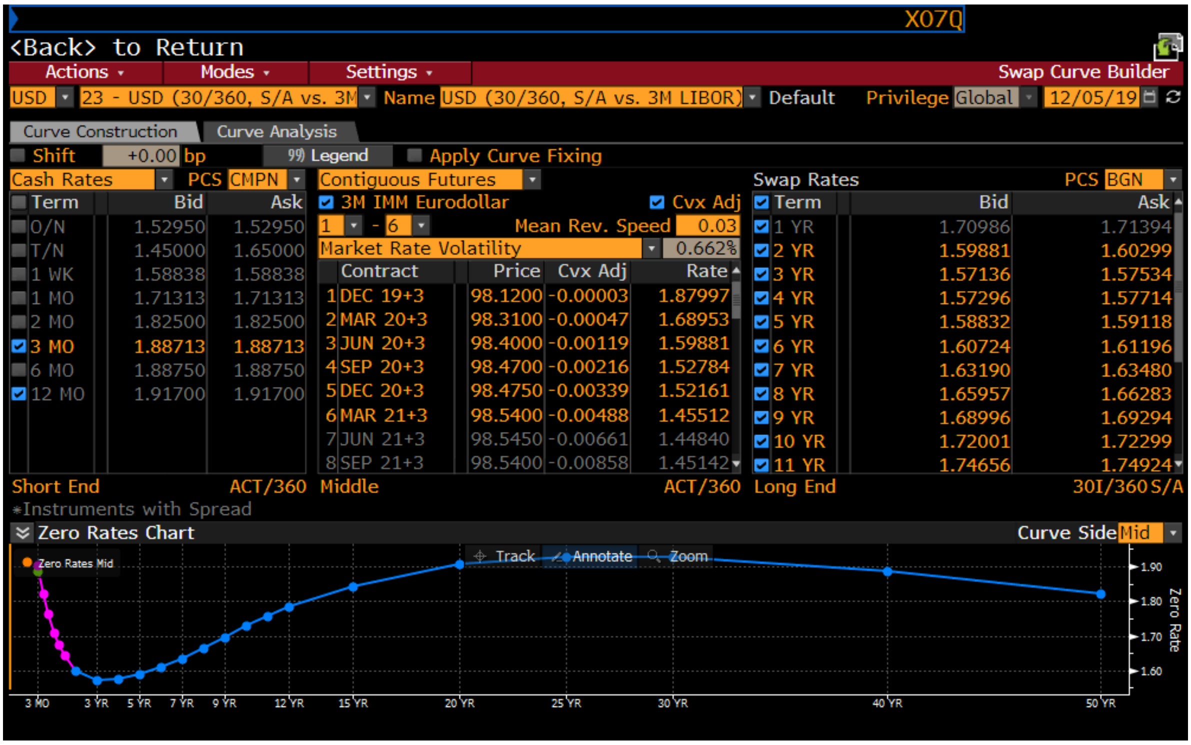


Fig. 1a. Example of Bloomberg screen to choose a set of instruments for building USD IR curve S23.

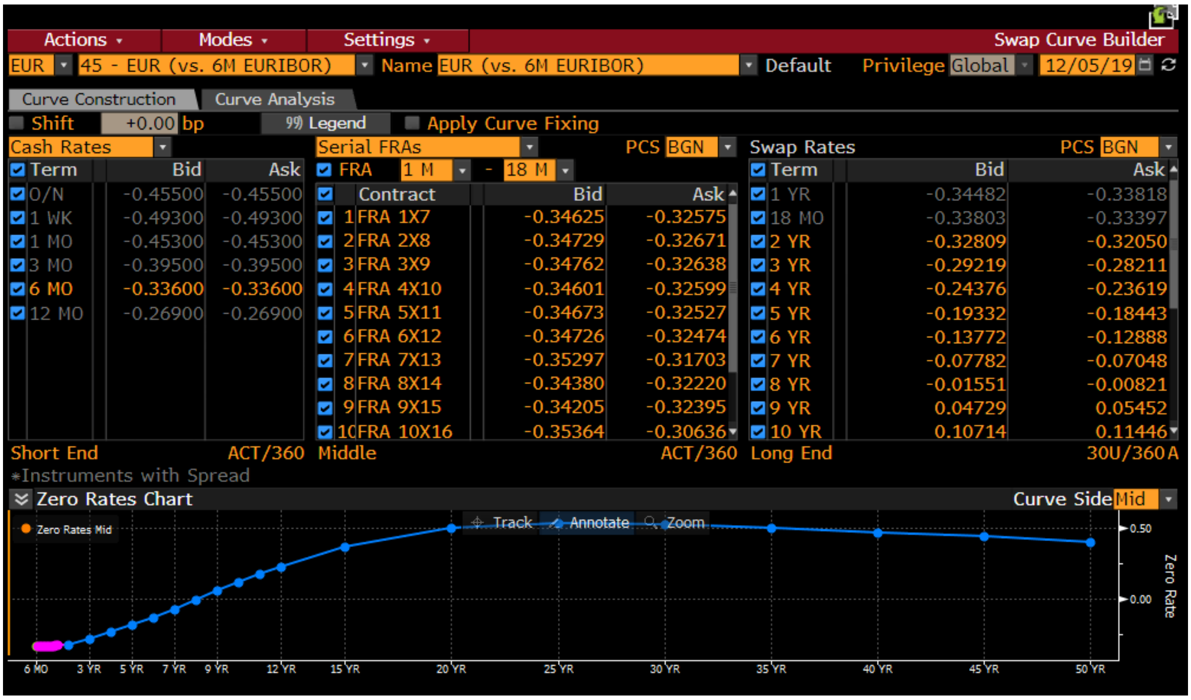


Fig. 1b. Example of Bloomberg screen showing instruments in EUR serial FRA IR curve S45.

The first group usually covers the short-end part of the curve; the cash/deposit securities are typically available up to 1 year tenor, but they are most commonly used for curve construction up to 3 or 6 months. Bloomberg recommends only including a single cash instrument with tenor equal to that of the curve being built. This is why in the example screen shot of S45 above only 6Mo has been selected.

The second group picks up where the first group ends. Futures strips will overlap with swaps, so the decision of where to switch from futures to swap rates is up to the user. As shown in the Fig.1a, the switch happens around 18 months for the default Bloomberg USD curve. One may only use IR futures (contiguous or serial) in USD curves, but for other curves, for instance, EUR curve S45, the user can choose between IR futures and FRAs, as shown in Fig. 1b.

The third group, swap rates, covers the time interval from where the user decides to end the use of futures/FRAs up to and including the final instrument used in the curve.

4. Interest Rate Instrument Pricing

Now let us consider each of these instruments in more detail, and understand what kind of constraint or mathematical equation they can provide to the IR curve.

1) Cash instruments:

Cash instruments are very straightforward. We are given a market quote for a given maturity. The correspondent equation is:

$$[1 + R \cdot \tau(t_s, t_e)] \cdot DF(t_e) = 1 \cdot DF(t_s) \qquad (5a)$$

Here R is the market quote for the cash instrument, t_s is accrual start time (usually the settlement time) and t_e is accrual end time (usually the maturity time) of this cash instrument, and $\tau(t_s, t_e)$ is the day count fraction for the rate. Thus $[1 + R * \tau(t_s, t_e)]$ is the total amount of money to be paid at payment time (usually the maturity time t_e). $DF(t)$ is the discount factor to be solved for at time t. Therefore, the left term in the above equation is the present value of the future payment and it must be equal to the right term which is the present value of the money we will pay for the deposit at the time t_s. Otherwise, there will be arbitrage which is not allowed.

2) Forward and Future instruments:

For a forward instrument, the equation based on the no arbitrage principal is very similar to that of a cash instrument. It can be readily expressed as:

$$[1 + R \cdot \tau(t_1, t_2)] \cdot DF(t_2) - DF(t_1) = 0 \qquad (5b)$$

here R is the market quote for the forward instrument, t_1 is the start time and t_2 is the maturity time. $\tau(t_1, t_2)$ is the day count fraction for the rate. Thus $[1 + R * \tau(t_1, t_2)]$ is the total amount of money to be paid at maturity time t_2. Here we can see there are two discount factors (at t_1 and t_2) that come into play in this equation.

As to the future instrument, the future rate can be converted into an IR forward rate using a so-called convexity adjustment as discussed in [3]. This conversion is based on a Vasicek model with given volatility σ and mean reversion speed a. The user can specify these two parameters in the middle segment on the screen shown in Fig. 1a.

3) Swap Instruments:

An IR Swap is an instrument which exchanges a stream of payments (fixed rate vs floating rate), on some pre-defined notional principal N, between two counterparties.

For the fixed leg, the payments are calculated using a fixed rate K defined at the inception of the swap. The present value of this leg can be expressed as:

$$PV(fixed\ leg) = \sum_{i=1}^{n} [N \cdot K \cdot \tau(i) \cdot DF(t_i)] + N \cdot DF(t_n)$$

here $\tau(i)$ is the day count fraction that determines the i^{th} payment amount and $DF(t_i)$ is the discount factor at i^{th} payment time t_i. If the principal needs to be repaid at maturity, this introduces an extra $N \cdot DF(t_n)$ term in the above PV expression.

For the floating leg, the payment amount is not determined from a known fixed rate, but relies on projected forward rates based on the forward curve. Without loss of generality, the present value can be readily expressed as:

$$PV(float\ leg) = \sum_{i=1}^{n} [N \cdot R(i) \cdot \hat{\tau}(i) \cdot DF(t_i)] + N \cdot DF(t_n)$$

Here $R(i)$ is an effective rate to cover i^{th} payment on the floating leg, and $N \cdot R(i) \cdot \hat{\tau}(i)$ is the expected amount of money paid for i^{th} period.

For a typical vanilla swap, the payments in the floating leg are calculated based on a (L)IBOR rate at the accrual start time of the correspondent time interval. These rates change as market instrument rates change, thus the name *floating leg*.

Under the no-arbitrage rule the equation for a swap, in general, can be expressed as:

$$PV(leg1) - PV(leg2) = 0$$

Different type of swaps (vanilla swaps, single currency basis swaps, cross currency basis swaps) will have different ways to compute present value. We will cover this in more detail in section 7.

5. Interpolation Methods

One can define curve stripping as a process to find a discount factor function $DF(t)$ for all $t \geq 0$, such that it prices all the instruments back to a set of market quotations. In other words, the solution must satisfy a system of equations based on the instruments used in the curve.

If there are N number of instruments on the curve, to ensure that we have a unique solution, the discount factor function $DF(t)$ should have also N degrees of freedom. Therefore, $DF(t)$ is typically defined as a parametric function with N independent parameters:

$$DF(t) = DF(t, p_1, p_2, \ldots p_N).$$

One way to achieve this is to break $DF(t)$ into N time intervals with the i^{th} interval covering period $[t_{i-1}, t_i]$. Here, t_i is the maturity time of the i^{th} instrument on the curve and $t_0 = 0$ is just the lower boundary of the discount factor function. The function will have N independent parameters, one for each segment. There are typically two methods to solve for this system to determine $DF(t)$:

1) *Bootstrapping* method, where each time interval on the curve has exactly one independent degree of freedom and it does not affect previous time intervals. In this case the curve is built by adjusting one piece at a time while moving from shorter maturities to longer maturities.

2) *Global* method, where all or at least some degrees of freedom of the curve affect its overall shape, and therefore one needs to solve a general system of N non-linear equations with N unknown variables at the same time.

Currently the Bloomberg terminal allows the user to choose one of the 4 functional forms for the IR curve (a.k.a. *interpolation methods*). By typing **{SWDF DFLT <GO>}**, one will see the following screen:

Fig. 2: Screen allowing user to choose curve interpolation method

On this screen one can choose 1 through 4 under the “Interpolation Method” section:

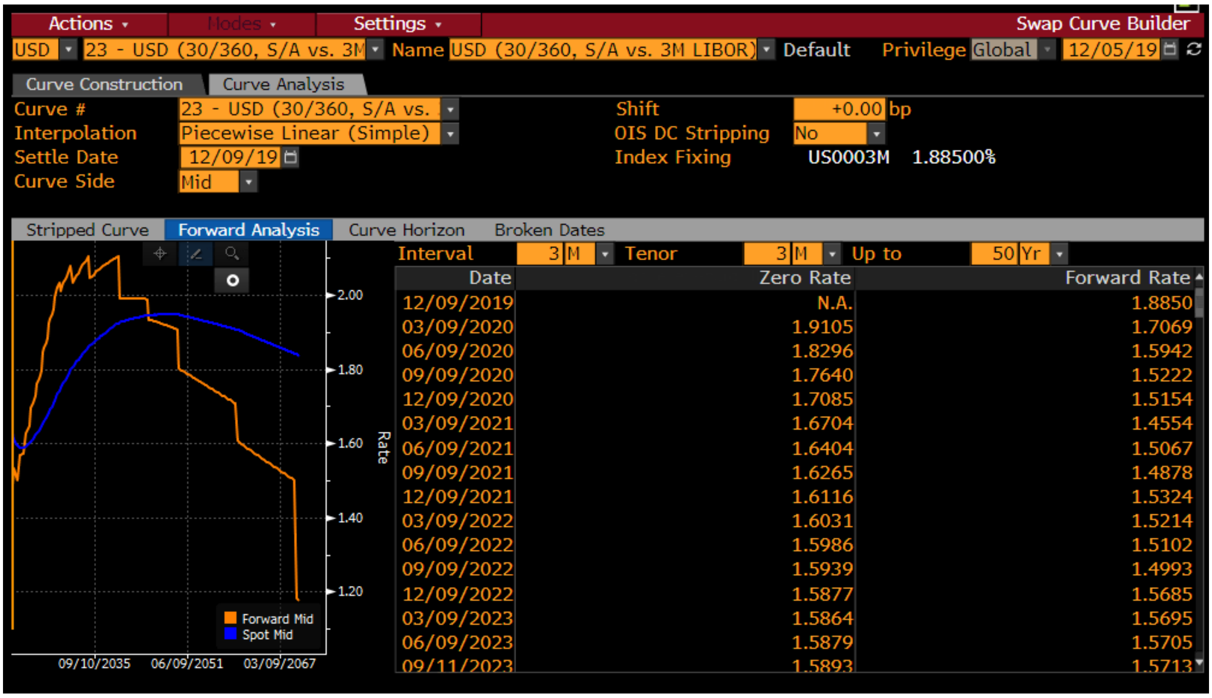
1) Piecewise linear
2) Smooth forward/Piecewise quadratic
3) Step-function forward
4) Piecewise linear

Interpolation method 1 - the simple zero rate r_{sz} defined by the formula (1) is a piecewise linear continuous function. A bootstrapping method is used to solve for all the simple zero rates at instrument maturities. The simple zero rate before the first instrument maturity is equal to the solved value from the first instrument. Similarly, in the extrapolated region after the last instrument maturity, the simple zero rate is equal to that of the last instrument. Thus, we have:

$$DF(t) = \frac{1}{1 + r_s(t_1) \cdot t} \qquad when \;\; t \in [0, \;\; t_1]$$

$$DF(t) = \frac{1}{1 + r_s(t_M) \cdot t} \qquad when \;\; t \in [t_M, \infty]$$

Here t_1 and t_M are the maturity time for the first instrument and last instrument respectively, and $r_s(t)$ is simple zero rate at time t. All dates are converted to time using ACT/360 convention. Thus, it produces a unique discount factor function $DF(t)$ that covers all the time $t \geq 0$ for this curve. Fig.3a shows an example of stripping results using interpolation method 1 for USD curve S23.



Date	Zero Rate	Forward Rate
12/09/2019	N.A.	1.8850
03/09/2020	1.9105	1.7069
06/09/2020	1.8296	1.5942
09/09/2020	1.7640	1.5222
12/09/2020	1.7085	1.5154
03/09/2021	1.6704	1.4554
06/09/2021	1.6404	1.5067
09/09/2021	1.6265	1.4878
12/09/2021	1.6116	1.5324
03/09/2022	1.6031	1.5214
06/09/2022	1.5986	1.5102
09/09/2022	1.5939	1.4993
12/09/2022	1.5877	1.5685
03/09/2023	1.5864	1.5695
06/09/2023	1.5879	1.5705
09/11/2023	1.5893	1.5713

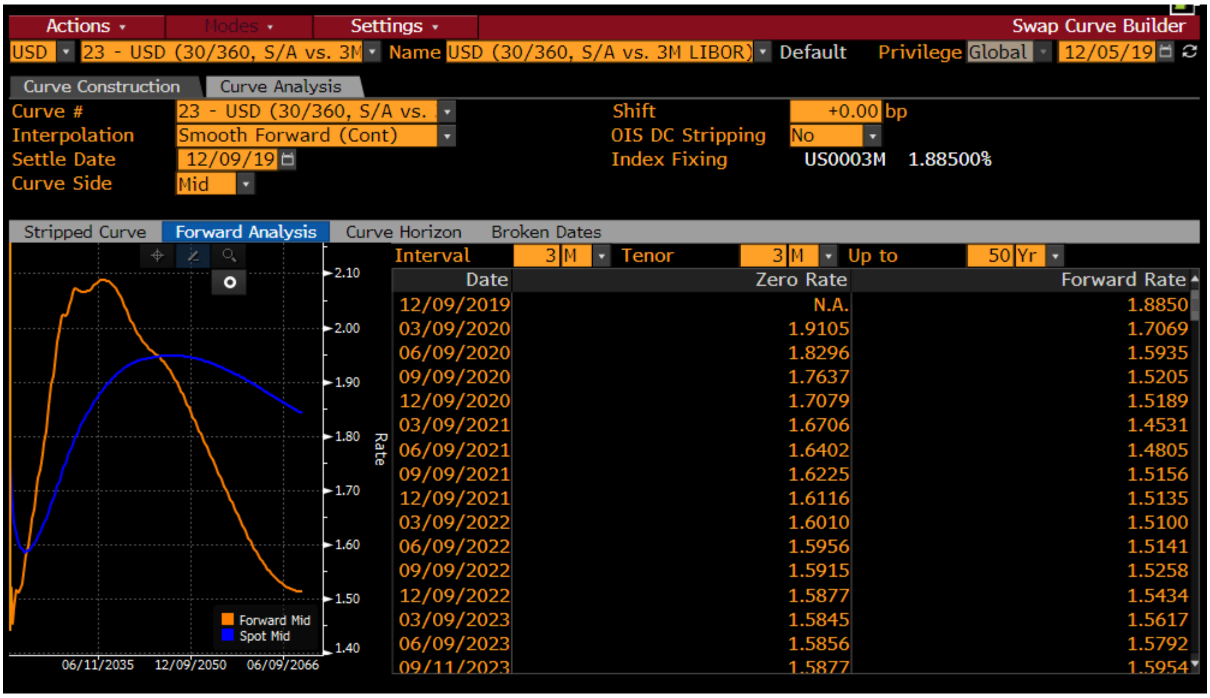
Fig. 3a: Spot rate (blue) and forward rate (orange) graphs for USD curve with interpolation method 1.

Interpolation method 2 - smooth forward/piecewise quadratic continuously-compounded forward rate. The forward rate r_{cf} defined by formula (3) is piecewise-quadratic. The neighboring sections of the forward curve are connected in such a way that the first order derivative is continuous, which is reflected in name ‘smooth’. In the extrapolated region after the last instrument maturity, the instantaneous forward rate is equal to that of the last instrument. Thus, we have:

$$DF(t) = exp[-L(t)] \qquad when\ t \in [0,\ t_1]$$

$$DF(t) = exp[-L(t)] = DF(t_M) * exp[-Fwd(t_M) \cdot (t - t_M)] \qquad when\ t \in [t_M, \infty]$$

Here $L(t)$ is a piecewise quadratic polynomial function and is determined after stripping (Refer to Appendix 1), t_M is the maturity time of the last instrument and $Fwd(t_M)$ is the instantaneous forward rate right at t_M. All dates are converted to time t using ACT/360 convention. Fig.3b shows an example of stripping results using interpolation method 2 for USD curve S23. To solve such a system, the aforementioned global method is required. For details of the functional form of the curve see Appendix 1.



Date	Zero Rate	Forward Rate
12/09/2019	N.A.	1.8850
03/09/2020	1.9105	1.7069
06/09/2020	1.8296	1.5935
09/09/2020	1.7637	1.5205
12/09/2020	1.7079	1.5189
03/09/2021	1.6706	1.4531
06/09/2021	1.6402	1.4805
09/09/2021	1.6225	1.5156
12/09/2021	1.6116	1.5135
03/09/2022	1.6010	1.5100
06/09/2022	1.5956	1.5141
09/09/2022	1.5915	1.5258
12/09/2022	1.5877	1.5434
03/09/2023	1.5845	1.5617
06/09/2023	1.5856	1.5792
09/11/2023	1.5877	1.5954

Fig. 3b: Spot rate (blue) and forward rate (orange) graphs for USD curve with interpolation method 2.

Interpolation method 3 - step-function forward continuously-compounded (CC) forward rate. The forward rate r_{cf} defined by the formula (3) is piecewise constant. A bootstrapping method is used to solve for all the CC forward rates between maturities of neighboring instruments. The CC forward rate before the first instrument maturity is equal to the solved value from the first instrument. Similarly, in the extrapolated region after the last instrument maturity, the CC forward rate is equal to that of the last instrument. Thus, we have:

$$DF(t) = exp[-ccFwd(0, t_1) \cdot t] \qquad when\ t \in [0,\ t_1]$$

$$DF(t) = DF(t_M) \cdot exp[-ccFwd(t_{M-1}, t_M) \cdot (t - t_M)] \qquad when\ t \in [t_M, \infty]$$

Here t_1 is the maturity time of the first instrument and $ccFwd(0, t_1)$ is the CC forward rate at t_1. Similarly, t_M is the maturity time of the last instrument and $ccFwd(t_{M-1}, t_M)$ is the CC forward rate between the last two instruments $(t_{M-1} \to t_M)$. All dates are converted to time t using ACT/360 convention. Thus, it produces a unique discount factor function $DF(t)$ that covers all the time $t \geq 0$ for this curve. Fig.3c shows an example of stripping results using interpolation method 3 for USD curve S23.

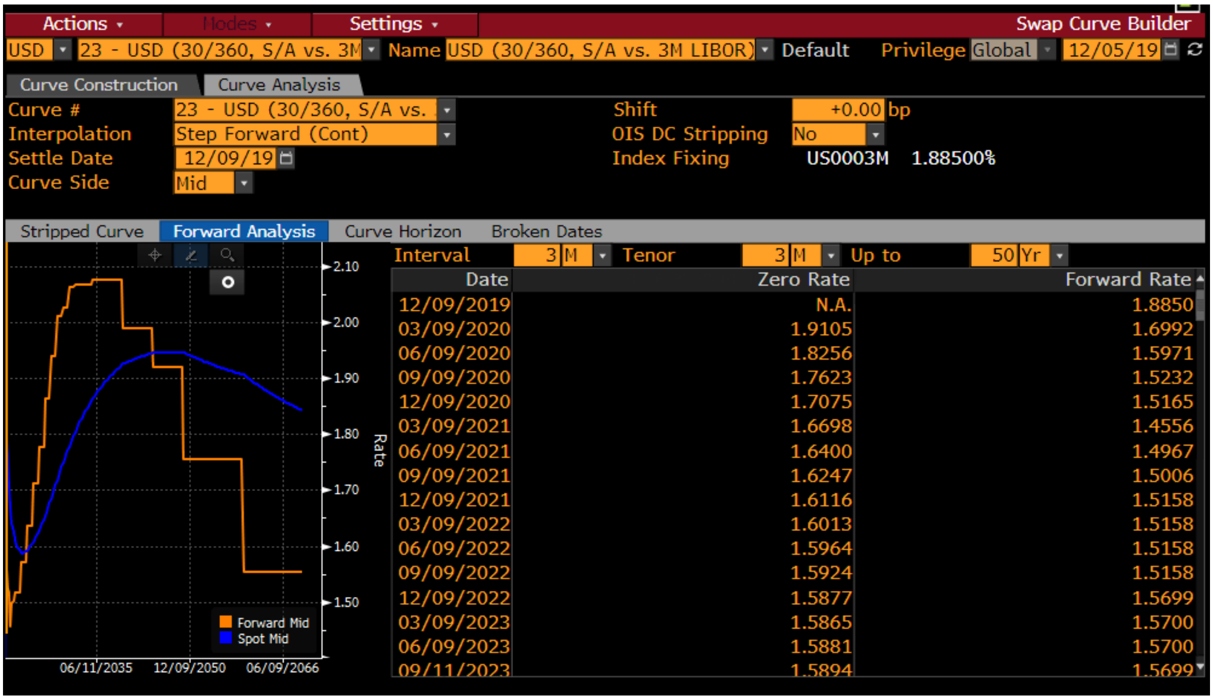


Fig. 3c: Spot rate (blue) and forward rate (orange) graphs for USD curve with interpolation method 3.

Interpolation method 4 - piecewise linear continuously-compounded zero rate. The zero rate r_{sz} as defined by the formula (4) is piecewise linear continuous function. A bootstrapping method is used to solve for all the CC zero rates between maturities of neighboring instruments. The CC zero rate before the first instrument maturity is equal to the solved value from the first instrument. Similarly, in the extrapolated region after the last instrument maturity, the CC zero rate is equal to that of the last instrument, Thus, we have:

$$DF(t) = exp[-ccZero(t_1) \cdot t] \qquad when\, t \in [0,\ t_1]$$

$$DF(t) = exp[-ccZero(t_M) \cdot t] \qquad when\, t \in [t_M, \infty]$$

Here t_1 is the maturity time of the first instrument, t_M is the maturity time of the last instrument, and $ccZero(t)$ is the CC zero rate right at time t. All dates are converted to time using ACT/360 convention. Thus, it produces a unique discount factor function $DF(t)$ that covers all the time $t \geq 0$ for this curve. Fig.3d shows an example of stripping results using interpolation method 4 for USD curve S23.

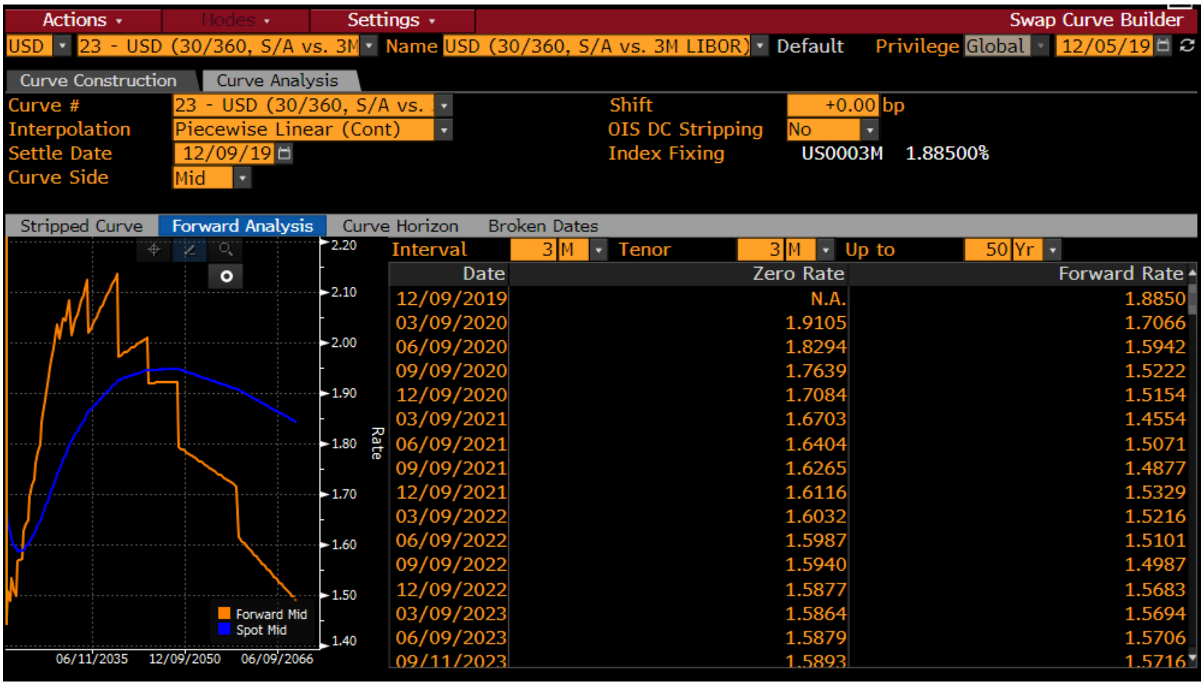


Fig. 3d: Spot rate (blue) and forward rate (orange) graphs for USD curve with interpolation method 4.

The three functional forms – 1, 3, and 4 – have one internal parameter per piece, allowing one to modify the behavior of one piece at a time while not changing the shape of the curve in previous pieces. This makes it possible to build curves of these types using the bootstrap method, as mentioned above. It means that we build the curve from left to right, one piece at a time by solving one equation at a time – we thus reduce the problem of solving a system of N equations with N variables to N consecutive solutions of one equation with one variable.

In case of a smooth curve (interpolation method 2) the situation is different. The curve is defined in such a way that changes in each internal parameter or degree of freedom affects the shape of the whole curve, and therefore requires a solution of a non-linear system of N equations with N variables. The functional form of this curve is described in Appendix 1.

6. IR Curve stripping

We detail the stripping process for various types of IR curve in this section. Please note that all the unknown variables (i.e. to be solved for) are marked in red in the formulas below.

6.1 Vanilla Swap Curve

A vanilla swap instrument comprises a fixed and floating leg with associated market quote being the rate on the fixed leg. Figure 4a shows a standard 5 year vanilla swap instrument on USD 3Mo LIBOR curve S23. Please note that after 2008 financial crisis, the pricing of vanilla swap starts shifting toward using OIS-discounting. But for the sake of simplicity, we still start with the pre-2008 standard vanilla swap first in this section. In Fig.4a, the fixed leg pays 1.634% interest every 6 months, while the floating leg pays a floating 3Mo LIBOR rate every 3 months.

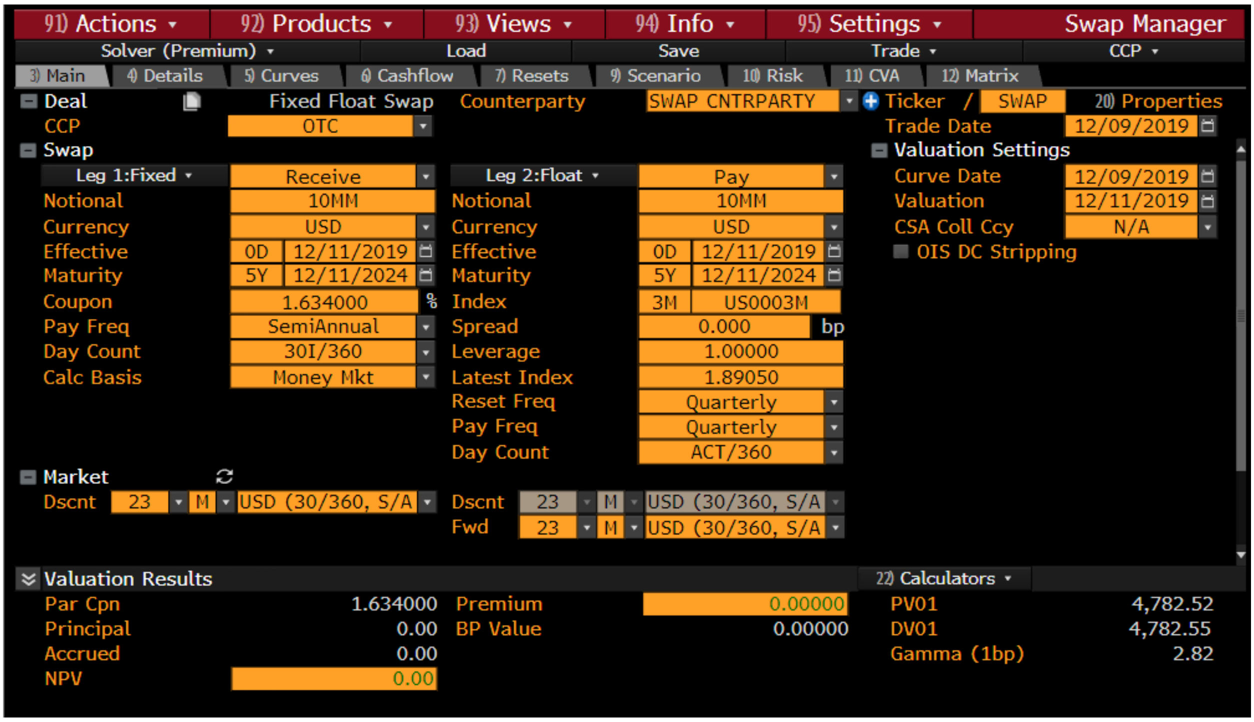


Fig. 4a. Standard 5Y swap instrument on curve S23 (pre-2008 standard)

The present value of both sides can be expressed as the summation of a series of cash flows in the future and discounted back to the start time:

$$PV(fixed) = \sum_i [N \cdot K \cdot \tau(i) \cdot df(t_i, S23)] + N \cdot df(t_M, S23)$$

$$PV(float) = \sum_j \left[N \cdot R(j, S23) \cdot \hat{\tau}(j) \cdot df(t_j, 23)\right] + N \cdot df(t_M, S23)$$

On the fixed leg, all the unknowns are discount factors from S23, which is the curve to be stripped. On the floating leg, the first effective rate is the quoted index instrument (e.g.

US0003M Index for S23) and all the remaining ones are the projected forward rates from S23, which in turn can be directly calculated based on the discount factors from S23. By solving $PV(fixed) = PV(float)$, we are able to determine a series of discount factors and thus strip S23.

6.2 OIS-discounting

OIS discounting means discounting the expected cash flows of a derivative using a nearly risk free curve such as an overnight index swap (OIS) curve. OIS-discounted IR curves are built using a dual-curve (DC) stripping technique. The OIS-discounting feature is enabled by default for all users. Bloomberg allows users to change this assumption by running **{SWDF DFLT<GO>}** and toggling the “Enable OIS Discounting / Dual Curve Stripping” option at the bottom. Note that this setting is global and applies to all currencies where an OIS DC Stripping has been enabled.

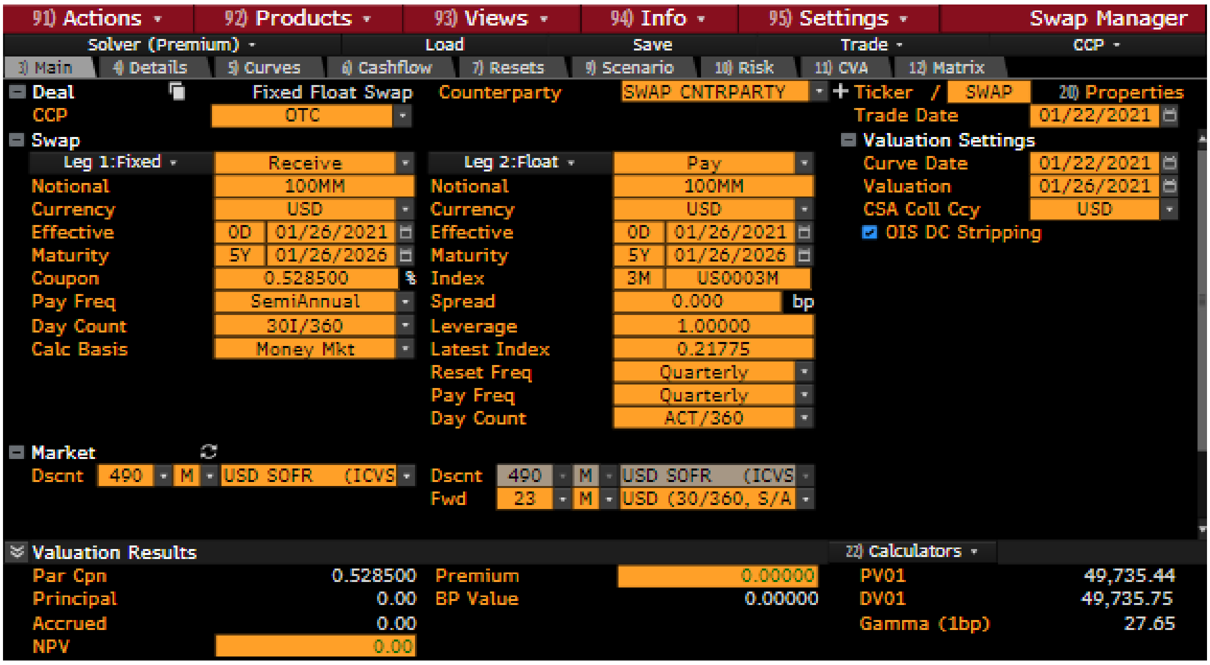


Fig. 4b. Standard 5Y swap instrument on curve S23 with OIS-discounting.

With OIS-discounting, swap rates are calculated using a different formula to its single-curve counterpart because a different curve is used to discount cash flows. Thus, the present value of each leg can be expressed as:

$$PV(fixed) = \sum_i [N \cdot K \cdot \tau(i) \cdot df(t_i, S490)] + N \cdot df(t_M, S490)$$

$$PV(float) = \sum_j \left[N \cdot R(j, S23) \cdot \hat{\tau}(j) \cdot df(t_j, 490)\right] + N \cdot df(t_M, S490)$$

Since OIS curve S490 has already been stripped before stripping S23 all the discount factors from S490 are known here. The only unknown value in the above two PV expressions are the payment values on the floating leg, which can be calculated based on the discount factors from S23. Therefore, $PV(fixed)$ is actually a constant and all the unknown variables are on the floating leg. The stripping process is to find proper discount factors for S23 to make the above two present values equal.

6.3 Single Currency Basis Curve

Single currency basis swaps (also known as tenor basis swaps) are financial instruments that exchange a series of floating rate payments. Both sides reference index instruments from the same currency but with different tenors. Fig. 4c displays a standard tenor basis swap on curve S51 **{SWPM -6MLB<GO>}**: one floating leg references a 3Month index (US0003M Index) while the other references a 6Month index (US0006M Index). Tenor basis swaps are typically quoted as a spread applied on one leg. For example, Fig. 4c shows a spread of 8.2bp needs to be applied to the 3Month leg in order to make its value equal to the 6Month leg.

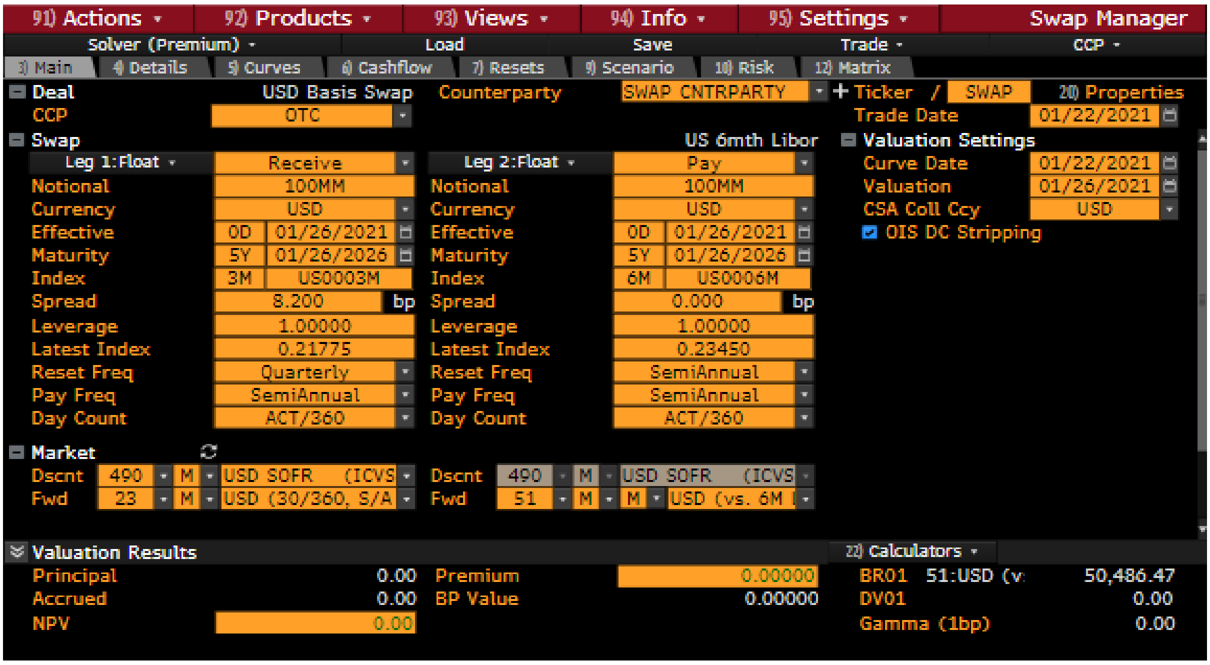


Fig. 4c. Standard 5Y tenor basis swap instrument on curve S51.

The present value of each leg can be expressed as the summation of a series of discounted cash flows:

$$PV(3Mo\ leg) = \sum_i [N \cdot (R(i, S23) + spread) \cdot \hat{\tau}(i) \cdot df(t_i, 490)] + N \cdot df(t_M, S490)$$

$$PV(6Mo\ leg) = \sum_j [N \cdot R(j, S51) \cdot \hat{\tau}(j) \cdot df(t_j, 490)] + N \cdot df(t_M, S490)$$

Here, $R(i)$ is the effective rate to cover the i^{th} payment period. Usually it is just the simple forward rate of the same period. However, on the 3Mo leg here, the floating rates are reset quarterly while the payment happens semiannually. Thus, $R(i)$ is a compounded rate (6Mo) based on two different reset rates (3Mo).

Please note that both curve S23 and S490 are stripped before stripping curve S51, thus the PV on the 3Mo leg can be readily calculated and becomes a constant. While on the 6Mo leg, the effective rates from S51 are directly linked to the unknown discount factors from S51, which is similar to the above OIS discounting case. Finally, solving $PV(3Mo\ leg) = PV(6Mo\ leg)$ can determine a series of discount factors and finalize the $DF(t)$ for S51.

As most of the liquidity in a given currency resides in the standard tenor IR swaps (3Mo LIBOR for USD), we often find we have less IR swap market data to define different tenor LIBOR curves. If one directly constructs a non-standard tenor LIBOR curve using non-standard tenor IR swaps with limited liquidity, then the fact that the spread between the curves is often relatively stable is not enforced, and can be violated if left unconstrained. An alternative approach to build the non-standard tenor LIBOR curves is to define the parameters of the calibration in terms of spreads over the existing curve. Bloomberg implements a so-called **step forward spread interpolation method**. This interpolation method is not configurable and the details of are coved in the Appendix 2.

6.4 Cross Currency Basis Curve Stripping (Non Mark-to-Market)

A cross currency (XCCY) basis swap is a financial instrument where each counterparty exchanges payments in a different currency. For example, entering **{SWPM –FLFL USD HKD <GO>}** in the Bloomberg terminal loads a 5-year cross currency HKD/USD XCCY basis swap.

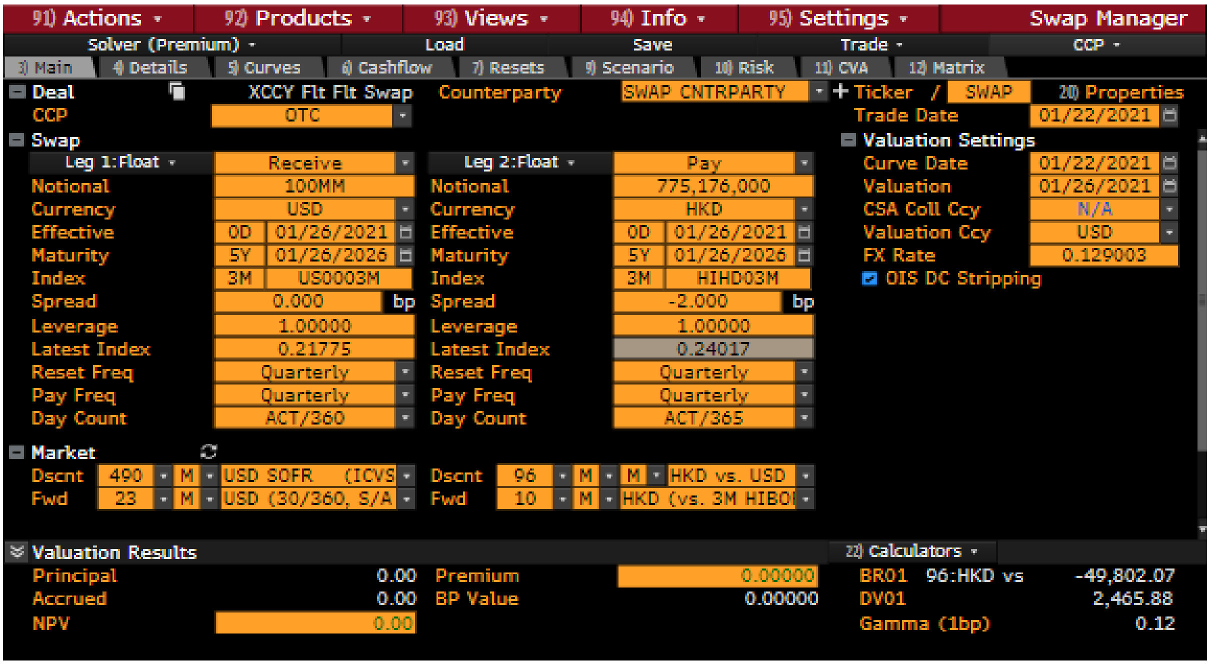


Fig. 4d. 5Y HKD/USD XCCY basis swap instrument on curve S96.

Please note that both legs are floating legs, the market quote (-2.0bps) for this swap is a spread applied to the HKD leg and there is no spread on the USD leg.

Theoretically, an XCCY basis swap should have a zero spread. However, in practice, due to supply and demand, spreads are generally non-zero. In the above example, the market is charging -2.0bp spread on the HKD leg for the right to swap floating payments into USD. Under the no-arbitrage principal, this market quote will make the present value of the HKD floating leg equal to the present value of the floating leg on the USD side. This forms the foundation to strip this curve.

It is straightforward to calculate the present value of the USD leg as:

$$PV(USD) = \sum_i [N_{USD} \cdot R(i, S23) \cdot \hat{\tau}(i) \cdot df(t_i, S490)] + N_{USD} \cdot df(t_M, S490)$$

Here N_{USD} is the principal on the USD leg and it is held constant throughout the life of the swap. Each payment can be calculated using the projected forward rate based on the USD forward curve (S23) and then discounted by the discount factor from the USD discount curve (S490). Finally, there is an extra term representing the repayment of principal at maturity.

On the HKD leg, the existence of a basis spread will change the way we compute the payment amount:

$$PV(HKD) = \sum_j \left[N_{HKD} \cdot [R(j, S10) + spread] \cdot \hat{\tau}(j) \cdot df\left(t_j, S96\right)\right] + N_{HKD} \cdot df(t_M, S96)$$

As can be seen, the spread is added to the projected forward rate based on the 3Mo HIBOR curve (S10) to determine the actual payment amounts.

During stripping, the mathematical equation representing this swap is simply $PV(USD) = PV(EUR)$. Please note that S10, S23 and S490 are already stripped, so all their derived values (forward rates / discount factors) are already determined. Principals on the USD/EUR legs are also determined at the inception of the swap. Thus $PV(USD)$ is actually a constant and the only unknowns are the discount factors from S96, which can be determined by solving the above equation.

Please note that the initial FX spot rate, i.e. the ratio N_{HKD}/N_{USD} does not impact the stripping results, because when setting equation $PV(USD) = PV(HKD)$, we are comparing the value of money, not the absolute number in each currency. This leads to the initial FX spot rate being cancelled on both sides. Additionally, not all basis spreads are quoted on the foreign currency side, there are cases (e.g. MXN/USD swap on curve S151) where the spread is actually applied on the domestic currency side.

6.5 Cross Currency Basis Curve Stripping (Mark-to-Market)

Mark-to-market (MTM) cross currency basis swaps are similar to the non-MTM XCCY basis swaps introduced above, but the MTM type is much more widely traded in financial markets, especially for major currencies. MTM XCCY basis swaps differ in that the principal on the leg paying (L)IBOR flat (generally the USD leg) is adjusted at the start of every payment period based on the spot foreign exchange (FX) rate. The difference between the principal used for the previous period and the next is paid or received. Here, the principal for the other currency (the leg with basis spread applied) is kept constant throughout the contract.

Basis swaps with this MTM feature are commonly referred to as resettable basis swaps, and an example can be viewed in the Bloomberg terminal by entering **{SWPM -FLFL USD EUR -MTM<GO>}**. In Fig.4e, please note the USD leg has a "*Notional Reset by FX" label at the top to indicate the mark-to-market notional resetting feature has been enabled for this leg of the swap. Also, in Fig 4f, we can see that the notional principal changes at every payment time on the USD leg.

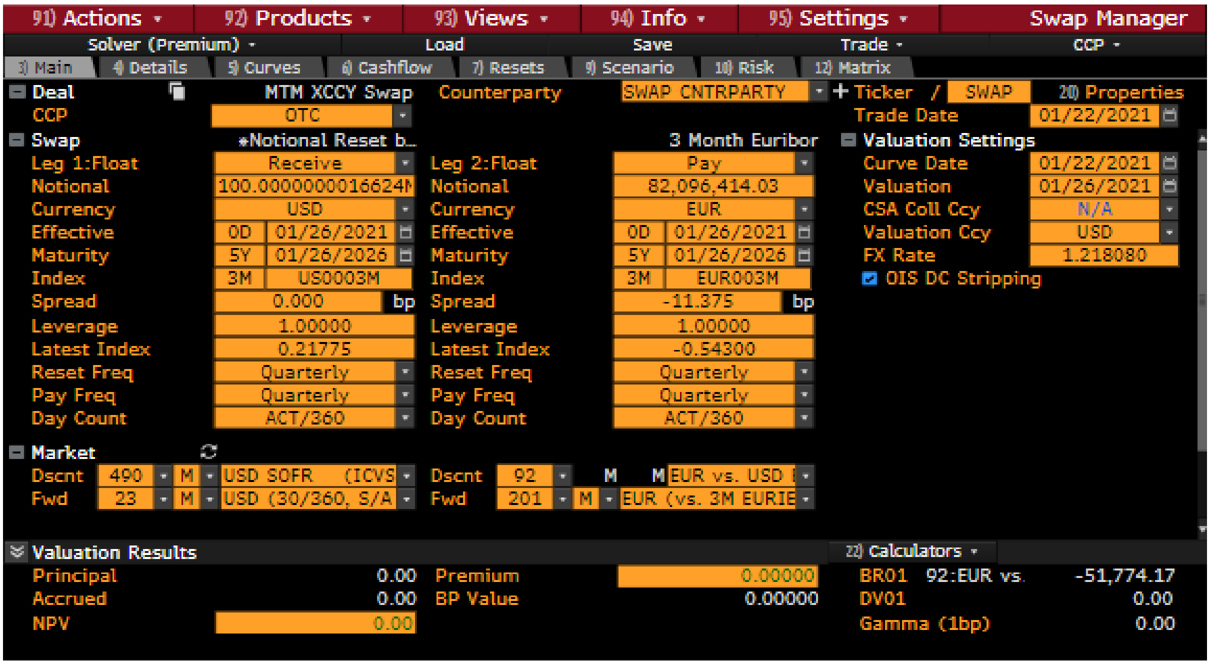


Fig. 4e. 5Y MTM cross currency EUR/USD basis swap instrument on curve S92.

Pay Date	Accrual Start	Accrual End	Da...	FX Fixing Date	FX Rate	Notional	Principal	Reset Date
04/26/2021	01/26/2021	04/26/2021	90	01/22/2021	0.82096	100,000,000.00	-191,183.76	01/22/2021
07/26/2021	04/26/2021	07/26/2021	91	04/22/2021	0.81940	100,191,183.76	-198,337.00	04/22/2021
10/26/2021	07/26/2021	10/26/2021	92	07/22/2021	0.81778	100,389,520.76	-202,985.26	07/22/2021
01/26/2022	10/26/2021	01/26/2022	92	10/22/2021	0.81613	100,592,506.02	-245,707.69	10/22/2021
04/26/2022	01/26/2022	04/26/2022	90	01/24/2022	0.81414	100,838,213.71	-208,685.62	01/24/2022
07/26/2022	04/26/2022	07/26/2022	91	04/22/2022	0.81246	101,046,899.33	-215,630.20	04/22/2022
10/26/2022	07/26/2022	10/26/2022	92	07/22/2022	0.81073	101,262,529.53	-226,416.65	07/22/2022
01/26/2023	10/26/2022	01/26/2023	92	10/24/2022	0.80892	101,488,946.18	-222,303.13	10/24/2022
04/26/2023	01/26/2023	04/26/2023	90	01/24/2023	0.80715	101,711,249.31	-251,244.95	01/24/2023
07/26/2023	04/26/2023	07/26/2023	91	04/24/2023	0.80516	101,962,494.26	-257,031.96	04/24/2023
10/26/2023	07/26/2023	10/26/2023	92	07/24/2023	0.80314	102,219,526.22	-260,515.15	07/24/2023
01/26/2024	10/26/2023	01/26/2024	92	10/24/2023	0.80110	102,480,041.37	-261,179.10	10/24/2023
04/26/2024	01/26/2024	04/26/2024	91	01/24/2024	0.79906	102,741,220.47	-325,334.53	01/24/2024
07/26/2024	04/26/2024	07/26/2024	91	04/24/2024	0.79654	103,066,555.00	-331,337.91	04/24/2024
10/28/2024	07/26/2024	10/28/2024	94	07/24/2024	0.79399	103,397,892.90	-336,061.80	07/24/2024
01/27/2025	10/28/2024	01/27/2025	91	10/24/2024	0.79141	103,733,954.71	-333,483.46	10/24/2024
04/28/2025	01/27/2025	04/28/2025	91	01/23/2025	0.78888	104,067,438.17	-396,589.91	01/23/2025
07/28/2025	04/28/2025	07/28/2025	91	04/24/2025	0.78588	104,464,028.08	-402,952.20	04/24/2025

Fig. 4f. Example of cash flows of a 5Y MTM cross currency EUR/USD basis swap instrument on curve S92.

Similar to the non-MTM XCCY basis curve stripping, the key to strip the MTM type of XCCY basis curve is to determine the discount factors that can make the net present value of the swap zero in order to satisfy the no-arbitrage principal.

Let's demonstrate this process using curve S92 (EUR/USD) as an example. As can be seen in Fig. 4e, the forward curve on USD side is S23 and it is S201 on EUR side, while the discounting curve on USD side is S490 and it is S92 on EUR side.

At the i^{th} payment time for each side, let's assume $Notional(i)$ is the principal for this period, $F(i)$ is the projected forward rate derived from curve S23 or S201, $df(i)$ is the corresponding discount factor and $spread$ is the basis spread quote applied (on EUR leg for curve S92). We also assume the final principals need to be exchanged at maturity.

On the EUR leg, the $Notional(i, EUR)$ is fixed and we can set its value to 1, i.e. $Notional(EUR) = 1$. Let's assume N is the total number of payments on this side. Also note that the i^{th}coupon payment is determined by projected forward rate and basis spread, we can write the present value of EUR leg as follows:

$$PV(EUR) = \sum_{i=1}^{N} [R(i, S201) + spread] \cdot \tau(i) \cdot df(t_i, S92) + df(t_N, S92)$$

On the USD leg, the principal needs to be adjusted to mark-to-market, thus the j^{th} payment has two components:

$$Coupon\ Payment\ CP(j) = \ Notional(j, USD) \cdot R(j, S23) \cdot \tau(j)$$

$$Principal\ Exchange\ PE(j) = Notional(j, USD) - Notional(j + 1, USD)$$

Please note the final principal exchange pays off the entire principal amount, and $Notional(j, USD)$ becomes zero after the last payment.

Based on Interest Rate Parity and the assumption that $Notional(EUR) = 1$, we have:

$$Notional(j, USD) = Notional(USD) \cdot \frac{df\left(t_{j-1}^{FX}, EUR\right)}{df\left(t_{j-1}^{FX}, USD\right)} = Notional(USD) \cdot \frac{df(t_{j-1}^{FX}, S92)}{df(t_{j-1}^{FX}, S490)}$$

Here t_{j-1}^{FX}means the FX-fixing time for j^{th}payment, and it is usually a few days ahead of the accrual start time t_{j-1} for j^{th} period $[t_{j-1}, t_j]$. Also note that $Notional(1, USD) = Notional(USD)$ since there is no notional resetting for the 1^{st} payment, thus we can set $t_0^{FX} = 0$ to automatically satisfy this since $df(0) \ = \ 1$ always holds.

Since the notional on EUR-leg is fixed to be 1, $Notional(USD)$ is just initial FX rate (USD:EUR). Thus, we can write $PV(USD)$ as follows (M is the number of payments on this leg):

$$PV(USD) = \sum_{j=1}^{M} [CP(j) + PE(j)] \cdot df\left(t_j, S490\right)$$

$$= \sum_{j=1}^{M-1} [CP(j) + PE(j)] \cdot df\left(t_j, S490\right) + [CP(M) + Notional(M, USD)] \cdot df(t_M, S490)$$

$$= \sum_{j=1}^{M-1} \left[\frac{df(t_{j-1}^{FX}, S92)}{df(t_{j-1}^{FX}, S490)} \cdot fwdRate(j, S23) \cdot cov(j) + \frac{df(t_{j-1}^{FX}, S92)}{df(t_{j-1}^{FX}, S490)} - \frac{df(t_{j}^{FX}, S92)}{df(t_{j}^{FX}, S490)} \right] \cdot df(t_j, S490) + \left[\frac{df(t_{M-1}^{FX}, S92)}{df(t_{M-1}^{FX}, S490)} \cdot fwdRate(M, S23) \cdot cov(M) + \frac{df(t_{M-1}^{FX}, S92)}{df(t_{M-1}^{FX}, S490)} \right] \cdot df(t_M, S490)$$

Since curve S23, S490 and S201 are already stripped, the only unknown variables are the discount factors from S92. By solving equation $PV(EUR) = PV(USD)$, we can determine the unique $DF(t)$ for S92.

6.6 Special Considerations for Curves with Serial FRAs

Incorporation of serial FRAs in curve building by itself does not require any special treatment from the stripping algorithm, in the sense that the stripper will succeed in producing a curve that matches all input instruments. However, converting FRAs to corresponding discount factors relies implicitly on the zero rates at the start dates of FRA instruments and these zero rates are often interpolated from the curve. For a curve with a few FRA instruments, the forward rates produced directly from stripping of the original curve can sometimes be quite bumpy, especially in the segment that covers the FRAs instruments.

In order to help solve this and make the stripping result smoother in the forward space, we adopt a special logic by adding a few so-called “implied cash rates” in the very front of any curve with serial FRAs before final stripping commences. These implied cash instruments are spaced monthly and cover the region before the first instrument on the curve. For example, EUR curve S45 usually starts with a 6Mo index instrument, and thus five extra cash instruments at 1Mo, 2Mo,…, 5Mo will be added to the front. The rates on these cash instruments are derived or implied based on the original curve S45.

In brief, we first partially strip the curve up to the first swap instrument (2Y for S45) to get all the discount factors in this region. Then a so-called “anchor chain” is constructed with the following instruments: 6Mo (Cash) / 6M X 12M (FRA) / 12M X 18M (FRA) / 18M X 24M (FRA*). Please note the last 18M X 24M FRA instrument is a derived one based on the discount factors at 18Mo and 2Y, while the rest are just the original instruments on curve S45. Let’s move on to consider another chain: 5Mo (Cash*) / 5M X 11M (FRA) / 11M X 17M (FRA) / 17M X 23M (FRA*). Here the 17M X 23M FRA rate is linearly interpolated based on the 12M X 18M (FRA) / 18M X 24M (FRA) from the anchor chain, and we are able to reverse calculate the implied cash rate at 5Mo based on the discount factor at 23M. All the other implied cash points at 1 - 4Mo can be determined in a similar manner. This way, the actual stripping result with these implied cash instrument included can produce much smoother forward rates across the FRA segment.

Since these implied cash instruments do not exist on the curve, they cannot be seen in **{SWDF 45 <GO>}**. Some users are puzzled when comparing zero rates before 6Mo since the result won’t match the interpolation (extrapolation, more accurately) method chosen. This is a direct consequence of the implied cash rates.

7. DV01 Calculation and Effects of DV01 and DC Settings

Currently the SWPM function calculates DV01 as follows: shift all curves up and down by 10 bps (all par rates except basis spreads), compute PVs using the up- and down-shifted curves, and scale the PV difference by 20.

In addition, two DV01 options are offered, in the **DV01/KRR Curve Setting Tab** after typing **{SWDF DFLT<GO>}** (see Fig.5), to control the behavior of the (L)IBOR fixing index that is embedded in the swap rates when the par curve is shifted:

1. *(L)ibor Fixing Shifts*: index fixing shifts with the curve.

2. *(L)ibor Fixing Constant*: index fixing remains constant.

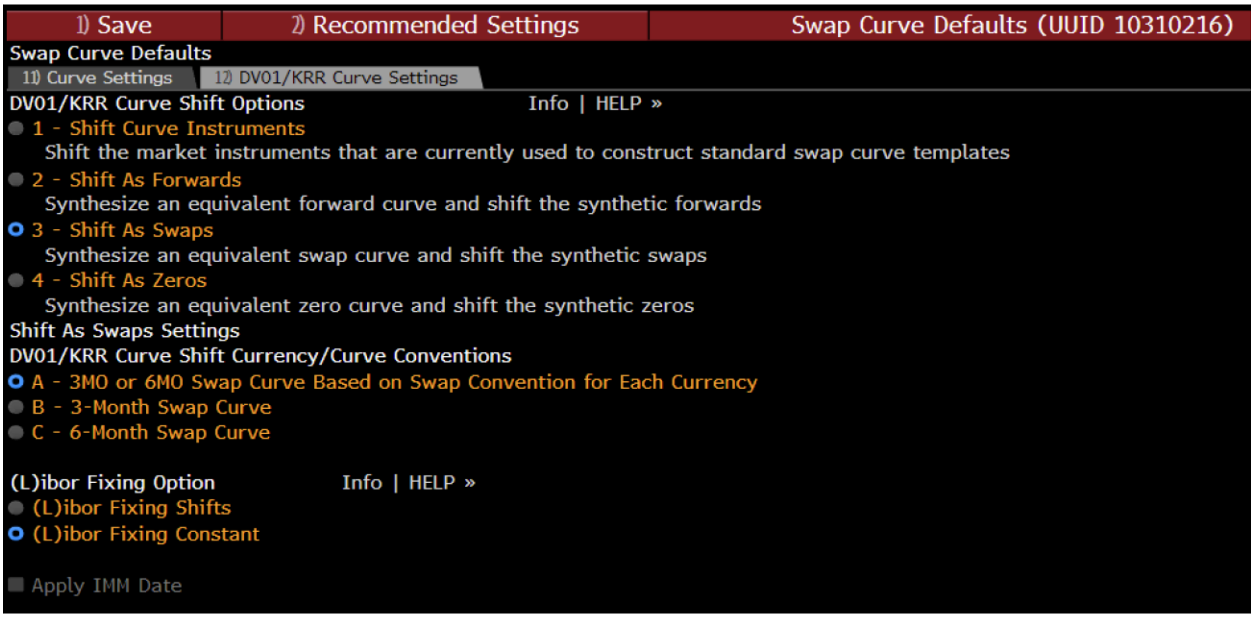


Fig. 5 Screen allowing user to choose (L)ibor Fixing Option (under {SWDF DFLT<GO>} command)

The current method of DV01 calculation of shifting the par curve can produce counter-intuitive values for some combinations of the DV01 and DC setting. The following table provides an example that summarizes DV01 values under different scenarios for EUR 6M fixed-float swaps entered on 10/3/2011, with those having counter-intuitive values shown in red. Note that DV01 values for 1 year swap are not impacted because the shortest swap maturity on the default curve S45 is 2 years.

The main cause for the observed counter-intuitive DV01 behavior is the fact that a parallel shift of a par curve does not necessarily result in a parallel shift of the forward curve. As illustrated in Fig. 6, parallel up-shift of the par curve almost always results in humps in forward rate shifts when instruments on the curve transition from cash/futures/FRA rates to swap rates (starting at year 2 in this example). Note that the magnitude of humps is much larger for DV01 option 2 of *LIBOR Fixing Constant* than for option 1 of *LIBOR Fixing Shifts*. With *constant fixing*, the first coupon remains constant as the curve shifts, hence the remaining forwards that are bootstrapped from this first swap rate need to have larger shifts to realize the shifted swap rate. The subsequent forwards humps are result of ripple-effects from the first one. Similar humps in forward shift are observed when DC is enabled.

Swap Maturity	LIBOR Discounting (Non-DC)				OIS Discounting (DC)			
	Shifting Index		Constant Index		Shifting Index		Constant Index	
	Fixed	Float	Fixed	Float	Fixed	Float	Fixed	Float
1 Yr	1000.82	-503.83	1008.82	-503.83	1011.82	-502.14	1011.82	-502.14
2 Yr	1961.81	-503.83	2465.64	-503.83	2035.87	-556.80	2035.87	-50.14
3 Yr	2914.05	-503.83	3417.89	-503.83	3037.66	-587.31	3037.66	-80.64
4 Yr	3846.66	-503.83	4350.49	-503.83	4020.64	-616.57	4020.64	-109.90
5 Yr	4758.04	-503.83	5261.88	-503.83	4989.53	-650.82	4989.53	-144.16
10 Yr	8928.94	-503.83	9432.81	-503.83	9487.16	-831.71	9487.16	-325.05
20 Yr	15516.27	-503.83	16020.19	-503.83	16721.22	-1174.81	16721.22	-668.15
30 Yr	20579.44	-503.83	21083.42	-503.83	22404.43	-1504.61	22404.43	-997.94

These humps on forward shifts will invariably be reflected in discount factors as well, when the shifted curve is used for discounting. Fig. 7 shows the ratio of discount factors before and after shifting the EUR 6M curve. For reference, discount ratios for the EONIA curve are also shown there. (Only one set of discount ratios is displayed, because EONIA resets daily and the effect of DV01 option is negligible). Note that the large hump in the forward rate observed in Fig.6 is now manifested here as a small decrease in discount factor after year 2. Effects of down-shifting the par curve are near mirror images of up-shifting.

With the help of Figs 6 and 7, we can explain why DV01 may behave in a counter-intuitive manner.

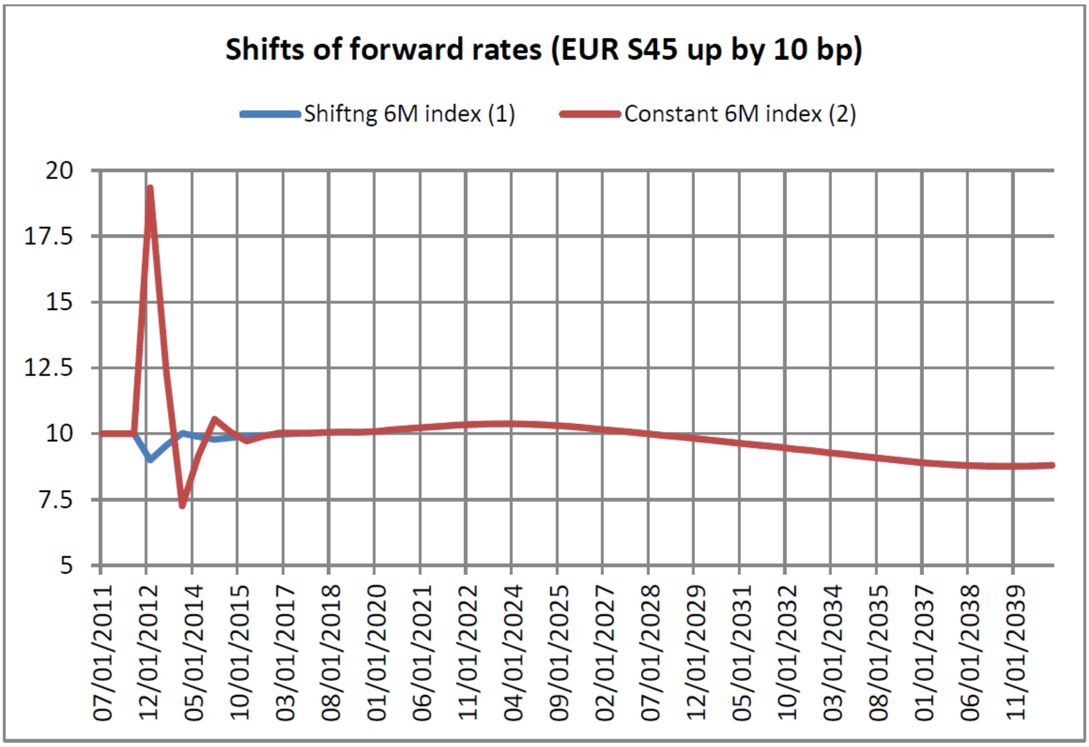


Fig. 6 Forward shifts as a result of par curve shifting (EUR 6Mo S45: mid/non-DC/curve date = 07/25/11)

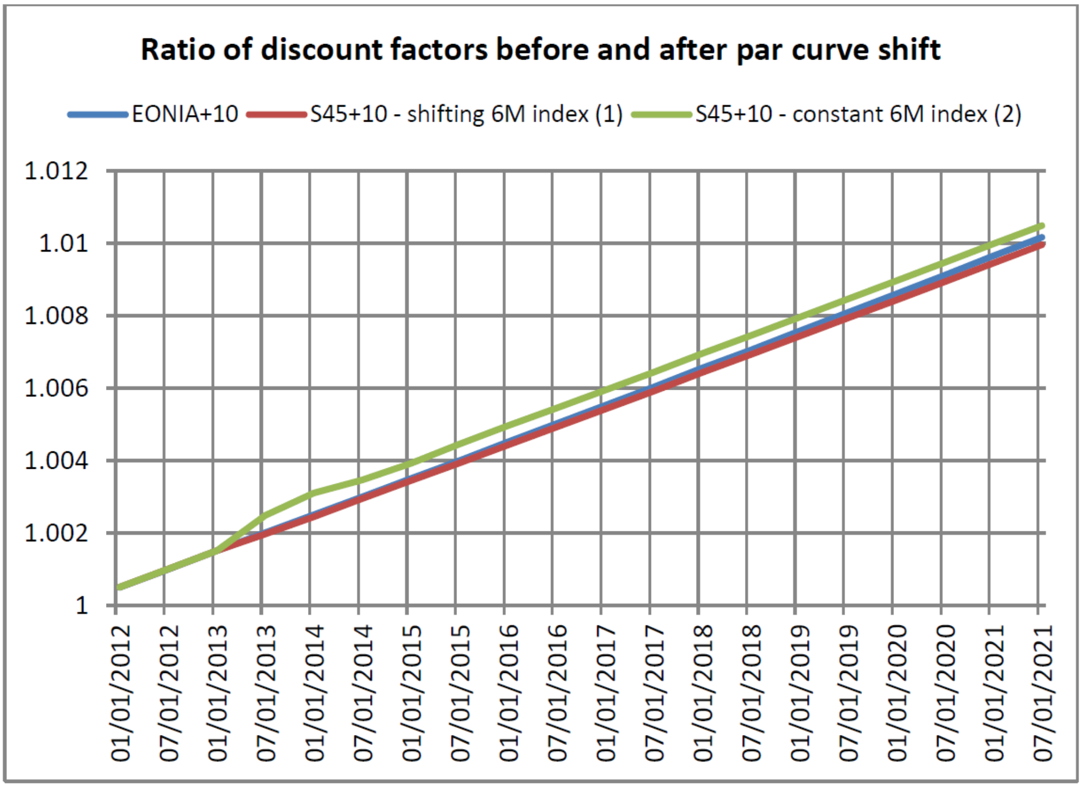


Fig .7 Ratio of discounts resulting from shifting EUR 6Mo curve & EONIA curve (mid on 07/25/11)

Bloomberg

Without OIS Discounting: A single curve is used in this case for both forward projection and discounting. Since the floating leg PV only depends on the first projected forward rate, its DV01 is hence immune from the aforementioned forward rate humps. But the fixed leg cannot escape the impact of the humps on forward shifts. Due to the bias in discount factors observed in Fig 7, option 2 (constant index) will produce a larger DV01 value than option 1 (shifting index). The difference is approximately equal to the equivalence of the LIBOR fixing index duration. For example, DV01 for 6M EUR swaps with 10M notional will differ by approximately 500 (~6Mo) between the two DV01 options.

With OIS Discounting: With OIS discounting and DC, the opposite effect is observed. Fixed legs no longer suffer from the DV01 step-up when switching from option 1 to option 2, because OIS resets daily and whether the fixing index shifts with the curve or not has virtually no impact on the resulting discount factors. However, the float leg PV valuation now requires all cash flows to be explicitly calculated up to the swap maturity and then discounted back to present. Since the humps in forwards can no longer be canceled out by the changes in discounts, they become exposed through the DV01 values in the form of unexpected bumps, especially when DV01 option 2 is selected.

Conclusion

This document describes the process of building the Bloomberg interest rate curves. We have discussed definitions, systems of equations to solve and algorithms used to solve them. The four different functional forms used in Bloomberg terminal for the IR curve are considered. We used an assumption that the IR curve describes risk-free present values. However, some of the instruments could have much lower liquidity (or frequency of quotes) than others, thus producing stale quotes and unreasonable stripping results. This indicates that the choice of the instruments used to build the curve needs to be revised. These considerations must be kept in mind by the user when choosing the instruments for IR curve construction.

Appendix 1: Functional Form of Smooth Curve (Piecewise Quadratic Forward Rate)

In this interpolation method, the functional form of the continuously compounded forward rate is a piecewise quadratic polynomial function $L(t)$, i.e.:

$$DF(t) = exp[-L(t)]$$

And L-polynomial is constructed as in its general form:

$$L(t) = (b \cdot t^2 + c \cdot t + d) + \frac{1}{6} \sum_{T_i < t} \lambda_i \cdot (t - T_i)^3 - \frac{1}{6} P \cdot t^3$$

Here T_i is the maturity of the i^{th} instrument that matures after t_c on the curve. The big summation is done for all the instruments at $i \in [1, \dots . N]$ such that $T_i < t$, and this term makes $L(t)$ a piecewise cubic polynomial function. At each time $= T_i$, a new cubic polynomial term is added in a manner that assures that first and second order derivatives of $L(t)$ are continuous. There are N lambdas together with four extra parameters (b, c, d and P) in the above formula, therefore we need four boundary conditions to reduce the number of freedoms back to N in the curve stripping.

Furthermore, in order to better support its application in multi-segment interpolation, i.e. mixed interpolation in different time intervals, we can extend the above discount factor function $DF(t)$ to allow a different interpolation method on the front end:

$$DF(t) = \begin{cases} df(t) & t \in [0, t_c] \\ exp[-L(t - t_c)] & t \in (t_c, +\infty] \end{cases}$$

where $df(t)$ is an arbitrary function that covers the region from zero to the joint time t_c. Based on the requirements, we can impose appropriate boundary conditions at time t_c to connect these two functions properly.

Let's consider the behavior of $DF(t)$ in the long end first, where the curve is extrapolated when $t \rightarrow +\infty$. We choose to hold the instantaneous forward rate $F(t)$ a constant beyond T_N and then

$$0 = \frac{dF(t)}{dt} = -\frac{d^2[\ln(DF(t))]}{dt^2} = \frac{d^2 L(t)}{dt^2}$$

This leads to two constraint equations:

$$\sum_{i=1}^{N} \lambda_i - P = 0 \qquad (eq. 1)$$

$$2 \cdot b - \sum_{i=1}^{N} \lambda_i \cdot T_i = 0 \qquad (eq. 2)$$

Let's move on to the short end side, where we need to satisfy the continuity condition at the joint point t_c. Please note this is the $t = 0$ point for $L(t)$ internally, and in this region L-polynomial is reduced to:

$$L(t) = b \cdot t^2 + c \cdot t + d - \frac{1}{6} P \cdot t^3$$

Let's consider two common boundary requirements here:

1) Continuity in $DF(t)$ only at t_c

Since $L(0) = d$, we can see that parameter d is pre-determined by the joint function $df(t)$ at t_c point. And there is one more freedom to eliminate, we usually set $b = 0$ for simplicity. It will not only make $b \cdot t^2$ term disappear, but also lead to a simpler form of the above eq.2. After renaming the unknown parameter c to λ_0, L-polynomial becomes:

$$L(t) = (\lambda_0 \cdot t + d) + \frac{1}{6} \sum_{T_i < t} \lambda_i \cdot (t - T_i)^3 - \frac{1}{6} P \cdot t^3$$

The ultimate unknown parameters to be solved are $[\lambda_0, \lambda_1, \dots \lambda_{N-1}]$ and both λ_N and P can be derived from eq.1 and eq.2 above.

In the extreme case where $t_c = 0$ and $df(0) = 1$, i.e. no constraint at all, L-polynomial will reduce to its traditional form that covers the whole $t \geq 0$ region:

$$L(t) = \lambda_0 \cdot t + \frac{1}{6} \sum_{T_i < t} \lambda_i \cdot (t - T_i)^3 - \frac{1}{6} P \cdot t^3$$

2) Continuity in both $DF(t)$ and $DF'(t)$ at t_c

Since $L(0) = d$ and $L'(0) = c$, both parameter d and c are pre-determined by the joint function $df(t)$ at t_c point. The corresponding constraint equations are:

$$df(t_c) = exp(-d) \qquad (eq.3)$$

$$df'(t_c) = df(t_c) \cdot (-c) \qquad (eq.4)$$

After renaming unknown parameter b to λ_0, L-polynomial becomes:

$$L(t) = (\lambda_0 \cdot t^2 + c \cdot t + d) + \frac{1}{6} \sum_{T_i < t} \lambda_i \cdot (t - T_i)^3 - \frac{1}{6} P \cdot t^3$$

Similarly, the ultimate unknown parameters to be solved are $[\lambda_0, \lambda_1, \dots \lambda_{N-1}]$ and both λ_N and P can be derived from eq.1 and eq.2 above.

Overall, the curve is 'smooth' e.g. the forward rate has a continuous first order derivative, and it has N degrees of freedom. It is easy to see that change in any parameter λ_i where $i \in [0, \dots . N-1]$ leads to changes in λ_N and P, thus affecting the value of the function everywhere. Therefore, to strip curve with this interpolation, one needs to solve a system of N non-linear equations with N variables. The Newton Raphson method is used, with an initial guess found using interpolation method 1 (linear simple zero rate).

Appendix 2: Step Forward Spread Interpolation Method

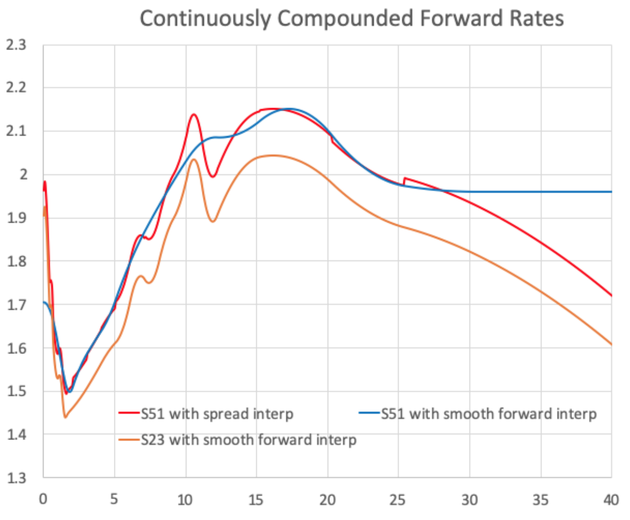
This interpolation takes in a so-called base curve and applies a constant continuous forward spread (for each segment) on top of the forward rate generated directly from base curve to get the final forward rate. So, the final continuously compounded forward rate can be expressed as following:

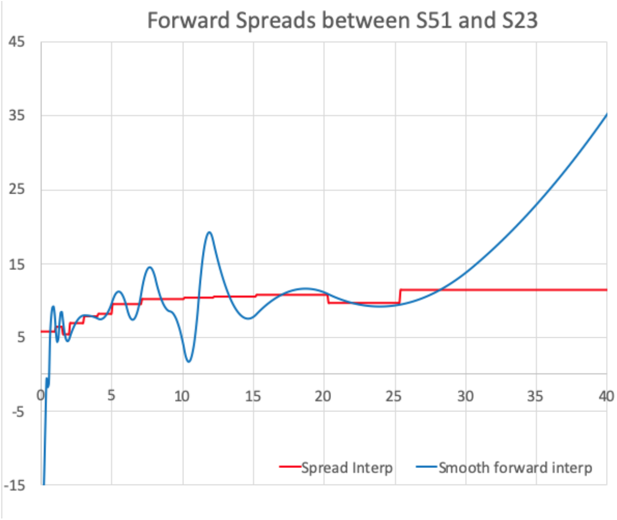
$$F(t_1, t_2) = fwdSpread + baseCCFwd(t_1, t_2)$$

Usually, t_1 is taken at the start time of this segment (t_s), which is the maturity time of the pervious instrument for most of the case. Thus, the discount factors at any time t within this segment can be readily calculated by:

$$DF(t) = DF(t_s) * \exp[-F(t_s, t) * (t - t_s)]$$

Since the base curve is given from outside and already stripped, the $baseCCFwd(t_1, t_2)$ can be directly computed at any period. The only unknown variables left are the above $fwdSpread(s)$, one for each segment. The stripping process will apply a bootstrapping solver to determine all of these values one by one under no-arbitrage principal. The biggest advantage of this interpolation method is to allow the user to capture the feature of the benchmark (base curve), and focus on the deviation from it. Below is a graph that plots the continuously compounded forward rates for USD 6Mo basis curve S51 under spread interpolation (red) as a function of time. It also displays the same forward rates for S51 using smooth forward interpolation, and the underlying curve S23 for comparison. It is quite obvious that the stripping results based on the spread interpolation are tracking the underlying curve much better than that from the smooth forward interpolation. Furthermore, smooth forward interpolation completely loses track of the underlying curve especially in the extrapolation region (beyond 30 years), while the spread interpolation still tracks it very well by keeping a constant distance above S23.





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