Hull White Simulation 로직

1. 이론적 기초

   HW 1Factor Process를 다음과 같이 가정한다.

$$\mathrm{dr_t} = [\theta_t - \kappa r_t]dt + \sigma_t dW$$

2. $\mathrm{e}^{\kappa t} \cdot r_t$에 대하여 $\mathrm{d}(\mathrm{e}^{\kappa t} \cdot r_t) = \kappa e^{\kappa t} r_t dt + e^{\kappa t} dr_t = \kappa e^{\kappa t} r_t dt + e^{\kappa t}[(\theta_t - \kappa r_t)dt + \sigma_t dW]$

$$= \kappa e^{\kappa t} r_t dt + e^{\kappa t}(\theta_t - \kappa r_t) + \sigma_t e^{\kappa t} dW = e^{\kappa t}\theta_t dt + \sigma_t e^{\kappa t} dW$$

   양 변을 T1에서 T2까지 적분하면

$$\mathrm{e}^{\kappa \mathrm{T}_2} \times r_{T_2} - \mathrm{e}^{\kappa \mathrm{T}_1} \times r_{T_1} = \int_{T1}^{T2} \theta_t e^{\kappa t} dt + \int_{T1}^{T2} \sigma_t e^{\kappa t} dW$$

$$\mathrm{r_{T2}} = \frac{e^{\kappa T_1}}{e^{\kappa T_2}} r_{T_1} + \int_{T1}^{T2} \theta_t e^{\kappa(t-T_2)} dt + \int_{T1}^{T2} \sigma_t e^{\kappa(t-T_2)}\, dW$$

   해당 모듈은 $\theta_t = 0$을 가정합니다.

   $\mathrm{r_{T_2}} = e^{-\kappa(T_2 - T_1)} \times r_{T_1} + \int_{T1}^{T2} \sigma_t e^{\kappa(t-T_2)}\, dW$

3. 따라서,

$$\mathrm{r(t)|r(s)} \sim N\left(e^{\kappa(t-s)} r(s), \int_s^t e^{-2\kappa(t-u)} du\right)$$

$$\mathrm{r(t_{i+1})} = \mathrm{XA(t_i)} \cdot r(t_i) + XV(t_i) \cdot \epsilon_i$$

$$\mathrm{XA(t_i)} = e^{-\kappa(t_{i+1} - t_i)}, \qquad \mathrm{XV(t_i)} = \left(\int_{t_i}^{t_{i+1}} e^{-2\kappa(t_{i+1} - t_i)} \sigma^2(\tau) d\tau\right)^{0.5}$$

4. $\mathrm{P_{HW}(0,t,T)} = \frac{P(0,T)}{P(0,t)} \exp\left(-r_{i+1} \times B(t,T) + \frac{1}{2} QVTerm(t,T)\right)$

$$B(t,T) = \frac{\left(1 - \exp\left(-\kappa(T-t)\right)\right)}{\kappa}$$

$$QVTerm(t,T) = \int_0^t \sigma^2 [B(s,t)^2 - B(s,T)^2 \;\;] ds$$

5. XA, XV, $B(t,T)$, $QVTerm(t,T)$는 시뮬레이션 전에 미리 Generate 해놓고 epsilon을 시뮬레

이션을 통해 산출하여 Short-Rate path 시뮬레이션한다.

6. 이후 기초금리를 계산한다.

$$\mathrm{R_{HW}}(0,\mathrm{t},\mathrm{T}) = \frac{1 - P_{HW}(0,t,T_N)}{\sum_1^N[\Delta T_i \times P_{HW}(0,t,T_i)]}$$

예를 들어 2년 만기 스왑금리를 기초금리라고 가정한다면,

1년 뒤의 시뮬레이션된 금리 산출과정은 다음과 같다.

$$\mathrm{P_{HW}}(0,1,1.25) = \frac{P(0,1.25)}{P(0,1)}\exp\left(-r_1 \times B(1,1.25) + \frac{1}{2}QVTerm(1,1.25)\right)$$

$$\mathrm{P_{HW}}(0,1,5) = \frac{P(0,1.5)}{P(0,1)}\exp\left(-r_1 \times B(1,5) + \frac{1}{2}QVTerm(1,5)\right)$$

$$\mathrm{P_{HW}}(0,1,1.75) = \frac{P(0,1.75)}{P(0,1)}\exp\left(-r_1 \times B(1,1.75) + \frac{1}{2}QVTerm(1,1.75)\right)$$

$$\mathrm{P_{HW}}(0,1,2) = \frac{P(0,2)}{P(0,1)}\exp\left(-r_1 \times B(1,2) + \frac{1}{2}QVTerm(1,2)\right)$$

$$\mathrm{P_{HW}}(0,1,2.25) = \frac{P(0,2.25)}{P(0,1)}\exp\left(-r_1 \times B(1,2.25) + \frac{1}{2}QVTerm(1,2.25)\right)$$

$$\mathrm{P_{HW}}(0,1,2.5) = \frac{P(0,2.5)}{P(0,1)}\exp\left(-r_1 \times B(1,2.5) + \frac{1}{2}QVTerm(1,2.5)\right)$$

$$\mathrm{P_{HW}}(0,1,2.75) = \frac{P(0,2.75)}{P(0,1)}\exp\left(-r_1 \times B(1,2.75) + \frac{1}{2}QVTerm(1,2.75)\right)$$

$$\mathrm{P_{HW}}(0,1,3) = \frac{P(0,3)}{P(0,1)}\exp\left(-r_1 \times B(1,3) + \frac{1}{2}QVTerm(1,3)\right)$$

$$\mathrm{R_{HW}^{\mathit{Swap}}}(0,1,3) = \frac{1 - P_{HW}(0,1,3)}{\sum_1^8[\Delta T_i \times P_{HW}(0,1,T_i)]}$$

7.