Contents

1 Mathematical Proof: Why n Balls In → n Balls Out

1.1 Setup and Assumptions

Consider Newton's Cradle with 5 identical balls, each of mass m. All collisions are perfectly elastic ($e = 1$). Ball 1 strikes the stationary line with velocity v_0.

1.2 Conservation Laws

For elastic collisions between identical masses, we must satisfy TWO fundamental conservation laws simultaneously:

Conservation of Momentum:

$$mv_0 = mv_1 + mv_2 + mv_3 + mv_4 + mv_5 \tag{1}$$

Simplifying:

$$v_0 = v_1 + v_2 + v_3 + v_4 + v_5 \tag{2}$$

Conservation of Kinetic Energy (elastic collision):

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 + \frac{1}{2}mv_3^2 + \frac{1}{2}mv_4^2 + \frac{1}{2}mv_5^2 \tag{3}$$

Simplifying:

$$v_0^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 \tag{4}$$

1.3 Testing Alternative Scenarios

1.3.1 Scenario A: 1 ball in → 2 balls out (each with speed $v_0/2$)

Let $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = v_0/2$.

Momentum check:

$$v_0 \overset{?}{=} 0 + 0 + 0 + \frac{v_0}{2} + \frac{v_0}{2} = v_0 \quad \checkmark \tag{5}$$

Energy check:

$$v_0^2 \overset{?}{=} 0 + 0 + 0 + \left(\frac{v_0}{2}\right)^2 + \left(\frac{v_0}{2}\right)^2 = \frac{v_0^2}{4} + \frac{v_0^2}{4} = \frac{v_0^2}{2} \quad \times \tag{6}$$

Energy is **NOT conserved**. This scenario violates physics.

1.3.2 Scenario B: 1 ball in → 1 ball out (with speed v_0)

Let $v_1 = v_2 = v_3 = v_4 = 0$ and $v_5 = v_0$.

Momentum check:

$$v_0 \overset{?}{=} 0 + 0 + 0 + 0 + v_0 = v_0 \quad \checkmark \tag{7}$$

Energy check:

$$v_0^2 \overset{?}{=} 0 + 0 + 0 + 0 + v_0^2 = v_0^2 \quad \checkmark \tag{8}$$

Both laws are satisfied! This scenario is **physically valid**.

1.3.3   Scenario C: 2 balls in (each with v_0) $\rightarrow$ 1 ball out with speed $2v_0$

Initial momentum: $2v_0$. Let $v_1 = v_2 = v_3 = v_4 = 0$ and $v_5 = 2v_0$.

Momentum check:
$$2v_0 \overset{?}{=} 0 + 0 + 0 + 0 + 2v_0 = 2v_0 \quad \checkmark \tag{9}$$

Energy check:
$$\text{Initial: } v_0^2 + v_0^2 = 2v_0^2 \tag{10}$$
$$\text{Final: } 0 + 0 + 0 + 0 + (2v_0)^2 = 4v_0^2 \tag{11}$$
$$2v_0^2 \neq 4v_0^2 \quad \times \tag{12}$$

Energy is **NOT conserved**. This scenario violates physics.

1.3.4   Scenario D: 2 balls in (each with v_0) $\rightarrow$ 2 balls out (each with v_0)

Initial momentum: $2v_0$. Let $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = v_0$.

Momentum check:
$$2v_0 \overset{?}{=} 0 + 0 + 0 + v_0 + v_0 = 2v_0 \quad \checkmark \tag{13}$$

Energy check:
$$\text{Initial: } v_0^2 + v_0^2 = 2v_0^2 \tag{14}$$
$$\text{Final: } 0 + 0 + 0 + v_0^2 + v_0^2 = 2v_0^2 \tag{15}$$
$$2v_0^2 = 2v_0^2 \quad \checkmark \tag{16}$$

Both laws are satisfied! This scenario is **physically valid**.

1.4   General Proof

Claim: If n balls approach with the same speed v, then exactly n balls must leave with speed v.

Proof:

Let n balls approach with velocity v_0, and suppose k balls leave with velocities $u_1, u_2, \ldots, u_k$ (the remaining balls are stationary).

From conservation of momentum:
$$nv_0 = u_1 + u_2 + \cdots + u_k = \sum_{i=1}^{k} u_i \tag{17}$$

From conservation of energy:
$$nv_0^2 = u_1^2 + u_2^2 + \cdots + u_k^2 = \sum_{i=1}^{k} u_i^2 \tag{18}$$

Key Mathematical Insight: Apply the Cauchy-Schwarz inequality:
$$\left(\sum_{i=1}^{k} u_i \right)^2 \leq k \sum_{i=1}^{k} u_i^2 \tag{19}$$

3

with equality if and only if all u_i are equal.

Substituting equations (17) and (18):

$$(nv_0)^2 \leq k(nv_0^2) \tag{20}$$

$$n^2 v_0^2 \leq k n v_0^2 \tag{21}$$

Dividing by nv_0^2:

$$n \leq k \tag{22}$$

However, physically we cannot have more balls leaving than we have total:

$$k \leq n \tag{23}$$

Therefore, we must have:

$$\boxed{k = n} \tag{24}$$

Moreover, for Cauchy-Schwarz equality to hold, all outgoing velocities must be equal:

$$u_1 = u_2 = \cdots = u_n \tag{25}$$

From equation (17):

$$nv_0 = nu \quad \Rightarrow \quad u = v_0 \tag{26}$$

2 Derivation for Energy Loss Mechanisms in a Newton's Cradle

2.1 Total Energy Loss from Restitution

For a head-on collision between identical spheres, the total mechanical energy loss can be expressed in terms of the coefficient of restitution e:

$$\Delta E_{\text{total}} = (1 - e^2)E_0 \tag{27}$$

where the initial kinetic energy is

$$E_0 = \frac{1}{2}mv_0^2. \tag{28}$$

This result follows from conservation of momentum and the definition of restitution [2, 3].

2.2 Hertzian Contact Force

The normal contact force between two elastic spheres is given by Hertzian contact theory:

$$F(\delta) = k\delta^{3/2} \tag{29}$$

where δ is the compression and

$$k = \frac{4}{3} E^* \sqrt{R^*}. \tag{30}$$

The effective elastic modulus and radius are

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \tag{31}$$

$$\frac{1}{R^*} = \frac{1}{R_1} + \frac{1}{R_2}. \tag{32}$$

For identical spheres:

$$E^* = \frac{E}{1 - \nu^2}, \qquad R^* = \frac{R}{2}. \tag{33}$$

These expressions are standard results from elastic contact mechanics [1].

2.3 Heat Loss: Viscoelastic Dissipation

Real materials exhibit internal damping. The Hertzian force law is modified to include a dissipative term:

$$F(\delta, \dot{\delta}) = k\delta^{3/2} + \gamma \delta^{1/2} \dot{\delta} \tag{34}$$

where γ is a viscoelastic damping coefficient [4].

The energy dissipated as heat during the collision is

$$E_{\text{heat}} = \int_0^\tau \gamma \delta^{1/2} \dot{\delta}^2 \, dt. \tag{35}$$

Using Hertzian scaling relations for maximum compression and collision duration [1, 4], the heat loss scales as

$$E_{\text{heat}} \sim \gamma m^{1/2} k^{-3/2} v_0^{6/5}. \tag{36}$$

2.4 Sound Radiation

The accelerating sphere surface acts as an acoustic dipole source. The radiated acoustic power is approximately [5]

$$P \sim \frac{\rho_{\text{air}}}{c_{\text{air}}} A^2 a^2 \tag{37}$$

where ρ_{air} is the air density, c_{air} is the speed of sound in air, A is the effective radiating area, and a is the characteristic surface acceleration.

The total sound energy radiated during a collision is

$$E_{\text{sound}} = \int_0^\tau P(t) \, dt. \tag{38}$$

5

Using Hertzian collision scaling, this gives

$$E_{\text{sound}} \sim \frac{\rho_{\text{air}}}{c_{\text{air}}} R^{21/5} (E^*)^{2/5} m^{-2/5} v_0^{7/5}. \tag{39}$$

2.5 Energy Balance

The complete energy balance for one collision is therefore

$$E_0 = E_{\text{after}} + E_{\text{heat}} + E_{\text{sound}}, \tag{40}$$

with

$$E_{\text{after}} = e^2 E_0. \tag{41}$$

In typical steel Newton's cradle systems, $E_{\text{heat}} \gg E_{\text{sound}}$.

3 Derivation: Unequal Mass Collision

3.1 Setup

Consider two balls: ball 1 (mass m_1, initial velocity v_0) strikes ball 2 (mass m_2, initially at rest). After collision, the velocities are v_1' and v_2' respectively.

3.2 Conservation Laws

Conservation of momentum:
$$m_1 v_0 = m_1 v_1' + m_2 v_2' \tag{42}$$

Conservation of kinetic energy (elastic collision):
$$\frac{1}{2} m_1 v_0^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \tag{43}$$

Simplifying equation (2):
$$m_1 v_0^2 = m_1 v_1'^2 + m_2 v_2'^2 \tag{44}$$

3.3 Solving for v_2'

From equation (1), isolate v_1':
$$v_1' = \frac{m_1 v_0 - m_2 v_2'}{m_1} \tag{45}$$

Substitute equation (4) into equation (3):

$$m_1 v_0^2 = m_1 \left(\frac{m_1 v_0 - m_2 v_2'}{m_1} \right)^2 + m_2 v_2'^2 \tag{46}$$

Simplify:
$$m_1 v_0^2 = \frac{(m_1 v_0 - m_2 v_2')^2}{m_1} + m_2 v_2'^2 \tag{47}$$

Multiply through by m_1:
$$m_1^2 v_0^2 = (m_1 v_0 - m_2 v_2')^2 + m_1 m_2 v_2'^2 \tag{48}$$

Expand the squared term:
$$m_1^2 v_0^2 = m_1^2 v_0^2 - 2 m_1 m_2 v_0 v_2' + m_2^2 v_2'^2 + m_1 m_2 v_2'^2 \tag{49}$$

Cancel $m_1^2 v_0^2$ from both sides:
$$0 = -2 m_1 m_2 v_0 v_2' + m_2^2 v_2'^2 + m_1 m_2 v_2'^2 \tag{50}$$

Factor out $m_2 v_2'$:
$$0 = -2 m_1 m_2 v_0 v_2' + m_2 v_2'^2 (m_2 + m_1) \tag{51}$$

Rearrange (taking non-trivial solution $v_2' \neq 0$):
$$2 m_1 m_2 v_0 v_2' = m_2 v_2'^2 (m_1 + m_2) \tag{52}$$

Divide both sides by $m_2 v_2'$:
$$2 m_1 v_0 = v_2' (m_1 + m_2) \tag{53}$$

Solve for v_2':
$$\boxed{v_2' = \frac{2 m_1}{m_1 + m_2} v_0} \tag{54}$$

3.4 Solving for v_1'

Substitute equation (13) back into equation (4):
$$v_1' = \frac{m_1 v_0 - m_2 \cdot \frac{2 m_1}{m_1 + m_2} v_0}{m_1} \tag{55}$$

Simplify:
$$v_1' = v_0 \left(1 - \frac{2 m_2}{m_1 + m_2} \right) \tag{56}$$

Combine fractions:
$$v_1' = v_0 \left(\frac{m_1 + m_2 - 2 m_2}{m_1 + m_2} \right) \tag{57}$$

$$\boxed{v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_0} \tag{58}$$

3.5 Introducing Mass Ratio $\mu = m_2 / m_1$

Divide numerator and denominator of equations (13) and (17) by m_1:

$$v_1' = \frac{1 - \mu}{1 + \mu} v_0 \tag{59}$$

$$v_2' = \frac{2}{1 + \mu} v_0 \tag{60}$$

3.6 Energy Transfer Efficiency

The energy transfer efficiency is the fraction of kinetic energy transferred to ball 2:

$$\eta = \frac{E_2'}{E_0} = \frac{\frac{1}{2}m_2 v_2'^2}{\frac{1}{2}m_1 v_0^2} \tag{61}$$

Substitute v_2' from equation (19):

$$\eta = \frac{m_2}{m_1} \cdot \frac{v_2'^2}{v_0^2} = \frac{m_2}{m_1} \cdot \left(\frac{2}{1+\mu}\right)^2 \tag{62}$$

Since $\mu = m_2/m_1$:

$$\eta = \mu \cdot \frac{4}{(1+\mu)^2} \tag{63}$$

$$\boxed{\eta = \frac{4\mu}{(1+\mu)^2}} \tag{64}$$

3.7 Maximizing Energy Transfer

To find the maximum, take the derivative with respect to μ and set to zero:

$$\frac{d\eta}{d\mu} = \frac{d}{d\mu}\left[\frac{4\mu}{(1+\mu)^2}\right] \tag{65}$$

Using the quotient rule:

$$\frac{d\eta}{d\mu} = \frac{4(1+\mu)^2 - 4\mu \cdot 2(1+\mu)}{(1+\mu)^4} \tag{66}$$

Simplify:

$$\frac{d\eta}{d\mu} = \frac{4(1+\mu) - 8\mu}{(1+\mu)^3} = \frac{4 + 4\mu - 8\mu}{(1+\mu)^3} \tag{67}$$

$$\frac{d\eta}{d\mu} = \frac{4(1-\mu)}{(1+\mu)^3} \tag{68}$$

Set equal to zero:

$$\frac{4(1-\mu)}{(1+\mu)^3} = 0 \quad \Rightarrow \quad 1 - \mu = 0 \quad \Rightarrow \quad \boxed{\mu = 1} \tag{69}$$

At $\mu = 1$:

$$\eta_{\text{max}} = \frac{4 \cdot 1}{(1+1)^2} = \frac{4}{4} = \boxed{1} \tag{70}$$

Therefore, energy transfer is maximized when masses are equal ($\mu = 1$), achieving perfect energy transfer ($\eta = 1$).

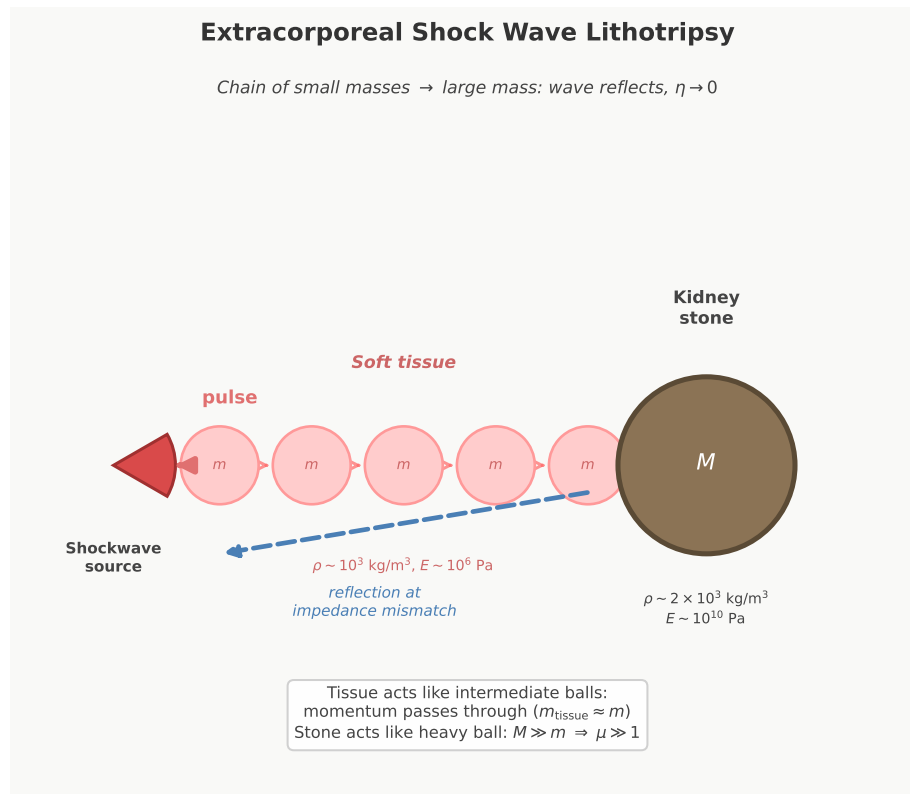## 3.8 Image to show relation Lithotripsy and Newtons Cradle

Figure 1: Lithotripsy to Newton Cradle Comparison Image

4 Code

4.1 Collision Graph

```python
# -*- coding: utf-8 -*-
"""AMA3020 Solo Code.ipynb

Automatically generated by Colab.

Original file is located at
    https://colab.research.google.com/drive/12iCTg8Mar2_BobOjqOpDo9ZL-uFcHuYG
"""

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.ticker as ticker

#       Parameters


mu = np.linspace(0.01, 10, 2000)    # mass ratio mu = m2/m1

#       Equations (6), (7), (8)


v1_norm =  (1 - mu) / (1 + mu)          # v1' / v0
v2_norm =   2       / (1 + mu)          # v2' / v0
eta     = 4 * mu   / (1 + mu)**2        # energy transfer efficiency

#       Figure setup     single plot, dual y-axes


```

```
24
25 fig, ax1 = plt.subplots(figsize=(10, 6))
26 plt.xticks(fontsize = 16)
27 plt.yticks(fontsize = 16)
28 ax2 = ax1.twinx()
29
30 C_BG    = "#f9f9f7"
31 C_V1    = "#4a7fb5"
32 C_V2    = "#5aa67a"
33 C_ETA   = "#c45050"
34 C_GRID  = "#dddddd"
35 C_ANNOT = "#555555"
36
37
38 ax1.spines["top"].set_visible(False)
39 ax1.yaxis.grid(True, linestyle="--", linewidth=0.6, color=C_GRID, zorder=0)
40 ax1.xaxis.grid(True, linestyle="--", linewidth=0.6, color=C_GRID, zorder=0)
41 ax1.set_axisbelow(True)
42 ax2.spines["top"].set_visible(False)
43
44
45 #        Velocity curves on left axis
46 l1, = ax1.plot(mu, v1_norm, color=C_V1, linewidth=2.2,
47                label=r"$v_1' / v_0$")
48 l2, = ax1.plot(mu, v2_norm, color=C_V2, linewidth=2.2,
49                label=r"$v_2' / v_0$")
50
51 ax1.axhline(0, color="#aaaaaa", linewidth=0.8, linestyle=":")
52 ax1.axvline(1, color="#aaaaaa", linewidth=1.0, linestyle="--", zorder=1)
53
54 # mu=1 markers and annotations
55
56
57 ax1.set_ylabel(r"Normalised velocity  $v'/v_0$", fontsize=16, color="#333333")
58 ax1.set_ylim(-1.3, 2.1)
59 ax1.yaxis.set_major_locator(ticker.MultipleLocator(0.5))
60 ax1.tick_params(axis="y", labelcolor="#333333")
61 #        Efficiency curve on right axis
62 l3, = ax2.plot(mu, eta, color=C_ETA, linewidth=2.5, linestyle="-.",
63                label=r"$\eta $")
64
65
66
67
68
69 ax2.set_ylabel(r"Energy transfer efficiency  $\eta$", fontsize=16, color=C_ETA
       )
70 ax2.set_ylim(0, 1.25)
71 ax2.yaxis.set_major_locator(ticker.MultipleLocator(0.2))
72 ax2.tick_params(axis="y")
73 ax2.spines["right"].set_edgecolor(C_ETA)
74 #        Axes, title, legend
75 ax1.set_xlabel(r"Mass ratio  $\mu$", fontsize=16)
76 ax1.set_xlim(0, 3)
77 ax1.xaxis.set_major_locator(ticker.MultipleLocator(1))
78
79
```

```
80 lines = [l1, l2, l3]
81 labels = [l.get_label() for l in lines]
82 ax1.legend(lines, labels, fontsize=16, loc="lower left",
83            framealpha=0.9)
84 plt.xticks(fontsize = 16)
85 plt.yticks(fontsize = 16)
86 plt.savefig("mass_ratio_figure.pdf")
87 plt.show()
```

4.2   Energy Transfer graph

```
1  # -*- coding: utf-8 -*-
2  """AMA3020 Solo Code1.ipynb
3
4  Automatically generated by Colab.
5
6  Original file is located at
7      https://colab.research.google.com/drive/1T07CHyvjhxL2mY8AKBVYiMR3HSEhSDn3
8  """
9
10 import numpy as np
11 import matplotlib.pyplot as plt
12 import matplotlib.ticker as ticker
13
14 #        Parameters
15 E0   = 1.0              # Initial kinetic energy (J)
16 Nmax = 40               # Number of collisions to simulate
17
18 materials = [
19     {"name": "Steel",  "e": 0.98, "color": "#4a90c4", "ls": "-"},
20     {"name": "Glass",  "e": 0.95, "color": "#5ab490", "ls": "--"},
21     {"name": "Brass",  "e": 0.90, "color": "#c4a020", "ls": "-."},
22     {"name": "Rubber", "e": 0.70, "color": "#c45050", "ls": ":"},
23 ]
24
25 #        Energy decay: E_n = e^(2n) * E0  (from eq. 41 applied repeatedly)
26 n = np.arange(0, Nmax + 1)
27
28 def energy(e, n):
29     return E0 * e**(2 * n)
30
31 #        Plot setup
32 fig, ax = plt.subplots(figsize=(10, 6))
33
34
35 #        Plot each material
36 for mat in materials:
37     E = energy(mat["e"], n)
38     loss_pct = (1 - mat["e"]**2) * 100        # % lost per collision
39     half_life = int(np.ceil(                  # collision where E = E0/2
40         np.log(0.5) / (2 * np.log(mat["e"]))))
41
42     label = (f"{mat['name']}  (e = {mat['e']:.2f}) ")
43
44     ax.plot(n, E,
```

11

```python
                linewidth=2.2,
                linestyle=mat["ls"],
                label=label)

    # Dot markers every 5 collisions
    ax.plot(n[::5], energy(mat["e"], n[::5]),
                "o", color=mat["color"], markersize=4, zorder=5)


#       Axes and labels


ax.set_xlabel("Collision number  $n$", fontsize=16)
ax.set_ylabel("Kinetic Energy  $E_n$  (J)", fontsize=16)
ax.set_xlim(0, 15)
ax.set_ylim(0, E0 * 1.08)
ax.xaxis.set_major_locator(ticker.MultipleLocator(5))
ax.xaxis.set_minor_locator(ticker.MultipleLocator(1))
ax.yaxis.set_major_locator(ticker.MultipleLocator(0.2))

ax.grid(True, which="major", linestyle="-",  linewidth=0.5, alpha=0.4)
ax.grid(True, which="minor", linestyle="--", linewidth=0.3, alpha=0.2)


#       Legend


leg = ax.legend(fontsize=16, loc="upper right",
                framealpha=0.9,
                title="Material  (e = coeff. of restitution)",
                title_fontsize=16)

plt.tight_layout()
plt.xticks(fontsize = 16)
plt.yticks(fontsize = 16)
plt.savefig("energy_decay.pdf", dpi=150, bbox_inches="tight")
plt.show()

print("Saved: energy_decay.png")

#       Console summary table


print(f"\n{'Material':<10} {'e':>6} {'Half-life (n)':>15}")
print("-" * 52)
```

References

[1] K. L. Johnson, *Contact Mechanics*, Cambridge University Press, 1985.

[2] W. Goldsmith, *Impact: The Theory and Physical Behaviour of Colliding Solids*, Dover, 2001.

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[4] N. V. Brilliantov and T. Pöschel, *Kinetic Theory of Granular Gases*, Oxford University Press, 2004.

[5] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, Princeton University Press, 1986.