Pricing of forward start swaptions

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1 Introduction

A forward start swaption is an option to enter into a swap. The difference between an ordinary swaption and a forward start swaption is in the timing between the expiry of the option and the start of the swap: in an ordinary swaption these will coincide (or, at most, the option will expire two days before the start of the swap), while in a forward start swap the option expiration date can be way before the start of the swap.

To fix notations, we will assume that the option expires at time t_a, and that the underlying swap starts at time t_b and ends at time t_c; we will write a swap as $\{b, c\}$ and a swaption as $a \rightarrow \{b, c\}$. The payment dates of the fixed leg of the swap are t_1 to $t_n = t_c$. The year fraction between t_{i-1} and t_i is τ_i (where $t_0 = t_b$). At time t, the zero coupon bond maturing at T is $P(t, T)$.

With these notation, at expiry the value V of the $a \rightarrow \{b, c\}$ swpation is

$$V(a) = L_{bc}(a)(S_{bc}(a) - K)^+$$

where the level L_{bc} and the swap rate S_{bc} are

$$\begin{aligned} L_{bc}(t) &= \sum_{i=1}^{n} \tau_i P(t, t_i) \\ S_{bc}(t) &= \frac{P(t, t_0) - P(t, t_n)}{L_{bc}(t)} \end{aligned}$$

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Forward start swaptions are not quoted in the market. We provide two different approaches to price forward start swaptions. The first one, Black vol, is simple and uses only quoted data, and we suggest to use it on screen wake up. The second method, frozen levels, has one parameter that must be estimated from historical data. We suggest, after screen startup, to allow users to use this method to get a more refined handle on the pricing.

2 Pricing with Black formula

Swaptions are quoted using the Black formula, which assumes that the swap rate follows a log normal dynamic with constant volatility. While in general this should only be thought of as a quoting mechanism, we can take this assumption seriously and price forward start swaptions accordingly. In that case the volatility of the swap rate is the same as the vol of the (ordinary) $b \to \{b, c\}$ swaption, as the underlying is the same in both cases. This vol is quoted in the market, so the time 0 price of the forward start swap is simply given by the Black formula:

$$\begin{aligned} V(0) &= L_{bc}(0)(S_{bc}(0)\Phi(d_+) - K\Phi(d_-)) \\ d_\pm &= \frac{\ln(S_{bc}(0)/K) \pm \sigma^2 u/2}{\sigma\sqrt{u}} \end{aligned}$$

where σ is the quoted vol of the $b \to \{b, c\}$ swaption.

While this pricing methodology is very simple, it is clear that assuming a *constant* volatility is a strong hypothesis. The studies made in IDOC 2044163 show that this is in fact a reasonable assumption.

3 Pricing with frozen levels

The $\{b, c\}$ swap can be synthesized as the difference of the two swaps $\{a, c\} - \{a, b\}$, so we have the relations between the prices of these swaps:

$$S_{bc}L_{bc} = S_{ac}L_{ac} - S_{ab}L_{ab}$$

The levels also add up:

$$L_{ac} = L_{ab} + L_{bc}$$

Putting this together, we can write the S_{bc} swap rate as:

$$\begin{aligned} S_{bc} &= \frac{L_{ac}S_{ac} - L_{ab}S_{ab}}{L_{ac} - L_{ab}} \\ &= w_c S_{ac} - w_b S_{ab} \end{aligned} \tag{1}$$

where we have defined the weights

$$\begin{aligned} w_b &= \frac{L_{ab}}{L_{ac} - L_{ab}} \\ w_c &= \frac{L_{ac}}{L_{ac} - L_{ab}} \end{aligned}$$

We assume, as in the Black model, that the swap rates follow a lognormal dynamic under their level payment measure:

$$dS = \sigma S dW$$

Expressing the dynamics of the three swap rates in a common measure will add drift terms, but not change the variance nor the correlations. If we freeze the level payments we can therefore compute the instantaneous variance of S_{bc} from equation (1):

$$(\sigma_{bc}S_{bc})^2 = (w_c\sigma_{ac}S_{ac})^2 + (w_b\sigma_{ab}S_{ab})^2 - 2\rho(w_c\sigma_{ac}S_{ac})(w_b\sigma_{ab}S_{ab})$$

where ρ is the correlation between W_{ac} and W_{ab}, the Brownian motion driving S_{ac} and S_{ab}.

From this formula we can easily extract S_{bc}'s volatility:

$$\sigma_{bc}^2 = \frac{(w_c\sigma_{ac}S_{ac})^2 + (w_b\sigma_{ab}S_{ab})^2 - 2\rho(w_c\sigma_{ac}S_{ac})(w_b\sigma_{ab}S_{ab})}{(S_{bc})^2}$$

We then price the forward start swaption with Black's formula. In the expression of the vol, the swap rates and levels can be computed from zero coupon prices, and we use quoted implied vols for σ_{ab} and σ_{ac}[1]. The only variable that is not quoted is the correlation ρ.

First we obtain daily time values for the swap rates of the $\{t, t+b-a\}$ and $\{t, t+c-a\}$ (we will write these time series as B and C). In general

[1] or more generally, implied vol obtained from the vol cube, if there are not market quotes for the exact dates a, b, and c.

there will not be market quotes for swaps with the exact tenors $b-a$ and $c-a$. In that case, the swap rates can be obtained by linearly interpolating swap rates with tenors that bracket $b-a$ and $c-a$. We then compute the log changes of these rates and remove outliers. To do so, we compute the variance of the returns of the time series, and removing all the data points that are more than 3 standard deviation from the mean. Note that given an outlier in one time series, the corresponding data point in the other time series should also be removed. We then use exponential moving averages to compute the correlation. We first recursively compute the conditional variances ν and covariance q^{bc}:

$$\begin{aligned}
\nu_t^b &= \lambda \nu_{t-1}^b + (1-\lambda)(\Delta B_t)^2 \\
\nu_t^c &= \lambda \nu_{t-1}^c + (1-\lambda)(\Delta C_t)^2 \\
q_t^{bc} &= \lambda q_{t-1}^{bc} + (1-\lambda)\frac{\Delta B_t \Delta C_t}{\sqrt{\nu_t^b \nu_t^c}}
\end{aligned}$$

and we then get the correlation

$$\rho_t^{bc} = \frac{q_t^{bc}}{\sqrt{q_t^b q_t^c}}$$

where

$$\begin{aligned}
q_t^b &= \lambda q_{t-1}^b + (1-\lambda)\frac{(\Delta B_t)^2}{\nu_t^b} \\
q_t^c &= \lambda q_{t-1}^c + (1-\lambda)\frac{(\Delta C_t)^2}{\nu_t^c}
\end{aligned}$$

The parameter λ controls how fast old data is “forgotten”. A low value will place more emphasis on recent data, adapting quickly to changing market conditions, but being also more affected by noise or spurious correlations. A value of $\lambda = 0.995$ should be adequate (for daily data), and would require at least three years of data, and preferably four.