Algorithmic Trading
: Optimal Execution

KAIST College of Business

Wooyeon Kim, Ph.D.
Meritz Securities Co. Ltd.

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Optimal Execution

<u>Question</u> : How do we trade in or out of a large position?

- Liquidation or acquisition
- Price impact (temporary and/or permanent)
- Inventory control (For example, a fund should buy a prespecified amount of a target stock within trading hours of a specific trading day)

In this lecture, the following topics are covered:

- Optimal **liquidation** with **temporary** price impact
- Optimal **acquisition** with **temporary** price impact
- Optimal **liquidation** with **temporary** and **permanent** price impact

Model Setup - Variables

Control variable

- $v = (v_t)_{\{0 \le t < T\}}$ is the trading rate, the speed at which the agent is liquidating or acquiring shares (using MOs only)

State variables

- $Q^v = (Q_t^v)_{\{0 \le t \le T\}}$ is the agent's inventory, which is clearly affected by how fast she trades
- $S^v = (S_t^v)_{\{0 \le t \le T\}}$ is the midprice process, and is also affected by the speed of the agent's trading
- $\hat{S}^v = \left(\hat{S}_t^v\right)_{\{0 \le t \le T\}}$ is the execution price, at which the agent can buy or sell the asset by walking the LOB
- $X^v = (X_t^v)_{\{0 \le t \le T\}}$ is the agent's cash process resulting from the agent's trading plan

Model Setup - Dynamics

Inventory

- Liquidation : $dQ_t^v = -v_t dt, \quad Q_0^v = q$
- Acquisition : $dQ_t^v = v_t dt, \quad Q_0^v = 0$

Midprice

- Liquidation : $dS_t^v = -g(v_t)dt + \sigma dW_t, \quad S_0^v = S$
- Acquisition : $dS_t^v = g(v_t)dt + \sigma dW_t, \quad S_0^v = S$
- Function g:R+ -> R+ denotes the **permanent price impact** that the agent's trading action incrus on the midprice

Execution price

- Liquidation : $\hat{S}_t^v = S_t^v - \left(\frac{1}{2}\Delta + f(v_t)\right), \quad \hat{S}^v = \hat{S}$
- Acquisition : $\hat{S}_t^v = S_t^v + \left(\frac{1}{2}\Delta + f(v_t)\right), \quad \hat{S}^v = \hat{S}$
- Function f:R+ -> R+ denotes the **temporary price impact** that the agent's trading action has on the price they can execute the trade at
- $\Delta \geq 0$ denotes the bid-ask spread, assumed to be constant

Price Impact

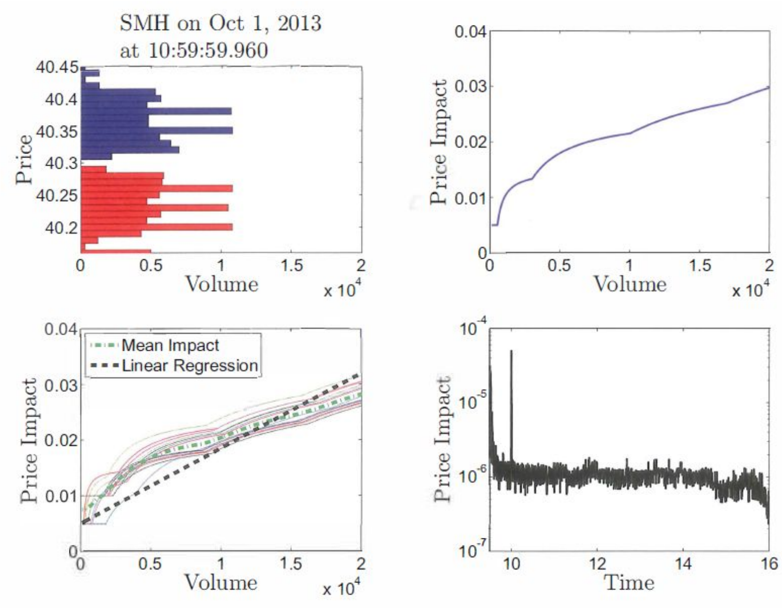

Figure 2-1. An illustration of how the temporary impact may be estimated from snapshots of the LOB using SMH on Oct 1, 2013

- A snapshot of the LOB for SMH on Oct 1, 2013 at 11 am
- Temporary price impact per share that an MO of various volumes would fase as it walks through the buy side of the LOB
- Price impact every second from 11:00 to 11:01 with the average of those curves (dash-dotted line)
- Slope of the linear regression model throughout the entire day

Optimal Liquidation with Temporary Impact

Liquidation

Assume $f(v) = kv, \quad g(v) = 0$; that is, only temporary price impact resulting from walking the LOB

Our objective is to liquidate N shares by time T with profit maximization

<u>Problem</u> : $\max_{v \in A} E_0\left[\int_{u=0}^{T}(S_u^v - kv_u)v_u du\right] \quad s.t. \quad Q_0^v = N$

The **value function** : $H(t, S, q) = \max_{v \in A} E_t\left[\int_{u=t}^{T}(S_u - kv_u)v_u du\right]$

By the **DPP (Dynamic Programming Principle)**, the value function should satisfy the following **HJB equation (or DPE, Dynamic Programming Equation)** :

- $\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \max_v((S - kv)v - v\partial_q H) = 0$
- The agent must liquidate all the inventory by time T, impose $H(t, S, q) \to -\infty \quad as \quad t \to T \quad for \quad q > 0$

By the **FOC (First Order Condition)** :

- $v^\star = \frac{1}{2k}(S - \partial_q H)$

Finally, we have the following **PDE for the value function** :

- $\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \frac{1}{4k}(S - \partial_q H)^2 = 0$

Liquidation (cont'd)

<u>Trial solution</u> : $H(t,S,q) = qS + h(t,q)$; the book value of current inventory plus the excess value due to optimally liquidating the remaining shares

The PDE is simplified to $\partial_t h + \frac{1}{4k}(\partial_q h)^2 = 0$

Again, make ansatz $h(t,q) = q^2 h_2(t)$; the excess value represented in a quadratic term in q

Thus $\partial_t h_2 + \frac{1}{k} h_2^2 = 0 \implies k\frac{dh_2}{dt} = -h_2^2$: **solvable!!!**

By integrating between t and T, we obtain $h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{k}(T-t)\right)^{-1}$

Liquidation (cont'd)

The FOC can be reduced to $v_t^\star = -\frac{1}{k} h_2(t) Q_t^{v^\star}$

By the dynamics of inventory ($dQ_t^{v^\star} = -v_t^\star dt$), we obtain $\frac{dQ_t^{v^\star}}{Q_t^{v^\star}} = \frac{h_2(t)}{k} dt$: **solvable!!!**

By integrating between t and T, $Q_t^{v^\star} = \frac{(T-t)-k/h_2(T)}{T-k/h_2(T)} N$

The condition of the zero terminal inventory is equivalent to requiring $h_2(t) \to -\infty \;\; as \;\; t \to T$

To satisfy the terminal inventory condition,

- $h_2(t) = -\frac{k}{T-t}$
- $Q_t^{v^\star} = \left(1 - \frac{t}{T}\right) N$
- $v_t^\star = \frac{N}{T}$

The result is **TWAP**!!! (Time-Weighted Average Price)

Optimal Acquisition with Temporary Impact

Acquisition

Assume $f(v) = kv, \quad g(v) = 0$; that is, only temporary price impact resulting from walking the LOB

Our objective is to acquire *N* shares by time *T* with cost minimization

The agent's expected costs : $EC^v = E\left[\int_{u=t}^{T}(S_u^v + kv_u)v_u du + (N - Q_T^v)S_T + \alpha(N - Q_T^v)^2\right]$

- Terminal cash
- Terminal execution at mid (closing price)
- Terminal penalty ~ quadratic in the remaining shares

To simplify notation, let $Y_t^v = N - Q_t^v$; the shares remaining to be purchased

Problem : $\min_{v \in A} E_0\left[\int_{u=0}^{T}(S_u^v + kv_u)v_u du + Y_T^v S_T + \alpha(Y_T^v)^2\right] \quad s.t. \quad Y_0^v = N$

The value function : $H(t, S, y) = \min_{v \in A} E_t\left[\int_{u=t}^{T}(S_u^v + kv_u)v_u du + Y_T^u S_T + \alpha(Y_T^v)^2\right]$

Acquisition (cont'd)

Applying the DPP, the DPE is $0 = \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H + \min_{v \in A}\{(S + kv)v - v\partial_y H\}$

At the terminal time T, $H(T, S, y) = yS + \alpha y^2$

By the FOC, optimal trading rate : $v^\star = \frac{1}{2k}(\partial_y H - S)$

And upon substitution into the DPE above, the resulting PDE is $\partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \frac{1}{4k}(\partial_y H - S)^2 = 0$

<u>Trial solution</u> : $H(t, S, y) = yS + h_0(t) + h_1(t)y + h_2(t)y^2$

- At time T, $h_0(T) = 0,\ h_1(T) = 0,\ h_2(T) = \alpha$

Upon substituting the ansatz into the above PDE, we obtain

- $0 = \left\{\partial_t h_2 - \frac{1}{k}h_2^2\right\}y^2 + \left\{\partial_t h_1 - \frac{1}{k}h_2 h_1\right\}y + \left\{\partial_t h_0 - \frac{1}{4k}h_1^2\right\}$

Acquisition (cont'd)

$$0 = \{\partial_t h_2 - \frac{1}{k}h_2^2\}y^2 + \{\partial_t h_1 - \frac{1}{k}h_2 h_1\}y + \{\partial_t h_0 - \frac{1}{4k}h_1^2\}$$

- Equation should be valid for each y, so each term in curly braces should individually vanish
- Due to the terminal condition $h_1(T) = 0$, we get $h_1(t) = 0$ by setting the second term in braces to 0
- Similarly, due to $h_0(T) = 0$ and $h_1(t) = 0$, we get $h_0(t) = 0$ by setting the third term in braces to 0
- Setting the first term in braces to 0 allows us to get $h_2(t) = \left(\frac{1}{k}(T-t) + \frac{1}{\alpha}\right)^{-1}$

Putting this together, the optimal trading speed is $v_t^\star = \left((T-t) + \frac{k}{\alpha}\right)^{-1} Y_t^{v^\star}$

The dynamics of {Y} is $dY_t^{v^\star} = -v_t^\star dt = -\left((T-t) + \frac{k}{\alpha}\right)^{-1} Y_t^{v^\star} dt$: **solvable!!!**

By integrating between t and T, we obtain $Y_t^{v^\star} = \frac{(T-t)+k/\alpha}{T+k/\alpha} N$

Hence, $Q_t^{v^\star} = \frac{t}{T+k/\alpha} N$

Acquisition (cont'd)

Inventory path

- $Q_t^{v^\star} = \frac{t}{T+k/\alpha} N$
- For $\alpha > 0$, $k > 0$, it is always optimal to leave some shares to be executed at the terminal date, and the fraction of shares left to execute at the end decreases with the relative price impact, k/α

Trading speed

- $v_t^\star = \frac{N}{T+k/\alpha}$
- The optimal trading speed decreases with the relative price impact, k/α

Optimal Liquidation with Temporary and Permanent Price Impact

Liquidation with Permanent Price Impact

Our objective is to liquidate N shares by time T with profit maximization

The agent's performance criterion is $H^v(t,x,S,q) = E_t\left[X_T^v + Q_T^v(S_T^v - \alpha Q_T^v) - \phi\int_{u=t}^{T}(Q_u^v)^2 du\right]$

- Terminal cash
- Terminal execution; Our trade at the terminal time impacts the terminal price
- Inventory penalty

Problem : $\max_{v\in A} E_0\left[X_T^v + Q_T^v(S_T^v - \alpha Q_T^v) - \phi\int_{u=0}^{T}(Q_u^v)^2 du\right] \quad s.t. \quad Q_0^v = N$

The value function : $H(t,x,S,q) = \max_{v\in A} H^v(t,x,S,q)$

The DPP implies the following DPE : $0 = \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 + \max_{v\in A}\{(v(S - f(v))\partial_x - g(v)\partial_S - v\partial_q)H\}$

- The terminal condition : $H(T,x,S,q) = x + Sq - \alpha q^2$
- Now we assume the linear permanent price impact in addition : $g(v) = bv, \quad b \geq 0$
- As usual, the linear temporary price impact is assumed : $f(v) = kv, \quad k \geq 0$

Liquidation with Permanent Price Impact (cont'd)

By the FOC, the optimal trading speed is $v^{\star} = \frac{1}{2k}\frac{(S\partial_x - b\partial_S - \partial_q)H}{\partial_x H}$

Thus, we have the following PDE : $0 = \left(\partial_x + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 + \frac{1}{4k}\frac{[(S\partial_x - b\partial_S - \partial_q)H]^2}{\partial_x H}$

Does this PDE with the above terminal condition have the analytic solution?

- YES!! It can be solved analytically, though its procedure is extremely complicated...
- The same approach we have used works well in this problem
- Make a trial solution of reasonable functional form, and solve it

The solution looks like...

Liquidation with Permanent Price Impact (cont'd)

Solution

- The optimal trading speed : $v_t^\star = -\sqrt{\frac{\phi}{k}}\frac{1+\zeta\exp(2\gamma(T-t))}{1-\zeta\exp(2\gamma(T-t))}Q_t^{v^\star}$
- The optimal inventory process : $Q_t^{v^\star} = \frac{\zeta\exp(\gamma(T-t))-\exp(-\gamma(T-t))}{\zeta\exp(\gamma T)-\exp(-\gamma T)}N$
- Parameters : $\gamma = \sqrt{\frac{\phi}{k}}$ and $\zeta = \frac{\alpha-\frac{1}{2}b+\sqrt{k\phi}}{\alpha-\frac{1}{2}b-\sqrt{k\phi}}$

In the limit in which the quadratic liquidation penalty goes to infinity, i.e., $\alpha \to \infty$,

- $Q_t^{v^\star} \to \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)}N$
- $v_t^\star \to \gamma\frac{\cosh(\gamma(T-t))}{\sinh(\gamma T)}N$
- Both are independent of b

Liquidation with Permanent Price Impact (cont'd)

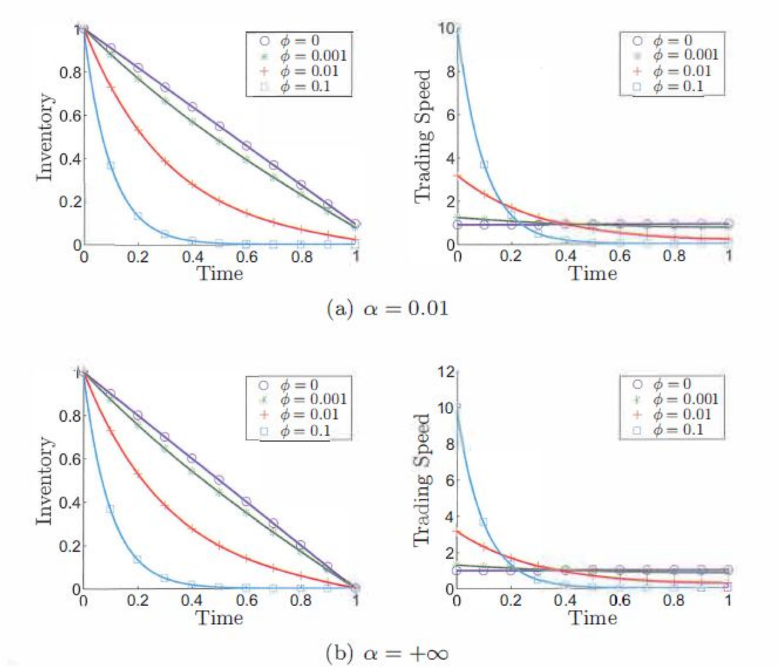

Figure 2-2. An illustration of the investor's inventory along the optimal path for various levels of the running penalty ϕ

- With no running penalty ($\phi = 0$), the strategies are straight lines and in particular, with the strategy is equivalent to a TWAP
- As the running penalty increases, the trading curve become more convex, i.e., the optimal strategy aims to sell more assets sooner
- As long as α is finite, it is optimal to leave some inventory executed at the terminal time

Optimal Execution When Order-Flow Affects Prices

Price Impact

<u>Question</u> : How do we trade in or out of a large position **when order-flow affects prices**?

Permanent impact

- Perform the regression $\Delta S_n = b\mu_n + \varepsilon_n$ where $\Delta S_n = S_{n\tau} - S_{(n-1)\tau}$ is the change in the midprice, μ_n is net order-flow defined as the difference between the volume of buy and sell MOs during $[(n-1)\tau, n\tau]$, ε_n is the error term assumed normal, and $\tau = 5\,\mathrm{min}$ is chosen in the empirical analysis

Temporary impact

- Take a snapshot of the LOB each second
- Determine the price per share for various volumes by walking through the LOB
- Compute the difference between the price per share and the best quote at that time
- Perform a linear regression

Price Impact (cont'd)

Table 2-1. Permanent and temporary price impact parameters for NASDAQ stocks (FARO, SMH, NTAP) in 2013.

	FARO		SMH		NTAP	
	mean	stdev	mean	stdev	mean	stdev
ADV	23,914	14,954	233,609	148,580	1,209,628	642,376
midprice	40.55	6.71	37.90	2.44	38.33	3.20
σ	0.151	0.077	0.067	0.039	0.078	0.045
b	1.41E-04	9.61E-05	5.45E-06	4.20E-06	5.93E-06	2.31E-06
k	1.86E-04	2.56E-04	8.49E-07	8.22E-07	3.09E-06	1.75E-06
b/k	1.02	0.83	7.43	6.24	2.04	0.77
λ^+	16.81	9.45	47.29	28.13	300.52	144.48
$\mathbb{E}[\eta^+]$	103.56	21.16	377.05	118.05	308.45	53.09
λ^-	17.62	10.69	46.37	27.62	293.83	136.13
$\mathbb{E}[\eta^-]$	104.00	21.79	381.70	126.74	312.81	49.86

Variables

- Average volume of MOs
- Average midprice
- Volatility (hourly) of price changes
- Mean arrival (hourly) of MOs
- Average volume of MOs

<u>Order-flow affects prices in a way of permanent price impact</u>

Price Impact (cont'd)

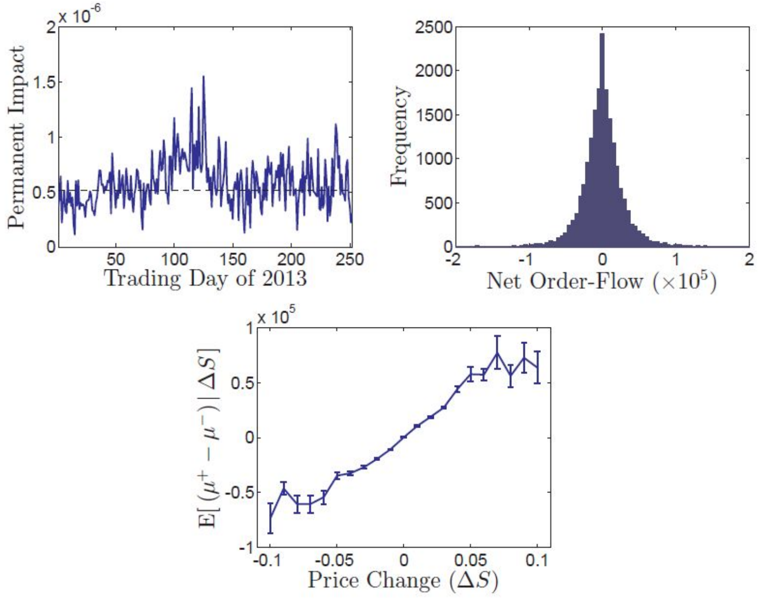

Figure 2-3. An illustration of order-flow and effect on the drift of midprice of INTC in 2013

- The first panel depicts the estimate of b for each day
- The second panel shows a histogram of the five-minute net order flow
- The last panel shows the expected net order flow conditional on a given price change being observed => Linearity is a reasonable assumption!

Price Impact (cont’d)

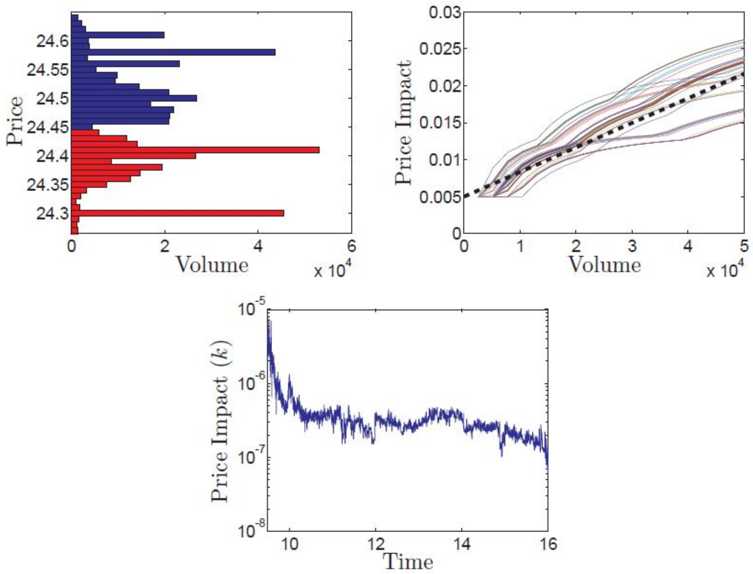

Figure 2-4. An illustration of how the temporary impact may be estimated from snapshots of the LOB using INTC on Nov 1, 2013

- The first panel depicts the LOB at 11:00am
- The second panel shows the estimated temporary price impact from 11:00am to 11:01am
- The last panel shows the estimated of k in the entire day

Price Impact (cont'd)

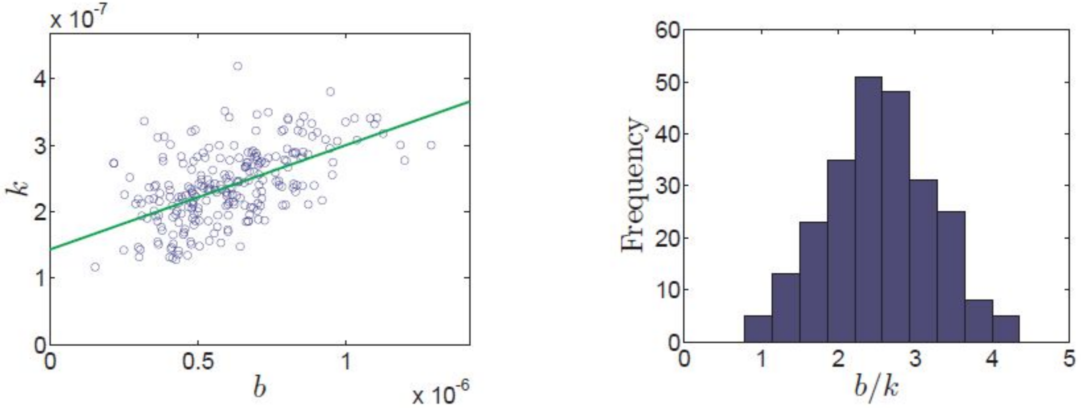

Figure 2-4. An illustration of the relationship between permanent and temporary price impact coefficients, *b* and *k*, respectively

The Model

Want to find the optimal trading plan for liquidating N shares by time T when order-flow affects prices

Inventory (investor is liquidating) : $dQ_t^v = -v_t dt, \quad Q_0^v = N$

Midprice : $dS_t^v = b(\mu_t - v_t)dt + \sigma dW_t$

- $\mu_t^{\pm}$ is the rate of buy/sell MOs of other market participants, and $\mu_t = \mu_t^+ - \mu_t^-$ is their net order-flow
- Our selling pressure incurs a negative impact on midprice

Execution price : $\hat{S}_t^v = S_t^v - \left(\frac{1}{2}\Delta + kv_t\right)$

- Δ is the bid-ask spread
- The temporary price impact is $f(v_t) = kv_t$

Cash : $dX_t^v = \hat{S}_t^v v_t dt, \quad X_0^v = x$

Value Function, DPE

The agent's performance criterion is $H^v(t,x,S,\mu,q) = E_t\Big[X_T^v + Q_T^v\big(S_T^v - \frac{1}{2}\Delta - \alpha Q_T^v\big) - \phi\int_{u=t}^{T}(Q_u^v)^2 du\Big]$

Then the value function is $H(t,x,S,\mu,q) = \max_{v\in A} H^v(t,x,S,\mu,q)$

The DPP suggests that the value function satisfies the following DPE :

$$0 = \big(\partial_t + \tfrac{1}{2}\sigma^2\partial_{SS}\big)H + L^\mu H - \phi q^2 + \max_{v\in A}\Big\{\Big(v\big(S - \tfrac{1}{2}\Delta - kv\big)\partial_x + b(\mu - v)\partial_S - v\partial_q\Big)H\Big\}$$

subject to the terminal condition $H(T,x,S,\mu,q) = x + q\big(S - \frac{1}{2}\Delta\big) - \alpha q^2$ and where L^μ is the infinitesimal generator for the net order flow acting on the value function

Solving the DPE

The DPE admits the following solution :

$$H(t,x,S,\mu,q) = x + q\big(S - \tfrac{1}{2}\Delta\big) + h_0(t,\mu) + qh_1(t,\mu) + q^2 h_2(t)$$

where

- $h_2(t) = \sqrt{k\phi}\frac{1+\zeta\exp(2\gamma(T-t))}{1-\zeta\exp(2\gamma(T-t))} - \frac{1}{2}b$
- $h_1(t,\mu) = b\int_t^T \left(\frac{\exp(-\gamma(T-u))-\zeta\exp(\gamma(T-u))}{\exp(-\gamma(T-t))-\zeta\exp(\gamma(T-t))}\right)E_t[\mu_u]du$
- $h_0(t,\mu) = \frac{1}{4k}\int_t^T E_t\big[h_1^2(t,\mu_u)\big]du$
- with constants $\gamma = \sqrt{\frac{\phi}{k}}$, $\zeta = \frac{\alpha-\frac{1}{2}b+\sqrt{k\phi}}{\alpha-\frac{1}{2}b-\sqrt{k\phi}}$

Limiting Cases

If all shares should be liquidated, i.e., $\alpha \to \infty$, the optimal trading speed simplifies to

$$\lim_{\alpha\to\infty} v_t^\star = \gamma \frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))} Q_t^{v^\star} - \frac{b}{2k} \int_t^T \frac{\sinh(\gamma(u-t))}{\sinh(\gamma(T-t))} E_t[\mu_u] du$$

- The first term is proportional to the current inventory Q; the more assets to sell, the faster the agent liquidates
- The agent adjusts the trading speed with the second term which represents the perturbations due to excess order flow; when the expected excess order flow is tilted to the buy side ($E_t[\mu_u] > 0$), the agent slows down trading since he anticipates that expected excess order flow will push the price upwards - and therefore will receive better prices when he eventually speeds up trading to sell assets later on

Additionally, if the agent does not penalize inventory, i.e., $\phi \to 0$, the optimal trading speed is

$$\lim_{\phi\to 0} \lim_{\alpha\to\infty} v_t^\star = \frac{1}{T-t} Q_t^{v^\star} - \frac{b}{2k} \int_t^T \frac{T-u}{T-t} E_t[\mu_u] du$$

Model for Order-Flow

Order-flow $\mu_t^{\pm}$ satisfy the SDEs : $d\mu_t^{\pm} = -\kappa\mu_t^{\pm}dt + \eta_{1+L_{t-}}^{\pm} dL_t^{\pm}$ where $L_t^{\pm}$ are independent Poisson process with equal intensity , λ $\{\eta_1^{\pm}, \eta_2^{\pm}, \dots\}$ is the process of an increase in order flow rate which is assumed to be non-negative i.i.d. random variable, independent from all process

The solution to these SDEs are $\mu_s^{\pm} = e^{-\kappa^{\pm}(s-t)}\mu_t^{\pm} + \int_t^s e^{-\kappa^{\pm}(s-u)}\eta_{1+L_{t-}}^{\pm} dL_u^{\pm}, \quad for\ s > t$

so that $E_t\left[\mu_s^{\pm}\right] = e^{-\kappa^{\pm}(s-t)}\left(\mu_t^{\pm} - \psi^{\pm}\right) + \psi^{\pm} \quad where \quad \psi^{\pm} = \frac{1}{\kappa^{\pm}}\lambda^{\pm}E\left[\eta^{\pm}\right]$

Therefore, under this particular model for order-flow, we follow the optimal trading strategy as follows :

$$\lim_{\alpha\to\infty} v_t^{\star} = \gamma\frac{\cosh(\gamma(T-t))}{\sinh(\gamma(T-t))}Q_t^{v^{\star}} - \frac{b}{2k}\left[l_1^{+}(t)\left(\mu_t^{+} - \psi^{+}\right) - l_1^{-}(t)\left(\mu_t^{-} - \psi^{-}\right) + l_0(t)(\psi^{+} - \psi^{-})\right]$$

- $l_0(t) = \frac{1}{\gamma}\frac{\cosh(\gamma(T-t))-1}{\sinh(\gamma(T-t))}$
- $l_1^{\pm}(t) = \frac{1}{2}\left(\frac{\exp(\gamma(T-t))-\exp\left(-\kappa^{\pm}(T-t)\right)}{\kappa^{\pm}+\gamma} - \frac{\exp(\gamma(T-t))-\exp\left(-\kappa^{\pm}(T-t)\right)}{\kappa^{\pm}-\gamma}\right)/\sinh\left(\gamma(T-t)\right)$

Simulations

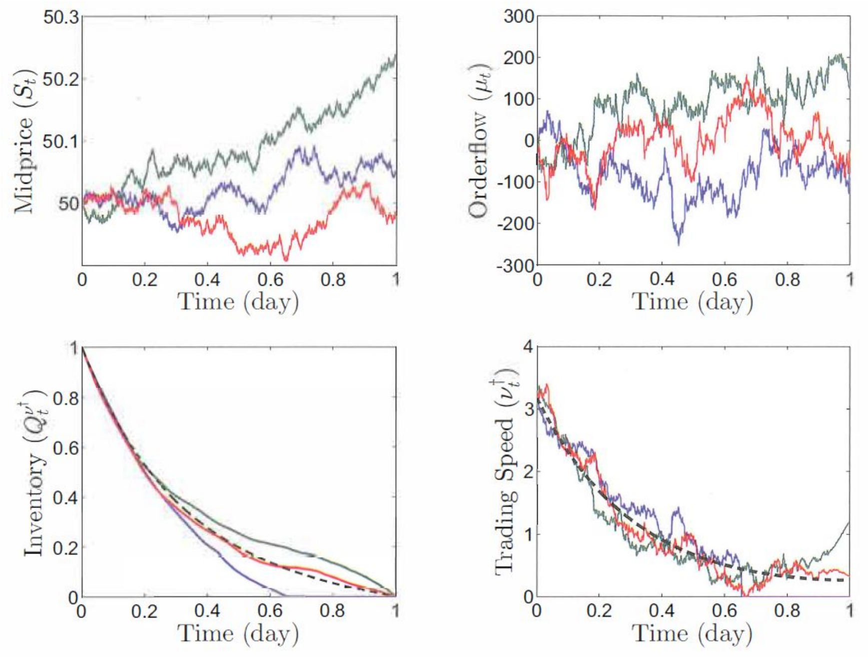

Figure 2-5. An illustration of the optimal trading in the presence of order flow. The dashed lines show the classical AC solution, i.e. the case of no order flow.

In the simulations, use the following parameters:

$$T = 1\,day,\ k = 10^{-3},\ b = 10^{-4},\ \phi = 0.01$$

$$\lambda = 1000,\ \kappa = 10,\ \eta \sim \exp(5),\ \sigma = 0.1$$

Simulations (cont'd)

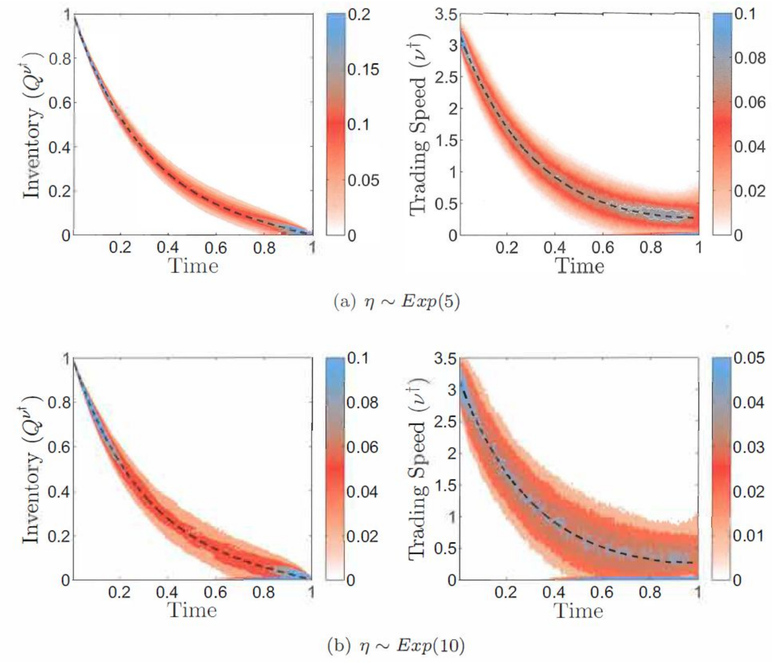

Figure 2-6. An illustration of heat-maps of the optimal trading in the presence of order-flow for two volatility levels. The dashed lines show the classical AC solution.