강의노트 (21.12.02)

- Market making with adverse selection
  - stock price dynamics

$$dS_t = (\nu + \alpha_t)\,dt + \sigma dW_t$$

long-term drift
: public info.

short-term drift
(predictable)

- 만약 MM가 이 숏텀 시그널을 관찰하지 못하거나, 의사결정 주기가 이 숏텀시그널 time scale 보다 큰 경우, MM가 의사결정시 α_t를 확인 X 호가 제출

  ⇒ α_t를 알고있는 정보거래자에게 호가를 먹혀서 ("picked off by informed traders") 결과적으로 돈을 잃음.

- ______ " ______ 관찰할 수 있는 경우,

  이를 의사결정에 반영해서 호가 제출할 것이다.

  이 경우 평균적으로 정보거래자에게 picked-off 되는 확률 ↓, 돈을 잃을 확률 ↓

$M^{\pm}$: counting process of buy / sell MOs arrival

- Dynamics of short-term alpha

$$d\alpha_t = \underbrace{-\zeta\, \alpha_t\, dt}_{\text{mean-reverting time drift}} + \underbrace{\eta\, dW_t^{\alpha}}_{\text{noise (uncertainty)}} + \underbrace{\varepsilon^{+}_{1+M_t^-} dM_t^{+}}_{\alpha_t \text{ jumps up when buy MOs arrive}} - \underbrace{\varepsilon^{-}_{1+M_t^-} dM_t^{-}}_{\alpha_t \text{ jumps down when sell MOs arrive}}$$

$\{\varepsilon_1^{\pm}, \varepsilon_2^{\pm} \dots\}$: size of the buy/sell MO impact on α_t

Buy / sell MOs arrive at a constant rate of $\lambda^{\pm}$ (poisson dist'ns)

- Problem formulation

(1) $$\max_{\ell^{\pm} \in \{0,1\}} \quad E_0\left[X_T^{\ell} + Q_T^{\ell}\left(S_T^{\ell} - \left(\frac{\Delta}{2} + \varphi Q_T^{\ell}\right)\right) - \phi \int_0^T (Q_u^{\ell})^2\, du \right]$$

s.t. $\begin{cases} dQ_t^{\ell} = dN_t^{-} - dN_t^{+} \\ dX_t^{\ell} = \underbrace{(S_t + \frac{\Delta}{2})}\, dM_t^{\ell,+} - (S_t - \frac{\Delta}{2})\, dM_t^{\ell,-} \\ dS_t = \alpha_t\, dt + \sigma\, dW_t \\ d\alpha_t = -\zeta \alpha_t\, dt + \eta\, dW_t^{\alpha} + \varepsilon^{+}_{1+M_t^-}\, dM_t^{+} - \varepsilon^{-}_{1+M_t^-}\, dM_t^{-} \end{cases}$

② Value func. : $\underbrace{H(t, \overbrace{x, S, \alpha, q}^{\text{state variables}})}_{\text{value func.}} = \max_{\boxed{\ell}\ \text{control var.}} \overbrace{H^{\ell}(t, x, S, \alpha, q)}^{\text{performance criterion}}$

where $H^{\ell}(t, x, S, \alpha, q) = E_t\left[X_T^{\ell} + Q_T^{\ell}\left(S_T^{\ell} - \left(\frac{\Delta}{2} - \varphi Q_T^{\ell}\right)\right) - \phi \int_t^T (Q_u^{\ell})^2\, du \right]$

PDE for H : HJB eq. (by DPP)

- HJB eq.

The value function H should satisfy :

$$0 = \max_{\ell^{\pm}} \Big[\partial_t H + \alpha\, \partial_S H + \frac{1}{2}\sigma^2 \partial_{SS} H - \zeta \alpha\, \partial_\alpha H + \frac{1}{2}\eta^2 \partial_{\alpha\alpha} H$$

$$-\phi q^2$$

$$+\ \underbrace{\lambda^+}_{①}\Bigg(\underbrace{\mathbb{1}_{\{q>\underline{q}\}}\cdot\ell^+}_{②}\,\mathbb{E}\Big[\underbrace{H\big(t,\ x+(s+\tfrac{\Delta}{2}\ell^+)\ell^+,\ s,\ \alpha+\epsilon^+,\ q-\ell^+\big)-H}_{③}\Big]$$

① buy MO가 λ^+의 강도로 시장에 도착

② MM의 잔고조건 만족하고 sell 호가를 제출한 경우

③ 가치함수의 변화량

$$+\ \underbrace{\big(1-\mathbb{1}_{\{q>\underline{q}\}}\cdot\ell^+\big)}_{②'}\,\mathbb{E}\Big[\underbrace{H\big(t,x,s,\alpha+\epsilon^+,q\big)-H}_{③}\Big]\Bigg)$$

②' 잔고조건 만족 X or sell 호가 제출 X

③ 체결 X 가치함수 변화량

Expectations on ϵ^+ or ϵ^-

$$+\ \lambda^-\Bigg(\mathbb{1}_{\{q<\bar{q}\}}\cdot\ell^-\,\mathbb{E}\Big[H\big(t,\ x-(s-\tfrac{\Delta}{2}\ell^-)\ell^-,\ s,\ \alpha-\epsilon^-,\ q+\ell^+\big)-H\Big]$$

$$+\ \big(1-\mathbb{1}_{\{q<\bar{q}\}}\cdot\ell^-\big)\,\mathbb{E}\Big[H\big(t,x,s,\alpha-\epsilon^-,q\big)-H\Big]\Bigg)$$

간단하게 정리

$$\cdots -\tfrac{1}{2}\sigma^2\partial_{s}H - \zeta\alpha\,\partial_\alpha H + \tfrac{1}{2}\eta^2\partial_{\alpha\alpha}H - \phi q^2$$

$$: \quad 0 = \partial_t H + \alpha \partial_S H + \frac{1}{2}\sigma \partial_{SS} H \cdots$$

$$+ \lambda^+ \max_{\ell^+ \in \{0,1\}} \Big[\ell^+ \mathbb{1}_{\{q > \underline{q}\}} \, \mathbb{E}\Big[H\big(t, x + (S + \tfrac{\Delta}{2}\ell^+)\ell^+, S, \alpha + \varepsilon^+, q - \ell^+\big) - H \Big]$$

$$+ \big(1 - \ell^+ \mathbb{1}_{\{q > \underline{q}\}}\big) \, \mathbb{E}\Big[H(t, x, S, \alpha + \varepsilon^+, q) - H \Big] \Big]$$

$$+ \lambda^- \max_{\ell^- \in \{0,1\}} \Big[\ell^- \mathbb{1}_{\{q < \bar{q}\}} \, \mathbb{E}\Big[H\big(t, x - (S - \tfrac{\Delta}{2}\ell^-)\ell^-, S, \alpha - \varepsilon^-, q + \ell^-\big) - H \Big]$$

$$+ \big(1 - \ell^- \mathbb{1}_{\{q < \bar{q}\}}\big) \, \mathbb{E}\Big[H(t, x, S, \alpha - \varepsilon^-, q) - H \Big] \Big]$$

with BC $H(T, x, S, \alpha, q) = x + q\big(S - (\tfrac{\Delta}{2} + \varphi q)\big)$

- Solving HJB eq.

  - Trial sol: $H(t, x, S, \alpha, q) = x + qS + h(t, \alpha, q)$

  - PDE for h

$$0 = \left(\partial_t - \zeta \alpha \partial_\alpha + \tfrac{1}{2}\eta^2 \partial_{\alpha\alpha}\right) h + \alpha q - \phi q^2$$

$$+ \lambda^+ \max_{\ell^+ = 0 \text{ or } 1} \left[\mathbb{1}_{\{q > \underline{q}\}} \cdot E\left[\ell^+ \frac{\Delta}{2} + h(t, \alpha + \varepsilon^+, q - \ell^+) - h(t, \alpha + \varepsilon^+, q) \right] \right]$$

$$+ \lambda^- \max_{\ell^- = 0 \text{ or } 1} \left[\mathbb{1}_{\{q < \bar{q}\}} \cdot E\left[\ell^- \frac{\Delta}{2} + h(t, \alpha - \varepsilon^-, q + \ell^-) - h(t, \alpha - \varepsilon^-, q) \right] \right]$$

$$+ \lambda^+ E\left[h(t, \alpha + \varepsilon^+, q) - h(t, \alpha, q) \right]$$

$$+ \lambda^- E\left[h(t, \alpha - \varepsilon^-, q) - h(t, \alpha, q) \right]$$

with BC $h(T, \alpha, q) = -q\left(\frac{\Delta}{2} + \phi q\right)$

in feedback form

— Solution in feasible

$$\ell^{+,*} = \mathbb{1}\left\{ \frac{\Delta}{2} + \mathbb{E}\left[h(t, \alpha+\epsilon^{+}, q-1) - h(t, \alpha+\epsilon^{+}, q) \right] > 0 \right\} \cap \left\{ q > \underline{q} \right\}$$

$$\ell^{-,*} = \mathbb{1}\underbrace{\left\{ \frac{\Delta}{2} + \mathbb{E}\left[h(t, \alpha-\epsilon^{-}, q+1) - h(t, \alpha-\epsilon^{-}, q) \right] > 0 \right\}}_{\text{MO가 시장에 도달할 때 MM의 호가가 체결됐을 때 가치함수 변화량} > -\frac{\Delta}{2} \text{ 이면 호가제출}} \cap \underbrace{\left\{ q < \bar{q} \right\}}_{\text{inventory constraint}}$$