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Algo. Trading and Quant. Trading (2021 Fall)

Homework #1

Due date: 11/25 18:59

1. (Optimal execution under stock price with drift)

Let the stock midprice dynamics satisfy

$$dS_t = \mu dt + \sigma dW_t$$

where $\sigma > 0$, μ is a constant and W_t is a standard Brownian motion. The agent wishes to liquidate N shares and his trades create a temporary adverse move in prices so that the price at which he transacts is

$$\hat{S}_t^v = S_t - k v_t$$

with $k > 0$ and the inventory satisfies

$$dQ_t^v = -v_t dt$$

where v_t is the liquidation rate. Any outstanding inventory at time T is liquidated at the midprice and picks up a penalty of αQ_T^2 where $\alpha \geq 0$ is a constant.

(a) Denote the agent's value function as $H(t, S, q)$. Write down $H(t, S, q)$ using parameters above.

(b) Show that the optimal liquidation rate in feedback form is

$$v^* = \frac{\partial_q H - S}{-2k}$$

Setup)

1. $dS_t = \mu \cdot dt + \sigma dW_t$

2. $\hat{S}_t^V = S_t - k \cdot V_t$

3. $dQ_t^V = -V_t dt$ $\qquad Q_0^V = N$

4. $Q_T^V \sim (S_T - \alpha \cdot Q_T^{V2})$ 으로 판다.

(a)

Value function : $H(t, S(t), Q(t))$

$$= \max_{\substack{V_u \\ t \le u \le T}} E_t \left[\int_t^T (S_u - kV_u) V_u \, du + Q_T^V (S_T - \alpha Q_T^{2V}) \right]$$

(b)

- HJB EQ)

$$0 = \max_{V_t} \left[(S_t - kV_t) V_t + dH/dt \right]$$

$$dH = \frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial S} dS + \frac{1}{2} \frac{\partial^2 H}{\partial S^2} (dS)^2 + \frac{\partial H}{\partial Q} dQ$$

$$0 = \max_{V_t} \left[(S_t - k \cdot V_t) V_t + \frac{\partial H}{\partial t} + \frac{\partial H}{\partial S} \mu + \frac{1}{2} \frac{\partial^2 H}{\partial S^2} \sigma^2 - \frac{\partial H}{\partial Q} V_t \right]$$

$$0 = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial S} \mu + \frac{1}{2} \frac{\partial^2 H}{\partial S^2} \sigma^2 + \max_{V_t} \left[(S_t - k \cdot V_t) V_t - \frac{\partial H}{\partial Q} V_t \right]$$

$$\rightsquigarrow \quad S_t - k V_t^* - k \cdot V_t^* - \frac{\partial H}{\partial Q} = 0$$

$$-2k V_t^* = \frac{\partial H}{\partial Q} - S_t$$

$$\therefore \quad V_t^* = \frac{\frac{\partial H}{\partial Q} - S_t}{-2k}.$$

(c) Use the trial solution $H(t,S,q) = qS + h(t,S,q)$ to show that the optimal liquidation rate is given by

$$v_t^* = \frac{Q_t^{v^*}}{(T-t) + \frac{k}{\alpha}} - \frac{1}{4k}\mu(T-t)\frac{(T-t) + \frac{2k}{\alpha}}{(T-t) + \frac{k}{\alpha}}$$

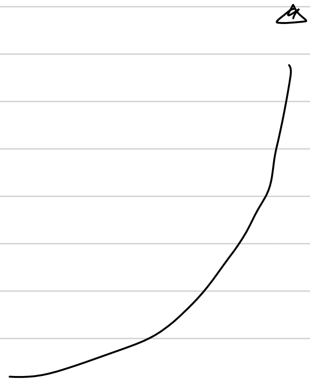
Discuss the relation between μ and the liquidation rate v_t^*.

(d) Let $\alpha \to \infty$ and show that the inventory along the optimal strategy is given by

$$Q_t^{v^*} = (T-t)\left(\frac{N}{T} + \frac{\mu}{4k}t\right)$$

$$0 = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 H}{\partial S^2}\sigma^2 + \max_{V_t}\left[(S_t - k\cdot V_t)V_t - \frac{\partial H}{\partial Q}V_t\right]$$

$$\rightsquigarrow \quad S_t - kV_t^* - k\cdot V_t^* - \frac{\partial H}{\partial Q} = 0$$



$$-2kV_t^* = \frac{\partial H}{\partial Q} - S_t$$

$$\therefore \quad V_t^* = \frac{\frac{\partial H}{\partial Q} - S_t}{-2k}$$

$$\frac{\partial H}{\partial t} + \frac{\partial H}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 H}{\partial S^2}\sigma^2 + \left[S_t + \frac{1}{2}\left[\frac{\partial H}{\partial Q} - S_t\right]\right]\left[-\frac{1}{2k}\right]\left[\frac{\partial H}{\partial Q} - S_t\right]$$

$$+ \frac{\partial H}{\partial Q}\frac{1}{2k}\left[\frac{\partial H}{\partial Q} - S_t\right]$$

$$\partial_t H + \partial_S H\cdot\mu + \frac{1}{2}\partial_{SS}H\sigma^2 + \left[\frac{1}{2}\right]\left[\frac{\partial H}{\partial Q} + S_t\right]\left[-\frac{1}{2k}\right]\left[\frac{\partial H}{\partial Q} - S_t\right]$$

$$+ \frac{\partial H}{\partial Q}\cdot\frac{1}{2K}\left[\frac{\partial H}{\partial Q} - S_t\right]$$

$$-\frac{1}{4K}\left(\partial_Q H^2 - S_t^2\right) + \frac{1}{2K}\left(\partial_Q H^2 - \partial_Q H \cdot S_t\right)$$

$$-\frac{1}{4K}\left(\partial_Q H^2 - S_t^2 - 2\partial_Q H^2 + 2\,\partial_Q H \cdot S_t\right)$$

$$-\frac{1}{4K}\left(-\partial_Q H^2 + 2 \cdot \partial_Q H \cdot S_t - S_t^2\right)$$

$$\frac{1}{4K}\left(\partial_Q H - S_t\right)^2$$

$$\Rightarrow \partial_t H + \partial_S H \cdot \mu + \frac{1}{2}\partial_{SS} H \cdot \sigma^2 + \frac{1}{4K}\left(\partial_Q H - S_t\right)^2 = 0$$

- $H(t,S,q) = qS + h(t,S,q)$

- $\partial_t H + \partial_S H \cdot \mu + \frac{1}{2}\partial_{SS} H \cdot \sigma^2 + \frac{1}{4k}(\partial_q H - S_t)^2 = 0$

$\Rightarrow \partial_t h + [Q + \partial_S h]\mu + \frac{1}{2}\partial_{SS} h \cdot \sigma^2 + \frac{1}{4k}[\partial_q h]^2 = 0$

$\partial_t h + Q\mu + \partial_S h \cdot \mu + \frac{1}{2}\partial_{SS} h\, \sigma^2 + \frac{1}{4k}[\partial_q h]^2 = 0$

- $h(t,S,q) = q S + h_1(t,S,q)$

$\partial_t h_1 + Q\mu + Q\mu + \frac{1}{2}\partial_{SS} h_1\, \sigma^2 + \frac{1}{4k}[S + \partial_q h_1]^2 = 0$?

해를 찾지 못하였습니다.

(c)

(c) Use the trial solution $H(t,S,q) = qS + h(t,S,q)$ to show that the optimal liquidation rate is given by

$$v_t^* = \frac{Q_t^{v^*}}{(T-t)+\frac{k}{\alpha}} - \frac{1}{4k}\mu(T-t)\frac{(T-t)+\frac{2k}{\alpha}}{(T-t)+\frac{k}{\alpha}}$$

Discuss the relation between μ and the liquidation rate v_t^*.

$\Rightarrow \quad \mu \uparrow \quad \Rightarrow \quad v_t^* \downarrow \qquad (\because \ k > 0, \ \alpha > 0)$

시간에 따라 주가가 drift term 으로 더 빨리 증가하기 때문에 liquidation rate 을 줄여야 더 큰 cash를 향유할 수 있습니다.

(d) Let $\alpha \to \infty$ and show that the inventory along the optimal strategy is given by

$$Q_t^{v^*} = (T-t)\left(\frac{N}{T} + \frac{\mu}{4k}t\right)$$

1) $a \to \infty$ $\quad V_t^* = \dfrac{Q_t^{v*}}{(T-t)} - \dfrac{1}{4k}\mu(T-t)$

$$V_t^* = \frac{1}{T-t} \cdot Q_t^{v*} - \frac{1}{4k}\mu(T-t)$$

$$dQ_t^v = -V_t \, dt$$

$$\frac{dQ_t^v}{dt} = -V_t$$

$$\frac{dQ_u^v}{du} = -V_u.$$

$$Q_t^v - Q_0^v = -\int_0^t V_u \, du.$$

$$= -\int_0^t \left[\frac{1}{T-u} Q_u^v - \frac{1}{4k}\mu(T-u)\right] du \; ?$$