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Pricing K^{th} to Default Basket CDS

Avtar Sehra

January 2014

Contents

1. Introduction

The focus of this work is to price (determine fair spreads) and analyse k^{th} to default basket CDS written on 5 reference names with total notional of $10million ($2million notional principal each) and maturity of T = 5years. A k^{th} to default basket swap has two legs; the premium leg and default leg. The premium leg contains a stream of periodic payments (spread payments), which are paid by the purchaser of protection until either the k^{th} default or maturity, whichever is earlier. If the k^{th} CDS in the basket defaults then the purchaser receives the notional minus the recovery rate and any accrued spread payment. This can be summarised as follows:

1st to Default Swap

- If there is no default the protection seller receives $Spread\ \ \$10m\,\frac{1}{5}$ per year.
- If any one of the reference names defaults before year 5 the protection buyer receives $\$10m\,\frac{1}{5}$ less recovery value and the contract terminates.

K^{th} to Default Swap

- If the K^{th} default occurs the protection buyer receives $\$10m\,\frac{1}{5}$ less recovery value and the contract terminates.
- The reference entities that have defaulted before the K^{th} default, are removed from the portfolio and the notional is reduced by $\frac{1}{5}$ each time, hence reducing the premium leg notional and the resulting payments.

To price the k^{th} to default swap we model the joint distribution of default times using Monte Carlo sampling (using both pseudo random numbers and low discrepancy sequences) with both Gaussian and Students T Copula. The fair spread is then found by calculating the discounted expectation over the joint loss distribution of the Premium and Defaults legs of the swap.

Section 2 gives an overview of the financial and mathematical preliminaries required for pricing K^{th} to default basket swaps.

Section 3 details the methodology used for pricing and provides an overview of the historical market data and calculation of implied model parameters: Probability of Survival and Hazard Rates, as well as correlations and degrees of freedom.

The results of pricing k^{th} to default swap under both Gaussian and Student T Monte Carlo are discussed in section 4.

Sensitivity Analysis is presented in section 5, detailing sensitivity to correlation, CDS levels, recovery rate and interest rates.

2.Background Financial and Mathematical Concepts

In this section a short overview of key mathematical and financial concepts is provided used in the pricing and calibration methodology.

2.1 Poisson Process

All processes and random variables in this work are defined on a complete filtered probability space $(\Omega, (F_t)_{t\geq 0}, \mathbb{P})$. The arrival of a credit event is modelled as a random point in time $\tau \in \mathbb{R}_+$ using the Poisson process. A Poisson process is a stochastic process which counts the number of events and the time that these events occur in a given time interval. The time between each pair of consecutive events has an exponential distribution with parameter λ and each of these interarrival times is assumed to be independent of other interarrival times. The probability of k events (such as defaults) in an interval can be written as:

$$\mathbb{P}[N(t+\tau) - N(t) = k] = e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!}, \tag{1}$$

Where $N(t_n)$ is the number of defaults up to time t, λ is the rate parameter (default intensity), and $\tau = t_n - t_{n-1}$ is the interarrival time, where

$$t_n = \sum_{i=1}^{n} \tau_i \tag{2}$$

As $\tau \to 0$ we find that Eqn. (1) becomes

- $For\ k = 0\ \mathbb{P}(0\ Events) = 1 - \lambda\tau$
- $For\ k = 1\ \mathbb{P}(1\ Event) = \lambda\tau$
- $For\ k > 1\ \mathbb{P}(> 1\ Event) = 0$

We can now denote $F(\tau)$ as the Cumulative Distribution Function (CDF) of τ, and define the probability of no event occurring in the interval τ as the survival function

$$S(\tau) = 1 - F(\tau). \tag{3}$$

Therefore, we can write the probability of surviving up to time t and immediately defaulting

$$\mathbb{P}(t < \tau \leq t + dt | \tau > t) = (1 - \lambda dt)^{n_t} \lambda dt = \lambda dt + O(dt^2). \tag{4}$$

Then using the relationship from conditional probability, $P(B|A) = \frac{P(A\cap B)}{P(A)}$, Eqn. (4) can be written as

$$\begin{aligned}\lambda &= \lim_{dt\to 0} \frac{\mathbb{P}(t < \tau \leq t + dt)}{dt\mathbb{P}(\tau > t)} \\ &= \lim_{dt\to 0} \frac{[1 - S(t + dt)] - [1 - S(t)]}{dtS(t)} \\ &= \lim_{dt\to 0} \frac{S(t) - S(t + dt)}{dtS(t)}\end{aligned}$$

$$= -\frac{d \log S(t)}{dt}$$

Solving the above ODE for S with boundary condition $S(0) = 1$, we obtain the survival probability function as

$$S(t) = e^{-\int_0^t \lambda(u)du}, \tag{5}$$

And the probability of default as

$$F(t) = 1 - S(t) = 1 - e^{-\int_0^t \lambda(u)du}. \tag{6}$$

2.2 Default Time

From Eqn. (6) we have $U = F(t) = 1 - S(t) = 1 - e^{-\int_0^t \lambda(u)du}$, then rearranging

$$\Rightarrow \int_0^t \lambda(u)du = -\log(1 - U). \tag{7}$$

Using this form the default time τ can be determined as

$$\tau = inf\left\{t > 0: \log(1 - U) \geq -\int_0^t \lambda(u)du,\right\} \tag{8}$$

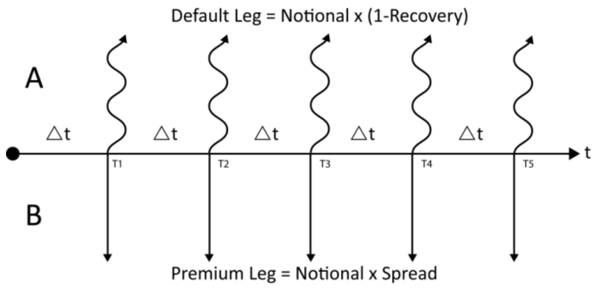
Discretising this and setting Δt to a constant over 0 to t, we have

$$\tau = inf\left\{t > 0: \log(1 - U) \geq -\Delta t \sum_{i=1}^{t} \lambda_i\right\} \tag{9}$$

We would use this inequality to calculate the default time iteratively, by summing up λ_i until the inequality begins to hold, i.e. if the inequality holds after adding λ_n then default would occur between t_{n-1} and t_n.

2.3 CDS Pricing

A buyer of a CDS, A, purchases protection against default of a reference security, and in return pays a periodic payment to the seller B, see figure below. Periodic payments continue until maturity or until reference entity defaults, in which event seller pays to buyer the loss on default of the reference security. The payment spread is chosen to make the premium and default legs present values equal to each other $PV\ (Premium) = PV\ (Default)$.



CDS Cashflows over 5 periods between protection buyer, A, and protection seller, B.

For N periods we can write the expected present value of the premium leg payments as

$$PV(Premium) = \sum_{i=1}^{N} Spread_N \;\; Notional\, D(0,T_i)\; S(T_i)\; \Delta t_i,$$

In a similar way we can write the expected presented value of the default leg payments as

$$PV(Default) = \sum_{i=1}^{N} (1-R)\; D(0,T_i)\; [S(T_{i-1}) - \mathrm{S}(\mathrm{T_i})],$$

where R is the recovery rate.

Now equating the Premium and Default legs we can determine the fair spread for a Nperiod CDS as

$$Spread_N = \frac{\sum_{i=1}^{N}(1-R)\; D(0,T_i)\; [S(T_{i-1}) - \mathrm{S}(\mathrm{T_i})]}{\sum_{i=1}^{N}\; Notional\, D(0,T_i)\; S(T_i)\; \Delta t_i}, \qquad (10a)$$

Written in terms of the default probability, $F(T_i)$, this can be written as

$$Spread_N = \frac{\sum_{i=1}^{N}(1-R)\; D(0,T_i)\; [F(T_i) - \mathrm{F}(\mathrm{T_{i-1}})]}{\sum_{i=1}^{N}\; Notional\, D(0,T_i)\; \big(1-F(T_i)\big)\; \Delta t_i}, \qquad (10b)$$

This will be the structure used in pricing the Kth to default swap.

2.4 Bootstrapping CDS Survival Probabilities and Intensities

CDS spreads and discount factors are directly observable quantities, therefore Eqn. (10) can be used to bootstrap the survival/default probabilities, which are not directly observable. For N periods you would use N CDS's of increasing maturity, and each with a spread S_N for the bootstrapping process, which yields the following iterative equation

$$S(T_N) = \frac{\sum_{i=1}^{N-1} D(0,T_i)\; [L\, S(T_{i-1}) - (L + \Delta t_i\; Spread_N)\; S(T_i)]}{\sum_{i=1}^{N}\; Notional\; D(0,T_N)\; (L + \Delta t_i\; Spread_N)} + \frac{S(T_{N-1})L}{L + \Delta t_N Spread_N}, \quad (11)$$

Where $L = 1 - R$ is the loss given default. So using Eqn. (11) we can start with $S(T_1)$ and go through the bootstrapping process to arrive at $S(T_N)$.

Using the survival probabilities, Eqn. (5) can now be utilised to back out the default intensity term structure $\lambda_i \;\; i = 1, \dots, N$. Which can be done by discretising Eqn. (5) to iteratively determine piecewise constant intensities by starting with $S(T_1)$ and moving up to $S(T_N)$, giving

$$\lambda_N = \frac{1}{(\mathrm{T_i} - T_{i-1})}\left(-\ln S(T_N) - \sum_{i=1}^{N-1} \lambda_i (\mathrm{T_i} - T_{i-1})\right). \qquad (12)$$

2.5 Copula Overview

For n uniform random variables, $U_1, U_2, \cdots, U_n$, with correlation $\mathbf{\Sigma}$ the joint distribution '*copula function'*, $\boldsymbol{C}$**,** is defined as

$$\boldsymbol{C}(U_1, U_2, \cdots, U_n|\mathbf{\Sigma}) = \mathbb{P}(u_1 \leq U_1, u_2 \leq U_2, \ldots, u_n \leq U_n), \tag{13}$$

A copula function can link univariate marginal distributions to their full multivariate distribution. Skalars theorem shows that if $F(x_1, x_2, \cdots, x_m)$ is a joint multivariate distribution function with univariate marginal distribution functions $U_1 = F_1(x_1)$, $U_2 = F_2(x_2), \cdots, U_n = F_n(x_n)$, then there exists a copula function $\boldsymbol{C}(U_1, U_2, \cdots, U_m|\mathbf{\Sigma})$ such that

$$\boldsymbol{F}(x_1, x_2, \cdots, x_n) = \boldsymbol{C}(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)|\mathbf{\Sigma}), \tag{14}$$

Where $\mathbf{\Sigma}$ is the correlation between the univariate distributions; also if each F_i is continuous then $\boldsymbol{C}$ is unique. Thus, copula functions provide a unifying and flexible way to work with multivariate distributions. That is if we know the marginal distributions and their correlations we can construct the joint distribution through use of an appropriate copula function. In this work the joint default probabilities are modelled using Gaussian and Student T copula functions.

2.5.1 Gaussian Copula

The multivariate Gaussian Copula Function can be expressed as:

$$\boldsymbol{C}(U_1, U_2, \ldots, U_n|\mathbf{\Sigma}) = \mathbf{\Phi}_n(\Phi^{-1}(U_1), \Phi^{-1}(U_2), \ldots, \Phi^{-1}(U_n)|\mathbf{\Sigma}), \tag{15}$$

Where $\mathbf{\Phi}_n$ is the CDF for the multivariate standard Normal distribution. The n dimensional Gaussian Copula density (in terms of uniform variables U) is given by

$$\boldsymbol{C}(U_1, U_2, \ldots, U_n|\mathbf{\Sigma}) = \frac{1}{\sqrt{|\mathbf{\Sigma}|}} e^{-\frac{1}{2}\mathbf{\Phi}^{-1}(\boldsymbol{U})'(\mathbf{\Sigma}^{-1}-\mathbf{1})\,\mathbf{\Phi}^{-1}(\boldsymbol{U})}. \tag{16}$$

Using the Gaussian Copula preserves the underlying distribution of the individual random variables but the joint distribution is a multidimensional Gaussian.

2.5.2 Student T Copula

In reality tail events occur in the financial markets more frequently than are modelled by the normal distribution. The Student T Copula provides a joint distribution which has fatter tails but preserves the same bell shaped, non-skewed) characteristics of the Gaussian. With heavier tails and parameterized with degrees of freedom v the multivariate Student T Copula function can be expressed as

$$\boldsymbol{C}(U_1, U_2, \ldots, U_n; \nu; \mathbf{\Sigma}) = \mathbf{T}_v(\mathrm{T}_v{}^{-1}(U_1), \mathrm{T}_v{}^{-1}(U_2), \ldots, \mathrm{T}_v{}^{-1}(U_n); \mathbf{\Sigma}), \tag{17}$$

Where $\mathbf{T}_v$ and T_v is the multivariate and univariate CDFs for the standard Student's T distribution, with the degrees of freedom parameter v. The n dimensional Student T Copula density (in terms of uniform variables U) is given by

$$\boldsymbol{C}(U_1, U_2, \dots, U_n; v; \boldsymbol{\Sigma}) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}|}} \frac{\Gamma\left(\frac{v+n}{2}\right)}{\Gamma\left(\frac{v}{2}\right)} \left(\frac{\Gamma\left(\frac{v}{2}\right)}{\Gamma\left(\frac{v+1}{2}\right)}\right)^n \frac{\left(1 + \frac{\boldsymbol{T}_v^{-1}(\boldsymbol{U})' \boldsymbol{\Sigma}\, \boldsymbol{T}_v^{-1}(\boldsymbol{U})}{v}\right)^{-\frac{v+n}{2}}}{\prod_{i=1}^{n} \left(1 + \frac{T_v^{-1}(U_i)^2}{v}\right)^{-\frac{v+1}{2}}}, \qquad (18)$$

Where $\Gamma(v)$ is a Gamma Function.

2.5.3 Estimating Student T Degrees of Freedom (d.f.)

The degrees of freedom, v, are estimated using the Maximum Likelihood procedure. This is achieved by the two stage process:

1. Calculate log-likelihood function of Eqn. (18) for a daily set of observations, N_{obs}.
2. Sum the log-likelihood functions and maximise w.r.t v.

This can be summarised as

$$\max_{v} \left\{ \sum_{i=1}^{N_{obs}} \log c(\boldsymbol{U}_i; v; \boldsymbol{\Sigma}) \right\} \qquad (19)$$

2.6 Copulas and Joint Default Probability

If an entity i defaults within time $\tau_i \leq t_i$ with probability

$$U_i = F_i(t_i), \qquad (20)$$

The joint probability of default for n entities can be written as

$$\boldsymbol{F}(t_1, t_2, \dots, t_n) = \mathbb{P}(\tau_1 \leq t_1, \tau_2 \leq t_2, \dots, \tau_n \leq t_n). \qquad (21)$$

Combining Eqns. (20) and (21) we have

$$\boldsymbol{F}\left(F_1^{-1}(U_1), F_2^{-1}(U_2), \dots, F_n^{-1}(U_n)\right) = \mathbb{P}(u_1 \leq U_1, u_2 \leq U_2, \dots, u_n \leq U_n), \qquad (22)$$

And using the definition of copula in Eqn. (13) we can write

$$\boldsymbol{F}\left(F_1^{-1}(U_1), F_2^{-1}(U_2), \dots, F_n^{-1}(U_n)\right) = \boldsymbol{C}(U_1, U_2, \dots, U_n | \boldsymbol{\Sigma}), \qquad (23)$$

So the joint probability of default can be modelled using an appropriate copula function $\boldsymbol{C}$.

3.Pricing Methodology and Model Calibration

The k[th] to default basket CDS swap with T=5year maturity consists of 5 reference entities: IBM, HP, Dell, Sony and Samsung. The default intensity term structure, correlation matrix and student T d.f. for these entities is determined from historical CDS spreads, and these parameters will are used as key inputs into the pricing model along with other market data such as discount rates and recovery rates.

3.1 Pricing Methodology

Pricing a K[th] to default swap follows a similar mathematical structure to pricing a CDS, as was shown in section 2.3, Eqn. (10b). The spread of a K[th] to default swap is computed by equating the expected valued of the discounted Premium Leg (PL) with the expected value of the discounted Default Leg (DL), under the risk neutral measure

$$PL = Spread_k\ Notional\ \Delta t\ \sum_{i=1}^{m} Z(0,\boldsymbol{t}_i)\big(1 - F_k(\boldsymbol{t}_i)\big),$$

$$DL = (1-R) Notional\ \sum_{i=1}^{m} Z(0,t_i)\big(F_k(\boldsymbol{t}_i) - F_k(\boldsymbol{t}_{i-1})\big).$$

Assuming the instruments exist over m periods, the fair spread of the k[th] to default swap is

$$Spread_k = \frac{DL}{PL} = \frac{(1-R)\ Notional\ \sum_{i=1}^{m} Z(0,\boldsymbol{t}_i)\big(F_k(\boldsymbol{t}_i) - F_k(\boldsymbol{t}_{i-1})\big)}{Notional\ \Delta t\ \sum_{i=1}^{m} Z(0,\boldsymbol{t}_i)\big(1 - F_k(\boldsymbol{t}_i)\big)}, \qquad (24)$$

Here $F_k(\boldsymbol{t}) = F_k(t_1, t_2, \dots, t_n)$ is the unknown function giving the k[th] to default probability for the n reference entities. To simplify the calculation process we can use the expected loss function

$$L_k = Notional\ \mathbb{E}[F_k(\boldsymbol{t})], \qquad (25)$$

Which then gives the fair spread for the K[th] to default swap as

$$Spread_k = \frac{DL}{PL} = \frac{(1-R)\ \ \sum_{i=1}^{m} Z(0,\boldsymbol{t}_i)({L^k}_i - {L^k}_{i-1})}{\Delta t\ \sum_{i=1}^{m} Z(0,\boldsymbol{t}_i)(Notional - {L^k}_i)}, \qquad (26)$$

To satisfy the loss distribution expectation operator in Eqn. (25), the fair spread calculation needs to be performed using a large number of simulated default times, and averaged across simulations. For each Monte Carlo simulation, where default times $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ are randomly sampled, Eqn. (26) gives the K[th] to default swap spread as

$$Spread_1 = \frac{(1-R)Z(0,\tau_1)\,\frac{1}{5}}{\tau_1\ Z(0,\tau_1)\frac{5}{5}} = \frac{(1-R)\,\frac{1}{5}}{\tau_1}$$

$$Spread_2 = \frac{(1-R)Z(0,\tau_2)\,\frac{1}{5}}{\tau_1\ Z(0,\tau_1)\frac{5}{5} + (\tau_2 - \tau_1)\ Z(0,\tau_2)\frac{4}{5}}, \qquad (27)$$

And so forth for S_3, S_4 and S_5.

In terms of implementing this as a Monte Carlo model the following algorithm is used

1. Uniform random variables are generated using either Excel Rand or Halton low discrepancy sequences, $\boldsymbol{U} = (U_1, U_2, U_3, U_4, U_5)$.
2. Uniform random variables are converted to standard normal variables using the Peter Acklam Inverse Norm algorithm **,** $\boldsymbol{Z} = (Z_1, Z_2, Z_3, Z_4, Z_5)$.
3. Correlated standard normal variables are generated using Cholesky decomposition of the market data correlation matrix, $\boldsymbol{Z}_c = \boldsymbol{A}\,\boldsymbol{Z}$.
4. Correlated uniform random variables are generated using either Gaussian distribution or Student T distributions (with d.f. ν), $\boldsymbol{U}_c = \Phi(\boldsymbol{Z}_c)$ or $\boldsymbol{U}_c = \mathrm{t}_\upsilon(\boldsymbol{Z}_c)$.
5. Correlated uniform random variables are converted to default times using the method described in section 2.2 – Eqn. (9), $\tau_i = F_i^{-1}(U_{c_i})$.
6. If τ_i is greater than the maturity of the basket swap ($T = 5$) it is equated to 999 as an indicator it does not default within maturity of the product.
7. An ordered default time vector is constructed using the calculated default times, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$.
8. Kth to default spread vector is calculated using the method discussed in the previous section, $\boldsymbol{S} = (S_1, S, S_3, S_4, S_5)$.
9. Above steps are performed a large number of times and the average of the spreads is determined

3.2 Model Calibration

In order to execute the above algorithm a number of key input parameters are required to appropriately calibrate the model for pricing Kth to default basket CDS swaps. For the recovery rate, R, the market standard of 40% was used, and US treasury curve from 15 August 2013 is used for discounting. However, this data can be changed by the user in the 'Market Data Input' worksheet.

The other key input parameters for the reference entities, Probability of Default, Default Intensity Term Structure, correlation Matrix, and the Student T Degrees of Freedom are calculated/estimated from market data.

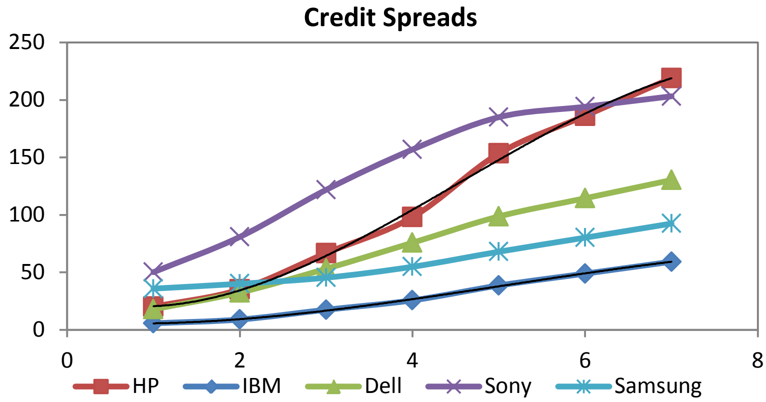
3.2.1 Default Intensity Term Structure

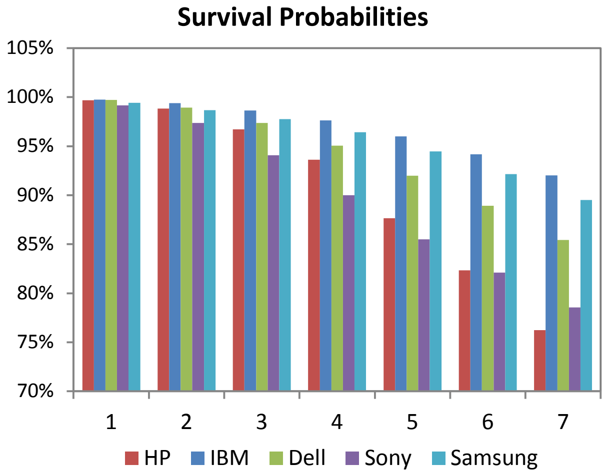
Reference entities used in this work were from the technology industry: IBM, HP, Dell, Sony and Samsung. 1 to 7 year credit spreads were used from 15 August 2013 – extra 2 years included to add more versatility to pricing model. This data is presented in the table below along with the appropriate discount factors for each maturity.

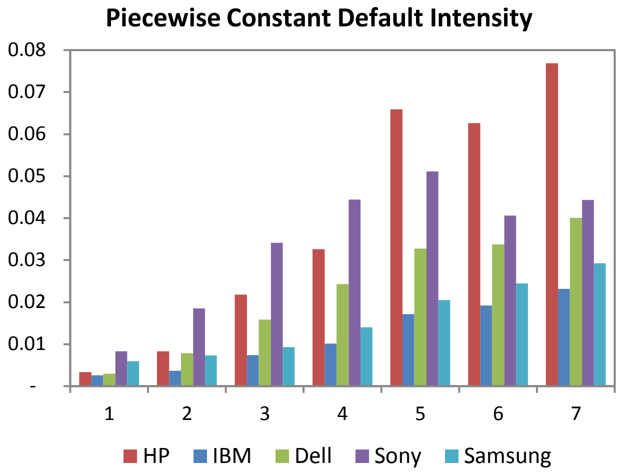
Tenor	CDS Credit Spreads With Applied Shift					Discount Factor With Applied Shift
	IBM	HP	Dell	Sony	Samsung	
0	0	0	0	0	0	1.00000
1	15.60	20.00	17.80	50.10	35.80	0.99871
2	18.90	35.00	32.40	80.80	40.00	0.99571
3	27.40	66.70	53.10	121.80	45.30	0.98810
4	35.60	98.00	75.70	156.90	54.80	0.97366
5	48.50	153.40	98.60	185.00	67.90	0.95249
6	58.85	186.20	114.45	194.10	80.15	0.92579
7	69.20	219.00	130.30	203.20	92.40	0.89909

CSD Spreads for Reference Entities from 15 August 2013

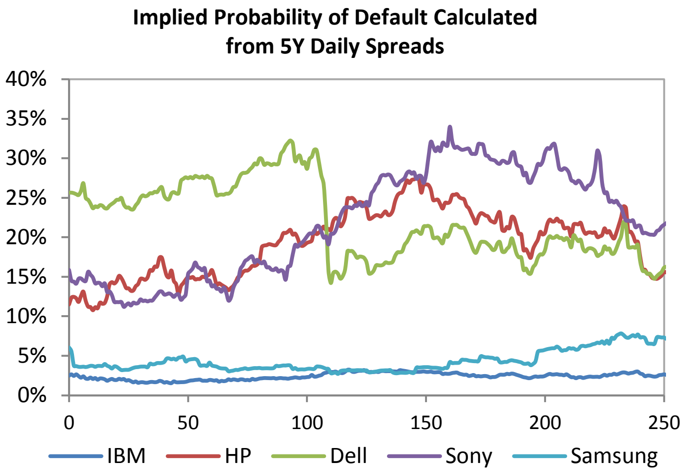
Using the methods described in section 2.4 the default probabilities and piecewise constant intensity term structures were bootstrapped. The output of these steps is summarised below, and the function used for the initial survival probability calculation is in the Excel Workbook – Module 1: `Pt(r As Double, S As Double, T As Integer, p As Variant, d As Variant)`. A full summary of the data and calculation steps can be reviewed in the 'Market Data Input' worksheet.



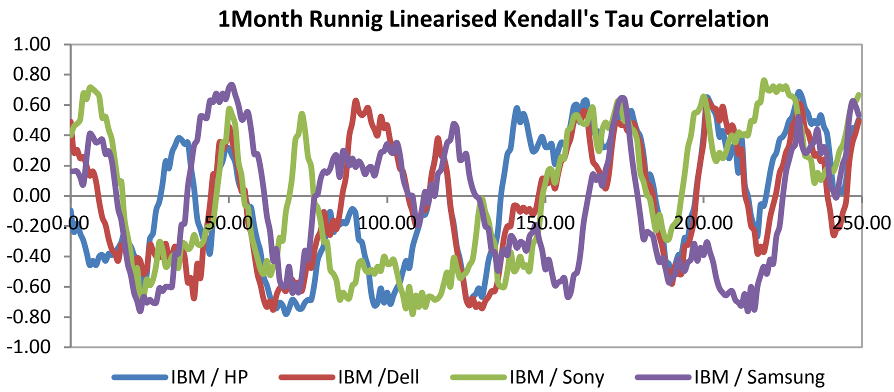




For the next steps in determining the data correlations and estimating the Student T d.f. daily spreads were used for the historical period 15 August 2012 – 15 August 2013. The daily survival/default probabilities for historical data were bootstrapped using the bootstrapping algorithm applied above. This was done for all maturities, and the data corresponding to the 5year spreads was used for the estimation of parameters. The bootstrapped implied probabilities are shown in the chart below.



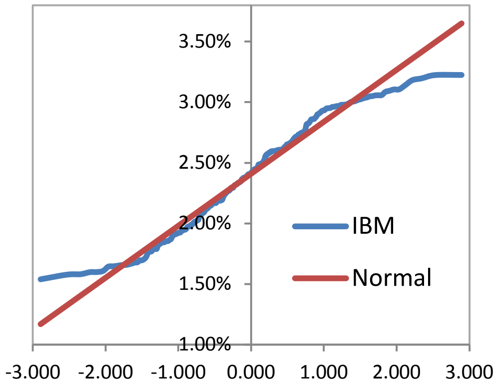
Running correlation analysis (over 1month) was performed on the data, example chart shown below with the 1month running Kendall's Tau correlation. This was performed across all reference name pairs and a summary of this is in the below table. As can be seen there is very little running correlation between the pairs, however this fluctuates with some minor seasonality behaviour – cycling approximately every 50 days.



Running Linearised Kendall's Tau Data	IBM / HP	IBM /Dell	IBM / Sony	IBM / Samsung	HP / Dell	HP / Sony	HP / Samsung	Dell / Sony	Dell / Samsung	Sony / Samsung
Average	-0.03	-0.03	-0.01	-0.06	0.01	0.00	0.01	-0.01	-0.04	-0.04
Standard Deviation	0.41	0.39	0.45	0.41	0.30	0.30	0.29	0.40	0.37	0.37
Max	0.69	0.63	0.76	0.73	0.67	0.64	0.70	0.74	0.67	0.61
Min	-0.78	-0.75	-0.78	-0.76	-0.75	-0.77	-0.65	-0.71	-0.71	-0.70

3.2.2 Student T Degrees of Freedom Estimation

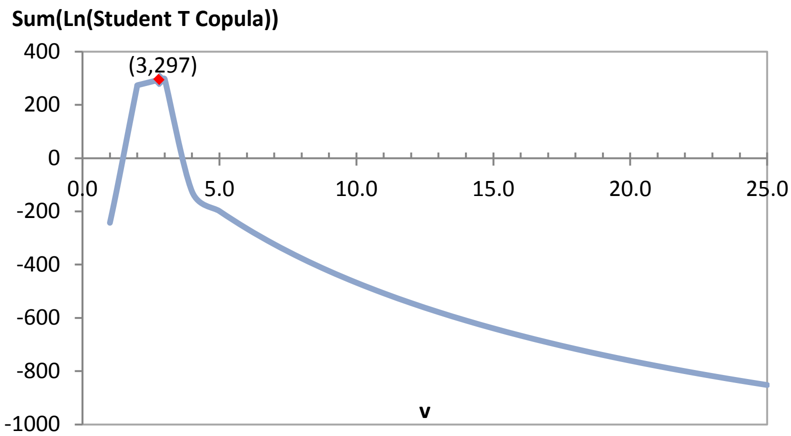
One of the key reasons to utilise the Student T distribution is that the reference entity data used for this work is not strongly correlated, as previously seen, and the data deviates from normality. An example of the deviation from normality can be seen in the QQ plot for IBM show to the left. Edges of the IBM QQ plot deviate significantly from the normal distribution while the middle part of the plot follows the normal line reasonably well. From this we can deduce that while the IBM probability of default data follows the shape of a normal distribution the data has lower kurtosis i.e. fatter tails. A summary of the rest of the data is given in the table below (Note: In Excel Kurtosis for a normal distribution is standardised to 0 so < 0 implies fatter tales than a normal distribution).



	IBM	HP	Dell	Sony	Samsung
Mean	0.024	0.191	0.218	0.217	0.044
Median	0.024	0.201	0.202	0.214	0.038
Standard Deviation	0.004	0.043	0.048	0.067	0.014
Kurtosis	-0.970	-1.094	-1.100	-1.420	-0.328
Skew	-0.117	-0.054	0.376	-0.007	0.993
Minimum	0.015	0.108	0.142	0.112	0.027
Maximum	0.032	0.274	0.322	0.340	0.082
Sum of All Data	6.312	50.076	57.093	56.929	11.631
Total Count of Data	262	262	262	262	262

Reference entity probability of default time series data summary

Using the method described in section 2.5.3 the d.f. for implementing the Student T distribution was determined, which was estimated to be $\nu \approx 3$. The curve generated from this method is shown below, and estimation workings can be seen in the Worksheet titled: 'Degrees of Freedom Workings.



Sample results of Student T Copula d.f. estimation by MLE

3.2.3 Correlation Matrix

As seen in the previous sections the probability of default data stripped from reference entity daily spreads deviates from normal behaviour due to fat tails. As a result linear correlation is not a statistically correct measure to use. For the purposes of this work Kendall's Tau correlation was used. Kendall's Tau is a Rank correlation measure, based on concordance and discordance. It measures the degree to which the large (small) values of one variable associate with the large (small) values of another variable.

Kendall's Tau is defined as

$$\begin{aligned} \rho_k &= \mathbb{P}[(x-x')(y-y')>0]-\mathbb{P}[(x-x')(y-y')<0] \\ &= 4\mathbb{E}[F(x,y)]-1 \end{aligned} \tag{28}$$

Where the pairs (x, y) and (x', y') drawn from a joint distribution $F(x, y)$ are concordant if $x > x'$ and $y > y'$. To estimate Kendall's Tau for n pairs of observations(x_i, y_i), we define a sign function as

$$A_{ij} = sgn(x_i - x_j)(y_i - y_j),$$

Kendall's Tau can then be determined as

$$\rho_k = \mathbb{E}[\boldsymbol{A}] = \frac{2}{n(n-1)}\sum_{i=1}^{n}\sum_{j>i}^{n} A_{ij}. \tag{29}$$

Transformation from Rank to linear correlation can be made using

$$\rho = \sin\left(\frac{\pi}{2}\rho_K\right). \tag{30}$$

A function has been implemented in Excel to determine the Kendall's Tau Correlation for any two sets of data, which is called `KendallTau(xRange As Range, yRange As Range)`. The correlation matrix determined using this function is shown below, along with its linearised version. These highlight the low correlation between most reference entity pairs.

Kendall's Tau	IBM	HP	Dell	Sony	Samsung
IBM	1.00	0.56	-0.36	0.50	-0.02
HP	0.56	1.00	-0.13	0.59	-0.10
Dell	-0.36	-0.13	1.00	-0.23	-0.18
Sony	0.50	0.59	-0.23	1.00	0.09
Samsung	-0.02	-0.10	-0.18	0.09	1.00

Linearised Kendall's Tau	IBM	HP	Dell	Sony	Samsung
IBM	1.00	0.77	-0.54	0.71	-0.04
HP	0.77	1.00	-0.21	0.80	-0.16
Dell	-0.54	-0.21	1.00	-0.35	-0.27
Sony	0.71	0.80	-0.35	1.00	0.14
Samsung	-0.04	-0.16	-0.27	0.14	1.00

Kendall's Tau Correlation Matrices for pairs of Reference Entities

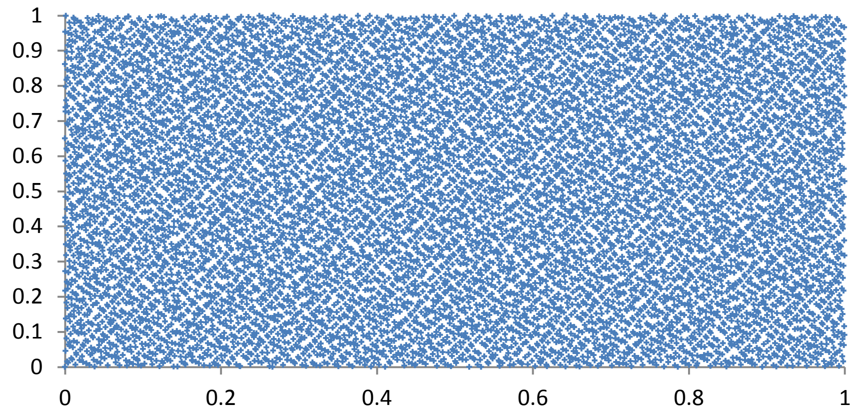
The linearised correlation matrix is decomposed using a Cholesky decomposition, $\boldsymbol{\Sigma} = \boldsymbol{A}^{\boldsymbol{T}}\,\boldsymbol{A}$, and the matrix $\boldsymbol{A}$ is then used to correlate the random variables in the pricing process.

4. Numerical Results

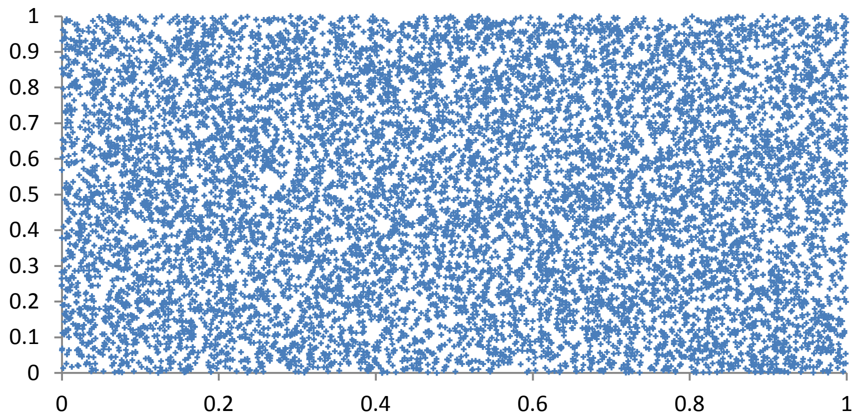
4.1 Generating Uniform Random Variables

The uniform random variables are generated using Excel Rand() and Halton (Generating Function is `HaltonBaseb(b As Long, N As Long)`). It can be observed, in the charts below, the Halton method avoids the clustering effect by filling in the space homogenously compared to Excel's random number generator.

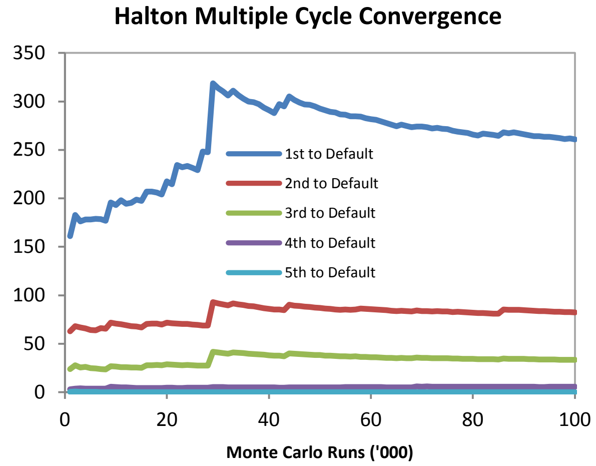
Halton Base 3 and 7

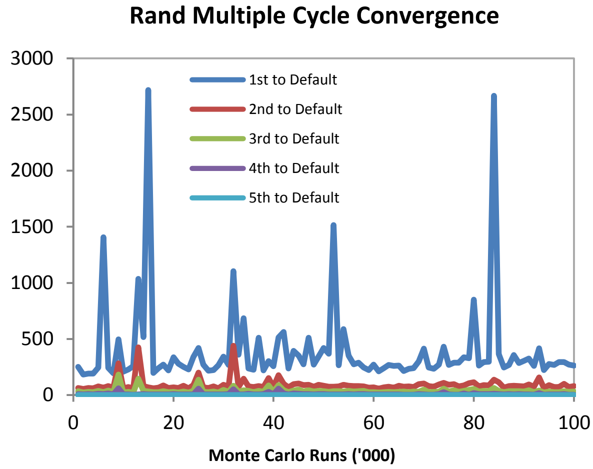


Excel RAND()



We can also note that convergence is significantly more pronounced and stable for Halton, as highlighted in the chart below. Using Halton to calculate the K^{th} to default fair spreads leads to stable convergence within 50,000 Monte Carlo runs, however with Rand even though a long run convergence is observed, it has some instability and results fluctuate significantly. The standard deviation of results for 1^{st} to default using Rand is 310bp, while for Halton it is 39bp (Other results are shown in table below, and they highlight a similar increase in standard deviation between Rand and Halton). Due to the convergence and stability benefits of Halton over Excel Rand as highlighted here all further work will be conducted using the Halton method. However, the simulation in Excel is implemented so that that the random number generator method can be easily changed by the user.



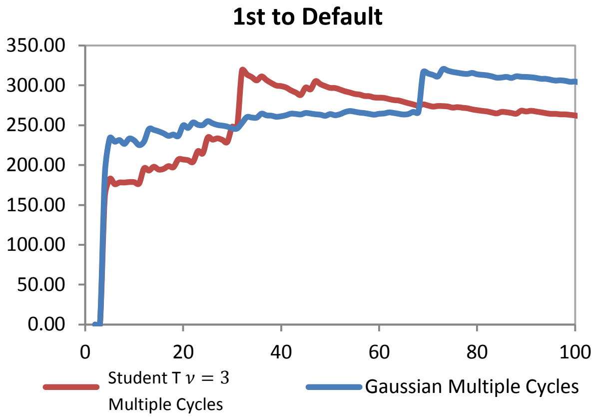


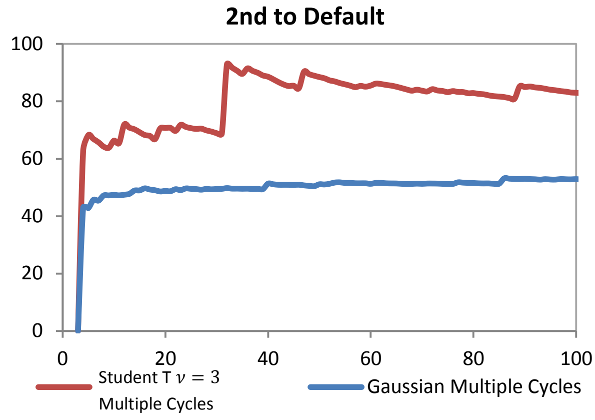
	Halton: Spread for K^{th} to Default					Rand: Spread for K^{th} to Default				
	1st	2nd	3rd	4th	5th	1st	2nd	3rd	4th	5th
Average	259.32	80.66	33.66	5.09	0.14	366.75	93.28	39.61	9.00	0.16
Standard Deviation	39.20	7.95	4.88	0.58	0.09	310.48	52.13	30.20	9.60	0.59
Max	318.79	92.93	41.63	5.90	0.74	2816.21	354.70	222.65	57.56	5.88
Min	160.91	62.76	23.63	2.95	0.10	137.52	48.16	17.84	2.07	0.00

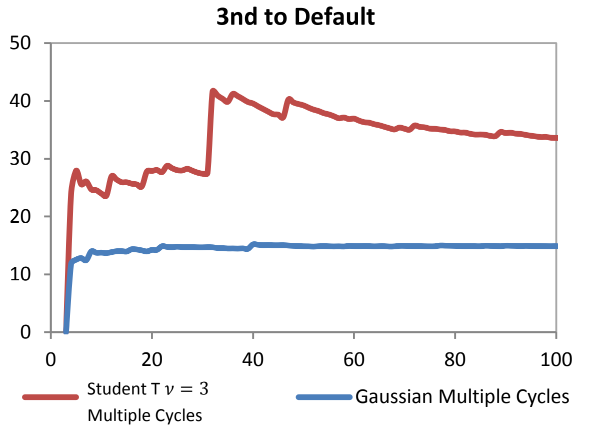
Summary of Convergence Results over 100,000 Monte Carlo Cycles

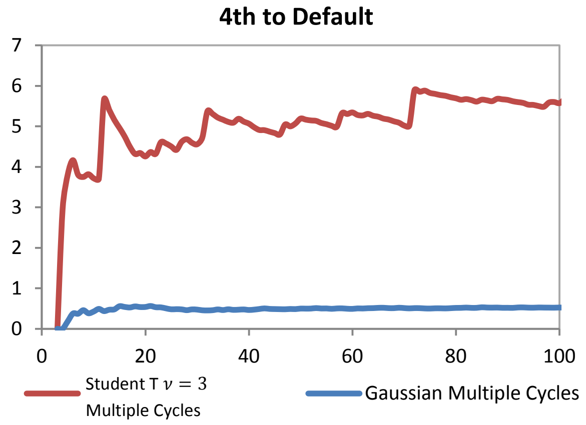
4.2 Pricing K^{th} to Default Basket Swaps

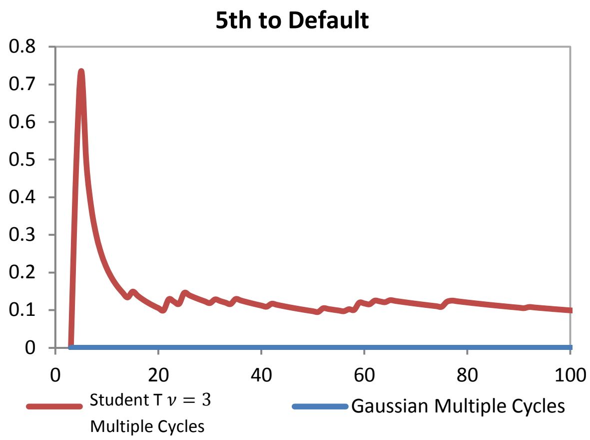
Pricing the K^{th} to default basket swaps using Gaussian and Student T ($\nu = 3$) copulae results in convergence at 50,000 Monte Carlo runs, as shown in the charts below. However, we can note that the spreads determined with both the Gaussian and Student T copula are close but different. Gaussian copula method prices the 1^{st} to default swap higher than the Student T, however the remaining k^{th} to default swaps are priced higher under the Student T. One point of issue is that under the Gaussian method the spread for the 5^{th} to default swap has an almost zero spread. These discrepancies between Gaussian and Student T copula methods is likely due to Student T's fatter tails, which provide a higher number of multiple defaults than a Gaussian distribution. This was tested by increasing the degrees of freedom, which showed the Student T pricing coming closer to the Gaussian Pricing, see chart below. Therefore, to price k^{th} to default basket swaps as precisely as possible, and ensure market dynamics are captured, the Student T copula is the optimal solution to model correlated default times as it provides a more effective method of capturing unlikely multiple default time events.











	Spread for K^{th} to Default Calculated over 100,000 Runs				
	1st to Default	2nd to Default	3rd to Default	4th to Default	5th to Default
Gaussian	304.36	52.82	14.99	0.53	0.00
Student T	261.02	82.41	33.34	5.60	0.10

	d.f.	1st to Default	2nd to Default	3rd to Default	4th to Default	5th to Default
Student T	3	261.0	82.4	33.3	5.6	0.1
Student T	10	305.4	66.6	22.8	1.3	0.0
Student T	20	299.3	60.6	19.1	0.8	0.0
Student T	30	292.5	57.1	17.5	0.7	0.0
Gaussian		304.4	52.8	15.0	0.5	0.0

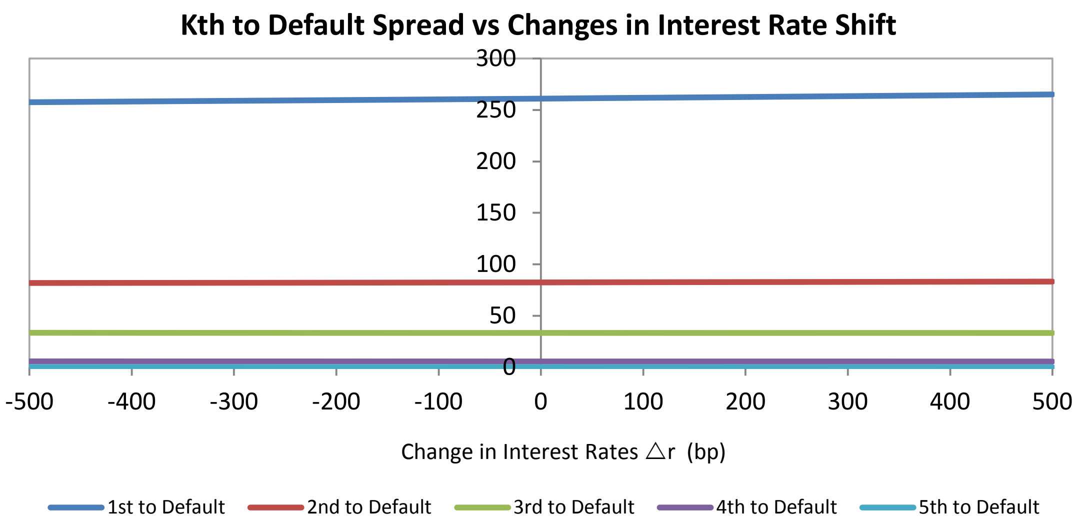
Increasing d.f. in Student T method reduces tale events, thus approaching Gaussian method

4.3 Sensitivity Analysis

The Excel implementation was developed to make sensitivity analysis as simple as possible. In the 'Simulator' worksheet a set of simulation options have been implemented to perform shifts in interest rates, CDS spreads, pair wise correlations and recovery rates. As a result the sensitivity analysis below was performed by calculating the K^{th} to default basket CDS spreads for various values of interest rates, reference entity spreads, correlations, and recovery rates. For each run 100,000 Monte Carlo simulations were performed, as this was the required amount to establish stable convergence using Halton low discrepancy random numbers. The baseline kth to default basket spread parameters and market data (the point with zero shifts) were kept as in the previous section, which resulted in pricing the spread at S=(261.02, 82.41, 33.34, 5.60, 0.10).

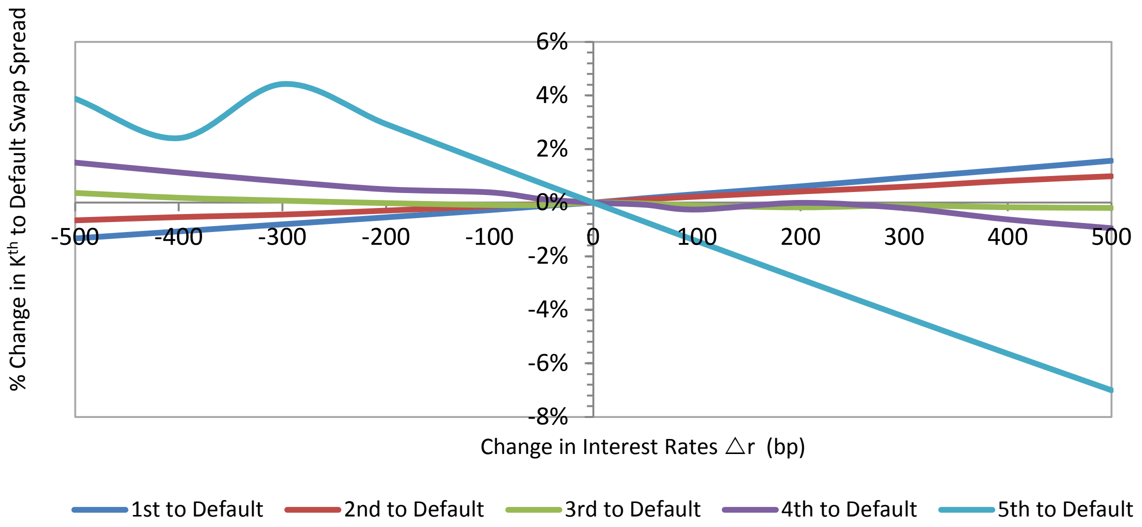
4.3.1 K^{th} to Default Basket Swap Sensitivity to Interest Rate Shifts

Keeping all parameters as in previous section parallel shifts were applied to the interest rate curve, with shifts equal to $\Delta r = (0, {}^{+}/{-}\,50, {}^{+}/{-}\,100, {}^{+}/{-}\,200, {}^{+}/{-}\,300, {}^{+}/{-}\,400, {}^{+}/{-}\,500)$. The impact of these shifts on the swap spreads was small, as shown in the chart below.



The chart below shows percentage changes in the k^{th} to default basket spreads, from the un-shifted spreads S=(261.02, 82.41, 33.34, 5.60, 0.10). It can be noted that only the 5^{th} to default swap has a significant percentage change with respect to changes in interest rates. Another point to note is that the gradient of change switches after 3^{rd} to default swap.

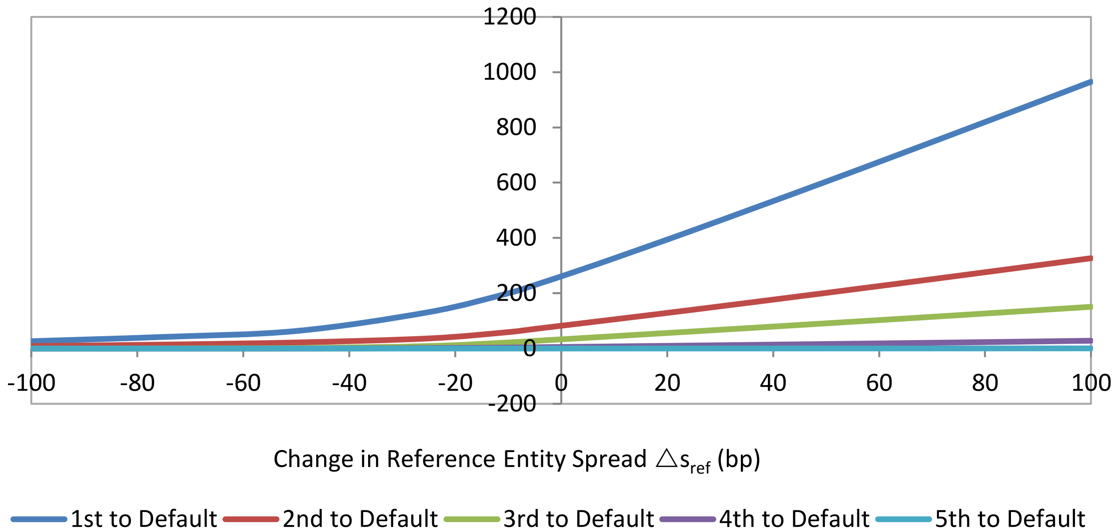
% Change in Kth to Default Spread vs Changes in Interest Rate Shift



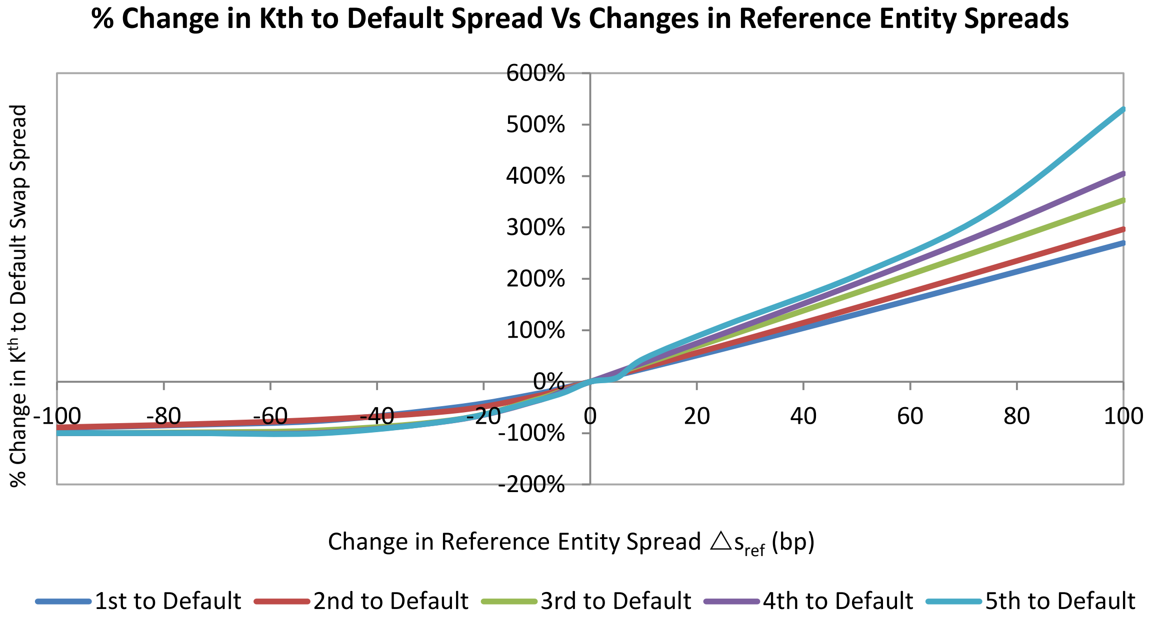
4.3.2 Kth to Default Basket Swap Sensitivity to CDS Spread Shifts

In this case equal parallel shifts were applied to all the reference entity spread, with shifts equal to $\Delta s_{reference} = (0, {}^{+}/{-}\,5, {}^{+}/{-}\,10, {}^{+}/{-}\,25, {}^{+}/{-}\,50, {}^{+}/{-}\,70, {}^{+}/{-}\,100)$. The impact of these shifts on the swap spreads was significant, in particular for 1st, 2nd and 3rd to default swaps, as shown below.

Kth to Default Spread Vs Changes in Reference Entity Spreads

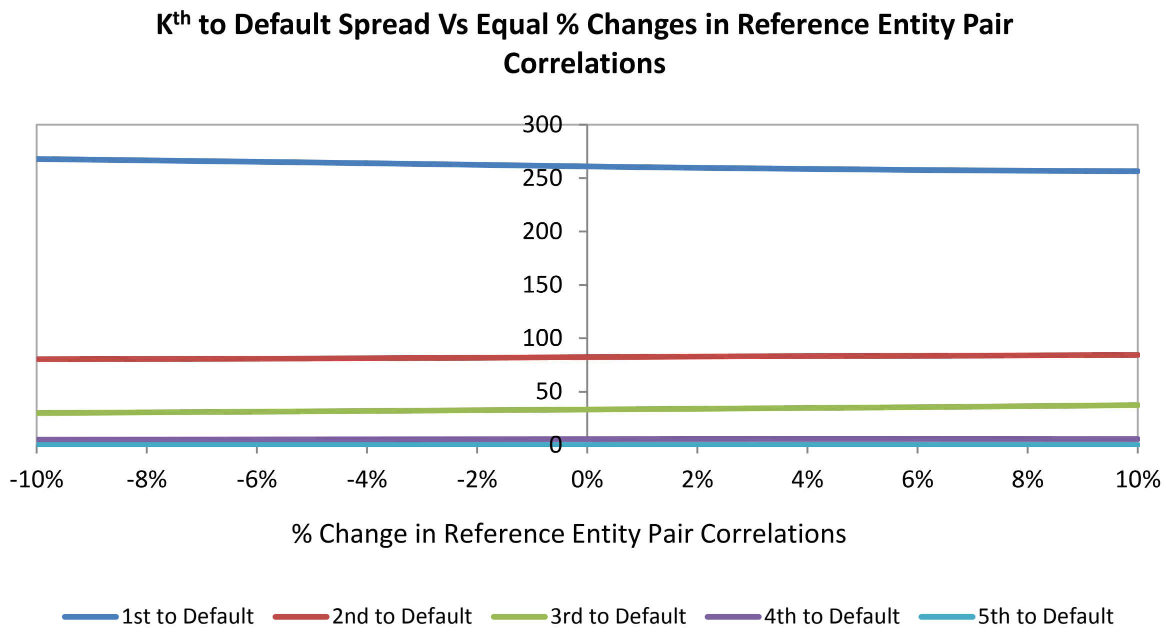


The sensitivity of the kth to default swap spreads to changes in underlying reference entity spreads can be clearly seen in the percentage change chart below. This highlights that for negative shifts in the underlying reference entity spreads the percentage change for the kth to default baskets is approximately 100% for all baskets. However, the percentage change increase linearly for all the baskets with increasing positive shift size.



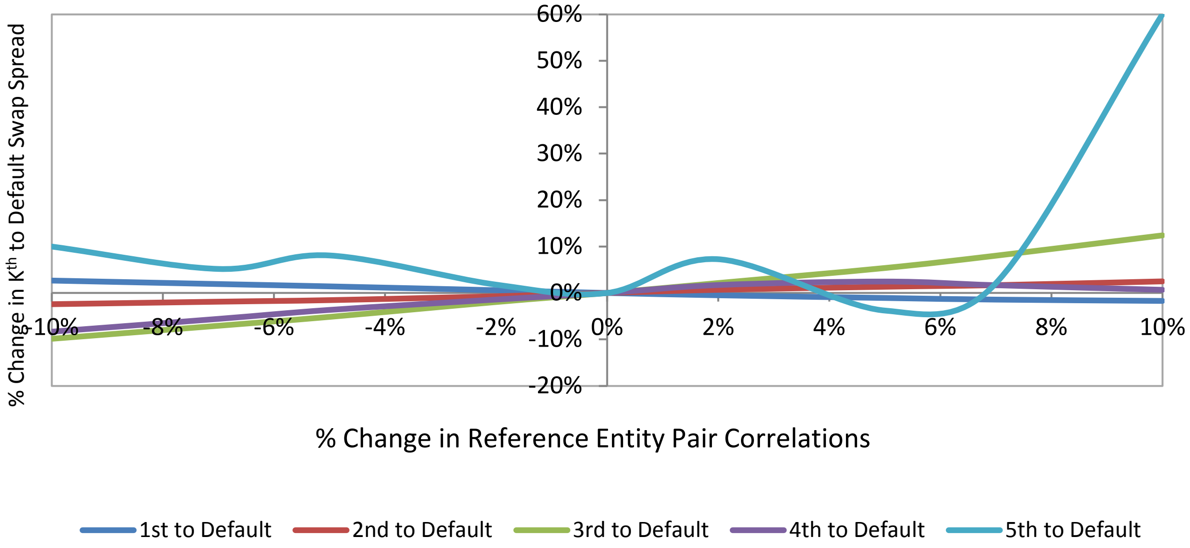
4.3.3 Kth to Default Basket Swap Sensitivity to Correlation Shifts

Correlations between reference entity pairs was shifted by $\Delta\rho = (0, {}^{+}/\!-2\%,\ {}^{+}/\!-5\%,\ {}^{+}/\!-7\%,\ {}^{+}/\!-10\%)$. From the first chart below it can be noted that the kth to default spreads do not change dramatically.



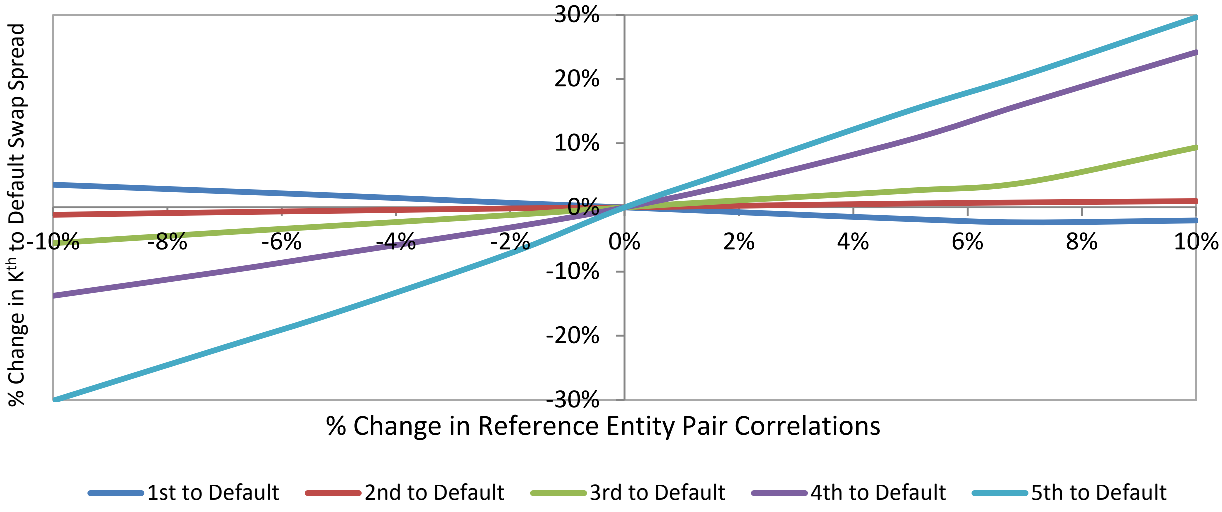
However, from the percentage changes in the spreads it can be seen that the significant percentage changes in spreads ($> 10\%$ changes in the kth to default spread) start taking place after an ${}^{+}/\!-8\%$ shift in the correlations. After this point the 5th to default spread changes dramatically. This is due to the fact that as correlations become increasingly +ve the 5th to default event becomes more likely.

% Change in Kth to Default Spread Vs % Changes in Reference Entity Pair Correlations



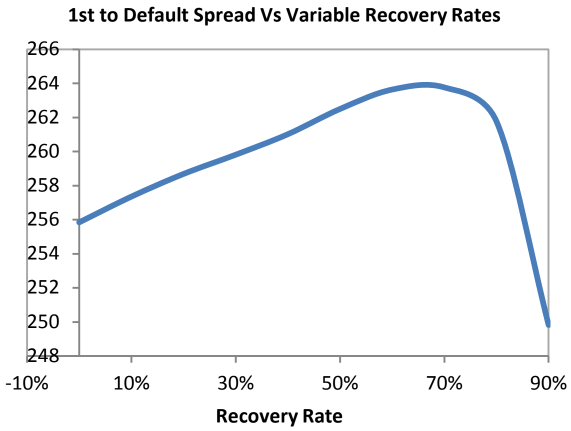
The above correlation pair shifts are performed while keeping the correlation signs constant (e.g. +ve or –ve correlation, see section 3.2.3). However, below the same shifts are applied while forcing positive correlations. The baseline (with zero shift) but with forced +ve correlation between all pairs changes from $S_k = (261.02,\ 82.41,\ 33.34,\ 5.60,\ 0.10)$ to $S_k = (226.81,\ 90.84,\ 41.74,\ 19.98,\ 5.85)$. It can be seen here that the 4th and 5th to default basket swap spreads increase significantly, when correlation between all pairs becomes +ve. This could be the case in market conditions undergoing recession as was seen in the last 5 years. In addition, performing the correlation shifts on this baseline gives the chart below, highlighting linear % changes in spreads with changes in correlation pairs.

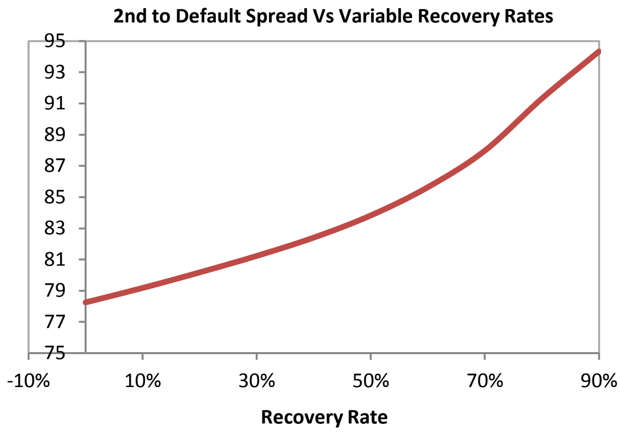
% Change in Kth to Default Spread Vs % Changes in Reference Entity Pair Correlations (with forced +ve Correlation)

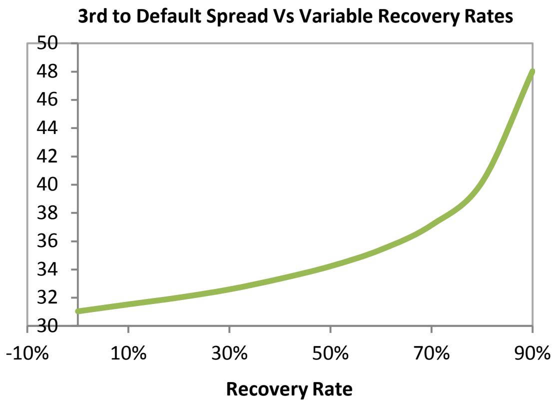


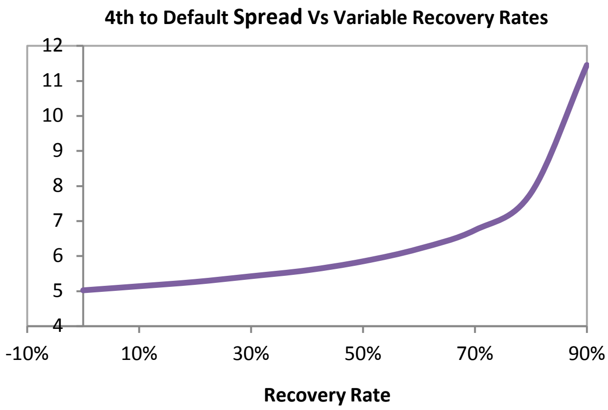
4.3.4 K^th to Default Basket Swap Sensitivity to Recovery Rate Shifts

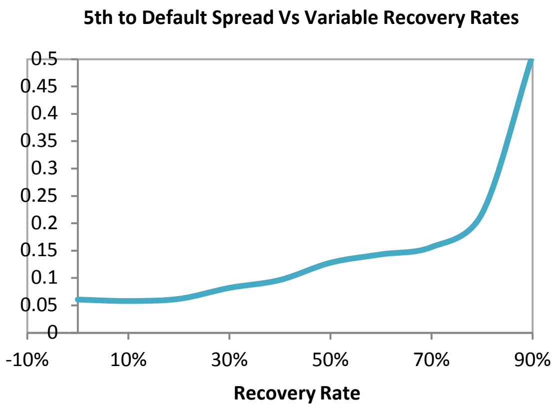
In this sensitivity analysis the recovery rate was changed from 0 to 90%. In general basket swap spreads increase as recovery rate increases. This is expected as implied survival probability of underlying reference entity reduces as recovery rates increase. This increases risk of default for each entity, thus increasing spreads of the basket swaps. However, an interesting aspect to note is that for the 1st to default swap the spread increase with increasing recovery rate, as expected, but it then starts to drop at a Recovery Rate of ~70%. This implied risk reduction is expected to be due to an interaction with correlation, between reference entities. When all pairwise correlations are forced to be +ve (highly correlated) this implied risk reduction is reduced.



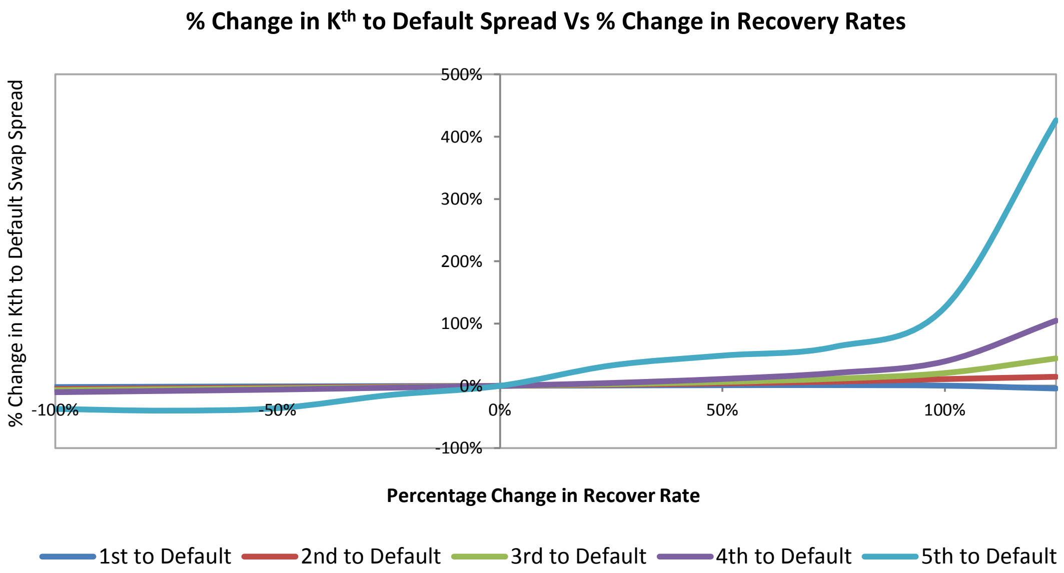








The final chart, below, shows the % change in the spreads for each k^{th} to default swap against percentage change in recovery rate relative to the baseline of 40%. It can be seen that after a 100% increase in the recovery rate (i.e. from Recovery Rate=40% to 80%) the spreads for all the swaps start to increase significantly, however the 5^{th} to default swap spread increases at the steepest rate. As already mentioned, in general an increasing recovery rate implies a reduced survival probability as given by the underlying CDS instruments. This reduction in survival translates to an increasing spread in the k^{th} to default swap.



5. Excel Implementation Overview

The Excel implementation consists of two key worksheets: Market Data Input, Simulator and Array Output.

The 'Market Data Input' sheet is used for inputting the current market data such as spreads, correlations, recover rates and discount curves, this is also where the default probabilities and intensities are calculated and held for use in the simulation.

The 'Simulator' worksheet is for setting the model parameters for pricing the k^{th} to defaults basket, this includes selecting the random number generation method, copula method, number of Monte Carlo Runs, and other factors such as applying shifts to the market data for sensitivity analysis. The Output Arrays worksheet allows the user to output data related to the simulation, such as step by step figures for the random numbers, default times and spreads; this data enables ease of trouble shooting.

Single or Multiple Cycle Simulation	
No of Runs	100000
Recovery Rate	90%
Interest Rate Parallel Shift (bps)	0.00
CDS Spread Parallel Shift (bps)	0.00
Correlation Shift (%) (Select '+' to force >0 Correlation)	0%
Random Number	Halton
Copula Method	Student T
Student T DF (v)	3.00
Output Array Data	N
dt (0=continuous)	0.00

The other key features of this worksheet are as follows:

The 'multiple cycles' feature allows a number of MC cycles to be run, with increasing number of runs, to show convergence. The Surface feature, when activated, turns off the multiple runs during activation. This allows pricing to be performed for multiple shifts in the market parameters i.e. interest rate, recovery rate, reference entity spreads and correlations.

Multiple Cycles (Switches off for Surface Calcs)	
Multiple Cycles	N
Number of Cycles	100
Number of Runs In 1st Cycle	1000
Increase Runs After Each Cycle by	1000

Surface	Y
Interest Rate Shift Surface	N
CDS Parallel Shift Surface	N
Correlation Shift Surface	N
Recovery Rate Shift Surface	Y

Two other sheets are also included which give the details of the calculations and functions used is estimating the correlations and the degrees of freedom for the Student T distribution.

6. References

1. Paul Wilmot, Introduction to Quantitative Finance
2. Tomas Bjork, Arbitrage Theory in Continuous Time
3. Alonso Pena, Credit Default Swaps Class Notes
4. Richard Diamond, Final Project Workshop Notes
5. Paul Glasserman, Monte Carlo Methods in Financial Engineering

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