

Consider a dividend paying stock. On an ex-dividend date, the stock prices before and after the dividend payment are related by

$$S(t+) = S(t-)(1 - y) - c, \quad (1)$$

where y and c are the relative and absolute dividends, respectively. Assuming constant risk free rate r and dividend yield q , the forward stock price in the presense of multiple dividends is

$$F(t, T) = S(t)e^{(r-q)(T-t)} \prod_{t \leq t_i \leq T} (1 - y_i) - \sum_{t \leq t_j \leq T} c_j e^{(r-q)(T-t_j)} \prod_{t_j < t_i \leq T} (1 - y_i), \quad (2)$$

where t_i 's and t_j 's are the relative and discrete dividend payment times.

Define the multiplicative forward factor between two times as

$$M(t, T) = e^{(r-q)(T-t)} \prod_{t \leq t_i \leq T} (1 - y_i), \quad (3)$$

and denote the price of a European call option written on the stock as a function of discrete dividends,

$$\text{Call}(C_i) \equiv \text{Call}(S(0), K; C_i), \quad (4)$$

the sensitivity of the European call price with respect to the discrete dividends in the vanishing limit is [1]

$$\begin{aligned} & \frac{\partial^k \text{Call}}{\partial C_{i_1} \cdots \partial C_{i_k}}(\mathbf{0}) \\ = & (-)^k \frac{\partial^k \text{Call}_{BS}}{\partial S^k} \left(S(0) e^{-\sigma^2 \sum_{p=1}^k T_{i_p}}, K, T \right) \prod_{p=1}^k M(0, T_{i_p})^{-1} e^{-\sigma^2 \sum_{p=2}^k (p-2) T_{i_p}}, \end{aligned} \quad (5)$$

where

$$\text{Call}_{BS}(S(0), K, T) = P(0, T)(S(0)M(0, T)N(d_+) - KN(d_-)), \quad (6)$$

is the vanilla call option price without any discrete dividend, with $P(0, T)$ as the discount factor between time 0 and T ,

$$d_{\pm} = \frac{1}{\sigma\sqrt{T}} \log \left(\frac{S(0)M(0, T)}{K} \right) \pm \frac{1}{2} \sigma \sqrt{T}, \quad (7)$$

and $N(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2} du$ is the standard normal cumulative distribution function.

Following [1], define a proxy European option price as

$$\text{Proxy}(C_1, \dots, C_k) = \text{Call}_{BS}(S^*(C_1, \dots, C_k), K^*(C_1, \dots, C_k), T), \quad (8)$$

where, up to second order,

$$S^*(C_1, \dots, C_k) = S(0) + \sum_{i=1}^k \left(a_i C_i + \frac{1}{2} a_{ii} C_i^2 \right) + \sum_{i=1}^k \sum_{j>i}^k a_{ij} C_i C_j, \quad (9)$$

and

$$K^*(C_1, \dots, C_k) = K + \sum_{i=1}^k \left(b_i C_i + \frac{1}{2} b_{ii} C_i^2 \right) + \sum_{i=1}^k \sum_{j>i}^k b_{ij} C_i C_j. \quad (10)$$

We can determine the coefficients by two relations. First, the sensitivity of the proxy in the vanishing dividend limit should coincide with that of the real option price, *i.e.*,

$$\frac{\partial^k \text{Proxy}}{\partial C_{i_1} \dots \partial C_{i_k}}(\mathbf{0}) = \frac{\partial^k \text{Call}}{\partial C_{i_1} \dots \partial C_{i_k}}(\mathbf{0}). \quad (11)$$

For both sides of the above equation, we have closed form expressions. The other relation is the call-put parity,

$$P(0, T) (S^* M(0, T) - K^*) = S(0) - P(0, T) K - \sum_{i=1}^k P(0, T_i) C_i. \quad (12)$$

References

- [1] Arnaud Gocsei and Fouad Sahel, *Analysis of the sensitivity to discrete dividends: A new approach for pricing vanillas*, arXiv:1008.3880.