Two-Factor Hull-White Model for Interest Rate Derivative Products in Bloomberg

May 1, 2015

Quantitative Analytics
Bloomberg L.P

Abstract

This document describes the detailed implementation of the Two-Factor Hull-White model for path-dependent interest rate derivatives

Keywords. Hull-White, Two-Factor, Monte Carlo, Dual Curve.

Contents

1 Two-factor Hull-White model

1.1 Model dynamics

1.1.1 Risk neutral measure

Under the risk neutral measure Q, numeraire is given by the money market account, i.e.

$$\mathcal{N}_{rn}(t, \mathcal{F}_t) := \exp\Big[\int_0^t r(s)ds\Big]. \tag{1}$$

The two-factor Hull-White model for short rate $r(t)$ is

$$r(t) = \theta(t) + \mathbf{1}^\top \boldsymbol{X}(t), \tag{2}$$

where $\theta(t)$ is a deterministic function to match the initial yield curve. $\boldsymbol{X}(t)$ is a two dimensional Ornstein-Uhlenbeck process,

$$\mathrm{d}\boldsymbol{X}(t) = -\boldsymbol{\kappa}\boldsymbol{X}(t)\mathrm{d}t + \boldsymbol{\sigma}(t)\mathrm{d}\boldsymbol{W}^Q(t), \qquad \boldsymbol{X}(0) = \mathbf{0}, \tag{3}$$

$$\Big(\mathrm{d}\boldsymbol{W}^Q(t)\Big)\Big(\mathrm{d}\boldsymbol{W}^Q(t)\Big)^\top = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \mathrm{d}t := \boldsymbol{\rho}\,\mathrm{d}t. \tag{4}$$

with parameters $\boldsymbol{\kappa}$, $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\rho}$.

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}, \tag{5}$$

$$\boldsymbol{\sigma}(t) = \begin{bmatrix} \sigma_1(t) & 0 \\ 0 & \sigma_2(t) \end{bmatrix}. \tag{6}$$

Constants κ_1 and κ_2 are speeds of mean-reversion for the first and second factor, respectively. We assume that the first factor captures long term dynamics, while the second factor captures short term dynamics. Therefore, $0 \leq \kappa_1 < \kappa_2$. $\sigma_u(t) \geq 0$ $(u = 1, 2)$ are volatility related parameters obtained by calibrating to the market volatilities. ρ is the correlation between the two factors.

The solution to the SDE specified in Eqs. 3 and Eqs. 4 is

$$\boldsymbol{X}(t) = \boldsymbol{h}(t - t_0)\boldsymbol{x} + \int_{t_0}^t \boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^Q(s), \qquad \text{for } \boldsymbol{X}(t_0) = \boldsymbol{x} \tag{7}$$

$$= \int_0^t \boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^Q(s) \qquad \text{for } \boldsymbol{X}(0) = \mathbf{0}, \tag{8}$$

where

$$\boldsymbol{h}(t) := e^{-\boldsymbol{\kappa} t} = \begin{bmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{bmatrix}, \tag{9}$$

$$h_u(t) := h(t; \kappa_u) := e^{-\kappa_u t}.$$

From Eqs. 7,

$$\Big[\boldsymbol{X}(t)|\boldsymbol{X}(t_0)=\boldsymbol{x}\Big]\sim\mathcal{N}\Big(\boldsymbol{\mu}^Q(t_0,t;\boldsymbol{x}),\boldsymbol{\Sigma}(t_0,t)\Big),$$

where

$$\boldsymbol{\mu}^Q(t_0,t;\boldsymbol{x})=\boldsymbol{h}(t-t_0)\boldsymbol{x} \tag{10}$$

$$\boldsymbol{\Sigma}(t_0,t)=\boldsymbol{\nu}(t)-\boldsymbol{h}(t-t_0)\boldsymbol{\nu}(t_0)\boldsymbol{h}(t-t_0). \tag{11}$$

where, $\boldsymbol{\nu}(t)$ is the co-variance of $\boldsymbol{X}(t)$, i.e.

$$\boldsymbol{\nu}(t):=\mathrm{Var}\big[\boldsymbol{X}(t)\boldsymbol{X}^\top(t)\big]=\int_0^t\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\boldsymbol{\rho}\boldsymbol{\sigma}(s)\boldsymbol{h}(t-s)\mathrm{d}s=\begin{bmatrix}\nu_{11}(t) & \nu_{12}(t)\\ \nu_{21}(t) & \nu_{22}(t)\end{bmatrix}, \tag{12}$$

$$\nu_{uv}(t)=\rho_{uv}\int_0^t\sigma_u(s)\sigma_v(s)e^{-(\kappa_u+\kappa_v)(t-s)}\mathrm{d}s. \tag{13}$$

1.1.2 Terminal forward measure

Under terminal forward measure Q_T, numeraire is zero coupon bond with maturity T, i.e.

$$\mathcal{N}_{Q_T}(t,\mathcal{F}_t):=\mathbb{P}\left(t,T\right). \tag{14}$$

Therefore, Eqs. 3 becomes

$$\mathrm{d}\boldsymbol{X}(t)=\Big(-\boldsymbol{\kappa}\boldsymbol{X}(t)-\boldsymbol{\sigma}(t)\boldsymbol{\rho}\boldsymbol{\sigma}(t)\boldsymbol{H}(T-t)\mathbf{1}\Big)\mathrm{d}t+\boldsymbol{\sigma}(t)\mathrm{d}\boldsymbol{W}^{Q_T}(t), \tag{15}$$

where,

$$\boldsymbol{H}(t):=\int_0^t\boldsymbol{h}(s)\mathrm{d}s=\begin{bmatrix}H_1(t) & 0\\ 0 & H_2(t)\end{bmatrix}. \tag{16}$$

The solution is,

$$\boldsymbol{X}(t)=-\boldsymbol{\gamma}^{Q_T}(t_0,t)+\boldsymbol{h}(t-t_0)\boldsymbol{x}+\int_{t_0}^t\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^{Q_T}(s),\qquad\text{for }\boldsymbol{X}(t_0)=\boldsymbol{x} \tag{17}$$

$$=-\boldsymbol{\gamma}^{Q_T}(0,t)+\int_0^t\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^{Q_T}(s)\qquad\text{for }\boldsymbol{X}(0)=\mathbf{0} \tag{18}$$

where,

$$\boldsymbol{\gamma}^{Q_T}(t_0,t):=\Big[\int_{t_0}^t\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\boldsymbol{\rho}\boldsymbol{\sigma}(s)\boldsymbol{H}(T-s)\,\mathrm{d}s\Big]\mathbf{1}. \tag{19}$$

The drift $\boldsymbol{\gamma}^{Q_T}(t_0,t)$ can be computed as

$$\boldsymbol{\gamma}^{Q_T}(0,t)=\Big[\boldsymbol{\nu}^h(t)+\boldsymbol{\nu}(t)\boldsymbol{H}(T-t)\Big]\mathbf{1}\qquad\text{for }0\le t\le T \tag{20}$$

$$\boldsymbol{\gamma}^{Q_T}(t_0,t)=\boldsymbol{\gamma}^{Q_T}(0,t)-\boldsymbol{h}(t-t_0)\boldsymbol{\gamma}^{Q_T}(0,t_0)\qquad\text{for }0\le t_0\le t\le T. \tag{21}$$

From Eqs. 17,

$$\Big[\boldsymbol{X}(t)|\boldsymbol{X}(t_0)=\boldsymbol{x}\Big] \sim \mathcal{N}\Big(\boldsymbol{\mu}^{Q_T}(t_0,t;\boldsymbol{x}),\boldsymbol{\Sigma}(t_0,t)\Big),$$

where

$$\boldsymbol{\mu}^{Q_T}(t_0,t;\boldsymbol{x}) = -\boldsymbol{\gamma}^{Q_T}(t_0,t)+\boldsymbol{h}(t-t_0)\boldsymbol{x} \tag{22}$$

and $\boldsymbol{\Sigma}(t_0,t)$ is the same as that in risk neutral measure in Eqs. 11.

1.1.3 Spot-Libor measure

Under spot-Libor measure Q_β, numeraire is given by the discretely rebalanced bank account (see [Brigo & Mercurio], Section 6.3)

$$\mathcal{N}_\beta(t,\mathcal{F}_t) := \frac{\mathbb{P}\left(t,T_{\beta(t)}\right)}{\prod_{j=1}^{\beta(t)}\mathbb{P}\left(T_{j-1},T_j\right)} \tag{23}$$

where $0=T_0<T_1<\cdots$ are fixed as our discrete tenor structure, $\beta(t)$ is the smallest index such that $t\le T_{\beta(t)}$. From Eqs. 14 and Eqs. 23, note that the stochastic part of $\mathcal{N}_\beta$ is the same as that in $\mathcal{N}_{Q_{\beta(t)}}$. Therefore, under the spot-Libor measure, the dynamics changes to

$$\mathrm{d}\boldsymbol{X}(t) = \Big(-\kappa\boldsymbol{X}(t)-\boldsymbol{\sigma}(t)\boldsymbol{\rho}\boldsymbol{\sigma}(t)\boldsymbol{H}(T_{\beta(t)}-t)\mathbf{1}\Big)dt+\boldsymbol{\sigma}(t)\mathrm{d}\boldsymbol{W}^{Q_\beta}(t). \tag{24}$$

Integrating Eqs. 24, we obtain

$$\boldsymbol{X}(t) = -\boldsymbol{\gamma}^{Q_\beta}(t_0,t)+\boldsymbol{h}(t-t_0)\boldsymbol{x}+\int_{t_0}^{t}\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^{Q_\beta}(s), \qquad \text{for } \boldsymbol{X}(t_0)=\boldsymbol{x} \tag{25}$$

$$= -\boldsymbol{\gamma}^{Q_\beta}(0,t)+\int_{0}^{t}\boldsymbol{h}(t-s)\boldsymbol{\sigma}(s)\mathrm{d}\boldsymbol{W}^{Q_\beta}(s) \qquad \text{for } \boldsymbol{X}(0)=\mathbf{0} \tag{26}$$

where $\boldsymbol{\gamma}^{Q_\beta}(t_0,t)$ can be calculated stepwise. When $\beta(s)=\beta(t)$ for $t_0<s<t$, we have $\boldsymbol{\gamma}^{Q_\beta}(t_0,t)=\boldsymbol{\gamma}^{Q_{T_{\beta(t)}}}(t_0,t)$, which is defined in Eqs. 19.

Therefore, the conditional distribution is

$$\Big[\boldsymbol{X}(t)|\boldsymbol{X}(t_0)=\boldsymbol{x}\Big] \sim \mathcal{N}\Big(\boldsymbol{\mu}^{Q_\beta}(t_0,t;\boldsymbol{x}),\boldsymbol{\Sigma}(t_0,t)\Big),$$

where

$$\boldsymbol{\mu}^{Q_\beta}(t_0,t;\boldsymbol{x}) = -\boldsymbol{\gamma}^{Q_\beta}(t_0,t)+\boldsymbol{h}(t-t_0)\boldsymbol{x}. \tag{27}$$

References

[Brigo & Mercurio] D. Brigo and F. Mercurio, Interest Rate Models: Theory and Practice, Springer, 2001.

[Glasserman2004] Paul Glasserman, Monte Carlo Methods in Financial Engineering, Springer, 2004.

[Henrard2007] M. Henrard, The Irony in the Derivatives Discounting, Wilmott Magazine, July 2007, 92-98.