COMP6210 Automated Software Verification

Symbolic Model Checking

Pavel Naumov

Intended Learning Outcomes

By the end of this lecture, you will be able to

- model systems with variables using Boolean functions
- model Boolean programs using Boolean functions
- understand how simple correctness properties can be verified using a symbolic representation of programs

Outline

Symbolic Modelling of Systems

Symbolic Model Checking

Modelling Software (Reminder)

- Running software has:
  - data, partitioned into variables, each taking values from a finite set; this includes:
    - a program counter per thread
    - variables on stack
    - variables on heap
  - instructions (sequential or concurrent) transforming data over time.
- We saw already how to model data and instructions using transition systems.

Explicit versus Symbolic Modelling of Software Systems

- transition systems model software systems explicitly
  - each node represents a system state
  - each edge represents an atomic state change
- this suffers from state space explosion
  - number of states grows exponentially with number of variables
  - a system with n Boolean variables can have 2^n states
  - approximately 10^{78} atoms in universe!
- transition systems are inadequate to model systems with more than a few variables
- in contrast, symbolic model checking avoids explicitly constructing the state space of a system, and instead represents it as a formula in propositional logic
  - this allows for compact representations, e.g. using Binary Decision Diagrams (BDDs)

Boolean Connectives and Notation

- boolean connectives: $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if)
- $V = \{v_1, \ldots, v_n\}$ set of boolean variables
- $\bigwedge_{j=1,\ldots,n} (v_j)$: $v_1 \wedge v_2 \wedge \ldots \wedge v_n$
- $\bigvee_{j=1,\ldots,n} (v_j)$: $v_1 \vee v_2 \vee \ldots \vee v_n$
- $\bigwedge_{j=1,\ldots,n, j\neq i} (v_j)$: $v_1 \wedge \ldots \wedge v_{i-1} \wedge v_{i+1} \wedge \ldots \wedge v_n$

Symbolic Representation of Systems

- finite set of variables $V = \{x_1, \dots, x_n\}$, over a finite domain D, to encode system states (nodes)
- describe the initial states with a formula over the variables in V:

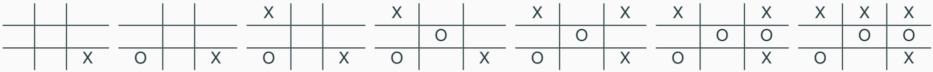
$$Init(x_1, \dots, x_n)$$

- represent transitions with a formula over the variables in V and the variables $x'_1, \dots, x'_n$ (which are a copy of the variables in V):

$$Trans(x_1, \dots, x_n, x'_1, \dots, x'_n)$$

  - variables $x_1, \dots, x_n$ describe the state *before* the transition
  - variables $x'_1, \dots, x'_n$ describe the state *after* the transition

Example: Noughts and Crosses



- variables and domain:

  $V = \{x_1, \ldots, x_9, t\}$ D = { –, X, O, A, B }
- variable x_i encodes the content of cell i
  - "–" stands for empty cell
  - "X" stands for marked by player A
  - "O" stands for marked by player B
- variable t encodes the player who moves next

状态空间的规模：

- 每个格子 x_i 都有 3 种可能的状态（"X", "O", "-"）。
- 9 个格子独立选择状态，所以这些格子的组合数为 3^9。
- 再加上变量 t（可能的值是 A 或 B），会使总状态空间大小变为 $3^9 \times 2 = 39366$ 种状态（约 40000 种状态）。

Note: An explicit representation of the state space would use $\approx$ 40000 states !

Example: Noughts and Crosses

		X

O		X

X		
O		X

X		
	O	
O		X

X		X
	O	
O		X

X		X
	O	O
O		X

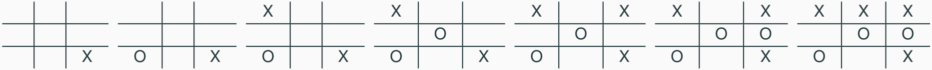
X	X	X
	O	O
O		X

Initial configurations:

Init: $(x_1 = -) \wedge (x_2 = -) \wedge \ldots \wedge (x_9 = -) \wedge (t = A \vee t = B)$

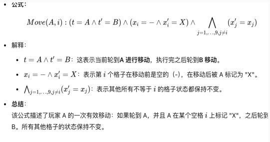
(all cells are empty and either player A or player B can start)

Example: Noughts and Crosses



Transition relation:

(either player A or player B can move, if it is her turn, and mark one of the empty cells)



- **公式：**

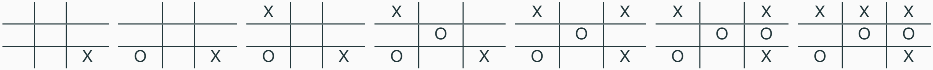
$$Move(A,i):(t=A\wedge t'=B)\wedge(x_i=-\wedge x_i'=X)\wedge\bigwedge_{j=1,\ldots,9,j\neq i}(x_j'=x_j)$$

- 解释：
  - $t=A\wedge t'=B$：这表示当前轮到**A 进行移动**，执行完之后轮到**B 移动**。
  - $x_i=-\wedge x_i'=X$：表示第 i 个格子在移动前是空的（-），在移动后被 A 标记为 "X"。
  - $\bigwedge_{j=1,\ldots,9,j\neq i}(x_j'=x_j)$：表示其他所有不等于 i 的格子状态都保持不变。
- **总结：**
  该公式描述了玩家 A 的一次有效移动：如果轮到 A，并且 A 在某个空格 i 上标记 "X"，之后轮到 B。所有其他格子的状态保持不变。

Trans : $\bigvee_{i=1,\ldots,9}(Move(A,i)\vee Move(B,i))$

- *Move(A, i)* :
  $$(t=A\wedge t'=B)\wedge(x_i=-\wedge x_i'=X)\wedge\bigwedge_{j=1,\ldots,9,j\neq i}(x_j'=x_j)$$
  (it's A's turn to move and B will move next; cell i was empty and now it is marked with X; all other cells remain unchanged)
- *Move(B, i)* is similar.

Example: Noughts and Crosses



Winning condition for player A:

$$
\begin{aligned}
\color{orange}{\mathit{WinA}} \quad : \quad & (x_1 = X \wedge x_2 = X \wedge x_3 = X) \\
& \vee (x_1 = X \wedge x_5 = X \wedge x_9 = X) \\
& \vee \; \ldots
\end{aligned}
$$

How to Represent a Program Symbolically?

```
int x,y = 0,0;
while (true)
    x,y = (x+1) mod 3, x+1;
```

- variables: x, y with domain int (bounded!)
- initial state:

$$Init : (x = 0) \wedge (y = 0)$$

- transition relation:

$$Trans : (x' = (x + 1) \text{ mod } 3) \wedge (y' = x + 1)$$

Boolean Programs

- in the previous examples, the domains for the variables are not Boolean
- to encode *Init* and *Trans* as *propositional logic formulas*, we would need to encode each program variable using *several* boolean variables
- to avoid doing this (for convenience), we restrict to Boolean programs:
    - only Boolean variables
    - only ASSIGNMENT, IF, and WHILE statements
    - no procedure calls

How to Represent a Boolean Program?

begin
L_1 : x_1 = false;
L_2 : **while** $(x_1 \wedge x_2)$ **do**
L_3 : x_1 = true;
L_4 : **endwhile**;
L_5 : **if** $(x_1 \vee x_2)$ **then**
L_6 : $x_1 = x_1 \leftrightarrow x_2$;
L_7 : **else**
L_8 : $x_2 = x_1 \leftrightarrow x_2$;
L_9 : **endif**
L_{10} : **end**

(program counters added as before)

How to Represent a Boolean Program?

$$
\begin{array}{ll}
 & \textbf{begin} \\
L_1: & \quad x_1 = \text{false}; \\
L_2: & \quad \textbf{while}\ (x_1 \wedge x_2)\ \textbf{do} \\
L_3: & \qquad x_1 = \text{true}; \\
L_4: & \quad \textbf{endwhile}; \\
L_5: & \quad \textbf{if}\ (x_1 \vee x_2)\ \textbf{then} \\
L_6: & \qquad x_1 = x_1 \leftrightarrow x_2; \\
L_7: & \quad \textbf{else} \\
L_8: & \qquad x_2 = x_1 \leftrightarrow x_2; \\
L_9: & \quad \textbf{endif} \\
L_{10}: & \textbf{end}
\end{array}
$$

- variables:

  $$V = \{x_1, x_2, PC\}$$

- domain:

  $$D = \{F, T, L_1, \ldots, L_{10}\}$$

  (still not Boolean!)

- transitions:

  $$Trans = ?$$

Encoding assignments

$$\boxed{\begin{array}{l} L_j : x_i = b(x_1, \ldots, x_n); \\ L_{j+1} : \ldots \end{array}}$$

(b is a boolean formula over the program variables)

$Trans_j$:

$$\begin{array}{rl} PC = L_j \wedge PC' = L_{j+1} & \text{(update program counter)} \\ \wedge\ x'_i \leftrightarrow b(x_1, \ldots, x_n) & \text{(update variable } x_i\text{)} \\ \wedge \bigwedge_{j=1,\ldots,n, j \neq i} (x'_j = x_j) & \text{(copy all the other variables)} \end{array}$$

Encoding if-then-else statements

L_i : **if** $(b(x_1, \ldots, x_n))$ **then**
L_{i+1} : first statement then-body
$\ldots$
L_j : **else**
L_{j+1} : first statement else-body
$\ldots$
L_k : **endif**
L_{k+1} : $\ldots$

(b is a boolean formula over the program variables)

$$
\begin{aligned}
\textit{Trans}_i : (\quad & (PC = L_i \wedge PC' = L_{i+1} \wedge b(x_1, \ldots, x_n)\ ) \\
& \vee (PC = L_i \wedge PC' = L_{j+1} \wedge \neg b(x_1, \ldots, x_n)\ ) \\
& \vee (PC = L_j \wedge PC' = L_{k+1}) && \text{(update PC)} \\
& \vee (PC = L_k \wedge PC' = L_{k+1})\ ) \\
& \wedge \bigwedge_{j=1,\ldots,n} (x_j' = x_j) && \text{(copy variables)}
\end{aligned}
$$

Encoding while statements

L_i : **while** $(b(x_1, \ldots, x_n))$ **do**
L_{i+1} : first statement while-body
$\ldots$
L_j : **endwhile**
L_{j+1} : $\ldots$

(b is a boolean formula over the program variables)

$$
\begin{array}{lll}
\mathit{Trans}_i : (& (PC = L_i \wedge PC' = L_{i+1} \wedge b(x_1, \ldots, x_n)\ ) & \text{(enter while)} \\
& \vee\ (PC = L_i \wedge PC' = L_{j+1} \wedge \neg b(x_1, \ldots, x_n)\ ) & \text{(exit while)} \\
& \vee\ (PC = L_j \wedge PC' = L_i)\ ) & \text{(back to condition)} \\
& \wedge \bigwedge_{j=1,\ldots,n} (x_j' = x_j) & \text{(copy variables)}
\end{array}
$$

How to Represent a Boolean Program?

	begin
L_1 :	x_1 = false;
L_2 :	**while** $(x_1 \wedge x_2)$ **do**
L_3 :	x_1 = true;
L_4 :	**endwhile**;
L_5 :	**if** $(x_1 \vee x_2)$ **then**
L_6 :	$x_1 = x_1 \leftrightarrow x_2$;
L_7 :	**else**
L_8 :	$x_2 = x_1 \leftrightarrow x_2$;
L_9 :	**endif**
L_{10} :	**end**

$$
\begin{aligned}
&\mathit{Trans}(x_1, x_2, PC, x_1', x_2', PC') = \\
&(PC = L_1 \wedge PC' = L_2 \wedge x_1' = F \wedge x_2' = x_2) \\
&\vee((\quad (PC = L_2 \wedge PC' = L_3 \wedge (x_1 \wedge x_2)) \\
&\qquad \vee(PC = L_2 \wedge PC' = L_5 \wedge \neg(x_1 \wedge x_2)) \\
&\qquad \vee(PC = L_4 \wedge PC' = L_2) \\
&\quad) \\
&\quad \wedge(x_1' = x_1) \wedge (x_2' = x_2)) \\
&\vee(PC = L_3 \wedge PC' = L_4 \wedge (x_1' = T) \wedge (x_2' = x_2)) \\
&\vee((\quad (PC = L_5 \wedge PC' = L_6 \wedge (x_1 \vee x_2)) \\
&\qquad \vee(PC = L_5 \wedge PC' = L_8 \wedge \neg(x_1 \vee x_2)) \\
&\qquad \vee(PC = L_7 \wedge PC' = L_{10}) \\
&\qquad \vee(PC = L_9 \wedge PC' = L_{10}) \\
&\qquad) \\
&\quad \wedge(x_1' = x_1) \wedge (x_2' = x_2)) \\
&\vee(PC = L_6 \wedge PC' = L_7 \wedge (x_1' = (x_1 \leftrightarrow x_2)) \wedge (x_2' = x_2)) \\
&\vee(PC = L_8 \wedge PC' = L_9 \wedge (x_2' = (x_1 \leftrightarrow x_2)) \wedge (x_1' = x_1))
\end{aligned}
$$

Encoding Program Counters using Boolean Variables

- representation on the previous slide is still not a Boolean formula!
- **Question:** how do we encode $PC = L_i$ as a boolean formula?
- **Answer:**
  - assume the values of the program counter ($L_1, \ldots, L_{10}$ in the previous example) can be represented using n bits (4 in the example)
  - use Boolean variables $pc_0, \ldots, pc_{n-1}$ to encode the value of the program counter
    - e.g. use variables pc_0, pc_1, pc_2, pc_3 to encode integers $i \in \{1, \ldots, 10\}$ (and thus the formula $PC = L_i$)
  - if i is represented as $c_{n-1} \ldots c_0$ in binary, $PC = L_i$ can be encoded using the formula $(pc_0 \leftrightarrow c_0) \wedge \ldots \wedge (pc_{n-1} \leftrightarrow c_{n-1})$
    - for $i = 6$ (i.e. 0110 in binary), the formula $PC = L_6$ is encoded as $(pc_3 \leftrightarrow 0) \wedge (pc_2 \leftrightarrow 1) \wedge (pc_2 \leftrightarrow 1) \wedge (pc_3 \leftrightarrow 0)$

Note: similar encodings can be done for programs with variables of types other than Boolean!

Encoding Program Counters using Boolean Variables

- representation on the previous slide is still not a Boolean formula!
- **Question:** how do we encode $PC = L_i$ as a boolean formula?
- **Answer:**
  - assume the values of the program counter ($L_1, \ldots, L_{10}$ in the previous example) can be represented using n bits (4 in the example)
  - use Boolean variables $pc_0, \ldots, pc_{n-1}$ to encode the value of the program counter
    - e.g. use variables pc_0, pc_1, pc_2, pc_3 to encode integers $i \in \{1, \ldots, 10\}$ (and thus the formula $PC = L_i$)
  - if i is represented as $c_{n-1} \ldots c_0$ in binary, $PC = L_i$ can be encoded using the formula $(pc_0 \leftrightarrow c_0) \wedge \ldots \wedge (pc_{n-1} \leftrightarrow c_{n-1})$
    - for $i = 6$ (i.e. 0110 in binary), the formula $PC = L_6$ is encoded as $(pc_3 \leftrightarrow 0) \wedge (pc_2 \leftrightarrow 1) \wedge (pc_2 \leftrightarrow 1) \wedge (pc_3 \leftrightarrow 0)$

Note: similar encodings can be done for programs with variables of types other than Boolean!

Outline

Symbolic Modelling of Systems

Symbolic Model Checking

Model Checking

- verification of (simple) safety properties: can we ever reach a "bad" state?
- verification of temporal properties (e.g. LTL)

Example

Check if the following program can ever reach a state satisfying $x \geq y$:

```
int x,y = 0,0;
while (true)
    x,y = (x+1) mod 3, x+1;
```

- initial states: $\mathit{Init}(x, y) : x = 0 \land y = 0$
- transition relation:

$$\mathit{Trans}(x, y, x', y') : (x' = (x + 1) \bmod 3) \land (y' = x + 1)$$

- states reachable in one step from the initial state:

$$\begin{aligned}
\mathit{Reach}_1(x', y') &= \exists x. \exists y. \mathit{Init}(x, y) \land \mathit{Trans}(x, y, x', y') \\
&= \exists x. \exists y. x = 0 \land y = 0 \land x' = (x + 1) \bmod 3 \land y' = x + 1 \\
&= x' = 1 \land y' = 1
\end{aligned}$$

Example (Cont'd)

- states reachable in one step from the initial state:

$$Reach_1(x', y') \quad = \quad x' = 1 \land y' = 1$$

- states reachable in two steps from the initial state:

$$\begin{aligned} Reach_2(x', y') &= \exists x. \exists y. Reach_1(x, y) \land Trans(x, y, x', y') \\ &= \exists x. \exists y. x = 1 \land y = 1 \land x' = (x + 1) \bmod 3 \land y' = x + 1 \\ &= x' = 2 \land y' = 2 \end{aligned}$$

- states reachable in *three* steps from the initial state:

$$Reach_3(x', y') = \ldots = x' = 0 \land y' = 3$$

- at each step i (incl. $i = 0$), we can check if $Reach_i(x, y) \land (x < y)$ is *satisfiable*!

Computing Reachable States

Inputs:

- $Init(x_1, \ldots, x_n)$ (boolean formula for initial states)
- $Trans(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ (boolean formula for transitions)

Output: $Reach(x_1, \ldots, x_n)$ (boolean formula for reachable states)

For $R(x_1, \ldots, x_n)$ (boolean formula encoding a *set* of states), define:

$$Next(R) := (\exists x_1. \ldots \exists x_n. R(x_1, \ldots, x_n) \wedge Trans(x_1, \ldots, x_n, x'_1, \ldots, x'_n))[x_1/x'_1, \ldots, x_n/x'_n]$$

Algorithm (in which variables are formulas over variables $x_1, \ldots, x_n$!):

```
Reach = ff; //empty set of states
R = Init; //set of initial states
while (¬ (R → Reach)) { //R ⊈ Reach
 Reach = Reach ∨ R; //add R to Reach;
 R = Next(R); //formula for successors of states in R
}
```

Checking Safety Properties

Input: safety property $\phi(x_1, \ldots, x_n)$ (system invariant)

- e.g. $(x_1 \vee x_2) \wedge \neg(x_1 \wedge x_2)$

Output: Does the system described by $Init(x_1, \ldots, x_n)$ and $Trans(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ satisfy the given property?

Algorithm:

1. Compute reachable states, as a formula $Reach(x_1, \ldots, x_n)$
2. Check if $Reach(x_1, \ldots, x_n) \wedge \neg\phi(x_1, \ldots, x_n)$ is *satisfiable* (i.e. it can be satisfied by an assignment of values to $x_1, \ldots, x_n$)
3. If Yes, return "Safety property violated".
   If No, return "Success".

Checking Safety Properties

Input: safety property $\phi(x_1, \ldots, x_n)$ (system invariant)

- e.g. $(x_1 \vee x_2) \wedge \neg(x_1 \wedge x_2)$

Output: Does the system described by $\mathit{Init}(x_1, \ldots, x_n)$ and $\mathit{Trans}(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$ satisfy the given property?

Algorithm:

1. Compute reachable states, as a formula $\mathit{Reach}(x_1, \ldots, x_n)$
2. Check if $\mathit{Reach}(x_1, \ldots, x_n) \wedge \neg\phi(x_1, \ldots, x_n)$ is *satisfiable* (i.e. it can be satisfied by an assignment of values to $x_1, \ldots, x_n$)
3. If Yes, return "Safety property violated".
   If No, return "Success".

Checking LTL Properties (Basic Idea)

Similar idea to explicit-state model checking for LTL:

- system described using $\mathit{Trans}(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$
- negation of LTL property ($\neg\phi$) described as tableau (boolean formula which encodes *paths* that satisfy $\neg\phi$)
- conjoin (using $\bigwedge$) the transition relations of
  (i) the system
  (ii) the tableau

  to find counter-example...
- algorithm computes the set of states from which a path satisfying $\neg\phi$ exists

BDDs and Symbolic Model Checking

- binary decision diagrams (BDDs) are a canonical form to represent Boolean functions
  - often more compact than traditional normal forms
  - can be manipulated efficiently
- reachable state space can be represented as a BDD
- property verification uses iterative computations on the reachable state space

Summary

- encoding systems with Boolean variables as boolean formulas
- encoding (Boolean) programs as boolean formulas
- symbolic model checking of simple properties

A Word on the State of the Art (not Examinable!)

Counter-example-Guided Abstraction Refinement (CEGAR)

- technique to further improve scalability
- not covered in this module

1. Model the system at hand.
2. Build more abstract (smaller!) version of the model, which *over-approximates* system behaviours.
3. Verify safety properties on the abstract model.
   3.1 If Success, the model satisfies safety (as an over-approximation was used).
   3.2 Otherwise, check if the counter-example is a valid one (i.e. it can be exhibited in the original model).
      - If yes, safety is violated.
      - If not, use the counter-example to guide a refinement of the abstract model, and go back to step 3.

Exercises

Encode the following programs:

```
begin
  while (x1 ∧ x2) do
     x1 = true;
     while (x1 ↔ x2) do
        x1 = x1 ∨ x2;
     endwhile;
   endwhile;
   if (x1 ∨ x2) then
        x3 = x1 ↔ x2;
        x2 = x1 ∨ x2;
   else
        x2 = x1 ∧ x2;
   endif
end
```

```
begin
   x1 = false;
  while (x1 ∧ x2) do
     x1 = true;
     while (x1 ↔ x2) do
       if (x1 ∨ x2) then
         x3 = x1 ∨ ¬x2;
       else
         x3 = ¬x1 ∧ x2;
       endif;
     endwhile;
   endwhile
end
```