# Algorithmic Trading
# : Market Making

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# Market Making

# Market Making

$dS_t = \sigma dW_t, \quad \sigma > 0$ and $(W_t)_{\{0 \le t \le T\}}$ is a standard Brownian motion

$\delta^{\pm}$ : depth at which the agent posts LOs; sell LOs are posted at a price of $S_t + \delta_t^+$ and buy LOs at $S_t - \delta_t^-$

$M^{\pm}$ : counting processes corresponding to the arrival of other participants' buy (+) and sell (-) MOs which arrive at Poisson times with intensities $\lambda^{\pm}$

$N^{\delta,\pm}$ : counting processes for the agent's filled sell (+) and buy(-) LOs

Conditional on an MO arrival, the LO is filled with probability $e^{-\kappa^{\pm}\delta_t^{\pm}}$

$X^{\delta}$ : the agent's cash process, $dX_t^{\delta} = \left(S_{t-} + \delta_t^+\right) dN_t^{\delta,+} - \left(S_{t-} - \delta_t^-\right) dN_t^{\delta,-}$

$Q^{\delta}$ : the agent's inventory process, $Q_t^{\delta} = N_t^{\delta,-} - N_t^{\delta,+}$

# Market Maker's Control Problem

The MM's performance criteria is

$$H^\delta(t,x,S,q) = E_t\left[X_T + Q_T(S_T - \alpha Q_T) - \phi\int_t^T (Q_u)^2 du\right]$$

where

- $\alpha \geq 0$ represents the fees for taking liquidity (i.e. using an MO) as well as the impact of the MO walking the LOB
- $\phi \geq 0$ is the running inventory penalty parameter

The MM's value function is

$$H(t,x,S,q) = \max_{\delta^\pm \in A} H^\delta(t,x,S,q)$$

The MM caps his inventory so that it is bounded above by $\bar{q} > 0$ and below by $\underline{q} < 0$

# DPE

A DPP holds and the value function satisfies the following DPE

$$
\begin{aligned}
0 = {} & \partial_t H + \frac{1}{2}\sigma^2 \partial_{SS} H - \phi q^2 \\
& + \lambda^+ \max_{\delta^+} \Big[ e^{-\kappa^+ \delta^+} \big( H\big(t, x + (S + \delta^+), q - 1, S\big) - H \big) \Big] 1_{\{q > \underline{q}\}} \\
& + \lambda^- \max_{\delta^-} \Big[ e^{-\kappa^- \delta^-} \big( H\big(t, x - (S - \delta^+), q + 1, S\big) - H \big) \Big] 1_{\{q < \bar{q}\}}
\end{aligned}
$$

with the terminal condition $H(T, x, S, q) = S + q(S - \alpha q)$

Recall that inventory is bounded, thus when $q = \bar{q}$ $(\underline{q})$ the optimal strategy is to post one-sided LOs which are obtained by solving the above DPE with the term proportional to $\lambda^-$ $(\lambda^+)$ absent as stated by the indicator function in the DPE

Alternatively, one can view these boundary cases as imposing $\delta^- = \infty$ *and* $\delta^+ = \infty$ when $q = \bar{q}$ *and* $q = \underline{q}$, respectively

# Solving HJB

Make an ansatz for $H$ : $H(t,x,S,q) = x + qS + h(t,q)$

- The first term is the accumulated cash, the second term is the book value of the inventory marked-to-market, and the last term is the added value from following an optimal market making strategy up to the terminal date

Hence,

$$\begin{aligned}\phi^2 q = \partial_t h &+ \lambda^+ \max_{\delta^+}\Big\{e^{-\kappa^+\delta^+}\big(\delta^+ + h(t,q-1)\big) - h\Big\}1_{\{q>\underline{q}\}} \\ &+ \lambda^- \max_{\delta^-}\Big\{e^{-\kappa^-\delta^-}\big(\delta^- + h(t,q+1)\big) - h\Big\}1_{\{q<\bar{q}\}}\end{aligned}$$

with the terminal condition $h(T,q) = -\alpha q^2$

# Optimal Controls

Then the optimal depths in feedback form are given by

$$\delta^{+,\star}(t,q) = \frac{1}{\kappa^+} - h(t,q-1) + h(t,q), \quad q \neq \underline{q}$$
$$\delta^{-,\star}(t,q) = \frac{1}{\kappa^-} - h(t,q+1) + h(t,q), \quad q \neq \bar{q}$$

and the boundary cases are $\delta^{+,\star}(t,q) = \infty$ *and* $\delta^{-,\star}(t,q) = \infty$ when $q = \underline{q}$ *and* $q = \bar{q}$, respectively

Substituting the optimal controls into the DPE, we obtain

$$\begin{aligned}\phi^2 q = \partial_t h(t,q) &+ \frac{\lambda^+}{\kappa^+} \exp(-1) \exp\left(-\kappa^+(-h(t,q-1) + h(t,q))\right) \cdot 1_{\{q>\underline{q}\}} \\ &+ \frac{\lambda^-}{\kappa^-} \exp(-1) \exp\left(-\kappa^-(-h(t,q+1) + h(t,q))\right) \cdot 1_{\{q<\bar{q}\}}\end{aligned}$$

# Symmetric Fill Probability

It is possible to find an analytical solution to the DPE if the fill probabilities of LOs is the same on both sides of the LOB; in this case of $\kappa = \kappa^+ = \kappa^-$ , then write

$$h(t,q) = \frac{1}{\kappa} \log w(t,q)$$

and stack *w(t,q)* into a vector

$$\vec{w}(t,q) = \left[w(t,\bar{q}),\, w(t,\bar{q}-1), \ldots, w\big(t,\underline{q}\big)\right]'$$

Now, let *M* denote the $(\bar{q} - \underline{q} + 1)$-square matrix whose rows are labeled from $\bar{q}$ to $\underline{q}$ and whose entries are given by

$$M_{i,q} = \begin{cases} -\phi\kappa q^2, & i = q \\ \lambda^+ e^{-1}, & i = q-1 \\ \lambda^- e^{-1} & i = q+1 \\ 0 & o.w. \end{cases}$$

with terminal and boundary conditions $w(T,q) = e^{-\alpha\kappa q^2}$

# Symmetric Fill Probability (cont'd)

Then the DPE becomes

$$\partial_t \vec{w}(t) + M\vec{w}(t) = \vec{0}$$

The solution of this matrix ODE is straightforward, and we finally have

$$\vec{w}(t) = e^{M(T-t)}\vec{z}$$

where $\vec{z}$ is a $(\bar{q} - \underline{q} + 1)$-dim vector where each component is $z_j = e^{-\alpha\kappa j^2}, \quad j = \bar{q}, \ldots, \underline{q}$

# Optimal Postings

<u>Simulation study</u> : Display the simulation results for parameters:

$$\lambda^{\pm} = 1,\ \kappa^{\pm} = 100,\ \bar{q} = -\underline{q} = 3,\ \phi = 0.001\ \mathit{or}\ 0.02,\ \alpha = 0.0001,\ \sigma = 0.01,\ S_0 = 100,\ T = 30$$

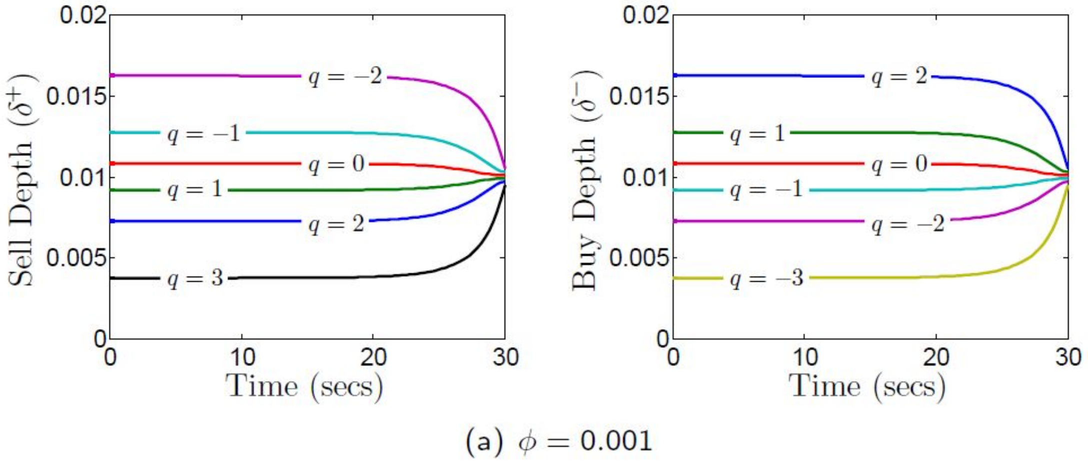


**Figure 4-1a**. The optimal depths as a function of time for various inventory levels with the inventory penalty 0.001

# Optimal Postings (cont’d)

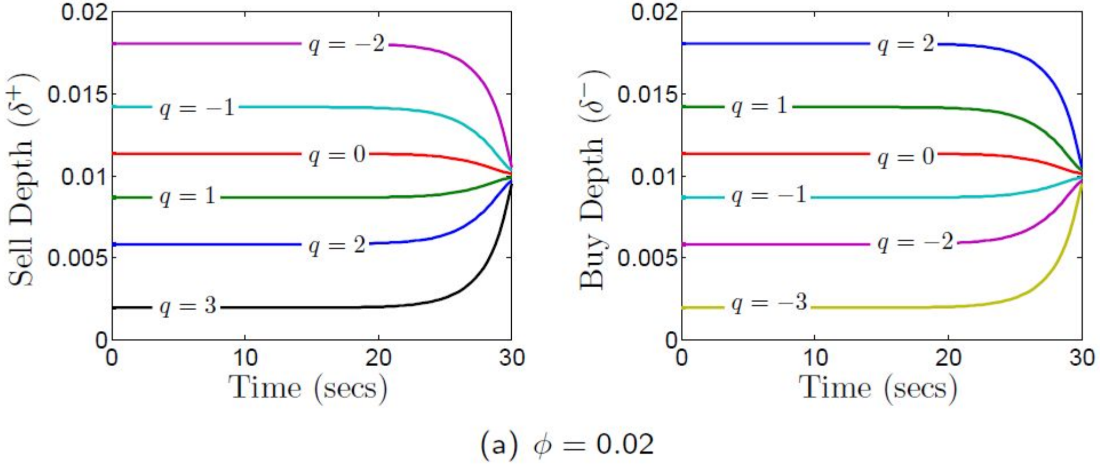


**Figure 4-1b**. The optimal depths as a function of time for various inventory levels with the inventory penalty 0.02

# Optimal Postings (cont'd)

<u>Observations</u>

- When the strategy is far away from expiry and inventories are close to the allowed minimum, the optimal sell posting is furthest away from the midprice because only at a very high price is the MM willing to decrease his inventories further, and at the same time the optimal buy posting is very close to the midprice because the strategy would like to complete round-trip trades and push inventories to zero
- As the strategy approaches $T$ and his inventory is short (long), i.e., $q < 0$ ($q > 0$), the optimal sell (buy) depth decreases (increases); To understand the intuition behind the optimal strategy note that if the terminal inventory $q_T < 0$ is liquidated at the price $S_T - \alpha q_T$, then when $\alpha$ is sufficiently low, as well as being fractions of a second away from expiry, it is optimal to post nearer the midprice to increase the chances of being filled (i.e., selling one more unit of the asset) because the price is not expected to move too much before expiry and the entire position will be unwound at the midprice - making a profit on the last unit of the asset that was sold

# Mean Reversion in Inventory

The optimal strategy induces mean reversion in inventories

- For example, if $q$ = 2 then the sell depth is lower than the buy depth so that it is more likely for the strategy to sell, than to buy, one unit of the asset
- This asymmetry in the optimal depths is what induces mean reversion to zero in the inventory
- The strategy's running inventory penalty is much higher and it is clear that the higher $\phi$ is, the quicker inventories will revert to zero

The expected drift in inventories is given by the difference in the arrival rates of filled orders

$$\begin{aligned}\mu(t,q) &= \lim_{s\to t}\frac{1}{s-t}E[Q_s - Q_t \,|\, Q_{t-} = q] \\ &= \lambda^- e^{-\kappa^-\delta^{-,\star}(t,q)} - \lambda^+ e^{-\kappa^+\delta^{+,\star}(t,q)}\end{aligned}$$

- It is clear that for the same level of inventory the speed will be different depending on how near or far the strategy is from the terminal date, because at time $T$ strategy tries to unwind all outstanding inventory

# Mean Reversion in Inventory (cont'd)

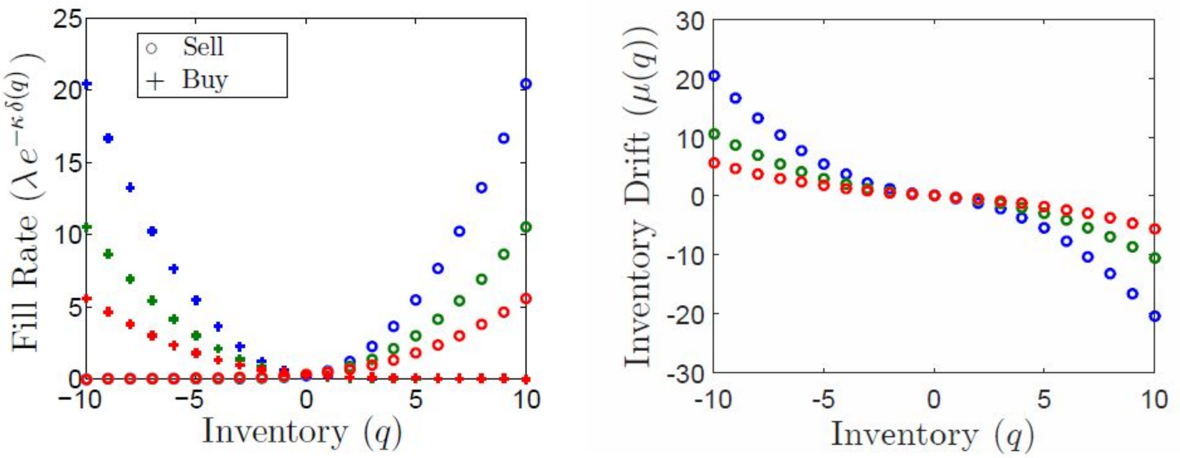


**Figure 4-2**. Arrival rates of filled orders (left) and inventory drift (right) for different inventory levels. Model parameters are :

$$\lambda^{\pm} = 1,\ \kappa^{\pm} = 100,\ \bar{q} = -\underline{q} = 10,\ \alpha = 0.0001,\ \sigma = 0.01,\ S_0 = 100,\ and\, \phi = \{2 \cdot 10^{-3}, 10^{-3}, 5 \cdot 10^{-4}\}$$

# Mean Reversion in Inventory (cont’d)

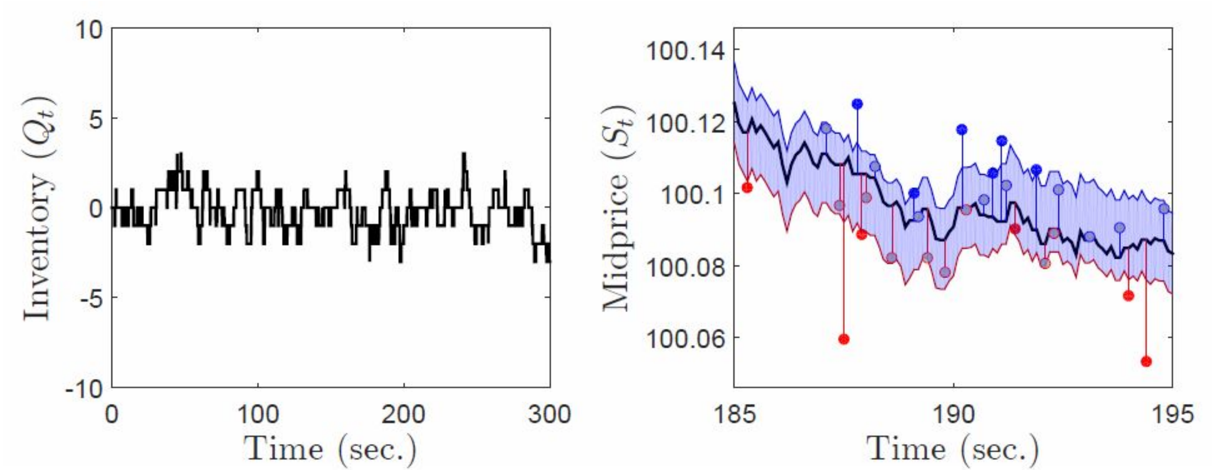


**Figure 4-3**. Inventory (left) and price path (right) for one simulation of the strategy. Model parameters are :

$$\lambda^{\pm} = 1,\ \kappa^{\pm} = 100,\ \bar{q} = -\underline{q} = 10,\ \alpha = 0.0001,\ \sigma = 0.01,\ S_0 = 100,\ and\, \phi = 0.02$$

# Profit and Loss

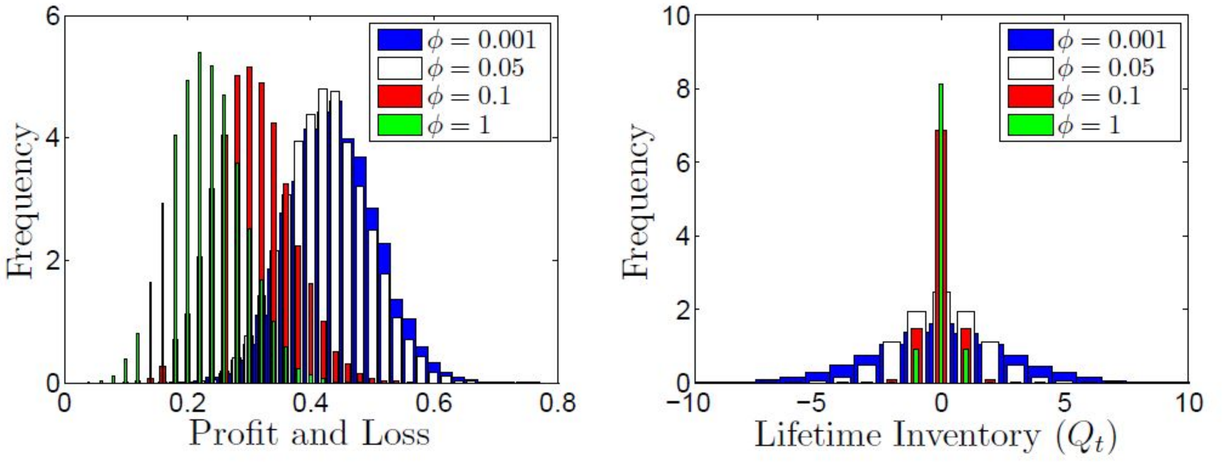


**Figure 4-4**. P&L (left) and lift inventory, i.e. how much time the strategy holds of an inventory *n*, (right) of the optimal strategy for 10,000 simulations. Model parameters are :

$$\lambda^{\pm} = 1,\ \kappa^{\pm} = 100,\ \bar{q} = -\underline{q} = 10,\ \alpha = 0.0001,\ \sigma = 0.01,\ S_0 = 100$$

# Profit and Loss (cont'd)

Observations

- When the inventory penalty increases, the histogram of P&L shifts to the left because the strategy does not allow inventory positions to stray away from zero, and hence expected profits decreases
- When the inventory penalty is large, the strategy heavily penalise deviations of running inventory from zero, so the strategy spends most of the time at inventory levels of -1, 0, 1

# Market Making with No Terminal Penalty

# Solving HJB with No Terminal Penalty

Assume no penalties for liquidating inventories at time $T$

Thus, the ansatz is $H(t,x,q,S) = x + qS + g(t)$

- Note that the function *g(t)* does not depend on q; In the previous problem with terminal penalty, the ansatz contained *h(t,q)* because the optimal strategy had to manage inventory risk which is something that is not a problem when $\alpha = 0$

Therefore,

$$0 = \partial_t g + \lambda^+ \max_{\delta^+} \left\{ e^{-\kappa^+ \delta^+} \delta^+ \right\} + \lambda^- \max_{\delta^-} \left\{ e^{-\kappa^- \delta^-} \delta^- \right\}$$

with terminal condition *h(T)* = 0

And the optimal postings are :

$$\delta^{+,\star} = \frac{1}{\kappa^+} \quad and \quad \delta^{-,\star} = \frac{1}{\kappa^-}$$

# Solving HJB with No Terminal Penalty (cont'd)

Note that :

- An MM who does not penalize inventories and who unwinds terminal inventory at the midprice, will make markets by maximizing the probability of his LOs being filled at every instant in time regardless of the inventory position or how close the terminal date is
- Hence, the MM's problem reduces to choosing $\delta^{\pm}$ to maximize the expected depth conditional on an MO hitting or lifting the appropriate side of the LOB, i.e. to maximize $\delta^{\pm} e^{-\kappa^{\pm}\delta^{\pm}}$
- The FOC of this optimization problem is

$$e^{-\kappa^{\pm}\delta^{\pm}} - \kappa^{\pm}\delta^{\pm}e^{-\kappa^{\pm}\delta^{\pm}} = 0$$

# Market Making At-The-Touch

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In very liquid markets, most orders do not walk the book and instead tend to only lift or hit LOs posted at-the-touch. To capture this market feature, we investigate the agent's optimal postings at-the-touch, i.e. at the best bid and best offer.

Throughout we assume that the spread is constant. Next, let $l_t^{\pm} \in \{0,1\}$ denote whether the agent is posted on the sell side (+) or buy side (-) of the LOB. In this way, the agent may be posted on both sides of the book, only the sell side, only the buy side, or not posted at all.

The performance criteria is

$$H^l(t,x,S,q) = E_t\Big[X_T^l + Q_T^l\Big(S_T^l - \left(\tfrac{\Delta}{2} + \varphi Q_T^l\right) - \phi \int_t^T \left(Q_u^l\right)^2 du\Big)\Big]$$

where his cash process now satisfies the SDE

$$dX_t^l = \left(S_t + \tfrac{\Delta}{2}\right) dN_t^{+,l} - \left(S_t - \tfrac{\Delta}{2}\right) dN_t^{-,l}$$

Assume that if he posted in the LOB, when a matching MO arrives his LO is filled with probability one. In this case, $N_t^{\pm,l}$ are controlled doubly stochastic Poisson process with intensity $l_t \lambda^{\pm}$

# Value Function and DPE

The value function is

$$H(t,x,S,q) = \max_{l\in A} H^l(t,x,S,q)$$

Applying the DPP, we find the agent's value function $H$ must satisfy the DPE :

$$\begin{aligned} 0 &= \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 \\ &+ \lambda^+ \max_{l^+\in\{0,1\}} \left[H\big(t, x + (S+\Delta/2)l^+, S, q - l^+\big) - H\right]1_{\{q>\underline{q}\}} \\ &+ \lambda^- \max_{l^-\in\{0,1\}} \left[H\big(t, x - (S-\Delta/2)l^-, S, q + l^+\big) - H\right]1_{\{q<\bar{q}\}} \end{aligned}$$

subject to $\quad H(T,x,S,q) = x + q\big(S - \big(\frac{\Delta}{2} + \varphi q\big)\big)$

# Solution

Ansatz :

$$H(t,x,S,q) = x + qS + h(t,q)$$

And on substituting this ansatz into the above DPE, we find that $h$ satisfies :

$$\begin{aligned} 0 = &\, \partial_t h - \phi q^2 \\ &+ \lambda^+ \max_{l^+ \in \{0,1\}} \left[ l^+ \frac{\Delta}{2} + \big(h\big(t, q - l^+\big) - h\big) \right] 1_{\{q > \underline{q}\}} \\ &+ \lambda^- \max_{l^- \in \{0,1\}} \left[ l^- \frac{\Delta}{2} + \big(h\big(t, q + l^-\big) - h\big) \right] 1_{\{q < \bar{q}\}} \end{aligned}$$

subject to $h(T,q) = -q\big(\frac{\Delta}{2} + \varphi q\big)$

Hence,

$$\begin{aligned} l^{+,\star}(t,q) &= 1_{\{\frac{\Delta}{2} + [h(t,q-1) - h(t,q)] > 0\} \cap \{q > \underline{q}\}} \\ l^{-,\star}(t,q) &= 1_{\{\frac{\Delta}{2} + [h(t,q+1) - h(t,q)] > 0\} \cap \{q < \bar{q}\}} \end{aligned}$$

The agent posts an LO on the LOB by ensuring that he only posts if the arrival of an MO, which hit/lifts his LO, produces a change in her value function larger than $-\frac{\Delta}{2}$

# Simulation

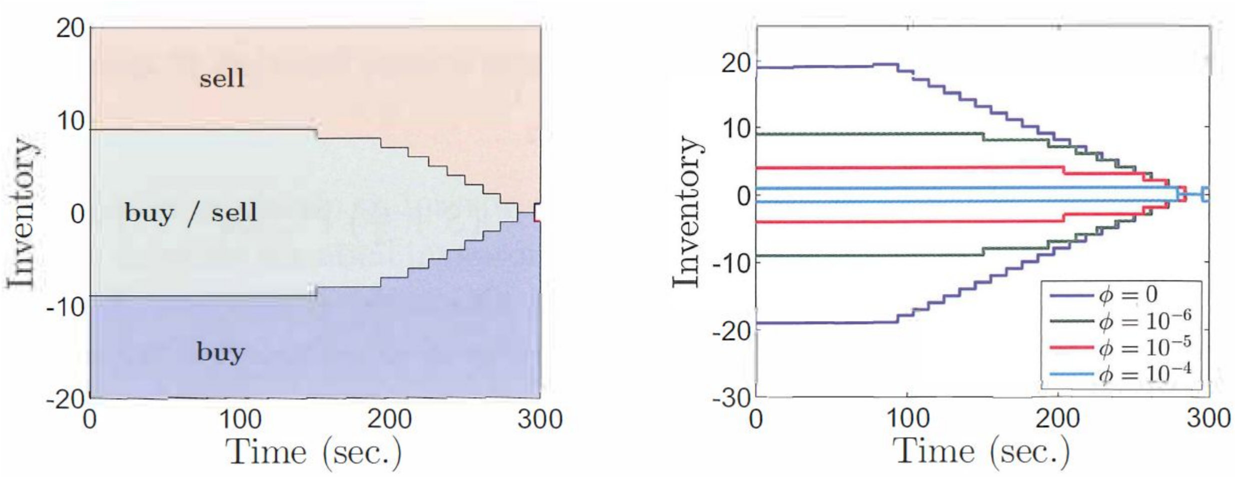



**Figure 4-5**. The optimal strategy for the agent who posts only at-the-touch. Model parameters are

$$T = 300s,\ \bar{q} = -\underline{q} = 20,\ \lambda^{\pm} = \frac{50}{300},\ \Delta = 0.01,\ \phi = 0.01,\ \sigma = 0.001$$

# Market Making with Adverse Selection

# Short-Term Alpha and Adverse Selection

Assume that the midprice of the asset follows :

$$dS_t = (v + \alpha_t)dt + \sigma dW_t$$

where the drift is given by a **long-term component** $v$ and by a **short-term component** $\alpha_t$ which is a predictable zero-mean reverting process

The long-term and short-term components are important when making markets

- If the agent is an MM who trades at time scales where he does not see the short-term component, then his strategy will not only be suboptimal, but will lose money to better informed traders - traders who are better informed will **pick-off the LOs** posted by the less informed MM
- On the other hand, if the MM has the ability to observe the short-term alpha then he will ensure that on average his strategy does not lose money to other traders, and also use this knowledge to execute more speculative trades when the alpha is different from zero

# Dynamics of Short-Term Alpha

Assume that the MM is operating at high-frequency and short-term alpha is driven by order flow. Thus, we model the short-term alpha as a zero-mean-reverting process which jumps by a random amount at the arrival times of MOs. The short-term drift jumps up when buy MOs arrive and jumps down when sell MOos arrive.

As such, the short-term alpha satisfies :

$$d\alpha_t = -\zeta \alpha_t dt + \eta dW_t^\alpha + \epsilon^+_{1+M^+_{t-}} dM_t^+ - \epsilon^-_{1+M^-_{t-}} dM_t^-$$

where

- $\{\epsilon_1^\pm, \epsilon_2^\pm, \dots\}$ are i.i.d. random variables independent of all process, representing the size of the sell/buy MO impact on the drift of the midprice
- $W_t^\alpha$ denotes a Brownian motion independent of all other process
- The MOs arrive at an independent constant rate of $\lambda^\pm$

# Value Function and DPE

The performance criteria as usual :

$$H^l(t,x,S,\alpha,q) = E_t\Big[X_T^l + Q_T^l\Big(S_T^l - \big(\tfrac{\Delta}{2} + \varphi Q_T^l\big) - \phi\int_t^T \big(Q_u^l\big)^2 du\Big)\Big]$$

And the value function is given as follows :

$$H(t,x,S,\alpha,q) = \max_{l\in A} H^l(t,x,S,\alpha,q)$$

Applying the DPP, the function $H$ must satisfy :

$$\begin{aligned}
0 = &\left(\partial_t + \alpha\partial_S + \frac{1}{2}\sigma^2\partial_{SS} - \zeta\alpha\partial_\alpha + \frac{1}{2}\eta^2\partial_{\alpha\alpha}\right)H - \phi q^2 \\
&+\lambda^+ \max_{l^+\in\{0,1\}}\left\{1_{\{q>\underline{q}\}}E\left[H\left(t, x + \left(S + \frac{\Delta}{2}l^+\right)l^+, S, \alpha+\epsilon^+, q - l^+\right) - H\right] + \left(1 - l^+1_{\{q>\underline{q}\}}\right)E\big[H\big(t,x,S,\alpha+\epsilon^+,q\big) - H\big]\right\} \\
&+\lambda^- \max_{l^-\in\{0,1\}}\left\{1_{\{q<\bar{q}\}}E\left[H\left(t, x + \left(S - \frac{\Delta}{2}l^-\right)l^-, S, \alpha-\epsilon^-, q + l^-\right) - H\right] + \left(1 - l^-1_{\{q<\bar{q}\}}\right)E\big[H\big(t,x,S,\alpha-\epsilon^-,q\big) - H\big]\right\}
\end{aligned}$$

subject to the terminal condition : $H(T,x,S,\alpha,q) = x + q\big(S - \big(\tfrac{\Delta}{2} + \varphi q\big)\big)$

# Solution

Ansatz :

$$H(t,x,S,\alpha,q) = x + qS + h(t,\alpha,q)$$

And on substituting this ansatz into the above DPE, we find that $h$ satisfies :

$$\begin{aligned}
0 = &\left(\partial_t - \zeta\alpha\partial_\alpha + \frac{1}{2}\eta^2\partial_{\alpha\alpha}\right)h + \alpha q - \phi q^2 \\
&+\lambda^+ \max_{l^+\in\{0,1\}}\left\{1_{\{q>\underline{q}\}} E\left[l^+\frac{\Delta}{2} + h\big(t,\alpha+\epsilon^+,q-l^+\big) - h\big(t,\alpha+\epsilon^+,q\big)\right]\right\} \\
&+\lambda^- \max_{l^-\in\{0,1\}}\left\{1_{\{q<\bar{q}\}} E\left[l^-\frac{\Delta}{2} + h\big(t,\alpha-\epsilon^-,q+l^-\big) - h\big(t,\alpha-\epsilon^-,q\big)\right]\right\} \\
&+\lambda^+ E\big[h\big(t,\alpha+\epsilon^+,q\big) - h(t,\alpha,q)\big] \\
&+\lambda^- E\big[h\big(t,\alpha-\epsilon^-,q\big) - h(t,\alpha,q)\big]
\end{aligned}$$

subject to $\quad h(T,\alpha,q) = -q\big(\frac{\Delta}{2} + \varphi q\big)$

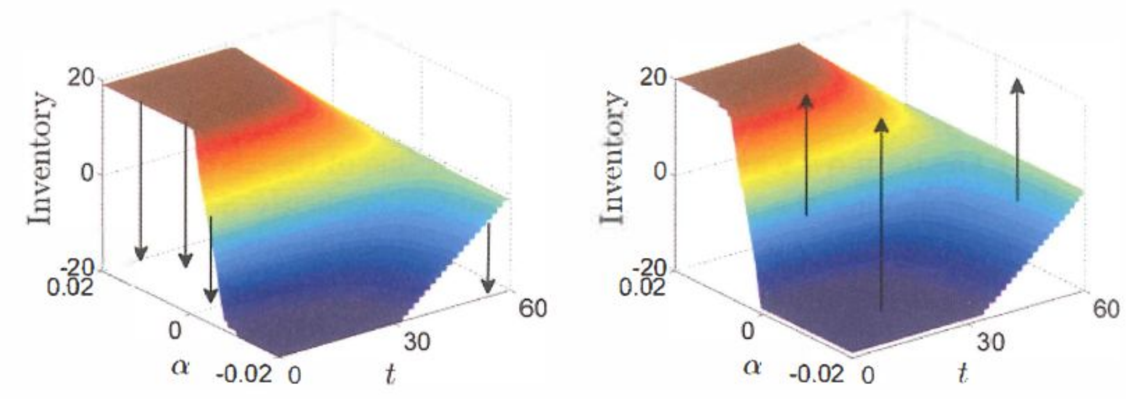
# Solution (cont'd)

Therefore,

$$l^{+,\star}(t,q) = 1_{\{\frac{\Delta}{2}+E[h(t,\alpha+\epsilon^+,q-1)-h(t,\alpha+\epsilon^+,q)]>0\}\cap\{q>\underline{q}\}}$$
$$l^{-,\star}(t,q) = 1_{\{\frac{\Delta}{2}+E[h(t,\alpha-\epsilon^-,q+1)-h(t,\alpha-\epsilon^-,q)]>0\}\cap\{q<\bar{q}\}}$$

The agent knows that when an MO arrives, $\alpha$ jumps up/down and therefore he compares the expected change in the value functions evaluated at $\alpha \pm \epsilon^{\pm}$, rather than at $\alpha$, with the half-spread

# Simulation




(a) Buy Side Posts (b) Sell Side Posts

**Figure 4-6**. The optimal postings when accounting for short-term-alpha. Model parameters are

$$\phi = 0,\ T = 60s,\ \bar{q} = -\underline{q} = 20,\ \lambda^{\pm} = 0.8333,\ \Delta = 0.01,$$
$$\varphi = 0.01,\ \eta = 0.001, \zeta = 0.5,\ E[\epsilon] = 0.005$$

# Simulation (cont'd)

**Observations**

- Due to the symmetry of rates of arrival of MOs, the surfaces are mirror reflections of one another.

- As maturity approaches his strategy becomes essentially independent of short-term-alpha, and instead induces him to sell when his inventory is positive and buy when inventory is negative. Therefore, the optimal strategy attempts to close the trading period with zero inventory.

- The optimal strategies become independent of time far from maturity.

- Far from maturity, the agent tends to post symmetrically when short-term-alpha and inventory are close to zero. As alpha increases, hi is willing to take on inventory, but keeps posting on both sides, until alpha becomes quite large, then he posts only buy LOs. the opposite holds when alpha decreases.