The Adjacency Matrix, Standard Laplacian, and Normalized Laplacian, and Some Eigenvalue Interlacing Results

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1 Introduction

As Richard Brualdi mentions in his *Combinatorial Matrix Theory* book with Herbert Ryser, "matrix theory and combinatorics enjoy a symbiotic relationship, that is, a relationship in which each has a beneficial impact on the other".

An example of the inter-play:
Let A be a square marix. Then A is "irreducible" iff $D(A)$ is strongly connected.

This talk is more specifically in the area of "spectral graph theory".

All graphs in this talk are simple graphs, namely, finite graphs without loops or parallel edges. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Two vertices are adjacent if they are two end-vertices of an edge and two edges are adjacent if they share a common end-vertex. A vertex and an edge are incident if the vertex is one end-vertex of the edge. For any vertex $v \in V(G)$, let d_v denote the degree of v, the number of edges incident to v.

Suppose that $V(G) = \{v_1, v_2, \ldots, v_n\}$. An $n \times n$ (0,1)-matrix $A := A(G) = (a_{ij})$ is called the *adjacency matrix* of G if

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

$A(G)$ is real symmetric - so all the eigenvalues are real. $\sum \lambda_i = \text{tr}(A) = 0$; so there is a $+$ and a $-$ ev (unless all vertices are isolated).

The eigenvalues of $A(G)$ have been studied extensively.

Books by Schwenk & Wilson, and Biggs, and others.

Also: *Spectra of Graphs*, by Cvetkovic, Doob, Sachs, 3rd ed, 1995.

The *standard Laplacian* $L := L(G) = (L_{ij})$ of a graph G of order n is the $n \times n$ matrix L defined as follows:

$$L_{ij} = \begin{cases} d_{v_i} & \text{if } v_i = v_j, \\ -1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $L = SS^T$ where S is the matrix whose rows are indexed by the vertices and whose columns are indexed by the edges of G such that each column corresponding to an edge $e = v_i v_j$ (with $i < j$) has entry 1 in the row corresponding to v_i, and entry -1 in the row corresponding to v_j, and has zero entries elsewhere. Since $L = SS^T$, L is positive semidefinite and has nonnegative eigenvalues. Furthermore, 0 is always an eigenvalue of L since the vector $(1, 1, \ldots, 1)^T$ is a corresponding eigenvector. In fact, the multiplicity of the eigenvalue 0 is equal to the number of connected components of the graph G (follows from the "Perron-Frobenius Theorem").

In "Algebraic connectivity of graphs", Czech Math J, 1973, Miroslav Fiedler defined the second smallest eigenvalue of L, $\mu = \mu(G)$, to be the *algebraic connectivity* of G. This paper has been referenced numerous times!!

(Fiedler also has a 1975 paper and others on the Laplacian.)

Fom the above, $\mu = 0$ iff G is disconnected.

The eigenvectors corresponding to μ are called *Fiedler vectors* of the graph G.

The smallest nonzero eigenvalue of L is called the spectral gap or *Fiedler value.*

There are very many graph theoretic relationships involving μ. Eg

$\mu \leq v(G) \leq e(G)$ if $G \neq K_n$,   MF (73)

For a tree T, $\mu(T) \leq 1$, with equality iff T is a star,

C. Maas, Discr. Appl. Math. (1987).

Papers by Mohar, Krzeminski/Signerska, and others, with an emphasis on μ.

There are many other applications of the Laplacian:

1) Kirkhoff's Theorem (Gustav Kirchoff - electrical circuit laws):

$$c(G) = \frac{1}{n}\lambda_1...\lambda_{n-1}$$

where $\lambda_1, ..., \lambda_{n-1}$ are the non-zero eigenvalues of L and $c(G)$ is the "complexity" of G, the number of spanning trees of G (connected).

Kirkhoff's Theorem essentially says that all cofactors of L are the same #, $c(G)$!

The Laplacian matrix is sometimes called the *Kirchoff matrix* or *admittance matrix.*

2) Predicting properties of chemical substances. The "distance matrix" of a connected graph is used: $\Delta(G) = (d(v_i, v_j))$, where $d(u, v)$ is the distance between u and v, the length of a shortest path from u to v.

(G. Chen our dept/chemistry dept GSU)

There is a strong relationship between $L(T)$ and $\Delta(T)$, T a tree. Eg: Suppose T has ≥ 3 vertices. Then for $L = L(T)$, if $i \neq j$, $d(v_i, v_j) = \det L(i, j)$, where $L(i, j)$ means to delete rows/columns i and j from L.

3) Many systems in biology, neuroscience, and engineering are connected via linear, diffusive coupling, defined by the Laplacian matrix. In neuroscience, for example, linear (Laplacian) coupling refers to electrical gap-junctions where the flux between two neighboring neurons is proportional to the difference between the membrane potentials. The basic idea here is the following: Given a pair of identical oscillators (neurons), one can identify the coupling required for their synchronization. Then, using that critical value, one can predict the synchronization threshold in ANY (large or small) Laplacian network, involving that oscillator. This quantity is dependent on the eigenvalues of the Laplacian matrix.
(Igor Belkyh of GSU has written several papers along these lines.)

4) For a simple spring network, the stiffness matrix (generalizing the stiffness of Hooke's law) is a Laplacian matrix.

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The *normalized Laplacian* of G is the $n \times n$ matrix $\mathcal{L} :=$ $\mathcal{L}(G) = (\mathcal{L}_{ij})$ given by

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } d(v_i) \neq 0, \\ -\dfrac{1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let T denote the diagonal matrix with the (i, i)-th entry having value d_{v_i}. We can write

$$\mathcal{L} = T^{-1/2} L T^{-1/2} = T^{-1/2} S S^T T^{-1/2}$$

with the convention that $T^{-1}(i, i) = 0$ if $d_{v_i} = 0$. It can be easily seen that all eigenvalues of $\mathcal{L}$ are real and non-negative. In fact, if λ is an eigenvalue of $\mathcal{L}$, then $0 \leq \lambda \leq 2$. As pointed out by Fan Chung, *Spectral Graph Theory*, 1997, the eigenvalues of the normalized Laplacians are in a "normalized" form, and the spectra of the normalized Laplacians relate well to other graph invariants for general graphs in a way that the other two definitions fail to do. The advantages of this definition are perhaps due to the fact that it is consistent with the eigenvalues in spectral geometry and in stochastic processes.

For $\mathcal{L} = T^{-1/2}LT^{-1/2}$:

$T^{\frac{1}{2}}e$ is an eigenvector corresponding to the eigenvalue 0, where e is the all 1's vector.

The multiplicity of the eigenvalue 0 is equal to the number of connected components of the graph G.

2 is an eigenvalue of $\mathcal{L}$ iff there is a connected component of G which is nontrivial bipartite graph.

For a $k-$regular graph G, $L(G) = k\mathcal{L}$.

Book by Chung has many applications of $\mathcal{L}$.

Chung, Laplacians of graphs and Cheeger inequalities, in Combinatorics, Paul Erdos is Eighty, Vol 2, Janos Bolyai Math. Soc., 1996. Follow-up 2000 paper by Chung/Oden.

Steve Butler, PhD student of Fan Chung, wrote a paper on "weighted" normalized Laplacians - ELA, 2007.

The following work was done with

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Kinnari did her MS degree at GSU; has just finished her PhD at EMU working under Ron Gould.

One of our goals is to describe all eigenvalue interlacing results for $A(G)$, $L(G)$, and $\mathcal{L}(G)$, associated with the removal of an edge or vertex. Three of the six possible cases have been resolved: eigenvalue interlacing result on the adjacency matrix when a vertex is removed; eigenvalue interlacing result on the standard Laplacian when an edge is removed; and eigenvalue interlacing result on the normalized Laplacian when an edge is removed. We complete the picture by obtaining best possible interlacing results for the three remaining situations. Some other interesting related results are also given along the way.

2 Known Interlacing Results

The first two preliminary results can be found in Chapter 4 in the *Matrix Analysis* book by Horn & Johnson.

Theorem 2.1 *(Cauchy's interlacing theorem) Let A be a real $n \times n$ symmetric matrix and B be an $(n-1) \times (n-1)$ principal submatrix of A. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1}$$

are the eigenvalues of A and B, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1.$$

This says that

$$\lambda_1 \geq \theta_1 \geq \lambda_2 \geq \theta_2 \geq \ldots \geq \theta_{n-1} \geq \lambda_n.$$

Note: We used λ_1 for the largest eigenvalue in order to facilitate some proofs.

Now, instead of deleting one row and column, what if we delete q rows and the corresponding q columns? The following generalization can be proved by using Cauchy's interlacing theorem iteratively, or for a direct proof, use Courant-Fisher.

Theorem 2.2 *Let A be an $n \times n$ real symmetric matrix and B be an $r \times r$, $1 \le r \le n$, principal submatrix of A, obtained by deleting $n - r$ rows and the corresponding columns from A. If*

$$\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$$
$$\theta_1 \ge \theta_2 \ge \ldots \ge \theta_r$$

are the eigenvalues of A and B, respectively, then for each integer i such that $1 \le i \le r$

$$\lambda_i \ge \theta_i \ge \lambda_{i+n-r}.$$

Let G be a graph of order n and let $H = G - v$, where v is a vertex of G. Theorem 2.1 (Cauchy's) gives an interlacing property of the eigenvalues of $A(G)$ and the eigenvalues of $A(H)$, which we refer to as the vertex version of the interlacing property.

Theorem 2.3 *Let G be a graph and $H = G - v$, where v is a vertex of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1}$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1.$$

Cauchy's interlacing does not directly apply to the standard Laplacian (or the normalized Laplacian) of G and H since the principal submatrices of a standard Laplacian (or a normalized Laplacian) may no longer be the standard Laplacian (or the normalized Laplacian) of a subgraph. However, the following result given in van den Heuvel, LAA, 1995, or in Mohar, 1995, reflects an edge version of the interlacing property.

Theorem 2.4 *Let G be a graph and $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0 \quad and$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n = 0$$

are the eigenvalues of $L(G)$ and $L(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+1} \quad for\ each\ i = 1, 2, 3, 4, \ldots, n - 1.$$

Same interlacing!

Since the trace of $\mathcal{L}$ is n when there are no isolated vertices, it is impossible to have an exactly parallel result to Theorem 2.4. However, through the use of Harmonic eigenfunctions, the following result was established by CDHLPS, SIAM J Discrete Math, 2004. (also part of Kinnari's MS thesis)

Theorem 2.5 *Let G be a graph and let $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad and$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \quad for\ each\ i = 1, 2, 3, 4, \ldots, n,$$

where $\lambda_0 = 2$ and $\lambda_{n+1} = 0$.

Gap of 1 on <u>each</u> side - this also occurs in other instances!

Note: $i = 1 : 2 \geq \theta_1 \geq \lambda_2 \quad i = n : \lambda_{n-1} \geq \theta_n \geq 0$

3   New Interlacing Results

We first consider how the eigenvalues of the standard Laplacian of the graphs G and $H = G - v$ interlace, where the degree of the vertex v is r.

Theorem 3.1 *Let G be a graph of order n and $H = G - v$, where v is a vertex of G of degree r. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1} = 0$$

are the eigenvalues of $L(G)$ and $L(H)$, respectively, then

$$\lambda_i \geq \theta_i \geq \lambda_{i+r} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n - 1,$$

where $\lambda_i = 0$ for $i \geq n + 1$.

Proof: We know that $L(G) = S_G S_G^T$ and that the eigenvalues of $S_G S_G^T$ are nonnegative. Now, for any $n \times m$ matrix A, the spectrum of the matrices AA^T and $A^T A$ coincide except possibly for the multiplicity of the eigenvalue 0. In particular, the positive eigenvalues of $L(G)$ are the same as the positive eigenvalues of $S_G^T S_G$. Observe that $L(H) = S_H S_H^T$ can be obtained by deleting the r columns and rows corresponding to the edges incident on the deleted vertex, so that $S_H^T S_H$ is an order $m - r$ principal submatrix of $S_G^T S_G$ (where m is the number of edges in G). Hence, the result follows from Theorem 2.2 (generalized Cauchy). $\square$

In the above theorem, we cannot improve the gap on the right by reducing it from r to $r - 1$, as shown by considering the complete bipartite graph. From W. Anderson and T. Morley, LAMA, 1985, the eigenvalues of the standard Laplacian of the complete bipartite graph $K_{m,n}$ on $m + n$ vertices are

$$m + n, \ m, \ n, \ 0$$

with multiplicities

$$1, \ n - 1, \ m - 1, \ 1$$

respectively. Without loss of generality, assume $m \geq n$. Then the eigenvalues of the graph $K_{m,n-1}$ are

$$m + n - 1, \ m, \ n - 1, \ 0$$

with multiplicities

$$1, \ n - 2, \ m - 1, \ 1$$

respectively. Since $\theta_n = n - 1$, $\lambda_{n+1} = \lambda_{n+2} = \ldots = \lambda_{m+n-1} = n$, and $\lambda_{m+n} = 0$, we have $\theta_n = n - 1 \geq 0 = \lambda_{m+n}$, where the gap is m, which is the degree of the vertex removed from $K_{m,n}$.

We now consider how the eigenvalues of the normalized Laplacian of the graphs G and $H = G - v$ interlace, where the degree of the vertex v is r. First of all, we look at a few results that will be of great help in proving a main result. We can apply Theorem 2.5 ($\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$) iteratively r times and we have the following proposition.

Proposition 3.2 *Let G be a graph and let H be a subgraph of G obtained by deleting r edges. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r} \geq \theta_i \geq \lambda_{i+r}, \quad \text{for each } i = 1, 2, \ldots, n$$

with the convention of

$$\lambda_i = 2 \quad \text{for each} \quad i \leq 0,$$
$$\lambda_i = 0 \quad \text{for each} \quad i \geq n + 1.$$

To be heavily used:

Theorem 3.3 (Courant-Fischer) *For a real, symmetric $n \times n$ matrix A with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$$

we have

$$\lambda_k = \max_{g^{(k+1)},g^{(k+2)},\ldots,g^{(n)} \in \mathbb{R}^n} \quad \min_{\substack{g \perp g^{(k+1)}, g^{(k+2)}, \ldots, g^{(n)} \\ g \neq 0}} \frac{g^T A g}{g^T g}$$

and

$$\lambda_k = \min_{g^{(1)},g^{(2)},\ldots,g^{(k-1)} \in \mathbb{R}^n} \quad \max_{\substack{g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \\ g \neq 0}} \frac{g^T A g}{g^T g}.$$

Lemma 3.4 *Let G be a graph on n vertices, let $L = L(G)$ be the standard Laplacian of G, and let $f = (f_1, \ldots, f_n)^T$ be a column vector in $\mathbb{R}^n$. Then,*

$$f^T L f = \sum_{i \sim j} (f_i - f_j)^2$$

where $\sum_{i \sim j}$ runs over all unordered pairs $\{i, j\}$ for which v_i and v_j are adjacent.

Proof: Lemma 3.4 directly follows from the definition of L. Write $L = SS^T$, so that

$$f^T L f = (S^T f)^T (S^T f) = \sum_{i \sim j} (f_i - f_j)^2.$$

$\square$

Lemma 3.5 *Suppose that for real a, b and γ*

$$a^2 - 2\gamma^2 \geq 0, \qquad b^2 - \gamma^2 > 0, \qquad \text{and} \qquad \frac{a^2}{b^2} \leq 2.$$

Then

$$\frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} \leq \frac{a^2}{b^2}.$$

Proof. The result follows from

$$\frac{a^2 - 2\gamma^2}{b^2 - \gamma^2} = \frac{a^2}{b^2} \frac{1 - 2\gamma^2/a^2}{1 - \gamma^2/b^2} \leq \frac{a^2}{b^2}$$

since the final inequality holds when $\frac{\gamma^2}{b^2} \leq \frac{2\gamma^2}{a^2}$ which is equivalent to $a^2/b^2 \leq 2$. $\square$

In the special case where the edge removed is incident on a vertex of degree 1, we can improve on Theorem 2.5 ($\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$) by showing that the eigenvalues do not increase when an edge is removed. Note that the corresponding result for the standard Laplacian is a trivial special case of Theorem 2.4 ($\lambda_i \geq \theta_i \geq \lambda_{i+1}$). The proof of the theorem for the normalized Laplacian is more involved and must deal with the fact that removing an edge decreases the degree of an adjacent vertex, thereby increasing the magnitude of some (possibly many) off-diagonal elements.

Theorem 3.6 *Let G be a graph of order n and $H = G - e$, where e is an edge incident on a vertex of G of degree 1. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n = 0$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_i \geq \theta_i \quad \text{for each } i = 1, 2, 3, 4, \ldots, n.$$

Proof: We adapt the Courant-Fischer theorem to the Laplacian using harmonic eigenfunctions. Recall that

$$\mathcal{L} = T^{-1/2} L T^{-1/2}.$$

We can assume that $T^{1/2}$ is invertible, that is, there are no vertices of degree zero (can handle isolated vertices separately).

For vectors g and $g^{(j)}$ define the vectors

$$f = T^{-1/2} g \qquad \text{and} \qquad f^{(j)} = T^{1/2} g^{(j)}.$$

Note that

$$g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)}$$

if and only if

$$f \perp f^{(1)}, f^{(2)}, \ldots, f^{(k-1)}.$$

Applying the Courant-Fischer theorem to get the eigenvalues λ_k of $\mathcal{L}$ gives

$$\lambda_k = \min_{\substack{g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{g \perp g^{(1)}, g^{(2)}, \ldots, g^{(k-1)} \\ g \neq 0}} \frac{g^T T^{-1/2} L T^{-1/2} g}{g^T g}$$

$$= \min_{\substack{f^{(1)}, f^{(2)}, \ldots, f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)}, f^{(2)}, \ldots, f^{(k-1)} \\ f \neq 0}} \frac{f^T L f}{f^T T f}$$

$$= \min_{\substack{f^{(1)}, f^{(2)}, \ldots, f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)}, f^{(2)}, \ldots, f^{(k-1)} \\ f \neq 0}} \frac{\sum_{i \sim j} (f_i - f_j)^2}{\sum_j f_j^2 d_j}$$

where f_j is the j-th component of f, $d_j = d(v_j)$ is the degree of v_j, and $\sum_{i \sim j}$ runs over all unordered pairs $\{i, j\}$ for which v_i and v_j are adjacent. The second line depends on the invertibility of T so that maximizing over vectors $f^{(k)}$ is equivalent to maximizing over vectors $g^{(k)}$. The final line depends on Lemma 3.4. The vector f can be viewed as a function $f(v)$ on the set of vertices that maps v_j to f_j. The function $f(v)$ is a harmonic eigenfunction.

Without loss of generality we assume that an edge between the particular vertices v_1 and v_2 is removed, where the degree of the vertex v_1 is 1 and consider the eigenvalues θ_k of the Laplacian of the modified graph. Two changes occur in the Courant-Fischer theorem when an edge is removed. The degrees of v_1 and v_2 are decreased from 1 and $d(v_2)$ to 0 and $d(v_2) - 1$, respectively, so that

$$\sum_j f_j^2 d_j \to \sum_j f_j^2 d_j - f_1^2 - f_2^2.$$

Also, since v_1 and v_2 are no longer adjacent, the sum no longer includes the pair $\{1, 2\}$ so that

$$\sum_{i \sim j} (f_i - f_j)^2 \to \sum_{i \sim j} (f_i - f_j)^2 - (f_1 - f_2)^2$$

Note that the sum $\sum_{i \sim j}$ still runs over vertices that are adjacent in the *original graph*; in applying the theorem to the modified graph we explicitly subtract out $(f_1 - f_2)^2$ instead of modifying the index set of the sum.

Thus

$$\theta_k = \min_{\substack{f^{(1)},\ldots,f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)},\ldots,f^{(k-1)} \\ f \neq 0}} \frac{\sum_{i \sim j}(f_i - f_j)^2 - (f_1 - f_2)^2}{\sum_j f_j^2 d_j - f_1^2 - f_2^2}$$

$$= \min_{\substack{f^{(1)},\ldots,f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)},\ldots,f^{(k-1)} \\ f \neq 0 \\ f_1 = -f_2}} \frac{\sum_{i \sim j}(f_i - f_j)^2 - 4f_2^2}{\sum_j f_j^2 d_j - 2f_2^2}$$

$$\leq \min_{\substack{f^{(1)},\ldots,f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)},\ldots,f^{(k-1)} \\ f \neq 0 \\ f_1 = -f_2}} \frac{\sum_{i \sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j}$$

$$\leq \min_{\substack{f^{(1)},\ldots,f^{(k-1)} \in \mathbb{R}^n}} \max_{\substack{f \perp f^{(1)},\ldots,f^{(k-1)} \\ f \neq 0}} \frac{\sum_{i \sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j} = \lambda_k$$

In line 2, we use the fact that the expression does not depend on f_1. In line 3 we use Lemma 3.5, which applies since the inequality $\lambda_k \leq 2$ implies

$$\frac{\sum_{i \sim j}(f_i - f_j)^2}{\sum_j f_j^2 d_j} \leq 2.$$

Hence,

$$\lambda_k \geq \theta_k \quad \text{for each } k = 1, 2, 3, 4, \ldots, n.$$

We have assumed throughout that T is invertible (i.e., there is no vertex of degree zero). However, this is not restrictive; the inequality holds in general. If $d(v) = 0$ for q vertices then the normalized Laplacian can be permuted so that

$$P\mathcal{L}P^T = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & 0_{q \times q} \end{pmatrix}$$

for some permutation matrix P. Thus for $\mathcal{L}$,

$$\lambda_{n-q} = \lambda_{n-q+1} = \cdots = \lambda_n = 0.$$

The removal of an edge then only affects $\mathcal{L}_1$ so that the theorem can be applied to the submatrix. The additional zero eigenvalues, $\theta_{n-q+1} = \theta_{n-q+2} = \cdots = \theta_n = 0$ satisfy

$$\lambda_k \geq \theta_k$$

for $k = n - q + 1, \cdots, n$. Interlacing bounds for all other θ_k follow from the interlacing theorem applied to $\mathcal{L}_1$. $\qquad\square$

Now consider removing j edges from v_1 to get eigenvalues $\theta_k^{(j)}$ (where $\theta_k^{(j)}$ represents the k-th eigenvalue of the normalized Laplacian of the subgraph obtained by deleting j edges of the original graph from v_1). Thus, from Theorem 3.6 and Proposition 3.2, we have the following

$$\theta_k^{(d_1)} \leq \theta_k^{(d_1-1)} \leq \lambda_{k-(d_1-1)},$$

where $d_1 = d_{v_1}$. Using this fact and Proposition 3.2 on the other side, we have the following new result.

Theorem 3.7 *Let G be a graph of order n and H be a subgraph of G obtained by deleting the r edges from a vertex v of degree r. If*

$$\lambda_1 \ \geq \ \lambda_2 \geq \ldots \geq \lambda_n = 0$$
$$\theta_1 \ \geq \ \theta_2 \geq \ldots \geq \theta_n = 0$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r} \quad \textit{for each } i = 1, 2, 3, 4, \ldots, n,$$

where $\lambda_i = 2$ for $i \leq 0$ and $\lambda_i = 0$ for $i \geq n+1$.

Hence, if we remove a vertex of degree r from the graph G to obtain the subgraph H of G, we will have $\theta_{n-1} = \theta_n = 0$, which will give us the following result.

Theorem 3.8 *Let G be a graph of order n and $H = G - v$, where v is a vertex of G of degree r. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = 0$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1} = 0$$

are the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}(H)$, respectively, then

$$\lambda_{i-r+1} \geq \theta_i \geq \lambda_{i+r} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1,$$

where $\lambda_i = 2$ for $i \leq 0$ and $\lambda_i = 0$ for $i \geq n+1$.

In the above theorem, we cannot improve the gap on left by reducing it from $r - 1$ to $r - 2$, as shown by considering the complete graph. It was noted by Chung that the eigenvalues of the normalized Laplacian of the complete graph K_n on n vertices are 0 and $\frac{n}{n-1}$ with multiplicities 1 and $n - 1$ respectively. Hence, the eigenvalue of the normalized Laplacian of the subgraph $K_{n-1} = K_n - v$ on $n - 1$ vertices are 0 and $\frac{n-1}{n-2}$ with multiplicities 1 and $n - 2$ respectively. Since $\theta_{n-2} = \frac{n-1}{n-2} \not\leq \frac{n}{n-1} = \lambda_1$, we have $\theta_{n-2} = \frac{n-1}{n-2} \leq 2 = \lambda_0 = \lambda_{n-2-(n-1)+1}$, where the gap is $n - 2$, which is 1 less than the degree of the vertex removed from K_n. Similarly, we cannot improve the gap on the right by reducing it from r to $r - 1$, as shown by considering the star graph. It was noted by Chung that the eigenvalues of the star graph S_n are 0, 1, 2 with multiplicities 1, $n - 2$, 1, respectively. Hence, if we remove the center vertex from the star graph S_n on n vertices, the eigenvalues of the new subgraph are 0 with multiplicity $n - 1$. Since $\theta_1 = 0 \not\geq 1 = \lambda_{n-1}$, we have $\theta_1 = 0 \geq 0 = \lambda_n = \lambda_{1+(n-1)}$, where the gap is $n - 1$, which is the degree of the vertex removed from S_n.

Finally, using Cauchy's interlacing result on two pairs of matrices, we can prove the following edge version of the interlacing result for the adjacency matrix.

Theorem 3.9 *Let G be a graph and let $H = G - e$, where e is an edge of G. If*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad and$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$$

are the eigenvalues of $A(G)$ and $A(H)$, respectively, then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1} \quad \text{for each } i = 2, 3, 4, \ldots, n-1,$$

$\theta_1 \geq \lambda_2$, *and* $\theta_n \leq \lambda_{n-1}$.

Proof: Let $P = G - v$, where v is a vertex of G that is incident to edge e. Let

$$\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{n-1}$$

be the eigenvalues of $A(P)$. Then $A(P)$ is a principal submatrix of both $A(G)$ and $A(H)$. So, from Cauchy,

$$\lambda_i \geq \gamma_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1 \quad \text{and}$$
$$\theta_i \geq \gamma_i \geq \theta_{i+1} \quad \text{for each } i = 1, 2, 3, 4, \ldots, n-1$$

Then,

$$\lambda_{i-1} \geq \gamma_{i-1} \geq \theta_i \geq \gamma_i \geq \lambda_{i+1}$$

Hence, the theorem follows. $\square$

In the above theorem, we cannot improve the gap, as shown by considering the Petersen graph G. The eigenvalues of $A(G)$ are 3, 1 (with multiplicity 5), and -2 (with multiplicity 4). The eigenvalues of $A(H)$, where the graph H is obtained by deleting any edge of G, are $2.8558, 1.4142, 1, 1, 1, 0.3216, -1.4142, -2, -2,$ and -2.1774. Hence, $\lambda_2 = 1 \ngeq 1.4142 = \theta_2$ and $\theta_1 = 2.8558 \ngeq 3 = \lambda_1$.

This completes the picture regarding the six cases of eigenvalue interlacing results for $A(G)$, $L(G)$, and $\mathcal{L}(G)$, associated with the removal of an edge or vertex.

Let G be a graph and $x \in V(G)$. The neighborhood of x is

$$N(x) = \{y \ : \ xy \in E(G)\}.$$

For any two vertices u and v of G, we use $G/\{u, v\}$ to denote the graph obtained from G by contracting u and v to one vertex, i.e., $G/\{u, v\}$ is the graph obtained from G by deleting the vertices u and v and adding a new vertex (uv) such that the neighborhood of (uv) is the union of the neighborhoods of u and v. When u and v are adjacent, $G/\{u, v\}$ is the graph obtained from G by contracting the edge uv. Contraction of edges and vertices has many applications in graph theory. By contracting two nonadjacent vertices with nonintersecting neighborhoods we obtain the following interlacing result.

Theorem 3.10 *Let G be a graph and let u and v be two distinct vertices of G. Define $H = G/\{u, v\}$ and let*

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \quad \text{and}$$
$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1}$$

be the eigenvalues of $A(G)$ and $A(H)$, respectively. Then

$$\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}, \quad \text{for each } i = 2, 3, 4, \ldots, n-2,$$

where $\theta_1 \geq \lambda_3$ and $\lambda_{n-2} \geq \theta_{n-1}$.

If we assume that $N(u) \cap (N(v) \cup \{v\}) = \emptyset$ then, depending on the sign of θ_i, the above inequalities can be strengthened in one of two ways. Let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. Then

$$\theta_i \geq \lambda_{i+1} \quad \text{for each } i = 1, 2, \ldots, k$$

and

$$\lambda_i \geq \theta_i \quad \text{for each } i = k+1, k+2, \ldots, n-1.$$

Proof: The matrix $A(H)$ can be obtained from $A(G)$ by removing the rows and columns associated with u and v and adding a new row and column for the contracted vertex. The inequality $\lambda_{i-1} \geq \theta_i \geq \lambda_{i+2}$ follows from Theorem 2.2.

Let us now suppose that $N(u) \cap (N(v) \cup \{v\}) = \emptyset$, and let k be such that $\theta_k \geq 0$ and $\theta_{k+1} < 0$. We assume without loss of generality that $u = v_1$ and $v = v_2$ and consider the

graph $\hat{H}$ obtained by deleting every edge from v_1 and adding a corresponding edge to v_2. The eigenvalues of the graph $\hat{H}$ differ from those of H only in that $\hat{H}$ has an additional vertex, v_1, of degree zero giving an additional zero eigenvalue. For $x \in \mathbb{R}^n$ we have

$$\frac{x^T A(G) x}{x^T x} = 2 \frac{\sum_{j \sim l} x_j x_l}{\sum_j x_j^2}.$$

If we let J be the set of indices of vertices adjacent to v_1 then

$$\frac{x^T A(\hat{H}) x}{x^T x} = 2 \frac{\sum_{j \sim l} x_j x_l + \sum_{j \in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2}.$$

The min-max part of the Courant-Fischer theorem gives

$$
\begin{aligned}
\hat{\theta}_i &= \min_{\substack{y^{(1)},\ldots,y^{(i-1)} \in \mathbb{R}^n}} \max_{\substack{x \perp y^{(1)},\ldots,y^{(i-1)} \\ x \neq 0}} 2 \frac{\sum_{j \sim l} x_j x_l + \sum_{j \in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2} \\[2ex]
&\geq \min_{\substack{y^{(1)},\ldots,y^{(i-1)} \in \mathbb{R}^n}} \max_{\substack{x \perp y^{(1)},\ldots,y^{(i-1)} \\ x_1 = x_2 \\ x \neq 0}} 2 \frac{\sum_{j \sim l} x_j x_l + \sum_{j \in J}(x_2 x_j - x_1 x_j)}{\sum_j x_j^2} \\[2ex]
&= \min_{\substack{y^{(1)},\ldots,y^{(i-1)} \in \mathbb{R}^n}} \max_{\substack{x \perp y^{(1)},\ldots,y^{(i-1)}, e_1 - e_2 \\ x \neq 0}} 2 \frac{\sum_{j \sim l} x_j x_l}{\sum_j x_j^2}
\end{aligned}
$$

$$\geq \min_{y^{(1)},\ldots,y^{(i)}\in\mathbb{R}^n} \max_{\substack{x \perp y^{(1)},\ldots,y^{(i)} \\ x \neq 0}} 2\frac{\sum_{j\sim l} x_j x_l}{\sum_j x_j^2} = \lambda_{i+1}$$

where in the above $i = 1, 2, \ldots, n-1$. The max-min part of the Courant-Fischer theorem can be used to show that $\hat{\theta}_i \leq \lambda_{i-1}$ for $i = 2, 3, \ldots, n$. Thus for the eigenvalues $\hat{\theta}_i$ of $A(\hat{H})$ we have $\lambda_{i-1} \geq \hat{\theta}_i \geq \lambda_{i+1}$ for $i = 2, 3, \ldots, n-1$ as well as $\hat{\theta}_n \leq \lambda_{n-1}$ and $\hat{\theta}_1 \geq \lambda_2$. Since the eigenvalues of $A(H)$ and $A(\hat{H})$ differ only in that the latter set includes an additional zero eigenvalue we have $\theta_i = \hat{\theta}_i$ for $i = 1, 2, \ldots, k$ from which we get $\theta_i = \hat{\theta}_i \geq \lambda_{i+1}$ for $i = 1, 2, \ldots, k$. Similarly we have $\theta_i = \hat{\theta}_{i+1}$ for $i = k+1, k+2, \ldots, n-1$ so that $\theta_i = \hat{\theta}_{i+1} \leq \lambda_i$ for $i = k+1, k+2, \ldots, n-1$. $\square$

We conclude by observing that the inequality $\lambda_{i-1} \geq \theta_i \geq \lambda_{i+1}$ holds in a surprising number of cases, including for $A(G)$ and $\mathcal{L}(G)$ in the case of removing an edge. In the case of contracting two vertices, the inequality holds for $\mathcal{L}(G)$ (proven by CDHLPS) and also for $L(G)$. Except for the adjacency matrix in the case of edge removal, none of these is an obvious consequence of Cauchy's interlacing theorem. This raises a natural question of what other operations on graphs might lead to similar interlacing results.

More Recent

The *normalized signless Laplacian* matrix of G is the $n \times n$ matrix $|\mathcal{L}| := |\mathcal{L}|(G) = (|\mathcal{L}|_{ij})$ given by

$$|\mathcal{L}|_{ij} = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } d(v_i) \neq 0, \\ \frac{1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Let T again denote the diagonal matrix with the (i, i)-th entry having value d_{v_i}. We can write

$$|\mathcal{L}| = T^{-1/2}|L|T^{-1/2} = T^{-1/2}SS^TT^{-1/2},$$

where $|L| = T + A$ is the signless Laplacian matrix (A is the adjacency matrix). It can be easily seen that here again all eigenvalues of $|\mathcal{L}|$ are real and non-negative.

For $|\mathcal{L}| = T^{-1/2}|L|T^{-1/2}$:

$T^{\frac{1}{2}}e$ is an eigenvector corresponding to the eigenvalue 2, where e is the all 1's vector.

Since $|\mathcal{L}| + \mathcal{L} = 2I$ (for G connected), we have that the spectrum of $|\mathcal{L}|$ is contained in $[0, 2]$.

The multiplicity of the eigenvalue 2 for $|\mathcal{L}|$ equals the multiplicity of the eigenvalue 0 for $\mathcal{L}$, which is the number of connected components of the graph G.

0 is an eigenvalue of $|\mathcal{L}|$ iff 2 is an eigenvalue of $\mathcal{L}$ iff there is a connected component of G which is a nontrivial bipartite graph.

Theorem 3.8 also holds for $|\mathcal{L}|$.