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FX Black-Scholes Model with Stochastic Rates using the Hull-White One-Factor Short-Rate Models

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Abstract

This document specifies in detail the implementation of a hybrid FX-IR model using a FX Black-Scholes model combined with two Hull-White one-factor interest rate models for domestic currency and foreign currency, respectively.

Keywords. Hull-White, Hull-White One-Factor, Black-Scholes, Stochastic Interest Rate, Monte Carlo, BS, HW1F, HW1-BS-CC.

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1 FX Black-Scholes Model with One-Factor Hull-White Short Rates

We combine two Hull-White one-factor (HW1F) short rate models with an FX Black-Scholes (BS) model to create a hybrid model which models a domestic short rate, foreign short rate, and FX spot underlying. This hybrid model is referred as “HW1-BS-CC” throughout the document, while in the DLIB screen it is referred to as “Hull-White BS”.

1.1 Model Dynamics

Domestic Risk-Neutral Measure

We recall the Hull-White one-factor model which simulates the short-rate $r(t)$ as a stochastic process decomposable into two components; a deterministic component $\theta(t)$, and a mean reverting stochastic process $x(t)$. Assuming both domestic and foreign interest rates $r_d(t)$ and $r_f(t)$ follow one-factor Hull-White dynamics, we may write, under their respective domestic and foreign money market measures Q^d, Q^f, where the associated domestic and foreign money market numeraires are defined as

$$\mathcal{N}_{Q^d}(t) := \exp\left[\int_0^t r_d(s)ds\right], \qquad \mathcal{N}_{Q^f}(t) := \exp\left[\int_0^t r_f(s)ds\right],$$

the following dynamics:

$$r_d(t) \quad := \quad \theta_d(t) + x_d(t), \tag{1.1a}$$

$$dx_d(t) \quad = \quad -\kappa_d x_d(t)dt + \sigma_d(t)dW_d^{Q^d}(t), \qquad x_d(0) = 0, \tag{1.1b}$$

$$r_f(t) \quad := \quad \theta_f(t) + x_f(t), \tag{1.2a}$$

$$dx_f(t) \quad = \quad -\kappa_f x_f(t)dt + \sigma_f(t)dW_f^{Q^f}(t), \qquad x_f(0) = 0. \tag{1.2b}$$

Next we consider the FX spot rate $Z(t)$, which represents the amount of domestic currency per unit of foreign currency, and whose dynamics under Q^d follows the Black-Scholes SDE:

$$\frac{dZ(t)}{Z(t)} = (r_d(t) - r_f(t))\, dt + \sigma_y(t)dW_y^{Q^d}(t), \tag{1.3}$$

where $\sigma_y(t)$ is the Black-Scholes volatility of $Z(t)$.

If one uses the change of variable $y(t) := \ln Z(t)$ to convert this spot process into logarithmic coordinates, one obtains

$$dy(t) \quad = \quad \left(r_d(t) - r_f(t) - \tfrac{1}{2}\sigma_y(t)^2\right) dt + \sigma_y(t)dW_y^{Q^d}(t). \tag{1.4}$$

We may then model its instantaneous correlation with domestic and foreign Hull-White factors, respectively, as the following:

$$dW_d^{Q^d}(t) \cdot dW_f^{Q^f}(t) =: \rho_{df}dt, \tag{1.5a}$$

$$dW_d^{Q^d}(t) \cdot dW_y^{Q^d}(t) =: \rho_{dy}dt, \tag{1.5b}$$

$$dW_f^{Q^f}(t) \cdot dW_y^{Q^d}(t) =: \rho_{fy}dt. \tag{1.5c}$$

In order to consolidate (1.1a), (1.2a), and (1.4) into a larger model using a common numeraire, we apply the change of numeraire from Q^f to Q^d,

$$dW_f^{Q^d}(t) \quad = \quad dW_f^{Q^f}(t) + \rho_{fy}\sigma_y(t)dt, \tag{1.6}$$

and rewrite (1.2a) as

$$dx_f \quad = \quad \left(-\kappa_f x_f - \sigma_f(t)\rho_{fy}(t)\sigma_y(t)\right)dt + \sigma_f(t)dW_f^{Q^d}(t), \tag{1.7}$$

where $W_f^{Q^d}(t)$ is a Q^d-Brownian motion. With this consolidation, the larger model is driven by a three-dimensional Brownian motion under the measure Q^d:

$$d\mathbf{W}^{Q^d}(t) \quad := \quad \left[dW_d^{Q^d}(t), dW_f^{Q^d}(t), dW_y^{Q^d}(t)\right], \tag{1.8}$$

where, using the notation $\boldsymbol{x} \wedge \mathbf{y} := \boldsymbol{x}' \cdot \mathbf{y}$ for row vectors $\boldsymbol{x}, \mathbf{y}$:

$$d\mathbf{W}^{Q^d}(t) \wedge d\mathbf{W}^{Q^d}(t) \quad =: \quad \begin{bmatrix} 1 & \rho_{df} & \rho_{dy} \\ \rho_{df} & 1 & \rho_{fy} \\ \rho_{dy} & \rho_{fy} & 1 \end{bmatrix} dt =: \boldsymbol{\rho}_M \, dt. \tag{1.9}$$

Note that the $\{\rho_{df}, \rho_{fy}, \rho_{dy}\}$ are precisely the model correlations that the user can set in the DLIB Correlation Tab.

Solutions and Zero Coupon Bonds

We denote by $\mathbb{P}_u(0,T)$ the current market price of a zero coupon bond maturing at time T, where $u = d, f$ for the domestic and foreign currencies, respectively. More generally, $\mathbb{P}_u(t,T)$ will denote the time-t price of a zero coupon bond of maturity T. Analytic formulas for zero coupon bonds in each currency may be derived as in [BM, §4.2]. For example, in the Q^d-measure, one obtains from the martingale property of the tradable $\mathbb{P}_d(t,T)$ with respect to the numeraire $B_d(t)$:

$$\mathbb{P}_d(t,T) \quad = \quad B_d(t) \cdot \mathbb{E}_t^{Q^d}\left[B_d(T)^{-1}\right] = \mathbb{E}_t^{Q^d}\left[e^{-\int_t^T r_d(s)ds}\right]. \tag{1.10}$$

Applying variation of parameters to (1.1a), and using the initial conditions, gives:

$$x_d(T) = \int_0^T e^{-\kappa_d(T-s)}\sigma_d(s)\, dW_d^{Q^d}(s),$$

from which we see that, letting $T \to \infty$, x_d mean reverts to its initial value of zero. Hence

$$r_d(T) \quad = \quad \theta_d(T) + \int_0^T e^{-\kappa_d(T-s)}\sigma_d(s)\, dW_d^{Q^d}(s) \tag{1.11}$$

mean reverts to $\theta_d(T)$. It is not difficult to deduce that the mean and variance of $r(T)$, conditional on the state of the world at time $t < T$, are given by:

$$\mathbb{E}_t^{Q^d}\left[r_d(T)\right] \quad = \quad \theta_d(T) + x_d(t)e^{-\kappa_d(T-t)}, \tag{1.12}$$

$$\mathrm{Var}_t^{Q^d}\left[r_d(T)\right] \quad = \quad \int_t^T e^{-2\kappa_d(T-s)}\sigma_d^2(s)\, ds \tag{1.13}$$

Note that (1.11) implies $r_d(t)$ is a Gaussian random variable, and so (1.10) and (1.12) imply:

$$\begin{aligned}
\mathbb{P}_d(t,T) &= \exp\left(-\mathbb{E}\left[\int_t^T r_d(s)\ ds\right] + \tfrac{1}{2}\mathrm{Var}\left[\int_t^T r_d(s)\ ds\right]\right) \\
&= \exp\Big(-\int_t^T \theta_u(s)ds - H_d(T-t)x_d(t) + \tfrac{1}{2}H_d(T-t)H_d(T-t)\nu_{dd}(s) \\
&\quad + \int_t^T H_d(T-s)\nu_{dd}(s)ds - H_d(T-t)\nu_d^h(t)\Big),
\end{aligned} \tag{1.14}$$

where in the first equality we have used $\mathbb{E}\left(e^{\mathcal{N}(\mu,\nu)}\right) = \exp\left(\mu + \frac{1}{2}\nu\right)$; and in the last equality (see [BM] and Appendices therein) we have introduced the notation:

$$\begin{aligned}
h_u(t) &:=^{1} \quad e^{-\kappa_u t}, \\
H_u(t) &:= \int_0^t h_u(s)ds = \frac{1-e^{-\kappa_u t}}{\kappa_u}, \\
\nu_{uv}(t;T) &:= Var\left(x_u(T)x_v(T)|x_u(t),x_v(t)\right) = \rho_{uv}\int_t^T \sigma_u(s)\sigma_v(s)h_u(T-s)h_v(T-s)ds, \\
\nu_u^h(t) &:= \int_0^t h_u(t-s)\nu_{uu}(s)ds,
\end{aligned}$$

where $u = d, f$ is used to indicate domestic and foreign quantities, respectively. Here $\rho_{dd} = \rho_{ff} = 1$.

Similar remarks apply to $\mathbb{P}_f(t,T)$, except account must be taken of the integration of the additional drift term $\nu_{fy} := \sigma_f(t)\rho_{fy}(t)\sigma_y(t)$ in (1.7), which will contribute an additional $\int_t^T h_u(T-s)\nu_{fy}(s)ds$ to the mean terms of (1.14). From (1.14) one observes how the price of the zero coupon bond depends on both a curve-dependent component, and also a model-dependent component.

Domestic Spot-Libor Measure

We consider the Domestic Spot Libor measure $Q^d_{\beta(t)}$, whose associated numeraire is the "discretely rebalanced domestic bank account" numeraire, which is defined as:

$$\mathcal{N}_{Q^d_\beta}(t) := \frac{\mathbb{P}_d\left(t, T_{\gamma(t)}\right)}{\prod_{j=1}^{\gamma(t)} \mathbb{P}_d\left(T_{j-1}, T_j\right)}, \tag{1.15}$$

where $0 = T_0 < T_1 < \cdots$ are fixed "grid-points" according to the discrete tenor structure, and $\gamma(t)$ is defined as the smallest index such that $t < T_{\gamma(t)}$: $\quad t \in [T_{k-1}, T_k) \Longleftrightarrow \gamma(t) = k$.

To distinguish between the index $\gamma(t)$ and the time $T_{\gamma(t)}$, we introduce the notation $\beta(t) := T_{\gamma(t)}$. Note that we always have $t < \beta(t)$, and that when t is a grid-point we have $t = T_{\gamma(t)-1}$. By applying the change-of-numeraire technique from Q^d to Q^d_β, the following Brownian motions are

[1] This is the "integrating factor" from ODE theory.

derived:

$$dW_d^{Q_\beta^d}(t) = dW_d^{Q^d}(t) + \rho_{dd}\sigma_{P_d}(t;\beta(t))dt, \tag{1.16a}$$

$$dW_f^{Q_\beta^d}(t) = dW_f^{Q^d}(t) + \rho_{df}\sigma_{P_d}(t;\beta(t))dt, \tag{1.16b}$$

$$dW_y^{Q_\beta^d}(t) = dW_y^{Q^d}(t) + \rho_{dy}\sigma_{P_d}(t;\beta(t))dt; \tag{1.16c}$$

where

$$\sigma_{P_d}(t;T) \quad := \quad \sigma_d(t)H_d(T-t) \tag{1.17}$$

is the volatility of the domestic zero coupon bond $\mathbb{P}_d(t,T)$ with maturity T, and from which we obtain the dynamics of the three-dimensional system $\boldsymbol{X}(t) = [x_d(t), x_f(t), y(t)]$ under Q_β^d:

$$dx_d(t) = [-\kappa_d x_d(t) + \sigma_d(t)\rho_{dd}\sigma_{P_d}(t)]\,dt + \sigma_d(t)dW_d^{Q_\beta^d}(t) \tag{1.18a}$$

$$dx_f(t) = [-\kappa_f x_f(t) - \sigma_f(t)\rho_{fy}\sigma_y(t) + \sigma_f(t)\rho_{df}\sigma_{P_d}(t)]\,dt + \sigma_f(t)dW_f^{Q_\beta^d}(t), \tag{1.18b}$$

$$dy(t) = \left[r_d(t) - r_f(t) - \tfrac{1}{2}\sigma_y(t)^2 + \sigma_y(t)\rho_{dy}\sigma_{P_d}(t)\right]dt + \sigma_y(t)dW_y^{Q_\beta^d}(t) \tag{1.18c}$$

where

$$\mathbf{W}^{Q_\beta^d} := \left[W_d^{Q_\beta^d}(t), W_f^{Q_\beta^d}(t), W_y^{Q_\beta^d}(t)\right] \tag{1.19}$$

is a three-dimensional Q_β^d-Brownian motion with covariance matrix

$$\mathbf{W}^{Q_\beta^d} \wedge \mathbf{W}^{Q_\beta^d} =: \boldsymbol{\rho}_M dt. \tag{1.20}$$

Accordingly, we may deduce the integrated solutions to (1.18) given $\boldsymbol{X}(t_0) = [x_d(t_0), x_f(t_0), y(t_0)]$:

$$\begin{aligned} x_d(T) &= h_d(T-s)x_d(t_0) + \int_{t_0}^T h_d(T-s)\sigma_d(s)\ dW_d^\beta(s) \\ &+ \int_{t_0}^T h_d(T-s)\left[\sigma_d(s)\sigma_{P_d}(s)\right]\ ds, \end{aligned} \tag{1.21a}$$

$$\begin{aligned} x_f(T) &= h_f(T-s)x_f(t_0) + \int_{t_0}^T h_f(T-s)\sigma_f(s)\ dW_f^\beta(s) \\ &+ \int_{t_0}^T h_f(T-s)\left[\sigma_f(s)\rho_{fy}\sigma_y(s) + \sigma_f(s)\rho_{df}\sigma_{P_d}(s)\right]\ ds, \end{aligned} \tag{1.21b}$$

$$\begin{aligned} y(T) &= y(t_0) + \int_{t_0}^T \sigma_y(s)\ dW_y^\beta(s) \\ &+ \int_t^T \left[r_d(s) - r_f(s) - \tfrac{1}{2}\sigma_y(s)^2 + \sigma_y(s)\rho_{dy}\sigma_{P_d}(s)\right]ds, \end{aligned} \tag{1.21c}$$

where the drift terms involving σ_{P_d} arise from the measure change from Q^d to Q_β^d.

2 Model Calibration

Calibration of HW1-BS-CC model requires the following steps:

1. user to select mean reversion levels $\{\kappa_d, \kappa_f\}$ for each HW1F component, and correlation components $\{\rho_{df}, \rho_{fy}, \rho_{dy}\}$ of the correlation matrix $\boldsymbol{\rho}_M$
2. calibrate independently the domestic HW1F component to its native domestic market, *i.e.* domestic rate curves and ATM swaptions
3. calibrate independently the foreign HW1F component to its native foreign market, *i.e.* foreign rate curves and ATM swaptions
4. calibrate the FX BS component to the FX market instruments, *i.e.* FX ATM options

In the first three steps, both domestic and foreign HW1F components are calibrated independently, each fitted to native market instruments in its respective currency. For example, in a USDJPY cross currency deal, the JPY rate model can be calibrated to one year expiry ATM swaptions using the JY0006M Libor curve, while the USD rate model can be calibrated to coterminal ATM swaptions using the US0003M Libor curve. For piecewise constant volatility structure, with user specified mean reversion parameter κ_u, the volatility parameters $\sigma_u(t \in [T_i, T_i + \tau_u])$ will result from the given calibration ($\tau_d = 6m, \tau_f = 3m$).

The calibrated domestic and foreign HW1F models are then used to perform the fourth step, which we now describe. The FX Black volatility $\sigma_y(t)$ is calibrated to market FX ATM options. The correlation parameters ρ_{df}, ρ_{dy}, ρ_{fy} are all specified by the user, whether based on historical data or other user input. For a piecewise volatility structure, the model parameters $\sigma_y(t \in [t_j, t_{j+1}])$ will result from the given FX calibration, where the coverages $(t_{i+1} - t_i)$ could range anywhere from one week in the near term, to five years in the long term.

In general, the T-forward FX process

$$\hat{Z}(t) = Z(t)\frac{\mathbb{P}_f\,(t,T)}{\mathbb{P}_d\,(t,T)}$$

has zero drift under the domestic T-forward measure Q_T^d. Its one-factor dynamics are given by:

$$\begin{aligned} d\hat{Z}(t) &= \hat{Z}(t)\left[\sigma_d(t)H_d(T-t)dW_d^{Q_T^d}(t) - \sigma_f(t)H_f(T-t)dW_f^{Q_T^d}(t) + \sigma_y(t)dW_y^{Q_T^d}(t)\right] \\ &= \hat{y}(t)\,\|\boldsymbol{\sigma}(t,T)\sqrt{\boldsymbol{\rho}_M}\|\,d\hat{W}_y^{Q_T^d}(t), \end{aligned}$$

where

$$\boldsymbol{\sigma}(t,T) := [\sigma_d(t)H_d(T-t), -\sigma_f(t)H_f(T-t), \sigma_y(t)].$$

Consider when $t \in [t_{i-1}, t_i)$, so $\gamma(t) = i$, and we set $T = \beta(t) = t_i$. In principle, given a set of market observed FX t_i-forward volatilities $\sigma^{mkt}(t_i)$, one may solve for $\sigma_y(t \in (t_{i-1}, t_i])$ by assuming $\sigma_y(t)$ is constant on each interval $(t_{i-1}, t_i]$, and then iterate over all i using a bootstrapping method. Specifically, we need only iteratively solve the following sequence of equations

$$[\sigma^{mkt}(t_i)]^2 t_i \quad = \quad \int_0^{t_i} \langle \boldsymbol{\sigma}(t,t_i), \boldsymbol{\rho}_M \boldsymbol{\sigma}(t,t_i)\rangle dt, \quad i = 1, 2, \ldots, \tag{2.1}$$

quadratic in $\sigma_y(t)$, and all quantities being known at each i except $\sigma_y(t \in (t_{i-1}, t_i])$.

More explicitly, (2.1) gives:

$$\begin{aligned}
[\sigma^{mkt}(t_i)]^2 t_i &= (t_i - t_{i-1})\sigma_y(t_i)^2 \\
&+ 2\left[\rho_{dy}\int_{t_{i-1}}^{t_i}\sigma_d(s)H_d(t_i - s)ds - \rho_{fy}\int_{t_{i-1}}^{t_i}\sigma_f(s)H_f(t_i - s)ds\right]\sigma_y(t_i) \\
&+ \int_{t_{i-1}}^{t_i}\sigma_d^2(s)H_d^2(t_i - s) + \sigma_f^2(s)H_f^2(t_i - s) - 2\sigma_d(s)H_d(t_i - s)\rho_{df}\sigma_f(s)H_f(t_i - s)\ ds \\
&+ [\sigma^{mkt}(t_{i-1})]^2 t_{i-1}.
\end{aligned}$$

It is important to note that the quadratic equation (2.1) need not always have a real solution. For example, assuming the market spot volatilities have no arbitrage, an unrealistic correlation between the domestic and foreign Hull-White factors ρ_{df} can result in an unsolvable quadratic equation. When this occurs, the Black-Scholes volatility $\sigma_y(t_i)$ is set to zero, resulting in a calibration error in the FX Options. The bias introduced by this calibration error will overprice the instrument relative to its market value.

It is a limitation of both the Hull-White one-factor and Black-Scholes models that, for a given expiry, only a single instrument's volatility can be fitted. On the other hand, there is no strike requirement on the selection of calibration instruments, and the user is free to select non-ATM options from the FX and interest rate markets.

Additionally, it should be noted that the calibration of the FX volatilities has no explicit[2] dependence on the interest rate curves.

3 Monte Carlo Simulation

The HW1-BS-CC model can be exactly sampled due to its known distribution.

3.1 Path Generation in the Domestic Spot-Libor Measure

The explicit, albeit coupled, solution to (1.18) has been presented in (1.21). Recalling

$$r_u(s) := \theta_u(s) + x_u(s),$$

[2]Inasmuch as the $\sigma^{mkt}(t_i)$ are quoted ATM forward volatilities, there is, however, an implicit dependence on the interest rate curves.

one may substitute the expressions for $x_d(s), x_f(s)$ from (1.21) to rewrite $y(T)$ in terms of initial conditions and known correlations and volatilities:

$$
\begin{aligned}
y(T) &= y(t_0) + \int_{t_0}^{T} \sigma_y(s)\ dW_y^{\beta}(s) + \int_{t}^{T} \left[\theta_d(s) - \theta_f(s) - \tfrac{1}{2}\sigma_y(s)^2 + \sigma_y(s)\rho_{dy}\sigma_{P_d}(s)\right] ds \\
&+ \int_{t_0}^{T} h_d(T-s)x_d(t_0)ds + \int_{t}^{T} h_f(T-s)x_f(t_0)ds \\
&+ \int_{t_0}^{T}\int_{t_0}^{z} h_d(z-s)\sigma_d(s)\ dW_d^{\beta}(s)\ ds\ dz + \int_{t_0}^{T}\int_{t_0}^{z} h_f(z-s)\sigma_f(s)\ dW_f^{\beta}(s)\ ds\ dz \\
&+ \int_{t_0}^{T}\int_{t_0}^{z} h_d(z-s)\left[\sigma_d(s)\sigma_{P_d}(s)\right]\ ds\ dz \\
&+ \int_{t_0}^{T}\int_{t_0}^{z} h_f(T-s)\left[\sigma_f(s)\rho_{fy}\sigma_y(s) + \sigma_f(s)\rho_{df}\sigma_{P_d}(s)\right]\ ds\ dz \\
&= y(t_0) + \int_{t_0}^{T} \sigma_y(s)\ dW_y^{\beta}(s) + \int_{t}^{T} \left[\theta_d(s) - \theta_f(s) - \tfrac{1}{2}\sigma_y(s)^2 + \sigma_y(s)\rho_{dy}\sigma_{P_d}(s)\right] ds \\
&+ H_d(T-t_0)x_d(t_0) + H_f(T-t_0)x_f(t_0) \qquad (3.1)\\
&+ \int_{t_0}^{T} H_d(T-s)\sigma_d(s)\ dW_d^{\beta}(s) + \int_{t_0}^{T} H_f(T-s)\sigma_f(s)\ dW_f^{\beta}(s) \\
&+ \int_{t_0}^{T} H_d(z-s)\left[\sigma_d(s)\sigma_{P_d}(s)\right]\ ds + \int_{t_0}^{T} H_f(T-s)\left[\sigma_f(s)\rho_{fy}\sigma_y(s) + \sigma_f(s)\rho_{df}\sigma_{P_d}(s)\right]\ ds,
\end{aligned}
$$

where the conversion of all double integrals to single integrals is achieved through a change in the order of integration. Specifically, the integration domain is the upper-left triangle of $(z, s) \in [t_0, T]^2$, which may be traversed horizontally and vertically, in either order[3]. For example,

$$
\begin{aligned}
\int_{t_0}^{T}\int_{t_0}^{z} h_d(z-s)\sigma_d(s)\ dW_d^{\beta}(s)\ dz &= \int_{t_0}^{T}\int_{s}^{T} h_d(z-s)\sigma_d(s)\ dz\ dW_d^{\beta}(s) \\
&= \int_{t_0}^{T} \sigma_d(s)\int_{s}^{T} h_d(z-s)dz\ dW_d^{\beta}(s) \\
&= \int_{t_0}^{T} \sigma_d(s)H_d(T-s)\ dW_d^{\beta}(s).
\end{aligned}
$$

Introducing the sample path $\boldsymbol{X}(t)$ with initial condition $\boldsymbol{x} = \boldsymbol{X}(t_0)$:

$$
\boldsymbol{X}(t) := [x_d(t), x_f(t), y(t)], \qquad \boldsymbol{x} := [x_d(t_0), x_f(t_0), y(t_0)],
$$

[3]The conversion to single integrals is used when applying Itô's isometry formula, as illustrated in (3.9) below.

we collect (1.21a), (1.21b), and (3.1) into deterministic and stochastic terms:

$$
\begin{aligned}
x_d(T) &= \mu_d(t_0,T;\boldsymbol{x}) + \int_{t_0}^{T} h_d(T-s)\sigma_d(s)\; dW_d^{\beta}(s) \\
&=: \mu_d(t_0,T;\boldsymbol{x}) + \boldsymbol{\sigma}_d(t_0,T) && \text{(3.2a)} \\
x_f(T) &= \mu_f(t_0,T;\boldsymbol{x}) + \int_{t_0}^{T} h_f(T-s)\sigma_f(s)\; dW_f^{\beta}(s) \\
&=: \mu_f(t_0,T;\boldsymbol{x}) + \boldsymbol{\sigma}_f(t_0,T) && \text{(3.2b)} \\
y(T) &= \mu_y(t_0,T;\boldsymbol{x}) + \int_{t_0}^{T} \sigma_y(s)\; dW_y^{\beta}(s) \\
&+ \int_{t_0}^{T} H_d(T-s)\sigma_d(s)\; dW_d^{\beta}(s) + \int_{t_0}^{T} H_f(T-s)\sigma_f(s)\; dW_f^{\beta}(s), \\
&=: \mu_y(t_0,T;\boldsymbol{x}) + \boldsymbol{\sigma}_y(t_0,T). && \text{(3.2c)}
\end{aligned}
$$

Further introducing the vector quantities:

$$
\begin{aligned}
\boldsymbol{\mu}_M^{Q_\beta^d}(t_0,T;\boldsymbol{x}) &:= [\mu_d(t_0,T;\boldsymbol{x}), \mu_f(t_0,T;\boldsymbol{x}), \mu_y(t_0,T;\boldsymbol{x})], && \text{(3.3)} \\
\boldsymbol{\sigma}_M^{Q_\beta^d}(t_0,T) &:= [\boldsymbol{\sigma}_d(t_0,T), \boldsymbol{\sigma}_f(t_0,T), \boldsymbol{\sigma}_y(t_0,T)], && \text{(3.4)}
\end{aligned}
$$

we may express the sample path $\boldsymbol{X}(T)$ more succinctly in terms of its drift and volatility terms:

$$
\boldsymbol{X}(T) = \boldsymbol{\mu}_M^{Q_\beta^d}(t_0,T;\boldsymbol{x}) + \boldsymbol{\sigma}_M(t_0,T), \qquad \boldsymbol{X}(t_0) = \boldsymbol{x}. \tag{3.5}
$$

Finally, using (3.5) we can describe the Gaussian distributions determined by (1.18), according to which the Monte Carlo paths are generated:

$$
[\boldsymbol{X}(t)|\boldsymbol{X}(t_0) = \boldsymbol{x}] \quad \sim \quad \mathcal{N}\left(\boldsymbol{\mu}_M^{Q_\beta^d}(t_0,T;\boldsymbol{x}), \boldsymbol{\Sigma}_M^{Q_\beta^d}(t_0,t)\right) \tag{3.6}
$$

where

$$
\begin{aligned}
\boldsymbol{\mu}_M^{Q_\beta^d}(t_0,T;\boldsymbol{x}) &:= [\mu_d(t_0,T;\boldsymbol{x}), \mu_f(t_0,T;\boldsymbol{x}), \mu_y(t_0,T;\boldsymbol{x})], && \text{(3.7)} \\
\boldsymbol{\Sigma}_M^{Q_\beta^d}(t_0,t) &:= \mathbb{E}_{t_0}^{Q_\beta^d}\left[\boldsymbol{\sigma}_M(t_0,T) \wedge \boldsymbol{\sigma}_M(t_0,T)\right]. && \text{(3.8)}
\end{aligned}
$$

Note that entries of the covariance matrix $\boldsymbol{\Sigma}_M(t_0,t)$ are easily computed using Itô's isometry formula:

$$
\mathbb{E}_{t_0}^{Q}\left[\int_{t_0}^{T} g_1(s)\; dW_1(s) \cdot \int_{t_0}^{T} g_1(s)\; dW_2(s)\right] = \int_{t_0}^{T} g_1(s)\rho_{12}(s)g_2(s)\; ds;
$$

where $W_1(s), W_2(s)$ are correlated Brownian motions in a common measure Q, with correlation

$$
\rho_{12}(s) := \mathbb{E}^{Q}\left[dW_1(s) \cdot dW_2(s)\right].
$$

For example, the lower right entry of $\boldsymbol{\Sigma}_M(t_0, t)$ is determined from (3.2c) as:

$$\begin{aligned}
\mathbb{E}_{t_0}^{Q_\beta^d}\left[\boldsymbol{\sigma}_y(t_0,T)\cdot\boldsymbol{\sigma}_y(t_0,T)\right] &= \int_{t_0}^T \sigma_y^2(s)\ ds + \int_{t_0}^T H_d^2(T-s)\sigma_d^2(s)\ ds + \int_{t_0}^T H_f^2(T-s)\sigma_f^2(s)\ ds \\
&+ \int_{t_0}^T \sigma_y(s)\rho_{yd}\sigma_d(s)H_d(T-s)\ ds + \int_{t_0}^T \sigma_y(s)\rho_{yf}\sigma_f(s)H_f(T-s)\ ds \\
&+ \int_{t_0}^T H_d(T-s)\sigma_d(s)\rho_{df}\sigma_f(s)H_f(T-s)\ ds. \qquad (3.9)
\end{aligned}$$

Moment Matching

Performing moment matching on Monte Carlo paths in a cross currency model is essentially the same as in the single currency case. Our goal is to adjust samples so that discount factors implied from Monte Carlo paths match those observed from the market. In other words, for each sampled path

$$\left\{\boldsymbol{x}(t_m,\omega_n), m=0,1,\ldots,M, n=1,2,\ldots,N\right\},$$

we would adjust the states according to:

$$\boldsymbol{x}(t_m,\omega_n) \longrightarrow \hat{\boldsymbol{x}}(t_m,\omega_n) = \boldsymbol{x}(t_m,\omega_n) + \boldsymbol{\delta}(t_m) \qquad (3.10)$$

where $\boldsymbol{\delta}(t_m) = [\delta_d, \delta_f, \delta_y](t_m)$ and $\boldsymbol{x} = [x_d, x_f, y]$ as above.

For domestic states, δ_d is determined in the same way as in the single currency case. Thus, from the theoretical price of a zero coupon bond in the domestic currency, we impose the following condition on δ_d:

$$\begin{aligned}
\mathbb{P}_d\left(0,t_m\right) &= \left\langle \text{adjusted } \mathcal{N}_{Q_\beta^d}(t_m)^{-1}\right\rangle \\
&\overset{\text{off-grid}}{=} \left\langle \mathbb{P}_d\left(t_m,\beta(t_m);x_d(t_m,\omega)+\delta_d(t_m)\right)^{-1}\prod_{j=1}^{\gamma(t_m)}\mathbb{P}_d\left(T_{j-1},T_j;x_d(T_{j-1},\omega)\right)\right\rangle \\
&= \left\langle \mathbb{P}_d\left(t_m,\beta(t_m);x_d(t_m,\omega)+\delta_d(t_m)\right)^{-1} P_d(\gamma(t_m);\omega)\right\rangle \qquad (3.11) \\
\mathbb{P}_d\left(0,\beta(t_m)\right) &\overset{\text{on-grid}}{=} \left\langle \mathbb{P}_d\left(t_m,\beta(t_m);x_d(T_{j-1},\omega)+\delta_d(t_m)\right)\prod_{j=1}^{\gamma(t_m)-1}\mathbb{P}_d\left(T_{j-1},T_j;x_d(T_{j-1},\omega)\right)\right\rangle, \\
&= \left\langle \mathbb{P}_d\left(t_m,\beta(t_m);x_d(t_m,\omega)+\delta_d(t_m)\right)^{-1} P_d(\gamma(t_m)-1;\omega)\right\rangle \qquad (3.12)
\end{aligned}$$

where the notation

$$\left\langle X(t,\omega)\right\rangle := \frac{1}{N}\sum_{n=1}^N X(t,\omega_n)$$

denotes the sample mean over all paths ω_n, and pathwise adjustments $\hat{x}_d(t)$ for $t < t_m$ have already been made to terms in the product-of-discounts:

$$\prod_{j=1}^{\gamma(t_m)} \mathbb{P}_d\left(T_{j-1}, T_j; x_d(T_{j-1}, \omega)\right) \quad =: \quad P_d(\gamma(t_m); \omega).$$

Since replacing x_d with $x_d + \delta_d$ introduces a single additional term $-H_d(\beta(t_m) - t_m)\delta_d$ into the exponent appearing on the right hand side of (1.14), we obtain the path-independent adjustment to each pathwise zero coupon bond:

$$\mathbb{P}_d\left(t_m, \beta(t_m); x_d(t_m, \omega) + \delta_d(t_m)\right) = e^{-H_d(\beta(t_m)-t_m)\delta_d} \cdot \mathbb{P}_d\left(t_m, \beta(t_m); x_d(t_m, \omega)\right).$$

The foregoing allows us to rewrite (3.11) as

$$\mathbb{P}_d\left(0, t_m\right) \quad = \quad e^{H_d(\beta(t_m)-t_m)\delta_d}\left\langle \mathbb{P}_d\left(t_m, \beta(t_m); x_d(t_m, \omega)\right)^{-1} \mathbb{P}_d\left(0, \beta(t_m); x_d(0, \omega)\right)\right\rangle,$$

which implies δ_d is uniquely determined as:

$$\begin{aligned} \delta_d(t_m) \quad &\overset{\text{off-grid}}{=} \quad \frac{-1}{H_d(\beta(t_m) - t_m)} \ln\left[\frac{1}{\mathbb{P}_d\left(0, t_m\right)}\left\langle \frac{P_d(\gamma(t_m); \omega)}{\mathbb{P}_d\left(t_m, \beta(t_m); x_d(t_m, \omega)\right)}\right\rangle\right] \\ &\overset{\text{on-grid}}{=} \quad \frac{+1}{H_d(\beta(t_m) - t_m)} \ln\left[\frac{1}{\mathbb{P}_d\left(0, t_m\right)}\left\langle \mathbb{P}_d\left(T_{\gamma(t_m)-1}, \beta(t_m); x_d(T_{j-1}, \omega) + \delta_d(t_m)\right) P_d(\gamma(t_m) - 1; \omega)\right\rangle\right]. \end{aligned} \tag{3.13}$$

For FX states, we want the price a foreign zero coupon bond (implied from the FX forwards curve and the domestic discount curve) to match its theoretical value. Applying the fundamental theorem to pricing a foreign zero coupon bond paid in domestic currency, namely

$$U(t) := \mathbb{P}_f\left(t, t_m\right) Z(t) = \mathbb{P}_f\left(t, t_m\right) Z(0) e^{Y(t)},$$

and using the Spot Libor numeraire, $\mathcal{N}_{Q_\beta^d}(t)$ gives:

$$\begin{aligned} \mathbb{P}_f\left(0, t_m\right) Z(0) \quad &= \quad \mathcal{N}_{Q_\beta^d}(0)\left\langle \frac{\mathbb{P}_f\left(t_m, t_m\right) Z(0) e^{[Y(t_m,\omega)+\delta_y(t_m)]}}{\mathcal{N}_{Q_\beta^d}(t_m, \omega)}\right\rangle \\ &= \quad Z(0) e^{\delta_y(t_m)}\left\langle e^{Y(t_m,\omega)} \mathbb{P}_d\left(t_m, \beta(t_m); \hat{x}_d(t_m, \omega)\right)^{-1} P_d(\gamma(t_m); \omega)\right\rangle. \end{aligned}$$

As we assume to have already adjusted $x_d(t_m, \omega_n) \to x_d(t_m, \omega_n) + \delta_d(t_m)$, the solution is

$$\delta_y(t_m) \quad = \quad -\ln\left[\frac{1}{\mathbb{P}_f\left(0, \beta(t_m)\right)}\left\langle e^{Y(t_m,\omega_n)} \frac{P_d(\gamma(t_m); \omega)}{\mathbb{P}_d\left(t_m, T_{\beta(t_m)}; x_d(t_m, \omega)\right)}\right\rangle\right]. \tag{3.14}$$

We currently do not employ a similar constraint on the foreign interest rate state, though we may utilize a moment matching approach in future implementations.

4 Numerical Tests and Results

When pricing all products, we use the Spot Libor Numeraire and employ the moment matching algorithm. The results are compiled into the tables below. Note that, as mentioned above, moment matching on the foreign curve under the Spot Libor domestic measure has not been implemented.

4.1 Convergence Results for Monte Carlo Pricing on Benchmark Products

We begin with a programmatically generated market using the following curve data:

$$\begin{aligned}\text{Domestic (USD) interest rate curve: } &\theta_d \sim \text{ constant rate of } 0.012\\ \text{Foreign (EUR) interest rate curve: } &\theta_f \sim \text{ constant rate of } 0.023\\ \text{FX rate (EURUSD): } &Z(0) = 1.1022\end{aligned}$$

Furthermore, we mandate the following HW1-BS-CC model parameters:

$$\begin{aligned}\text{Domestic Mean Reversion: } &\kappa_d = 0.03\\ \text{Foreign Mean Reversion: } &\kappa_f = 0.025\\ \text{Correlation Matrix : } &\rho_{df} = -0.051, \rho_{dy} = 0.007, \rho_{fy} = 0.017\end{aligned}$$

And finally, we specify the remaining HW1-BS-CC model parameters using the following data:

Domestic Hull-White Model:

$$\begin{aligned}\sigma_d(t) =&[0.01, 0.012, 0.02, 0.011, 0.013, 0.02]\\ t =&[0.001368925, 0.002737851, 0.004106776, 0.008213552, 0.013689254, 0.041067762]\end{aligned}$$

Foreign Hull-White Model:

$$\begin{aligned}\sigma_f(t) =&[0.0123, 0.0121, 0.0195, 0.0105, 0.013, 0.0115]\\ t =&[0.001368925, 0.002737851, 0.004106776, 0.008213552, 0.013689254, 0.041067762]\end{aligned}$$

FX Black-Scholes model:

$$\sigma_y(t) = 0.3$$

NPATHS	MV	STD
4000	1.351775	0.005435
8000	1.351775	0.003768
16000	1.351775	0.002672
32000	1.351775	0.001874
64000	1.351775	0.001320
128000	1.351775	0.000936
Exact	1.351775	

(a) Domestic Fixed Coupon Bond

NPATHS	MV	STD
4000	1.361725	0.020356
8000	1.361725	0.015298
16000	1.361725	0.010874
32000	1.361725	0.007520
64000	1.361725	0.005330
128000	1.361725	0.003732
Exact	1.361725	

(b) Foreign Fixed Coupon Bond

NPATHS	MV	STD
4000	1.046348	0.000139
8000	1.046336	0.000098
16000	1.046340	0.000069
32000	1.046336	0.000049
64000	1.046336	0.000035
128000	1.046337	0.000024
Exact	1.046341	

(c) Domestic Float Coupon Bond

NPATHS	MV	STD
4000	1.151352	0.019154
8000	1.148848	0.014044
16000	1.149084	0.009321
32000	1.148860	0.006704
64000	1.148425	0.004752
128000	1.149685	0.003314
Exact	1.150518	

(d) Foreign Float Coupon Bond

NPATHS	MV	STD
4000	-0.105326	0.015968
8000	-0.103779	0.011429
16000	-0.103266	0.007847
32000	-0.103069	0.005665
64000	-0.102989	0.00400
128000	-0.103384	0.002794
Exact	-0.104059	

(e) Cross-Currency Float-Float Swap

NPATHS	MV	STD
4000	0.301947	0.006021
8000	0.302869	0.004261
16000	0.303806	0.003015
32000	0.303521	0.002122
64000	0.303469	0.001499
128000	0.303481	0.001063

(f) Domestic European Swaption

NPATHS	MV	STD
4000	0.083612	0.009452
8000	0.088641	0.00652
16000	0.091422	0.004253
32000	0.090701	0.00306
64000	0.092321	0.002273
128000	0.088457	0.001563

(g) Cross-Currency European Swaption

NPATHS	MV	STD
4000	0.835027	0.012121
8000	0.835695	0.008302
16000	0.835552	0.005978
32000	0.836029	0.004118
64000	0.835551	0.002925
128000	0.835838	0.002102

(h) PRDC

Table 4.1: Convergence results for different products using moment-matching algorithm.

4.2 Regression Variables and Monte Carlo Pricing of Bermudan PRDC

We begin with a market using the following curve data taken from February 25, 2016:

$$\begin{aligned}\text{Domestic (JPY) interest rate curve: } & \theta_d \sim \text{JPY-OIS}\\ \text{Foreign (USD) interest rate curve: } & \theta_f(Z, \theta_d) \sim \text{implied from (USDJPY, JPY-OIS)}\end{aligned}$$

Furthermore, we mandate the following HW1-BS-CC model parameters:

$$\begin{aligned}\text{Domestic Mean Reversion: } & \kappa_d = 0.03\\ \text{Foreign Mean Reversion: } & \kappa_f = 0.03\\ \text{Correlation Matrix : } & \rho_{df} = 0.2, \rho_{dy} = 0.3, \rho_{fy} = 0.1\end{aligned}$$

And finally, we calibrate the remaining HW1-BS-CC model parameters using vol-cube swaption data and curve data from February 25, 2016, and specifying the following calibration curves:

Domestic Hull-White Model:

$$\begin{aligned}\text{Forward Curve} =& \text{JP0006M}\\ \text{Discount Curve} =& \text{JPY-OIS}\\ t =& [0.09034, 0.24640, 0.49828, 0.75017, 1.00752, 2.00410, 3.00068, 4.0, 5.0020, 105.002]\\ \sigma_d(t) =& [0.0054, 0.00517, 0.00500, 0.00448, 0.00470, 0.00466, 0.00531, 0.00000, 0.00843, 0.00630]\end{aligned}$$

Foreign Hull-White Model:

$$\begin{aligned}\text{Forward Curve} =& \text{US0003M}\\ \text{Discount Curve} =& \text{USD-OIS}\\ t =& [0.09034, 0.24640, 0.49828, 0.75017, 1.00752, 2.00410, 3.00068, 4.0000, 5.0020, 105.002]\\ \sigma_f(t) =& [0.00963, 0.00964, 0.00984, 0.00799, 0.00955, 0.00978, 0.01079, 0.01110, 0.01149, 0.01103]\end{aligned}$$

FX Black-Scholes model:

$$\begin{aligned}\text{FX Forward Curve} =& \text{USDJPY FX-Forward}\\ \sigma_y(t) =& [0.12086, 0.13797, 0.14328, 0.14660, 0.113806, 0.113806, 0.12157, 0.11420, 0.11090, 0.10555,\\ & 0.11588, 0.11588, 0.114337, 0.113478, 0.11347, 0.11170, 0.11170, 0.11170, 0.11170]\end{aligned}$$

We display in Table 4.2 results for pricing a particular Bermudan PRDC product. See [Blo] for more details regarding regression variables in DLIB.

NPATHS	MEAN	STDDEV
4000	1192917	6214
8000	1199259	4409
16000	1195409	3109
20000	1196233	2776
32000	1197456	2211
64000	1195997	1555
128000	1197182	1101

(a) Convergence results

DATE	Survival Probability (%)
Mar 24, 2022	55
Mar 24, 2023	51
Mar 21, 2024	48
Mar 24, 2025	47
Mar 24, 2026	46
Mar 19, 2027	45
Mar 24, 2028	44
Mar 22, 2029	43
Mar 22, 2030	43
Mar 24, 2031	42
Mar 22, 2032	40
Mar 24, 2033	39
Mar 24, 2034	37
Mar 20, 2035	35

(b) Option Survival Probability.

Table 4.2: Convergence results for a Bermudan PRDC using the three underlyings (USDJPY, US0003M, JPY0006M) as regression variables, and also resulting survival probabilities.

References

[Blo] Bloomberg. Overview of American Monte Carlo Pricing in DLIB. DOCS #2076964<GO>.

[BM] D. Brigo and F. Mercurio. *Interest Rate Models - Theory and Practice: With Smile, Inflation, and Credit.* Springer Verlag, Berlin, 2006.