1 Model Setup

Consider an underlying asset following a geometric Brownian motion with time dependent but deterministic parameters, in the risk-neutral measure $\mathbb{Q}$ induced by the money market numeraire $\beta(t) = \exp\left(\int_0^t r(u)du\right)$,

$$\frac{dS(t)}{S(t)} = (r(t) - q(t))\,dt + \sigma(t)dW^{\mathbb{Q}}(t), \tag{1}$$

where r, q, and σ are the risk free rate, the dividend yield, and the volatility, respectively. The forward function for the underlying can be explicitly calculated,

$$F(t) = \mathrm{E}^{\mathbb{Q}}\left[S(t)\right] = S(0)\exp\left(\int_0^t (r(u) - q(u))\,du\right). \tag{2}$$

Using this deterministic forward function, the underlying dynamics can be reformulated in terms of $X(t) = \log\left(S(t)/F(t)\right)$, which leads to

$$dX(t) = -\frac{1}{2}\sigma(t)^2 dt + \sigma(t)dW^{\mathbb{Q}}(t). \tag{3}$$

2 Early Exercise Premium

Consider an American option price, $V_A(t, S(t))$, with strike K and maturity T, for an underlying asset following the Black-Scholes dynamics in the risk neutral measure,

$$dS(t) = (r(t) - q(t))S(t)dt + \sigma(t)S(t)dW(t), \tag{4}$$

with $S(0) = S$. Define the forward curve as $F(t) = S\exp(\int_0^t (r(u) - q(u))du)$, and $x(t) = \log(S(t)/F(t)$, then the dynamics of $x(t)$ is given by

$$dx(t) = -\frac{1}{2}\sigma(t)^2 dt + \sigma(t)dW(t). \tag{5}$$

References

[1] Leif Andersen, Mark Lake and Dimitri Offengenden, *High-performance American option pricing*, Journal of Computational Finance 20(1):39-87 (2016).