Contents

1 Mathematical Proof: Why n Balls In → n Balls Out

1.1 Setup and Assumptions

Consider Newton's Cradle with 5 identical balls, each of mass m. All collisions are perfectly elastic ($e = 1$). Ball 1 strikes the stationary line with velocity v_0.

1.2 Conservation Laws

For elastic collisions between identical masses, we must satisfy TWO fundamental conservation laws simultaneously:

Conservation of Momentum:

$$mv_0 = mv_1 + mv_2 + mv_3 + mv_4 + mv_5 \tag{1}$$

Simplifying:

$$v_0 = v_1 + v_2 + v_3 + v_4 + v_5 \tag{2}$$

Conservation of Kinetic Energy (elastic collision):

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \frac{1}{2}mv_4^2 + \frac{1}{2}mv_5^2 \tag{3}$$

Simplifying:

$$v_0^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 \tag{4}$$

1.3 Testing Alternative Scenarios

1.3.1 Scenario A: 1 ball in → 2 balls out (each with speed $v_0/2$)

Let $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = v_0/2$.

Momentum check:

$$v_0 \stackrel{?}{=} 0 + 0 + 0 + \frac{v_0}{2} + \frac{v_0}{2} = v_0 \quad \checkmark \tag{5}$$

Energy check:

$$v_0^2 \stackrel{?}{=} 0 + 0 + 0 + \left(\frac{v_0}{2}\right)^2 + \left(\frac{v_0}{2}\right)^2 = \frac{v_0^2}{4} + \frac{v_0^2}{4} = \frac{v_0^2}{2} \quad \times \tag{6}$$

Energy is **NOT conserved**. This scenario violates physics.

1.3.2 Scenario B: 1 ball in → 1 ball out (with speed v_0)

Let $v_1 = v_2 = v_3 = v_4 = 0$ and $v_5 = v_0$.

Momentum check:

$$v_0 \stackrel{?}{=} 0 + 0 + 0 + 0 + v_0 = v_0 \quad \checkmark \tag{7}$$

Energy check:

$$v_0^2 \stackrel{?}{=} 0 + 0 + 0 + 0 + v_0^2 = v_0^2 \quad \checkmark \tag{8}$$

Both laws are satisfied! This scenario is **physically valid**.

1.3.3 Scenario C: 2 balls in (each with v_0) $\to$ 1 ball out with speed $2v_0$

Initial momentum: $2v_0$. Let $v_1 = v_2 = v_3 = v_4 = 0$ and $v_5 = 2v_0$.

Momentum check:

$$2v_0 \stackrel{?}{=} 0 + 0 + 0 + 0 + 2v_0 = 2v_0 \quad \checkmark \tag{9}$$

Energy check:

$$\text{Initial: } v_0^2 + v_0^2 = 2v_0^2 \tag{10}$$
$$\text{Final: } 0 + 0 + 0 + 0 + (2v_0)^2 = 4v_0^2 \tag{11}$$
$$2v_0^2 \neq 4v_0^2 \quad \times \tag{12}$$

Energy is **NOT conserved**. This scenario violates physics.

1.3.4 Scenario D: 2 balls in (each with v_0) $\to$ 2 balls out (each with v_0)

Initial momentum: $2v_0$. Let $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = v_0$.

Momentum check:

$$2v_0 \stackrel{?}{=} 0 + 0 + 0 + v_0 + v_0 = 2v_0 \quad \checkmark \tag{13}$$

Energy check:

$$\text{Initial: } v_0^2 + v_0^2 = 2v_0^2 \tag{14}$$
$$\text{Final: } 0 + 0 + 0 + v_0^2 + v_0^2 = 2v_0^2 \tag{15}$$
$$2v_0^2 = 2v_0^2 \quad \checkmark \tag{16}$$

Both laws are satisfied! This scenario is **physically valid**.

1.4 General Proof

Claim: If n balls approach with the same speed v, then exactly n balls must leave with speed v.

Proof:

Let n balls approach with velocity v_0, and suppose k balls leave with velocities $u_1, u_2, \ldots, u_k$ (the remaining balls are stationary).

From conservation of momentum:

$$nv_0 = u_1 + u_2 + \cdots + u_k = \sum_{i=1}^{k} u_i \tag{17}$$

From conservation of energy:

$$nv_0^2 = u_1^2 + u_2^2 + \cdots + u_k^2 = \sum_{i=1}^{k} u_i^2 \tag{18}$$

Key Mathematical Insight: Apply the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{k} u_i\right)^2 \leq k \sum_{i=1}^{k} u_i^2 \tag{19}$$

with equality if and only if all u_i are equal.

Substituting equations (17) and (18):

$$(nv_0)^2 \leq k(nv_0^2) \tag{20}$$

$$n^2v_0^2 \leq knv_0^2 \tag{21}$$

Dividing by nv_0^2:

$$n \leq k \tag{22}$$

However, physically we cannot have more balls leaving than we have total:

$$k \leq n \tag{23}$$

Therefore, we must have:

$$\boxed{k = n} \tag{24}$$

Moreover, for Cauchy-Schwarz equality to hold, all outgoing velocities must be equal:

$$u_1 = u_2 = \cdots = u_n \tag{25}$$

From equation (17):

$$nv_0 = nu \quad \Rightarrow \quad u = v_0 \tag{26}$$

2 Derivation for Energy Loss Mechanisms in a Newton's Cradle

2.1 Total Energy Loss from Restitution

For a head-on collision between identical spheres, the total mechanical energy loss can be expressed in terms of the coefficient of restitution e:

$$\Delta E_{\text{total}} = (1 - e^2)E_0 \tag{27}$$

where the initial kinetic energy is

$$E_0 = \frac{1}{2}mv_0^2. \tag{28}$$

This result follows from conservation of momentum and the definition of restitution [2, 3].

2.2 Hertzian Contact Force

The normal contact force between two elastic spheres is given by Hertzian contact theory:

$$F(\delta) = k\delta^{3/2} \tag{29}$$

where δ is the compression and

$$k = \frac{4}{3} E^* \sqrt{R^*}. \tag{30}$$

The effective elastic modulus and radius are

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}, \tag{31}$$

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{32}$$

For identical spheres:

$$E^* = \frac{E}{1-\nu^2}, \qquad R^* = \frac{R}{2}. \tag{33}$$

These expressions are standard results from elastic contact mechanics [1].

2.3 Heat Loss: Viscoelastic Dissipation

Real materials exhibit internal damping. The Hertzian force law is modified to include a dissipative term:

$$F(\delta, \dot{\delta}) = k\delta^{3/2} + \gamma\delta^{1/2}\dot{\delta} \tag{34}$$

where γ is a viscoelastic damping coefficient [4].

The energy dissipated as heat during the collision is

$$E_{\text{heat}} = \int_0^\tau \gamma\delta^{1/2}\dot{\delta}^2\, dt. \tag{35}$$

Using Hertzian scaling relations for maximum compression and collision duration [1, 4], the heat loss scales as

$$E_{\text{heat}} \sim \gamma m^{1/2} k^{-3/2} v_0^{6/5}. \tag{36}$$

2.4 Sound Radiation

The accelerating sphere surface acts as an acoustic dipole source. The radiated acoustic power is approximately [5]

$$P \sim \frac{\rho_{\text{air}}}{c_{\text{air}}} A^2 a^2 \tag{37}$$

where ρ_{air} is the air density, c_{air} is the speed of sound in air, A is the effective radiating area, and a is the characteristic surface acceleration.

The total sound energy radiated during a collision is

$$E_{\text{sound}} = \int_0^\tau P(t)\, dt. \tag{38}$$

Using Hertzian collision scaling, this gives

$$E_{\text{sound}} \sim \frac{\rho_{\text{air}}}{c_{\text{air}}} R^{21/5} (E^*)^{2/5} m^{-2/5} v_0^{7/5}. \tag{39}$$

2.5 Energy Balance

The complete energy balance for one collision is therefore

$$E_0 = E_{\text{after}} + E_{\text{heat}} + E_{\text{sound}}, \tag{40}$$

with

$$E_{\text{after}} = e^2 E_0. \tag{41}$$

In typical steel Newton's cradle systems, $E_{\text{heat}} \gg E_{\text{sound}}$.

3 Derivation: Unequal Mass Collision

3.1 Setup

Consider two balls: ball 1 (mass m_1, initial velocity v_0) strikes ball 2 (mass m_2, initially at rest). After collision, the velocities are v_1' and v_2' respectively.

3.2 Conservation Laws

Conservation of momentum:

$$m_1 v_0 = m_1 v_1' + m_2 v_2' \tag{42}$$

Conservation of kinetic energy (elastic collision):

$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{43}$$

Simplifying equation (2):

$$m_1 v_0^2 = m_1 v_1'^2 + m_2 v_2'^2 \tag{44}$$

3.3 Solving for v_2'

From equation (1), isolate v_1':

$$v_1' = \frac{m_1 v_0 - m_2 v_2'}{m_1} \tag{45}$$

Substitute equation (4) into equation (3):

$$m_1 v_0^2 = m_1 \left(\frac{m_1 v_0 - m_2 v_2'}{m_1} \right)^2 + m_2 v_2'^2 \tag{46}$$

Simplify:

$$m_1 v_0^2 = \frac{(m_1 v_0 - m_2 v_2')^2}{m_1} + m_2 v_2'^2 \tag{47}$$

Multiply through by m_1:

$$m_1^2 v_0^2 = (m_1 v_0 - m_2 v_2')^2 + m_1 m_2 v_2'^2 \tag{48}$$

Expand the squared term:

$$m_1^2 v_0^2 = m_1^2 v_0^2 - 2m_1 m_2 v_0 v_2' + m_2^2 v_2'^2 + m_1 m_2 v_2'^2 \tag{49}$$

Cancel $m_1^2 v_0^2$ from both sides:

$$0 = -2m_1 m_2 v_0 v_2' + m_2^2 v_2'^2 + m_1 m_2 v_2'^2 \tag{50}$$

Factor out $m_2 v_2'$:

$$0 = -2m_1 m_2 v_0 v_2' + m_2 v_2'^2 (m_2 + m_1) \tag{51}$$

Rearrange (taking non-trivial solution $v_2' \neq 0$):

$$2m_1 m_2 v_0 v_2' = m_2 v_2'^2 (m_1 + m_2) \tag{52}$$

Divide both sides by $m_2 v_2'$:

$$2m_1 v_0 = v_2'(m_1 + m_2) \tag{53}$$

Solve for v_2':

$$\boxed{v_2' = \frac{2m_1}{m_1 + m_2} v_0} \tag{54}$$

3.4 Solving for v_1'

Substitute equation (13) back into equation (4):

$$v_1' = \frac{m_1 v_0 - m_2 \cdot \frac{2m_1}{m_1+m_2} v_0}{m_1} \tag{55}$$

Simplify:

$$v_1' = v_0 \left(1 - \frac{2m_2}{m_1 + m_2}\right) \tag{56}$$

Combine fractions:

$$v_1' = v_0 \left(\frac{m_1 + m_2 - 2m_2}{m_1 + m_2}\right) \tag{57}$$

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_0} \tag{58}$$

3.5 Introducing Mass Ratio $\mu = m_2/m_1$

Divide numerator and denominator of equations (13) and (17) by m_1:

$$v_1' = \frac{1 - \mu}{1 + \mu} v_0 \tag{59}$$

$$v_2' = \frac{2}{1 + \mu} v_0 \tag{60}$$

3.6 Energy Transfer Efficiency

The energy transfer efficiency is the fraction of kinetic energy transferred to ball 2:

$$\eta = \frac{E_2'}{E_0} = \frac{\frac{1}{2}m_2 v_2'^2}{\frac{1}{2}m_1 v_0^2} \tag{61}$$

Substitute v_2' from equation (19):

$$\eta = \frac{m_2}{m_1} \cdot \frac{v_2'^2}{v_0^2} = \frac{m_2}{m_1} \cdot \left(\frac{2}{1+\mu}\right)^2 \tag{62}$$

Since $\mu = m_2/m_1$:

$$\eta = \mu \cdot \frac{4}{(1+\mu)^2} \tag{63}$$

$$\boxed{\eta = \frac{4\mu}{(1+\mu)^2}} \tag{64}$$

3.7 Maximizing Energy Transfer

To find the maximum, take the derivative with respect to μ and set to zero:

$$\frac{d\eta}{d\mu} = \frac{d}{d\mu}\left[\frac{4\mu}{(1+\mu)^2}\right] \tag{65}$$

Using the quotient rule:

$$\frac{d\eta}{d\mu} = \frac{4(1+\mu)^2 - 4\mu \cdot 2(1+\mu)}{(1+\mu)^4} \tag{66}$$

Simplify:

$$\frac{d\eta}{d\mu} = \frac{4(1+\mu) - 8\mu}{(1+\mu)^3} = \frac{4 + 4\mu - 8\mu}{(1+\mu)^3} \tag{67}$$

$$\frac{d\eta}{d\mu} = \frac{4(1-\mu)}{(1+\mu)^3} \tag{68}$$

Set equal to zero:

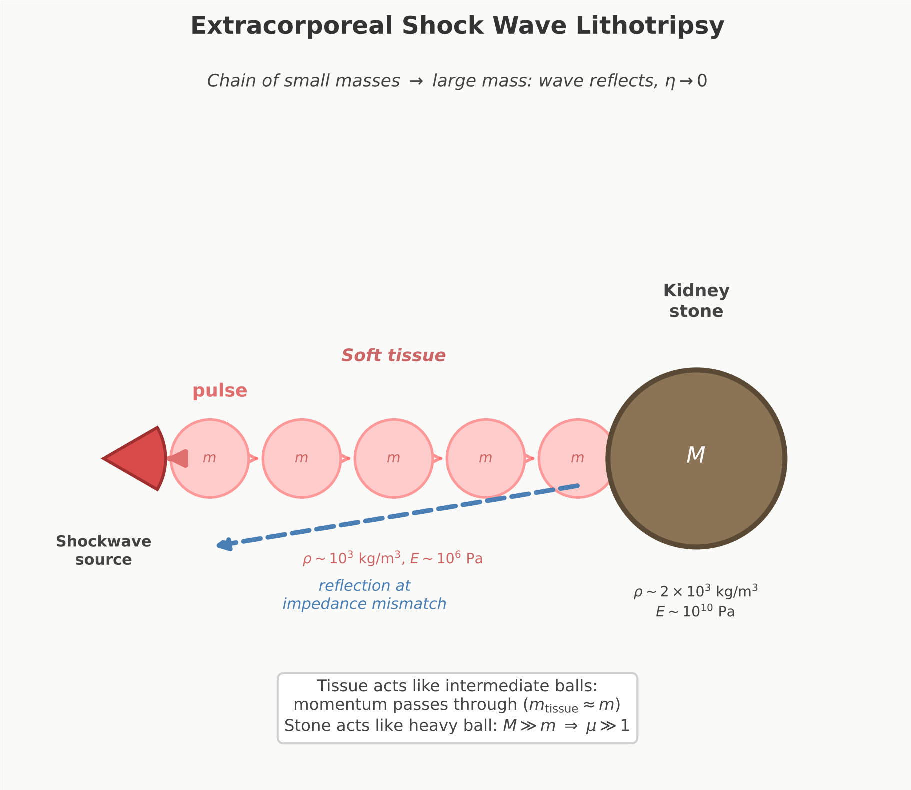
$$\frac{4(1-\mu)}{(1+\mu)^3} = 0 \quad \Rightarrow \quad 1 - \mu = 0 \quad \Rightarrow \quad \boxed{\mu = 1} \tag{69}$$

At $\mu = 1$:

$$\eta_{\max} = \frac{4 \cdot 1}{(1+1)^2} = \frac{4}{4} = \boxed{1} \tag{70}$$

Therefore, energy transfer is maximized when masses are equal ($\mu = 1$), achieving perfect energy transfer ($\eta = 1$).

4 Image to show relation Lithotripsy and Newtons Cradle



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