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Numerix Credit Curve Stripping

June 2015

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This document describes the Numerix methodology for constructing survival probabilities from a quoted credit default swap curve.

1 Introduction

A credit default swap (CDS) is an instrument that is designed to transfer default risk (for example, on a loan) from one party to another. Suppose one party has a defaultable contract with a third party, called the *reference entity*, and wishes to protect itself against the risk of the reference entity defaulting. This party may enter into a CDS (as the CDS buyer) with a counterparty (the CDS seller), who takes on the risk of default by the reference entity. To pay for this protection, the CDS buyer makes a series of payments to the seller over the life of the swap (e.g., quarterly payments for five years) or until default occurs. This stream of payments is known as the *premium leg*, and the size of the payments is called the *default swap spread*. If the reference entity defaults, then the premium leg ends and the CDS seller makes a payment to the CDS buyer, and this cashflow is known as the *protection leg*. The CDS seller compensates the CDS buyer for the loss based on some factor $0 \leq R \leq 1$ of the asset's face value, which represents the fraction of the asset value that can be recovered if default occurs and is called the *recovery rate*.

A CDS is valued like other swaps: by taking the expected value of all future cash flows. In order to calculate this expected value, we must have a way of measuring the probability that the reference entity will default at any given time. In general, the more likely a reference entity is to default, the more expensive the protection against that event will be. These probabilities, however, are not directly observed. What is observed is the prices of CDSs that are traded on the market, and these prices are assumed to implicitly contain an estimate of the default probabilities of the reference entities. We therefore obtain default probabilities from CDS quotes.

The purpose of this paper is to explain the method by which default probabilities can be computed from quoted CDS spreads. We first discuss how a CDS is valued in terms of the probabilities of default, and then we use this relationship to develop a bootstrapping algorithm where a survival probability curve is constructed from a set of CDS spread quotes.

2 Valuing Credit Default Swaps

A CDS consists of two streams of payments: the premium leg and the protection leg. To value a CDS, we take the present value of the expectation of each leg. The payments made depend on whether or not the reference entity defaults, and so we need a measurement of the probability of a default event. Let $\lambda(t)$ denote the *hazard rate* at time t, which is defined to be the function such that $\lambda(t)\Delta t$ is the probability at time t that the entity defaults in the interval $[t, t+\Delta t)$, conditional on the entity surviving until t. That is, if an entity has not defaulted before t, then it has a probability of $1-\lambda(t)\Delta t$ of staying alive until time $t+\Delta t$ and a probability of $\lambda(t)\Delta t$ of defaulting before $t+\Delta t$. The survival probability at time t, denoted *SP*(t), is the probability that the entity has survived until time t. The quantities *SP*(t) and $\lambda(t)$ are related by

$$SP(t) = e^{-\int_0^t \lambda(u)du}. \tag{2.1}$$

Consider first the premium leg, wherein the CDS buyer makes a stream of payments on dates $\{t_1, \ldots, t_N\}$,[1] as long as the reference entity has not defaulted before the payment date.

[1]Here t_N is the maturity date of the CDS.

We assume a constant spread s. The present ($t = 0$) value of the premium leg is then

$$\text{PV}_{\text{Premium Leg}} = s \sum_{n=1}^{N} \alpha(t_{n-1}, t_n) P(0, t_n) \textit{SP}(t_n), \tag{2.2}$$

where $\alpha(t_{n-1}, t_n)$ denotes the day-count fraction between t_{n-1} and t_n, and $P(0, t_n)$ is the time-zero price of a zero-coupon bond that matures at t_n. Note the presence of the term $\textit{SP}(t_n)$, which represents the probability that the reference entity has survived to time t_n.

Note that (2.2) does not take into account the accrual of premium payments between payment dates: if the reference entity defaults on some date $t_n < t < t_{n+1}$, then the last payment was made on t_n and the remaining time, $t - t_n$, is not included in the payments. A CDS contract can specify that the premium that would be accrued during this period be included, and one can modify (2.2) to include premium accrual in the valuation. See [1] for a discussion of this issue.

Consider now the protection leg. If the reference entity defaults, then the CDS seller pays the CDS buyer a factor (1-R) of the face value of the underlying asset. To value this payment, we realize that if a default occurs on some interval $[t, t + \Delta t]$, then the entity has survived until time t, which occurs with probability $\textit{SP}(t)$, and defaults in $[t, t+\Delta t)$, which occurs with probability $\lambda(t)\Delta t$. The probability of this event is therefore $\textit{SP}(t)\lambda(t)\Delta t$. In the case of default, a payment $(1 - R)$ is made, and this payment is discounted back to $t = 0$, resulting in a present value of

$$\text{PV}_{\text{Protection Leg}} = (1 - R) \int_0^{t_N} P(0, u) \textit{SP}(u) \lambda(u) du. \tag{2.3}$$

For simplicity in calculation, we approximate the integral in (2.3) by assuming that a default event can only occur on some number M of regularly spaced times a year (e.g., $M = 12$ for monthly intervals).[2] Then if the maturity of the swap is t_N, there are Mt_N possible default dates, which we label $\tau_1, \ldots, \tau_{Mt_N}$. Then (2.3) is approximated by

$$\text{PV}_{\text{Protection Leg}} \approx (1 - R) \sum_{m=1}^{Mt_N} P(0, \tau_m) \left(\textit{SP}(\tau_{m-1}) - \textit{SP}(\tau_m) \right), \tag{2.4}$$

where we take $\tau_0 = 0$.

2.1 Breakeven Default Swap Spread

The quoted value of a CDS is the spread that provides a fair value for the swap, which is the spread that equates the valuation of the two legs of the swap. This is called the breakeven spread, and we find it by equating (2.2) and (2.4). That is, we solve

$$s \sum_{n=1}^{N} \alpha(t_{n-1}, t_n) P(0, t_n) \textit{SP}(t_n) = (1 - R) \sum_{m=1}^{Mt_N} P(0, \tau_m) \left(\textit{SP}(\tau_{m-1}) - \textit{SP}(\tau_m) \right)$$

for s, yielding

$$s_{\text{breakeven}} = \frac{(1 - R) \sum_{m=1}^{Mt_N} P(0, \tau_m) \left(\textit{SP}(\tau_{m-1}) - \textit{SP}(\tau_m) \right)}{\sum_{n=1}^{N} \alpha(t_{n-1}, t_n) P(0, t_n) \textit{SP}(t_n)}. \tag{2.5}$$

[2] The minimum unit of time discretization is one day.

3 Bootstrapping Credit Default Swap Curves

Equation (2.5) shows how the quoted spread for a CDS depends on the survival probabilities. However, there are up to $N + Mt_N$ points on the survival probability curve (the values of *SP*(t_n) for $n = 1, \ldots, N$ and the values of *SP*(τ_m) for $m = 1, \ldots, Mt_N$), so a single CDS quote is not enough information to determine the curve *SP*. In fact, without any additional assumptions on *SP*, no finite set of information can completely determine the curve. However, assumptions on the form of *SP* allows us to reduce the degrees of freedom to a finite number, which allows us to use (2.5), together with a set of market quotes, as constraints to construct *SP*. A common practice is to choose some interpolation method on *SP* between nodes of the time grid, for example, loglinear. Using loglinear interpolation on *SP* means interpolating log *SP* linearly, and from (2.1), this is equivalent to using a piecewise constant hazard rate λ. We assume that the recover rate R and the zero-coupon bond curve P are given.

We first note that the sums in (2.5) go up to the maturity t_N. If we have, for example, a one-year swap and a two-year swap with the same inception date, then we can choose our time grid to have the nodes $\{0, 1, 2\}$ (in years). The unknown survival probabilities, $\{SP(1), SP(2)\}$,[3] will determine the entire survival probability curve up to $t = 2$ under loglinear interpolation. To determine the survival probabilities from the quoted spreads, we realize that the breakeven spread equation (2.5) for the two-year swap includes the same points along the survival probability curve as the equation for the one-year swap (i.e., *SP*$(\tau_0), \ldots,$ *SP*(τ_M)). Moreover, the equation for the one-year swap has a single unknown—the single constraining value *SP*(1) that determines the survival probability curve between $t = 0$ and $t = 1$. Therefore, we can use (2.5) for the one-year swap to solve for the value of *SP*(1), which constrains *SP*(t) for $t \in [0, 1]$ (again, under loglinear interpolation). This value can then be used in (2.5) for the two-year swap to solve for *SP*(2), which is the only unknown. If we added a third instrument, say a five-year swap, then we could add a third node $t = 5$ and use the values of the survival probability curve from the previous steps to solve (2.5) for *SP*(5), which determines the value of *SP*(t) for $t \in [2, 5]$. This process is known as bootstrapping, because the solution from the current step provides the means to find the solution in the next step.

To make the algorithm explicit, suppose we have L instruments with maturities $t_{N_1} < t_{N_2} < \cdots < t_{N_L}$ with quoted spreads $s_1, s_2, \ldots, s_L$. Assume that the recover rate R, the zero-coupon bond curve P (used for discounting), the number of possible default times per year M, and all payment dates $t_{k,1}, t_{k,2}, \ldots, t_{k,N_k}$ for $k = 1, \ldots, L$ are all given. We assume that log *SP* is interpolated linearly between the maturity nodes $\{t_{N_1}, \ldots, t_{N_L}\}$. We proceed by the following algorithm:

1. Compute *SP*(t_{N_1}) by numerically solving

$$s_1 = \frac{(1 - R)\sum_{m=1}^{Mt_{N_1}} P(0, \tau_m)\,(SP(\tau_{m-1}) - SP(\tau_m))}{\sum_{n=1}^{N_1} \alpha(t_{1,n-1}, t_{1,n}) P(0, t_{1,n}) SP(t_{1,n})},$$

   where $\tau_m = m/M$. Note here that all the values of *SP* are determined by loglinear interpolation between *SP*$(0) = 0$ and the unknown value *SP*(t_{N_1}).

2. For k from 1 to L:

[3] Note that *SP*$(0) = 1$.

Using the values of $\mathit{SP}(t)$ for $t \in [0, t_{N_{k-1}})$ from the previous steps, compute $\mathit{SP}(t_{N_k})$ by numerically solving

$$s_{k+1} = \frac{(1-R)\sum_{m=1}^{Mt_{N_k}} P(0,\tau_m)\left(\mathit{SP}(\tau_{m-1}) - \mathit{SP}(\tau_m)\right)}{\sum_{n=1}^{N_k} \alpha(t_{k,n-1}, t_{k,n}) P(0, t_{k,n}) \mathit{SP}(t_{k,n})},$$

where $\tau_m = m/M$. Note that $\mathit{SP}(t)$ is determined for $t \in [t_{N_{k-1}}, t_{N_k})$ by the value $\mathit{SP}(t_{N_{k-1}})$ (which is known by the previous step) and $\mathit{SP}(t_{N_k})$.

3. The curve *SP* is given by loglinear interpolation between the maturity nodes $t_{N_1}, \ldots, t_{N_L}$:

$$\mathit{SP}(t) = \mathit{SP}(t_{N_{k-1}})^{\frac{t-t_{N_k}}{t_{N_{k-1}}-t_{N_k}}}\, \mathit{SP}(t_{N_k})^{\frac{t-t_{N_{k-1}}}{t_{N_k}-t_{N_{k-1}}}}$$

for $t \in [t_{N_{k-1}}, t_{N_k}]$.

4 Numerix Example

We provide an example of stripping a CDS curve in Numerix CrossAsset. We consider four CDS instruments with maturities of one, two, five, and ten years. Figure 1 shows the data for the CDSs.

CDS	Tenor	Recovery	Payment Frequency	Market Spread Quote
CDS_1Y	1Y	0.3	3M	0.0099
CDS_2Y	2Y	0.3	3M	0.0089
CDS_5Y	5Y	0.3	3M	0.0061
CDS_10Y	10Y	0.3	3M	0.0045

ID	CDS_2Y
OBJECT	INSTRUMENT
TYPE	CDS
CURRENCY	USD
RECOVERY	0.3
END TENOR	2Y
SPREAD	0.0089
FREQ	3M
Updated	2355 @ 02:11:21 PM
Timer	6.54429E-05
ID	CDS_2Y

Figure 1: Data for the USD CDS instruments and a CrossAsset object for the 2Y CDS.

The CDS instruments are collected in an `Instrument Collection` object, and a `Credit Market Data` object is created to construct the survival probability curve by specifying the interpolation method and variable. Figure 2 shows an image of this object. The method specified is `LogLinear` and the interpolation variable is `SP`, which indicates that interpolate loglinearly on survival probabilities (or, equivalently, linearly on the log of survival probabilities). The yield curve used is a constant 2%.

ID	SPCurveFromSpreadQuotes
OBJECT	MARKET DATA
TYPE	CREDIT
INTERP METHOD	LogLinear
INTERP VARIABLE	SP
NOWDATE	6/1/2015
CURRENCY	USD
INSTRUMENTS	CDS_Collection
DOMESTIC YIELD CURVE	ConstYield
BASIS	ACT/360
RECOVERY FREQUENCY	3M
Updated	2369 @ 02:13:58 PM
Timer	0.001206459
ID	SPCURVEFROMSPREADQUOTES

Figure 2: A `Market Data` object in Numerix CrossAsset for stripping a survival probability curve from a collection of instruments. The `CDS_Collection` object contains the four CDS instruments described in Figure 1.

If we view this object, we can see the survival probabilities found for the maturity dates of the four input instruments, which are shown in Figure 3. To see the survival probability curve for other dates, we use the `nxLibGetSurvProb` function, whose call signature is `nxLibGetSurvProb(CreditCurveID, StartDate, EndDate)`. The input dates are computed with a `nxLibGetDCF` function with an ACT/360 basis for one-week increments from 01-Jan-2015 to 01-Jan-2025. A plot of the resulting survival probabilities is shown in Figure 4.

NAME	DATE	SURVIVAL PROBABILITY
CDS 1-jun-2015 1-jun-2016	01-Jun-2016	0.985749720753691
CDS 1-jun-2015 1-jun-2017	01-Jun-2017	0.974601441426151
CDS 1-jun-2015 1-jun-2020	01-Jun-2020	0.957408391971637
CDS 1-jun-2015 1-jun-2025	01-Jun-2025	0.939002153366197

Figure 3: Object viewer for the `SPCurveFromSpreadQuotes` object in Figure 2.

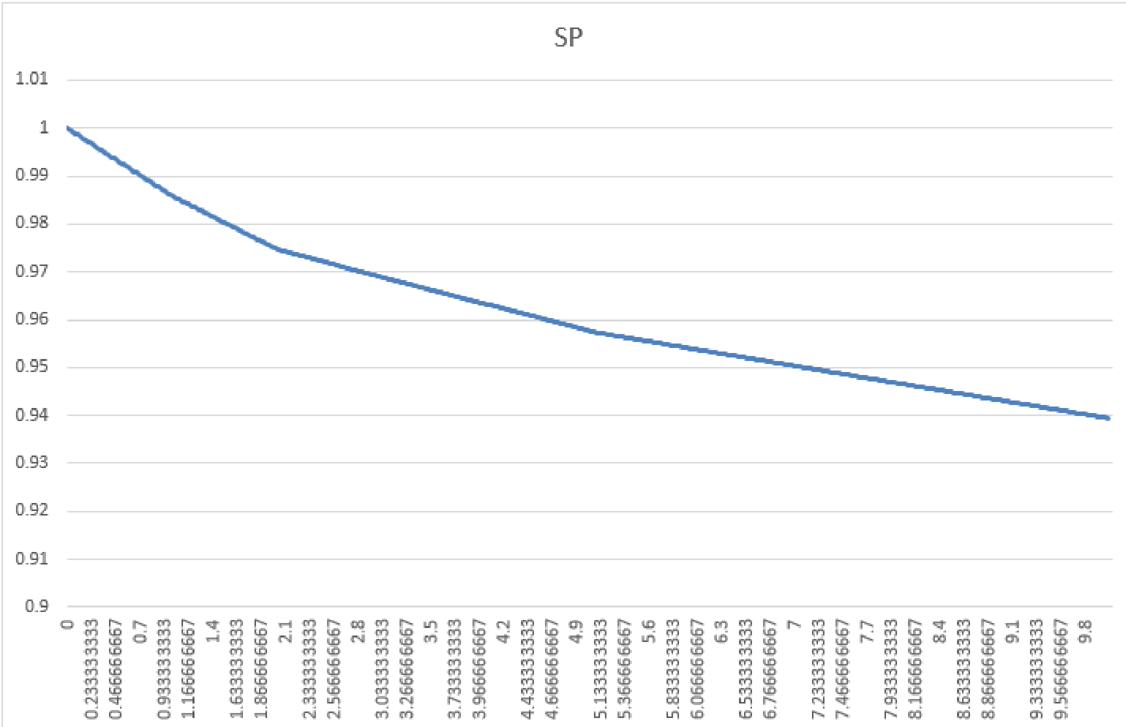

Figure 4: Plot of the survival probabilities obtained from the `SPCurveFromSpreadQuotes` object in Figure 2 up to 10 years.

References

[1] O'Kane, D. Credit derivatives explained. Structured credit research, Lehman Brothers, March 2001.

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