Algebra and Logic, Vol. 47, No. 1, 2008

CAYLEY POLYNOMIALS

Y. Yoshii

UDC 512.554

Dedicated to Bruce Allison
on the occasion of his 60th birthday

Keywords: *alternative algebra, ring of Cayley polynomials, octonion algebra.*

We consider a polynomial version of the Cayley numbers. Namely, a ring of Cayley polynomials is defined in terms of generators and relations in the category of alternative algebras. The ring turns out to be an octonion algebra over an ordinary polynomial ring. Also, a localization (a ring of quotients) of the ring of Cayley polynomials gives another description of an octonion torus. Finally, we find a subalgebra of a prime nondegenerate alternative algebra which is an octonion algebra over its center.

INTRODUCTION

Nonassociative analogs of Laurent polynomials appear naturally in classifying extended affine Lie algebras and Lie tori. These Lie algebras are a natural generalization of affine Kac–Moody Lie algebras (see [1-3]). Just as affine Kac–Moody Lie algebras are coordinatized by a ring of Laurent polynomials in one variable, so are extended affine Lie algebras or Lie tori coordinatized by nonassociative analogs of Laurent polynomials in several variables. Types of those Lie algebras are determined by finite irreducible root systems, and coordinate algebras depend on these types. In particular, such Lie algebras of type A_2 are coordinatized by alternative algebras; an alternative analog of Laurent polynomials (which is not associative) was found in [4]. A coordinate algebra is called an *octonion torus*. It turns out that the coordinate algebras of the extended affine Lie algebras or Lie tori of type A_2, C_3, or F_4 which are not associative are exactly octonion tori (see also [5-8]).

An *octonion torus* (an *octonion n-torus*) is defined via a Cayley–Dickson process over a ring of Laurent polynomials. More precisely, such is produced by applying the Cayley–Dickson process three times over $F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ with $n \geqslant 3$, where F is a field of characteristic not 2 and z_1, z_2, and z_3 are structure constants, i.e., in the standard notation for the Cayley–Dickson process (see Sec. 1),

$$(F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}], z_1, z_2, z_3).$$

To study the algebra structure, it suffices to consider an octonion n-torus $(F[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}], z_1, z_2, z_3)$, which is also called a *Cayley torus*, for the case $n = 3$. Our goal is to find a simple presentation of the Cayley torus, or, more exactly, a simple presentation of its subalgebra

$$D = (F[z_1, z_2, z_3], z_1, z_2, z_3).$$

Akita National College of Technology, 1-1 Iijima Bunkyocho Akita-shi, 011-8511 Japan; yoshiiakita-nct.jp. Translated from *Algebra i Logika*, Vol. 47, No. 1, pp. 54-70, January-February, 2008. Original article submitted October 11, 2006; revised July 30, 2007.

The algebra D also appears as a certain subalgebra of a free alternative algebra generated by three or more elements, which was brought in sight by G. V. Dorofeev (see Remark 3.2 below). The affine Krull dimension of this algebra was studied in [9].

First we define an algebra $F_C[t_1, t_2, t_3]$ over F by the relations

$$t_2 t_1 = -t_1 t_2, \quad t_3 t_1 = -t_1 t_3, \quad t_3 t_2 = -t_2 t_3, \quad (t_1 t_2) t_3 = -t_1 (t_2 t_3)$$

in the category of alternative algebras, and call it the ring of *Cayley polynomials*, or *universal octonion algebra* (since it covers all the octonion algebras over F). Then we show that $F_C[t_1, t_2, t_3]$ is isomorphic to D. In particular, $F_C[t_1, t_2, t_3]$ is an octonion algebra over the center $F[t_1^2, t_2^2, t_3^2]$ (ordinary commutative associative polynomials in three variables t_1^2, t_2^2, and t_3^2). Also, a Cayley torus can be viewed as a ring of quotients of $F_C[t_1, t_2, t_3]$ w.r.t. monomials in the center $F[t_1^2, t_2^2, t_3^2]$. As a consequence we obtain presentations of the Cayley torus, and also of any octonion algebra over F. Moreover, if the base field F is replaced by a ring Φ of scalars containing $1/2$, then the above-mentioned algebras can be treated over Φ. It is worth mentioning that free alternative algebras of rank 3 were dealt with in [10], and a finite basis of identities in a variety of octonion algebras over a field was taken up in [11] (although these papers do not have a direct bearing on ours).

Finally, we will look into *Cayley–Dickson rings*. Such a ring embeds in an octonion algebra over a field. However, the ring itself is not necessarily an octonion algebra in general. We note that the ring $F_C[t_1, t_2, t_3]$, or octonion torus, is a Cayley–Dickson ring and also an octonion algebra. In Proposition 4.3, we show that there always exists a subring B of a Cayley–Dickson ring R (or else of a prime nondegenerate alternative algebra R over Φ) such that B is an octonion algebra over the center of R, and the central closure $\overline{B}$ coincides with the central closure $\overline{R}$.

Throughout the paper, Φ is a unital commutative associative ring with $1/2$, and all algebras are assumed to be unital.

1. CAYLEY–DICKSON PROCESS

We recall the Cayley–Dickson process over a ring Φ of scalars (see [12] for details). For an algebra B over Φ, we assume that B is *faithful*, i.e., $\alpha 1 = 0 \Rightarrow \alpha = 0$ for all $\alpha \in \Phi$. Let $*$ be a *scalar involution* on B over Φ, i.e., an antiautomorphism of period 2 with $bb^* \in \Phi 1$. Let $\mu \in \Phi$ be a *cancellable scalar*, i.e., if $\mu b = 0$ for some $b \in B$ then $b = 0$. We say that an algebra $B \oplus B$ is derived from $B = (B, *)$ over Φ via the Cayley–Dickson process with structure constant μ if the product in $B \oplus B$ is given by the rule $(a, b)(c, d) = (ac + \mu db^*, a^* d + cb)$ for $a, b, c, d \in B$. If we put $v = (0, 1)$ we can write this algebra in the form $B + vB$ with multiplication

$$(a + vb)(c + vd) = (ac + \mu db^*) + v(a^* d + cb). \tag{1.1}$$

We call v the *basic generator*. Note that $v^2 = \mu$. The algebra $B + vB$ has a new involution $*$ defined by setting $(a + vb)^* = a^* - vb$. We can continue the process, and write, for example, (B, μ, ν), instead of $((B, \mu), \nu))$. Notice that B is faithful $\Leftrightarrow$ (B, μ) is faithful, and ν is cancellable in $B \Leftrightarrow \nu$ is cancellable in (B, μ).

Let μ_1, μ_2, μ_3 be any cancellable scalars in Φ. A *quaternion algebra* is the algebra (Φ, μ_1, μ_2) derived by applying the Cayley–Dickson process twice over Φ with trivial involution; such is a free 4-dimensional Φ-module and an associative but not commutative algebra. An *octonion algebra* is the algebra $(\Phi, \mu_1, \mu_2, \mu_3)$

derived by applying the Cayley–Dickson process thrice; such is an 8-dimensional free Φ-module and an alternative but not associative algebra. Note that quaternion algebras and octonion algebras are central, and if Φ is a field, then they are simple.

LEMMA 1.1. Let v_1, v_2, and v_3 be basic generators for an octonion algebra $(\Phi, \mu_1, \mu_2, \mu_3)$, with $v_1^2 = \mu_1$, $v_2^2 = \mu_2$, and $v_3^2 = \mu_3$. Then $v_2 v_1 = -v_1 v_2$, $v_3 v_1 = -v_1 v_3$, $v_3 v_2 = -v_2 v_3$, and $(v_1 v_2) v_3 = -v_1 (v_2 v_3)$.

Proof. The desired identities are readily derived from (1.1). □

In the next section, we will deal with quaternion algebras and octonion algebras over various scalar rings, not necessarily over Φ.

2. HAMILTON POLYNOMIALS

An associative algebra over Φ generated by elements t_1 and t_2 and defined by the relation $t_1 t_2 = -t_2 t_1$ is called the *ring of Hamilton polynomials*, or *universal quaternion algebra*, and is denoted by $\Phi_H[t_1, t_2]$. Note that the center of $\Phi_H[t_1, t_2]$ coincides with $\Phi[t_1^2, t_2^2]$ (ordinary commutative associative polynomials in two variables t_1^2 and t_2^2 over Φ), and $\Phi_H[t_1, t_2]$ is a quaternion algebra over $\Phi[t_1^2, t_2^2]$, i.e., $(\Phi[t_1^2, t_2^2], t_1^2, t_2^2)$ in the notation of Sec. 1. (If we consider, not Φ, but $\Phi[t_1^2, t_2^2]$, then t_1^2 and t_2^2 will be cancellable in $\Phi[t_1^2, t_2^2]$.) Clearly, any quaternion algebra over Φ is a homomorphic image of the ring $\Phi_H[t_1, t_2]$.

Since the multiplicative subset $S := \{t_1^r t_2^s\}_{r,s \in 2\mathbb{N}}$ of the center $\Phi[t_1^2, t_2^2]$ of a ring $\Phi_H[t_1, t_2]$ does not contain zero divisors of $\Phi_H[t_1, t_2]$, we can construct a ring $S^{-1}\Phi_H[t_1, t_2]$ of quotients; see, e.g., [13, p. 219]. Therefore, $S^{-1}\Phi_H[t_1, t_2]$ is again a quaternion algebra, that is,

$$S^{-1}\Phi_H[t_1, t_2] = (\Phi[t_1^{\pm 2}, t_2^{\pm 2}], t_1^2, t_2^2).$$

Let t_1, t_2, t_1^{-1}, and t_2^{-1} have respective degrees $(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$ (t_1 and t_2 are invertible in $S^{-1}\Phi_H[t_1, t_2]$, with $t_1^{-1} = t_1^{-2} t_1$ and $t_2^{-1} = t_2^{-2} t_2$). Then $S^{-1}\Phi_H[t_1, t_2]$ turns into a $\mathbb{Z}^2$-graded algebra, which we call the *quaternion* 2*-torus*, or *Hamilton torus*. Note that $\Phi_H[t_1, t_2]$ embeds in $S^{-1}\Phi_H[t_1, t_2]$.

If A is an associative algebra over Φ generated by invertible elements a and b such that $ab = -ba$, then A is a homomorphic image of $S^{-1}\Phi_H[t_1, t_2]$ via $t_1 \mapsto a$ and $t_2 \mapsto b$, in view of the universal property of a ring of quotients. An associative algebra L over Φ, which is generated by elements $t_1^{\pm 1}$ and $t_2^{\pm 1}$ and is defined by relations $t_1 t_1^{-1} = t_2 t_2^{-1} = 1$, has a natural $\mathbb{Z}^2$-grading, as above. Therefore, there exists a natural graded homomorphism from $S^{-1}\Phi_H[t_1, t_2]$ onto $L/(t_1 t_2 + t_2 t_1)$. On the other hand, since $S^{-1}\Phi_H[t_1, t_2]$ has defining relations of $L/(t_1 t_2 + t_2 t_1)$, there is a natural graded homomorphism from $L/(t_1 t_2 + t_2 t_1)$ onto $S^{-1}\Phi_H[t_1, t_2]$. Hence these homomorphisms are graded isomorphisms. Thus the Hamilton torus $S^{-1}\Phi_H[t_1, t_2]$ has a presentation in the category of associative algebras generated by $t_1^{\pm 1}$ and $t_2^{\pm 1}$ and defined by $t_1^{-1} t_1 = t_2 t_2^{-1} = 1$ and $t_1 t_2 = -t_2 t_1$.

With this fact in mind, we will assume that

$$S^{-1}\Phi_H[t_1, t_2] = \Phi_H[t_1^{\pm 1}, t_2^{\pm 1}].$$

A *quaternion* n*-torus* ($n \geqslant 2$) is defined by setting

$$\Phi_H[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] := \Phi_H[t_1^{\pm 1}, t_2^{\pm 1}] \otimes_\Phi \Phi[t_3^{\pm 1}, \ldots, t_n^{\pm 1}],$$

where $\Phi[t_3^{\pm 1}, \ldots, t_n^{\pm 1}]$ is an ordinary Laurent polynomial algebra in $n-2$ variables over Φ. Quaternion tori turn into coordinate algebras of extended affine Lie algebras of types A_ℓ, C_ℓ, and F_4.

Below is a proposition which is well known in ring theory. We can prove it following the same line of argument as will be used in proving Theorem 3.6 for octonion algebras.

PROPOSITION 2.1. Any quaternion algebra (Φ, μ_1, μ_2) over Φ with cancellable scalars μ_1, μ_2 in Φ is isomorphic to $\Phi_H[t_1, t_2]/(t_1^2 - \mu_1, t_2^2 - \mu_2)$. Hence (Φ, μ_1, μ_2) has a presentation in the category of associative algebras generated by elements t_1 and t_2 and defined by relations $t_1^2 = \mu_1$, $t_2^2 = \mu_2$, and $t_1 t_2 = -t_2 t_1$.

In particular, if Φ is a field, and A is an associative algebra over Φ generated by a_1 and a_2 satisfying $a_1 a_2 = -a_2 a_1$, $a_1^2 = \mu_1$, and $a_2^2 = \mu_2$, then A is isomorphic to (Φ, μ_1, μ_2).

3. CAYLEY POLYNOMIALS

Below we write $[a, b] = ab - ba$ for a commutator and write $(a, b, c) = (ab)c - a(bc)$ for an associator. Alternative algebras are defined by two identities $(a, a, b) = 0$ and $(b, a, a) = 0$. We have $(a, b, c) = -(b, a, c) = (b, c, a)$ (the alternative law) and $(a, b, a) = 0$ (the flexible law); therefore, the parentheses in $(ab)a = a(ba)$ can be omitted. In Proposition 3.1, use will be made of the middle Moufang identity $(ab)(ca) = a(bc)a$. Recall that the center of an alternative algebra A is defined to be $\{z \in A \mid [z, a] = (z, a, b) = 0 \text{ for all } a, b \in A\}$.

An alternative algebra over Φ generated by t_1, t_2, and t_3 and defined by the *Cayley relations*

$$t_2 t_1 = -t_1 t_2, \quad t_3 t_1 = -t_1 t_3, \quad t_3 t_2 = -t_2 t_3, \quad (t_1 t_2) t_3 = -t_1 (t_2 t_3) \tag{C}$$

is called the ring of *Cayley polynomials*, or *universal octonion algebra*, and is denoted by $\Phi_C[t_1, t_2, t_3]$. Note that any octonion algebra over Φ is a homomorphic image of $\Phi_C[t_1, t_2, t_3]$ by Lemma 1.2.

Let Z be the center of $\Phi_C[t_1, t_2, t_3]$. Our prime goal is to show that $\Phi_C[t_1, t_2, t_3]$ is an octonion algebra over Z.

CLAIM 1. In $\Phi_C[t_1, t_2, t_3]$, the identities $(t_i t_j) t_k = -t_i (t_j t_k)$ (antiassociativity) and $(t_i t_j) t_k = -t_k (t_i t_j)$ (anticommutativity) hold for any distinct $i, j, k \in \{1, 2, 3\}$.

Proof. By the anticommutativity of the identities in (C), it suffices to argue for just three anticommutativity identities $(t_i t_j) t_k = -t_k (t_i t_j)$ with $k = 1, 2, 3$. At the same time, we need to verify five antiassociativity identities of the form $(t_i t_j) t_k = -t_i (t_j t_k)$.

By the alternative law, $(t_1, t_2, t_3) = -(t_2, t_1, t_3)$. Therefore,

$$\begin{aligned} (t_2 t_1) t_3 + t_2 (t_1 t_3) &= (t_2 t_1) t_3 - (t_2, t_1, t_3) + (t_2 t_1) t_3 \\ &= (t_1, t_2, t_3) - 2(t_1 t_2) t_3 \quad (\text{since } t_2 t_1 = -t_1 t_2) \\ &= 0 \quad (\text{since } (t_1 t_2) t_3 = -t_1 (t_2 t_3)). \end{aligned}$$

Hence,

$$(t_2 t_1) t_3 = -t_2 (t_1 t_3) \quad (\text{antiassociativity}). \tag{a1}$$

Now $(t_1, t_2, t_3) = -(t_1, t_3, t_2)$ implies $(t_1 t_2) t_3 - t_1 (t_2 t_3) = -(t_1 t_3) t_2 + t_1 (t_3 t_2)$. Since $t_3 t_2 = -t_2 t_3$, we obtain

$$(t_1 t_2) t_3 = -(t_1 t_3) t_2. \tag{1}$$

Below the identities in (C) will be used without further comment. By (1), we have $(t_1 t_3) t_2 + t_1 (t_3 t_2) = -(t_1 t_2) t_3 + t_1 (t_3 t_2) = 0$. Consequently,

$$(t_1 t_3) t_2 = -t_1 (t_3 t_2) \quad (\text{antiassociativity}). \tag{a2}$$

By (a1) and (1), we obtain

$$t_2(t_1t_3) = -(t_1t_3)t_2 \quad \text{(anticommutativity for } k = 2\text{).} \tag{2}$$

Also, $(t_3t_1)t_2 + t_3(t_1t_2) = -(t_1, t_2, t_3) + 2(t_3t_1)t_2 = -(t_1, t_2, t_3) - 2(t_1t_3)t_2 = -(t_,t_2, t_3) + 2(t_1t_2)t_3 = 0$ by (1). Hence,

$$(t_3t_1)t_2 = -t_3(t_1t_2) \quad \text{(antiassociativity).} \tag{a3}$$

In view of (1) and (a3),

$$t_3(t_1t_2) = -(t_1t_2)t_3 \quad \text{(anticommutativity for } k = 3\text{).} \tag{3}$$

Now, $(t_3t_2)t_1 + t_3(t_2t_1) = (t_3, t_2, t_1) + 2t_3(t_2t_1) = -(t_1, t_2, t_3) - 2t_3(t_1t_2) = -(t_1, t_2, t_3) + 2(t_1t_2)t_3 = 0$ by (3). Consequently,

$$(t_3t_2)t_1 = -t_3(t_2t_1) \quad \text{(antiassociativity).} \tag{a4}$$

In virtue of (a4), the main involution on the free alternative algebra $\Phi\langle t_1, t_2, t_3\rangle$, i.e., an involution defined by $t_1 \mapsto t_1$, $t_2 \mapsto t_2$, and $t_3 \mapsto t_3$, preserves the relations specified in (C). Therefore, applying the induced involution to (a2) yields

$$(t_2t_3)t_1 = -t_2(t_3t_1) \quad \text{(antiassociativity).} \tag{a5}$$

Ultimately,

$$\begin{aligned}
(t_3t_2)t_1 + t_1(t_3t_2) &= -t_3(t_2t_1) + t_1(t_3t_2) \quad \text{(by (a4))} \\
&= t_3(t_1t_2) - t_1(t_2t_3) \\
&= -(t_1t_2)t_3 - t_1(t_2t_3) \quad \text{(by (3))} \\
&= 0.
\end{aligned}$$

Hence,

$$(t_3t_2)t_1 = -t_1(t_3t_2) \quad \text{(anticommutativity for } k = 1\text{).} \ \square$$

CLAIM 2. We have $t_1^2, t_2^2, t_3^2 \in Z$.

Proof. For all $i, j \in \{1, 2, 3\}$, $[t_1^2, t_i] = (t_1^2, t_i, t_j) = 0$. Indeed, except for the cases where (i, j) is equal to $(2, 3)$ and to $(3, 2)$ in the second identity, the required equality follows from (C) and Artin's theorem, which states that any subalgebra generated by 2 elements is associative. For the cases with (i, j) equal to $(2, 3)$ or $(3, 2)$, we can use the identity $(a^2, b, c) = (a, ab + ba, c)$, which holds for any alternative algebra (see [13, identity (17)]). Hence $(t_1^2, t_2, t_3) = (t_1^2, t_3, t_2) = 0$ by (C). Thus t_1^2 is central for a generating set of $\Phi_C[t_1, t_2, t_3]$, and $t_1^2 \in Z$, as follows by a theorem of Bruck and Kleinfeld (see [13, Lemma 16]). By the symmetry of the relations in (C) with respect to Claim 1, we also obtain $t_2^2, t_3^2 \in Z$. $\square$

CLAIM 3. We have $(t_1t_2)^2 = -t_1^2t_2^2$, $(t_1t_3)^2 = -t_1^2t_3^2$, $(t_2t_3)^2 = -t_2^2t_3^2$, and $(t_1(t_2t_3))^2 = t_1^2t_2^2t_3^2$. In particular, these elements are all in the center Z.

Proof. Again, by Artin's theorem, $(uv)^2 = uvuv$ (no parentheses are needed). Therefore, for example, $(t_1t_2)^2 = -t_1^2t_2^2$ and $(t_1(t_2t_3))^2 = t_1(t_2t_3)t_1(t_2t_3) = -(t_2t_3)t_1^2(t_2t_3)$ (by Claim 1) $= -t_1^2(t_2t_3)^2 = t_1^2t_2^2t_3^2$. $\square$

We can endow $\Phi_C[t_1, t_2, t_3]$ with a natural $\mathbb{N}^3$-grading, setting $\deg t_1 = (1, 0, 0)$, $\deg t_2 = (0, 1, 0)$, and $\deg t_3 = (0, 0, 1)$. This is possible since $\Phi_C[t_1, t_2, t_3]$ is defined by the homogeneous relations specified in (C).

PROPOSITION 3.1. Let $t = t_{i_1} \dots t_{i_k}$ be an element of degree (ℓ, m, n) in $\Phi_C[t_1, t_2, t_3]$ where all possible parentheses are omitted, and $t_{i_j} = t_1$, t_2, or t_3, i.e., the total degree of t is $\ell + m + n = k$. Then $t = \pm(t_1^\ell t_2^m)t_3^n$.

Proof. In view of Claim 1, the proposition is obvious for k equal to 1, 2, or 3. Suppose that the total degree k is greater than 3. Then t contains at least one of the following parts: (i) $a(bc)$, (ii) $(ab)c$, or (iii) $(ab)(cd)$ for some $a, b, c, d \in \{t_1, t_2, t_3\}$.

For (i), if two elements among a, b, c are equal, then $a(bc) = \pm t_p^2 t_q$ for some $p, q \in \{1, 2, 3\}$ by Claims 1 and 2. Hence $t = \pm t' t_p^2$ for some t', which is a product of elements t_{i_j} with total degree $k-2$, as follows by Claim 2. By induction, then, t' has the desired form, and so does t by Claim 2.

Otherwise $a(bc) = \pm(t_1 t_2)t_3$. Consider the next part s such that t has a part $s((t_1t_2)t_3)$ or $((t_1t_2)t_3)s$ (s can be one of t_1, t_2, t_3). By induction and Claims 1 and 2, s is equal to $\pm t_i z_1$, $\pm t_i t_j z_2$ ($i \neq j$), or $\pm(t_1t_2)t_3 z_3$, where z_1, z_2, z_3 are products of even powers of elements t_{i_j}, and so $z_1, z_2, z_3 \in Z$. If $z_i \neq 1$, then again we can use induction over the part produced by dropping z_i from t. Hence we may assume that s is equal to t_i, $t_i t_j$ ($i \neq j$), or $\pm(t_1t_2)t_3$. Then $s((t_1t_2)t_3)$ or $((t_1t_2)t_3)s$ has, respectively, the factor t_i^2, $(t_it_j)^2 = -t_i^2 t_j^2$, or $((t_1t_2)t_3)^2 = t_1^2 t_2^2 t_3^2$ (by Claims 1 and 3); so $t = \pm t' t_i^2$, $t = \pm t' t_i^2 t_j^2$, or $t = \pm t' t_1^2 t_2^2 t_3^2$ for some t' which is a product of t_{i_j} with total degree $k-2$, $k-4$, or $k-6$. Hence, by induction, t' has the desired form, and so does t by Claim 2.

Case (ii) follows from (i) in view of Claim 1.

For (iii), two elements among a, b, c, d should coincide, and by Claims 1, 2 and the middle Moufang identity, $(ab)(cd) = \pm t_p^2 (t_q t_r)$ for some $p, q, r \in \{1, 2, 3\}$. Hence, by the same argument as in the first part of (i), we can conclude that t has the desired form. □

Now, let $\Phi[z_1, z_2, z_3]$ be an ordinary polynomial algebra in three variables over Φ and let $D = (\Phi[z_1, z_2, z_3], z_1, z_2, z_3)$ be an octonion algebra, i.e., one that is derived by applying the Cayley–Dickson process three times over $\Phi[z_1, z_2, z_3]$ with structure constants z_1, z_2, and z_3 and trivial initial involution. Let v_1, v_2, and v_3 be basic generators for D at each stage of the process, with $v_1^2 = z_1$, $v_2^2 = z_2$, and $v_3^2 = z_3$. Then D is endowed with a natural $\mathbb{N}^3$-grading, if we set $\deg v_1 = (1, 0, 0)$, $\deg v_2 = (0, 1, 0)$, and $\deg v_3 = (0, 0, 1)$. It is easy to see that every homogeneous space is a 1-dimensional free Φ-module.

Remark 3.2. An algebra D appears as a subalgebra of a free alternative algebra (see [13, Thm. 13]). More precisely, let $\mathcal{F}$ be the free alternative algebra over Φ generated by distinct elements a, b, and c. Suppose $u = [a, b]$, $v = (a, b, c)$, and $w = (u, v, a)$. Then the subalgebra of $\mathcal{F}$ generated by u, v, and w is isomorphic to D via $u \mapsto v_1$, $v \mapsto v_2$, and $w \mapsto v_3$.

We now prove the main theorem.

THEOREM 3.3. An algebra $\Phi_C[t_1, t_2, t_3]$ is graded isomorphic to D. In particular, $Z = \Phi[t_1^2, t_2^2, t_3^2]$, which is an ordinary polynomial algebra in three variables t_1^2, t_2^2, and t_3^2 over Φ, and $\Phi_C[t_1, t_2, t_3]$ is an octonion algebra (Z, t_1^2, t_2^2, t_3^2).

Proof. By Lemma 1.2, there exists an epimorphism from $\Phi_C[t_1, t_2, t_3]$ onto D defined by $t_1 \mapsto v_1$, $t_2 \mapsto v_2$, and $t_3 \mapsto v_3$. Therefore, we need only show that every homogeneous space of the natural $\mathbb{N}^3$-grading of $\Phi_C[t_1, t_2, t_3]$ is generated by one element. This follows from Prop. 3.1. □

The multiplicative subset

$$S := \{z_1^p z_2^q z_3^r\}_{p,q,r \in \mathbb{N}}$$

of the center $\Phi[z_1, z_2, z_3]$ of D does not contain zero divisors of the octonion algebra D (which is 8-dimensional over the center). Therefore, a ring $S^{-1}D$ of quotients is also 8-dimensional over the center $\Phi[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$, with generators v_1, v_2, and v_3 the multiplication table relative to which is the same as is one for D. Hence the ring is the octonion algebra

$$S^{-1}D = (\Phi[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}], z_1, z_2, z_3).$$

We observe that v_1, v_2, and v_3 are invertible in $S^{-1}D$, with $v_1^{-1} = z_1^{-1}v_1$, $v_2^{-1} = z_2^{-1}v_2$, and $v_3^{-1} = z_3^{-1}v_3$. Set $\deg v_1^{-1} = (-1,0,0)$, $\deg v_2^{-1} = (0,-1,0)$, and $\deg v_3^{-1} = (0,0,-1)$. Then $S^{-1}D$ turns into a $\mathbb{Z}^3$-graded algebra, which we call the *octonion* 3*-torus*, or *Cayley torus*. Note that D embeds in $S^{-1}D$. Clearly, every homogeneous space of the Cayley torus is a 1-dimensional free Φ-module.

COROLLARY 3.4. Let $T = \{t_1^p t_2^q t_3^r\}_{p,q,r\in 2\mathbb{N}}$ be a subset of $\Phi_C[t_1,t_2,t_3]$. Then:

(1) T is a multiplicative subset of the center of $\Phi_C[t_1,t_2,t_3]$ which does not contain zero divisors of $\Phi_C[t_1,t_2,t_3]$, and $T^{-1}\Phi_C[t_1,t_2,t_3]$ can be identified with the Cayley torus $S^{-1}D$ via $t_1 \mapsto v_1$, $t_2 \mapsto v_2$, and $t_3 \mapsto v_3$.

(2) If A is an alternative algebra over Φ generated by a_1, a_2, and a_3 satisfying the Cayley relations $a_1a_2 = -a_2a_1$, $a_1a_3 = -a_3a_1$, $a_2a_3 = -a_3a_2$, and $(a_1a_2)a_3 = -a_1(a_2a_3)$, then A is a homomorphic image of $\Phi_C[t_1,t_2,t_3]$ via $t_1 \mapsto a_1$, $t_2 \mapsto a_2$, and $t_3 \mapsto a_3$, and a_1^2, a_2^2, and a_3^2 are central in A. If, moreover, a_1, a_2, and a_3 are invertible, then A is a homomorphic image of the Cayley torus $T^{-1}\Phi_C[t_1,t_2,t_3]$ via the same map.

(3) The Cayley torus has a presentation in the category of alternative algebras generated by $t_1^{\pm 1}$, $t_2^{\pm 1}$, and $t_3^{\pm 1}$ and defined by $t_i t_i^{-1} = 1$, $i = 1,2,3$, and the Cayley relations specified in (C).

Proof. (1) Follows from Theorem 3.3.

(2) Let $\varphi : \Phi_C[t_1,t_2,t_3] \longrightarrow A$ be an epimorphism defined by $t_1 \mapsto a_1$, $t_2 \mapsto a_2$, and $t_3 \mapsto a_3$. Then elements of $\varphi(T)$ are central in A by Theorem 3.3. Since $\varphi(T)$'s elements are invertible, φ extends to $T^{-1}\Phi_C[t_1,t_2,t_3]$ by the universal property of a ring of quotients.

(3) Let Q be an alternative algebra having a presentation such as in the assertion. Define $\deg t_1^{\pm 1} = (\pm 1, 0, 0)$, $\deg t_2^{\pm 1} = (0, \pm 1, 0)$, and $\deg t_3^{\pm 1} = (0, 0 \pm 1)$. Let Q^α be a space generated by monomials of degree $\alpha \in \mathbb{Z}^3$. Then $Q = \sum\limits_{\alpha\in\mathbb{Z}^3} Q^\alpha$. Since the Cayley torus $T^{-1}\Phi_C[t_1,t_2,t_3]$ has the alleged relations, there is a natural homomorphism from Q onto $T^{-1}\Phi_C[t_1,t_2,t_3]$ such that Q^α is mapped onto a homogeneous space of degree α in the Cayley torus. Hence $Q = \bigoplus\limits_{\alpha\in\mathbb{Z}^3} Q^\alpha$ (the sum is direct). On the other hand, it follows by (2) that there exists a natural graded homomorphism from $T^{-1}\Phi_C[t_1,t_2,t_3]$ onto Q. Consequently, these homomorphisms are graded isomorphisms. □

We note that item (3) in Corollary 3.4 was obtained independently by B. Allison (unpublished). In view of the presentation of the Cayley torus, it is natural to assume that

$$T^{-1}\Phi_C[t_1,t_2,t_3] = \Phi_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}].$$

Also, an *octonion n-torus* $(n \geqslant 3)$ $(\Phi[z_1^{\pm 1},\ldots,z_n^{\pm 1}], z_1, z_2, z_3)$ can be written in the form

$$\Phi_C[t_1^{\pm 1},\ldots,t_n^{\pm 1}] := \Phi_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \otimes_\Phi \Phi[t_4^{\pm 1},\ldots,t_n^{\pm 1}],$$

where $\Phi[t_4^{\pm 1},\ldots,t_n^{\pm 1}]$ is a Laurent polynomial algebra in $n-3$ variables over Φ. Letting $P := \Phi[t_4^{\pm 1},\ldots,t_n^{\pm 1}]$, we can conceive of the octonion torus $\Phi_C[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$ as a Cayley torus over P, i.e., $P_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$.

We start with an alternative algebra $\Phi_C[t_1,\ldots,t_n]$ over Φ generated by elements $t_1,\ldots,t_n$ $(n \geqslant 3)$ and defined by Cayley relations such as in (C) and the central relations

$$[t_i,t_k] = (t_i,t_j,t_k) = 0 \ \text{ for } i<j<k \text{ with } i,j = 1,\ldots,n \text{ and } k = 4,\ldots,n. \tag{Z}$$

By the above argument, we can prove the following:

THEOREM 3.5. An algebra $\Phi_C[t_1,\ldots,t_n]$ is graded isomorphic to $(\Phi[z_1,\ldots,z_n], z_1, z_2, z_3)$. In particular, the center Z is $\Phi[t_1^2, t_2^2, t_3^2, t_4, \ldots, t_n]$ and $\Phi_C[t_1,\ldots,t_n]$ is an octonion algebra (Z, t_1^2, t_2^2, t_3^2). Moreover, generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$, together with the Cayley relations in (C), the central relations in (Z), and the invertible relations $t_i t_i^{-1} = 1$ with $i = 1, \ldots, n$, yield a presentation for the octonion torus $\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in the category of alternative algebras.

An octonion torus was constructed in [4] (where it was called an alternative torus) in classifying extended affine Lie algebras (called quasi-simple Lie algebras). Octonion tori turn into coordinate algebras of extended affine Lie algebras of types A_2, C_3, and F_4. The generators and relations for an octonion torus will be useful in determining generators and relations for certain of the extended affine Lie algebras.

A presentation of the octonion algebra over Φ is given by

THEOREM 3.6. Any octonion algebra $(\Phi, \mu_1, \mu_2, \mu_3)$ over Φ with cancellable scalars $\mu_1, \mu_2, \mu_3 \in \Phi$ is isomorphic to $\Phi_C[t_1, t_2, t_3]/(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$. Hence $(\Phi, \mu_1, \mu_2, \mu_3)$ has a presentation in the category of alternative algebras generated by elements t_1, t_2, and t_3 and defined by Cayley relations such as in (C) and by $t_1^2 = \mu_1$, $t_2^2 = \mu_2$, and $t_3^2 = \mu_3$. In particular, if Φ is a field, and A is an alternative algebra over Φ generated by a_1, a_2, and a_3 satisfying $a_1^2 = \mu_1$, $a_2^2 = \mu_2$, and $a_3^2 = \mu_3$ and the Cayley relations $a_1a_2 = -a_2a_1$, $a_1a_3 = -a_3a_1$, $a_2a_3 = -a_3a_2$, and $(a_1a_2)a_3 = -a_1(a_2a_3)$, then A is isomorphic to $(\Phi, \mu_1, \mu_2, \mu_3)$.

Proof. Set $B := \Phi_C[t_1, t_2, t_3]/(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$. Let v_1, v_2, and v_3 be basic generators for $(\Phi, \mu_1, \mu_2, \mu_3)$, with $v_1^2 = \mu_1$, $v_2^2 = \mu_2$, and $v_3^2 = \mu_3$. Suppose $\varphi : \Phi_C[t_1, t_2, t_3] \longrightarrow (\Phi, \mu_1, \mu_2, \mu_3)$ is an epimorphism defined by $t_1 \mapsto v_1$, $t_2 \mapsto v_2$, and $t_3 \mapsto v_3$. Since the ideal $(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$ is contained in the kernel of φ, the φ induces an epimorphism $\overline{\varphi} : B \longrightarrow (\Phi, \mu_1, \mu_2, \mu_3)$. By Theorem 3.3, $\Phi_C[t_1, t_2, t_3]$ is an 8-dimensional free $\Phi[t_1^2, t_2^2, t_3^2]$-module with basis $\{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, (t_1t_2)t_3\}$. Therefore, $\{1, \overline{t_1}, \overline{t_2}, \overline{t_3}, \overline{t_1t_2}, \overline{t_1t_3}, \overline{t_2t_3}, \overline{(t_1t_2)t_3}\}$ generates B over Φ, where $\overline{}$ is the canonical map from $\Phi_C[t_1, t_2, t_3]$ onto B. Since $\overline{\varphi}(\overline{t_i}) = v_i$, $i = 1, 2, 3$, and $\{1, v_1, v_2, v_3, v_1v_2, v_1v_3, v_2v_3, (v_1v_2)v_3\}$ are linearly independent over Φ, we conclude that $\overline{\varphi}$ is injective and $B \cong (\Phi, \mu_1, \mu_2, \mu_3)$.

As to the second statement, we note that $A \cong B$ since A is a homomorphic image of the simple algebra B, and $A \neq 0$. □

4. CAYLEY–DICKSON RINGS

Let R be an alternative ring with nonzero center Z which is freed of zero divisors in R (e.g., R is prime). Then $Z^* = Z \setminus \{0\}$ is a multiplicative subset of Z, and we can construct a ring $(Z^*)^{-1}R$ of quotients, which is called the *central closure* of R and is denoted by $\overline{R}$. Note that R embeds in $\overline{R}$, $\overline{Z} = (Z^*)^{-1}Z$ is a field of quotients of Z, $\overline{R}$ is a central $\overline{Z}$-algebra, and $\overline{R} \cong \overline{Z} \otimes_Z R$. Moreover, R is a *Cayley–Dickson ring* if the central closure $\overline{R}$ is an octonion algebra over $\overline{Z}$ (see [13, p. 228]). For example, if Φ is an integral domain, then the ring $\Phi_C[t_1, t_2, t_3]$ of Cayley polynomials or the octonion torus $\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a Cayley–Dickson ring, since its central closure is $(\overline{Z}, t_1^2, t_2^2, t_3^2)$, where $\overline{Z} = \overline{\Phi}(t_1^2, t_2^2, t_3^2)$ or $\overline{\Phi}(t_1^2, t_2^2, t_3^2, t_4, \ldots, t_n)$ (rational function fields in 3 or n variables over the field $\overline{\Phi}$ of quotients), respectively. However, a Cayley–Dickson ring is not necessarily an octonion algebra over the center (see Example 4.4).

LEMMA 4.1. Let A be an alternative algebra with center Z over Φ which does not contain zero divisors of A. Assume that A is generated by a_1, a_2, and a_3 satisfying the Cayley relations $a_1a_2 = -a_2a_1$, $a_1a_3 = -a_3a_1$, $a_2a_3 = -a_3a_2$, and $(a_1a_2)a_3 = -a_1(a_2a_3)$. Then A is an octonion algebra over Z, isomorphic to (Z, a_1^2, a_2^2, a_3^2).

Proof. Since A embeds in $\overline{A}$, it follows that a_1, a_2, and a_3 also satisfy the Cayley relations in $\overline{A}$. Furthermore, $a_1^2, a_2^2, a_3^2 \in Z$ by Corollary 3.4(2). By Theorem 3.6, therefore, $\overline{A}$ is an octonion algebra over $\overline{Z}$, i.e., $\overline{A} = (\overline{Z}, a_1^2, a_2^2, a_3^2)$. In particular, A is an 8-dimensional free Z-module with basis

$\{1, a_1, a_2, a_3, a_1a_2, a_1a_3, a_2a_3, (a_1a_2)a_3\}$, which is also a basis of $\overline{A}$. Moreover, A and $\overline{A}$ have the same multiplication table relative to this basis, and so $A = (Z, a_1^2, a_2^2, a_3^2)$. □

THEOREM 4.2 (Slater; see [13, Thm. 9]). Any prime nondegenerate alternative algebra that is not associative is a Cayley–Dickson ring.

(It is also true that every Cayley–Dickson ring is a prime nondegenerate ring; see [13, Prop. 3].) Using this theorem, we obtain the following:

PROPOSITION 4.3. Let R be a prime, nondegenerate, nonassociative alternative algebra over Φ and $Z = Z(R)$ be its center. Then there exists a subalgebra A of R such that $Z(A) \subset Z$ and A is an octonion algebra over $Z(A)$. Moreover, the subalgebra $B := ZA$ is an octonion algebra over Z and the central closures of B and R coincide, i.e., $\overline{B} = \overline{R}$. Furthermore, $\overline{R}$ is a base field extension of $\overline{A}$, and namely, $\overline{R} \cong \overline{Z} \otimes_K \overline{A}$, where $K = \overline{Z(A)}$.

Proof. By Slater's theorem, $\overline{R}$ is an octonion algebra over the field $\overline{Z}$. Let v_1, v_2, and v_3 be basic generators satisfying the Cayley relations. Note that $v_1 = z_1^{-1}r_1$, $v_2 = z_2^{-1}r_2$, and $v_3 = z_3^{-1}r_3$ for some $z_1, z_2, z_3 \in Z^*$ and some $r_1, r_2, r_3 \in R$. Therefore, $a_1 := z_1z_2z_3v_1$, $a_2 := z_1z_2z_3v_2$, and $a_3 := z_1z_2z_3v_3$ also satisfy the Cayley relations, and they are in R. Let A be a subalgebra of R generated by a_1, a_2, and a_3. If $z \in Z(A)$, then z, in particular, is central for a generating set $\{a_1, a_2, a_3\}$ of A and for a generating set $\{v_1, v_2, v_3\}$ of $\overline{R}$. By the theorem of Bruck and Kleinfeld, z is central for $\overline{R}$; so $z \in Z$ (see [13, Lemma 16]). Thus $Z(A)$ is freed of zero divisors in A; hence $A = (Z(A), a_1^2, a_2^2, a_3^2)$ by Lemma 4.1.

Now, $Z \subset Z(B) \subset Z$, and so $Z = Z(B)$. Consequently, $B = (Z, a_1^2, a_2^2, a_3^2)$ by Lemma 4.1 again. Eventually, for any $r \in R$, there exists some $z \in Z^*$ such that $zr \in B$. In fact, $r = f(v_1, v_2, v_3) = f(z_1^{-1}z_2^{-1}z_3^{-1}a_1, z_1^{-1}z_2^{-1}z_3^{-1}a_2, z_1^{-1}z_2^{-1}z_3^{-1}a_3)$ for a polynomial f over $\overline{Z}$. Therefore, there exists $z \in Z^*$ such that $zf(z_1^{-1}z_2^{-1}z_3^{-1}a_1, z_1^{-1}z_2^{-1}z_3^{-1}a_2, z_1^{-1}z_2^{-1}z_3^{-1}a_3) = g(a_1, a_2, a_3)$ for a polynomial g over Z. Hence $zr = g(a_1, a_2, a_3) \in B$. Thus $r = z^{-1}b$ for some $b \in B$, and so $\overline{R} \subset \overline{B}$. The inverse inclusion being obvious, we obtain $\overline{R} = \overline{B}$.

In order to prove the last statement, we consider a K-linear map φ from $\overline{Z} \otimes_K \overline{A}$ to $\overline{R}$ defined by $\varphi(u_i \otimes w_j) = u_iw_j$, where $\{u_i\}$ is a basis of $\overline{Z}$ over K and $\{w_i\}$ is one of $\overline{A}$ over K. Then φ is a homomorphism, which is $\overline{Z}$-linear. Since $\overline{Z} \otimes_K \overline{A}$ and $\overline{R}$ are both 8-dimensional over $\overline{Z}$, it suffices to show that φ is surjective. As above, for $v^{-1}r \in \overline{R}$ ($v \in Z^*$ and $r \in R$), there exist $z \in Z^*$ and $b \in B$ such that $zr = b$; therefore, $zr = \sum\limits_k z_ka_k$ for some $z_k \in Z$ and some $a_k \in A$. Consequently, $v^{-1}r = v^{-1}z^{-1}\sum\limits_k z_ka_k = \varphi\left(\sum\limits_k v^{-1}z^{-1}z_k \otimes a_k\right)$, proving that φ is surjective. □

Finally, we give an example of a prime nondegenerate algebra which is not an octonion algebra over the center.

Example 4.4. For simplicity, let F be a field of characteristic not 2 and $F[z]$ be an ordinary polynomial algebra over F. Suppose $F[z]_C[t_1, t_2, t_3]$ is a ring of Cayley polynomials over $F[z]$ and R is an F-subalgebra of $F[z]_C[t_1, t_2, t_3]$ generated by t_1, t_2, t_3, and zt_1. Then $Z = Z(R) = F[t_1^2, t_2^2, t_3^2, z^2t_1^2, zt_1^2]$ and R is a 12-dimensional free Z-module with the basis

$$\{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, (t_1t_2)t_3, zt_1, zt_1t_2, zt_1t_3, zt_1(t_2t_3)\}.$$

Hence R is not an octonion algebra over Z; but the central closure $\overline{R}$ is an octonion algebra over $\overline{Z} = F(z, t_1^2, t_2^2, t_3^2)$, i.e., $\overline{R} = (\overline{Z}, t_1^2, t_2^2, t_3^2)$ by Theorem 3.6. Therefore, R is a Cayley–Dickson ring. As a subalgebra A of the algebra R in Proposition 4.3 we can take $F_C[t_1, t_2, t_3]$.

Acknowledgments. I am grateful to Profs. B. Allison and E. Neher for useful remarks.

REFERENCES

1. B. Allison, S. Azam, S. Berman, et al., *Extended Affine Lie Algebras and Their Root Systems, Mem. Am. Math. Soc.*, **126**, No. 603, Am. Math. Soc., Providence, RI (1997).
2. E. Neher, "Lie tori," *C. R. Math. Acad. Sci., Soc. R. Can.*, **26**, No. 3, 84-89 (2004).
3. Y. Yoshii, "Root systems extended by an abelian group and their Lie algebras," *J. Lie Th.*, **14**, No. 2, 371-394 (2004).
4. S. Berman, Y. Gao, Y. Krylyuk, and E. Neher, "The alternative torus and the structure of elliptic quasi-simple Lie algebras of type A_2," *Trans. Am. Math. Soc.*, **347**, No. 11, 4315-4363 (1995).
5. B. Allison and Y. Gao, "The root system and the core of an extended affine Lie algebra," *Sel. Math., New Ser.*, **7**, No. 2, 149-212 (2001).
6. G. Benkart and Y. Yoshii, "Lie G-tori of symplectic type," *Quart. J. Math., Oxford*, **57**, No. 4, 425-448 (2006).
7. Y. Yoshii, "Classification of division $\mathbb{Z}^n$-graded alternative algebras," *J. Alg.*, **256**, No. 1, 28-50 (2002).
8. Y. Yoshii, "Lie tori—a simple characterization of extended affine Lie algebras," *Publ. Res. Inst. Math. Sci.*, **42**, No. 3, 739-762 (2006).
9. S. V. Polikarpov and I. P. Shestakov, "Nonassociative affine algebras," *Algebra Logika*, **29**, No. 6, 709-723 (1990).
10. A. V. Il'tyakov, "Free alternative algebras of rank 3," *Algebra Logika*, **23**, No. 2, 136-158 (1984).
11. A. V. Il'tyakov, "Specht ideals of identities of certain simple nonassociative algebras," *Algebra Logika*, **24**, No. 3, 327-351 (1985).
12. K. McCrimmon, "Nonassociative algebras with scalar involution," *Pac. J. Math.*, **116**, No. 1, 85-109 (1985).
13. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings That Are Nearly Associative* [in Russian], Nauka, Moscow (1978).