Asymptotic Evaluation of Certain Markov Process Expectations for Large Time. IV

M. D. DONSKER AND S. R. S. VARADHAN

1. Introduction

In this paper we extend our asymptotic results [1], [2] to a more general setting. Let Ω be a space of functions $\omega(\cdot)$ on $-\infty < t < \infty$ with values in a Polish space X. We assume Ω consists of functions with discontinuities only of the first kind, normalized to be right continuous, and with convergence induced by the Skorohod topology on bounded intervals. In this case, Ω is itself a Polish space. Denote by Ω_t^+ the corresponding space of functions on $[t, \infty)$ with values in X. We denote by $\mathcal{F}_t^s$ the σ-field in Ω generated by $\omega(\sigma)$ for $s \leqq \sigma \leqq t$. We denote the translation map on Ω by θ_t, i.e., $(\theta_t\omega)(s) = \omega(s+t)$.

Let $P_{0,x}$ be a Markov process on Ω_0^+ starting from $x \in X$ satisfying the hypothesis that the mapping $x \to P_{0,x}$ is weakly continuous (which implies the Feller property for the process $P_{0,x}$).

Let $\omega \in \Omega$ and, for each $t > 0$, define ω_t by

$$\omega_t(s) = \omega(s), \qquad 0 \leqq s < t,$$

$$\omega_t(s+t) = \omega_t(s) \quad \text{for all} \quad s \in (-\infty, \infty).$$

Now, $(\theta_s\omega_t)(\tau) = \omega_t(s+\tau)$, $0 \leqq s \leqq t$, and for any set $A \subset \Omega$ we define

$$R_{t,\omega}(A) = \frac{1}{t}\int_0^t \chi_A(\theta_s\omega_t)\, ds. \tag{1.1}$$

If we let $\mathcal{M}_S(\Omega)$ denote the space of stationary processes on Ω, we see that, for each $\omega \in \Omega$ and each $t > 0$, $R_{t,\omega}(\cdot) \in \mathcal{M}_S(\Omega)$. This is because, for any $\sigma > 0$,

$$\begin{aligned} R_{t,\omega}(\theta_\sigma A) &= \frac{1}{t}\int_0^t \chi_{\theta_\sigma A}(\theta_s\omega_t)\, ds = \frac{1}{t}\int_0^t \chi_A(\theta_{s-\sigma}\omega_t)\, ds \\ &= \frac{1}{t}\int_{-\sigma}^{t-\sigma} \chi_A(\theta_s\omega_t)\, ds = \frac{1}{t}\int_0^t \chi_A(\theta_s\omega_t)\, ds = R_{t,\omega}(A). \end{aligned}$$

Thus, for fixed $t > 0$, $R_{t,\omega}$ is a mapping of $\Omega \to \mathcal{M}_S(W)$ and as such is $\mathcal{F}_t^0$ measurable. For each $t > 0$ and each $x \in X$, we use this mapping to induce a probability measure $\Gamma_{t,x}$ on $\mathcal{M}_S(\Omega)$ by defining $\Gamma_{t,x} = P_{0,x}R_{t,\omega}^{-1}$, i.e., if $B \subset \mathcal{M}_S(\Omega)$, then

$$\Gamma_{t,x}(B) = P_{0,x}\{\omega \in \Omega\colon R_{t,\omega} \in B\}. \tag{1.2}$$

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If $P_{0,x}$ is ergodic with invariant measure $\nu(dx)$ on X, and if $\bar{Q} \in \mathcal{M}_S(\Omega)$ is the stationary Markov process with ν as its marginal distribution, then, by the ergodic theorem, as $t \to \infty$,

$$\Gamma_{t,x} \Rightarrow \delta_{\bar{Q}} \quad \text{for all} \quad x. \tag{1.3}$$

Here, $\delta_{\bar{Q}}$ is the Dirac measure on $\mathcal{M}_S(\Omega)$ concentrated at $\bar{Q}$. What this paper is concerned with are the probabilities of large deviations for the measure $\Gamma_{t,x}$. If $G \subset \mathcal{M}_S(\Omega)$ is open and contains $\bar{Q}$, then, as just noted, $\lim_{t\to\infty} \Gamma_{t,x}(G) = 1$. On the other hand, if A is a set whose closure does not contain $\bar{Q}$, $\Gamma_{t,x}(A) \to 0$ as $t \to \infty$ and the question we are concerned with here is the rate at which it goes to 0.

In Sections 2 and 3 of this paper we define, for any $Q \in \mathcal{M}_S(\Omega)$, the entropy, $H(Q)$, of the stationary process Q with respect to the Markov process $P_{0,x}$. We then show that this entropy function governs the rate at which $\Gamma_{t,x}(G) \to 0$ as described above. In particular, we show, under suitable hypotheses to be detailed below, that for open sets $G \subset \mathcal{M}_S(\Omega)$,

$$\varliminf_{t\to\infty} \frac{1}{t} \log \Gamma_{t,x}(G) \geqq -\inf_{Q \in G} H(Q). \tag{1.4}$$

We also show under suitable hypotheses to be detailed below that, for closed sets $C \subset \mathcal{M}_S(\Omega)$,

$$\varlimsup_{t\to\infty} \frac{1}{t} \log \Gamma_{t,x}(C) \leqq -\inf_{Q \in C} H(Q). \tag{1.5}$$

Now, $H(Q) \geqq 0$ and it turns out that $H(\bar{Q}) = 0$ so that (1.4) and (1.5) are consistent with (1.3).

If $F(Q)$ is a continuous real-valued function on $\mathcal{M}_S(\Omega)$, then (1.4) and (1.5) imply (see [3])

$$\lim_{t\to\infty} \frac{1}{t} \log E^{\Gamma_{t,x}}\{e^{tF(Q)}\} = \sup_{Q \in \mathcal{M}_S(\Omega)} [F(Q) - H(Q)]. \tag{1.6}$$

An equivalent form of (1.6), which is responsible for the title of this paper, is

$$\lim_{t\to\infty} \frac{1}{t} \log E^{P_{0,x}}\{\exp\{tF(R_{t,\omega})\}\} = \sup_{Q \in \mathcal{M}_S(\Omega)} [F(Q) - H(Q)]. \tag{1.7}$$

Before giving examples of (1.7) we want to make two remarks which will relate (1.7) to our earlier results.

First of all, our earlier asymptotic results involved functions of the occupation distribution: for $\omega \in \Omega$, $t > 0$, and $A \subset X$,

$$L_{t,\omega}(A) = \frac{1}{t} \int_0^t \chi_A(\omega(s))\, ds,$$

i.e., $L_{t,\omega}(A)$ is the proportion of time up to t that the particular path $\omega(\cdot)$ occupies the set $A \subset X$. For fixed $t > 0$, and $\omega \in \Omega$, $L_{t,\omega}(\cdot)$ is then a probability measure on X. We denote the space of probability measures on X by $\mathcal{M}(X)$. Now, the relation between $L_{t,\omega}\in\mathcal{M}(X)$ and $R_{t,\omega}\in\mathcal{M}_S(\Omega)$ is made clear by observing that the marginal distribution of $R_{t,\omega}$ is precisely $L_{t,\omega}$.

Furthermore, in our earlier papers the asymptotic rate involving functionals of $L_{t,\omega}$ was governed by the I-function (for the Markov process $P_{0,x}$) which was defined on $\mathcal{M}(X)$ (see, e.g., [2]). To see the relation between that I-function and the entropy function $H(Q)$ of this paper, we prove in Section 6 that

$$\inf_{\substack{Q\in\mathcal{M}_S(\Omega)\\ q(Q)=\mu}} H(Q)=I(\mu), \tag{1.8}$$

where, in (1.8), the notation $q(Q)=\mu$ means the marginal of the stationary measure Q is μ. We refer to relation (1.8) as the contraction principle.

We now mention a few examples to illustrate (1.7) and the remarks just made.

EXAMPLE 1. Let $V(\cdot)$ be a continuous real-valued function on X and define

$$F(Q)=E^Q\{V(\omega(0))\}.$$

Then,

$$F(R_{t,\omega})=\int_\Omega V(\omega'(0))R_{t,\omega}(d\omega')=\frac{1}{t}\int_0^t V(\omega(s))\,ds,$$

and (1.7) becomes, in this case,

$$\lim_{t\to\infty}\frac{1}{t}\log E^{P_{0,x}}\left\{\exp\left\{\int_0^t V(\omega(s))\,ds\right\}\right\}=\sup_{Q\in\mathcal{M}_S(\Omega)}[E^Q\{V(\omega(0))\}-H(Q)]. \tag{1.9}$$

Using (1.8), this becomes

$$\lim_{t\to\infty}\frac{1}{t}\log E^{P_{0,x}}\left\{\exp\left\{\int_0^t V(\omega)(s))\,ds\right\}\right\}=\sup_{\mu\in\mathcal{M}(X)}\left[\int V(x)\mu(dx)-I(\mu)\right], \tag{1.10}$$

where, of course, $I(\mu)$ in (1.10) is the I-function associated with the Markov process $P_{0,x}$. Formula (1.10) agrees with earlier results of the authors.

EXAMPLE 2. Let $F(Q)=E^Q\{V(\omega(0),\ \omega(1))\}$, where $V:X\times X\to R$ is bounded and continuous. Now,

$$F(R_{t,\omega})=\frac{1}{t}\int_0^{t-1} V(\omega(s),\omega(s+1))\,ds+O\left(\frac{1}{t}\right)\quad\text{as}\quad t\to\infty,$$

and from this and (1.7) it follows that

$$\begin{aligned}\log_{t\to\infty}\frac{1}{t}\log E^{P_{0,x}}\left\{\exp\left(\int_0^t V(\omega(s),\omega(s+1))\,ds\right)\right\}\\ =\sup_{Q\in\mathcal{M}_S(\Omega)}[E^Q\{V(\omega(0),\omega(1))\}-H(Q)].\end{aligned} \tag{1.11}$$

EXAMPLE 3. Let P_t and $E^{P_t}\{\ \}$ denote, respectively, the probability measure and expectation with respect to three-dimensional Brownian motion $\omega(\cdot)$ tied down at both ends, i.e., $\omega(0)=\omega(t)=0$. For $\alpha>0$, let

$$G(\alpha,t)=E^{P_t}\left\{\exp\left\{\alpha\int_0^t\int_0^t\frac{e^{-|\sigma-s|}}{|\omega(\sigma)-\omega(s)|}\,d\sigma\,ds\right\}\right\}.$$

A long standing problem in statistical mechanics (cf. [4]), the "polaron problem" has been to show

$$\lim_{t\to\infty}\frac{1}{t}\log G(\alpha,t)=g(\alpha)\text{ exists,}\tag{1.12}$$

and, moreover, according to a conjecture of Pekar [5],

$$\lim_{\alpha\to\infty}\frac{g(\alpha)}{\alpha^2}=g_0\text{ exists,}\tag{1.13}$$

with

$$g_0=\sup_{\substack{\phi\in L_2(R_3)\\ \|\phi\|=1}}\left[2\int\int\frac{\phi^2(x)\phi^2(y)}{|x-y|}\,dx\,dy-\tfrac{1}{2}\int|\nabla\phi|^2\,dx\right].\tag{1.14}$$

Using the results of the present paper, the authors succeeded in proving (1.12) obtaining as an expression for $g(\alpha)$ a variational formula over all stationary processes as in (1.7). This infinite-dimensional variational formula turns out to be just explicit enough to allow us to also prove Pekar's conjecture (1.13) obtaining indeed the expression (1.14) for g_0. The details of this example will be found in a forthcoming paper [6].

Now, we wish to make precise the hypotheses imposed in order to prove (1.4) and (1.5). For the upper estimate (Theorem 4.6 below) we first comment that if the space X happens to be compact, then no further hypotheses on the Markov process $P_{0,x}$ need to be imposed in order to obtain (1.5), but if X is not compact we need to impose the following hypothesis on $P_{0,x}$.

There exists a sequence $\{u_n(x)\}$ of functions in $\mathcal{D}(L)$ (the domain of the infinitesimal generator L of the Markov process $P_{0,x}$) with the five properties:

(i) $u_n(x)\geqq c>0$ for all x and n.

(ii) There exists, for every compact set $K\subset X$, a constant C_K such that $\sup_{x\in K}\sup_n u_n(x)\leqq C_K$.

(iii) $-(Lu_n/u_n)(x)=V_n(x)\geqq -C$ for all n and all x.

(iv) There exists a function $V(x)$ such that, for all $x\in X$, $\lim_{n\to\infty}V_n(x)=V(x)$.

(v) $\{x: V(x)\leqq l\}$ is compact in X for every $l<\infty$.

In Section 7 of this paper we show that the Ornstein–Uhlenbeck process in d-dimensions, for example, satisfies these 5 properties. In the polaron problem we use the upper estimate (1.5) for precisely this case of the Ornstein–Uhlenbeck process.

For the lower estimate (1.4) (see Theorem 5.5 below), we impose two restrictions on $P_{0,x}$. Let $p(t, x, dy)$ be the transition probability associated with $P_{0,x}$. We shall assume that there exists a density function for $p(1, x, dy)$ with respect to a reference measure α on X, i.e., for all $x \in X$ there exists a function $p(1, x, y) > 0$ for almost all y (α measure) such that for all $x \in X$:

I: $p(1, x, dy) = p(1, x, y)\alpha(dy)$,

and

II: $p(1, x, \cdot)$ as a mapping from $X \to L_1(\alpha)$ is continuous.

In the polaron problem, for example, we need the lower estimate (1.4) for the case of ordinary three-dimensional Brownian motion and this satisfies I and II.

As for the asymptotic formula (1.6) or equivalently (1.7) we see that this holds for any Markov process $P_{0,x}$ satisfying both (i)–(v) and I–II above.

2. Some Properties of Entropy

Let (X, Σ) be a measurable space and let λ and μ be the probability measures on (X, Σ). Let $\mathscr{B}(\Sigma)$ be the space of bounded measurable functions on (X, Σ). We define the entropy of μ with respect to λ by

$$h(\lambda : \mu) = \inf\left\{c : \int \Phi(x)\mu(dx) \leqq c + \log \int e^{\Phi(x)}\lambda(dx) \quad \text{for all} \quad \Phi \in \mathscr{B}(\Sigma)\right\}. \tag{2.1}$$

If X is a Polish space and Σ is the Borel σ-field, then replacing $\mathscr{B}(\Sigma)$ by $C(X)$ in (2.1) gives the same infimum (see [2-III]).

From the definition (2.1) we see that, for fixed λ, $h(\lambda ; \mu)$ is a non-negative, convex function of μ and $0 \leqq h(\lambda ; \mu) \leqq \infty$. Moreover, if X is a Polish space, then $h(\lambda ; \mu)$ is lower semicontinuous in μ in the weak topology.

The following basic theorem was proved in our earlier work (see [2-III]).

THEOREM 2.1. *For any (X, Σ) and any two probability measures λ, μ on (X, Σ), $h(\lambda ; \mu)$ is finite if and only if*

(a) $\mu \ll \lambda$,

and

(b) *$d\mu/d\lambda = f(x)$ is such that $f(x) \log f(x) \in L_1(\lambda)$.*

Under these circumstances,

$$h(\lambda ; \mu) = \int \log f(x)\mu(dx) = \int f(x) \log f(x)\lambda(dx).$$

Remark. Suppose $\mu \ll \lambda$ and $d\mu/d\lambda = f(x)$. Since $f(x)\log f(x)$ is bounded for small $f(x)$, there is no difficulty when $\log f(x)$ is negative, i.e., $f(x)(\log(f(x))^-$ is always integrable with respect to λ and thus $f(x)\log f(x) \in L_1(\lambda)$ if and only if $f(x)(\log f(x))^+ \in L_1(\lambda)$.

We shall also use the following lemma (proved in [2-III]) which we state here for convenience.

LEMMA 2.2. *Let $h(\lambda:\mu) \leqq c < \infty$ and let $A \subset X$ be any set with $\lambda(A) = \varepsilon > 0$. Then, $\mu(A) \leqq \phi(c, \varepsilon)$, where $\phi(c, \varepsilon)$ is a universal function tending to 0 as $\varepsilon \to 0$ for every $c < \infty$.*

If $\mathscr{F} \subset \Sigma$ is a sub σ-field in the measurable space (X, Σ), and if we take $\Phi \in \mathscr{B}(\mathscr{F})$ instead of $\Phi \in \mathscr{B}(\Sigma)$ in definition (2.1), then we shall denote the corresponding entropy by $h_{\mathscr{F}}(\lambda;\mu)$. This is, of course, the same as restricting λ and μ to $(X, \mathscr{F})$.

LEMMA 2.3. *Let (X, Σ) be a Polish space and let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \Sigma$ be sub σ-fields. Denote by λ_ω, μ_ω the regular conditional probability distributions (r.c.p.d.) of λ and μ given $\mathscr{F}_1$. Then,*

$$h_{\mathscr{F}_2}(\lambda;\mu) = h_{\mathscr{F}_1}(\lambda;\mu) + E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)\}. \tag{2.2}$$

Proof: If $h_{\mathscr{F}_1}(\lambda;\mu) = \infty$, then since $h_{\mathscr{F}_2}(\lambda;\mu) \geqq h_{\mathscr{F}_1}(\lambda;\mu)$ we have $h_{\mathscr{F}_2}(\lambda;\mu) = \infty$ also and there is nothing to prove. Thus, we assume $h_{\mathscr{F}_1}(\lambda;\mu) < \infty$ and thus, from Theorem 2.1, $\mu \ll \lambda$ on $\mathscr{F}_1$. This means λ_ω is defined almost everywhere with respect to μ measure and therefore the expectation on the right of (2.2) is well defined. Let $\Phi \in \mathscr{B}(\mathscr{F}_2)$ so that from the definition of $h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)$ we have

$$\begin{aligned} E^{\mu}\{\Phi\} &= E^{\mu}\{E^{\mu_\omega}\{\Phi\}\}, \\ &\leqq E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega) + \log E^{\lambda_\omega}\{e^{\Phi}\}\}. \end{aligned} \tag{2.3}$$

If we let $\psi(\omega) = \log E^{\lambda_\omega}\{e^{\Phi}\}$, then $\psi(\omega) \in \mathscr{B}(\mathscr{F}_1)$ and so from the definition of $h_{\mathscr{F}_1}(\lambda;\mu)$ and (2.3) we get

$$\begin{aligned} E^{\mu}\{\Phi\} &\leqq E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)\} + E^{\mu}\{\psi(\omega)\} \\ &\leqq E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)\} + h_{\mathscr{F}_1}(\lambda;\mu) + \log E^{\lambda}\{e^{\psi(\omega)}\} \\ &= E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)\} + h_{\mathscr{F}_1}(\lambda;\mu) + \log E^{\lambda}\{e^{\Phi}\}. \end{aligned} \tag{2.4}$$

Using again the definition of $h_{\mathscr{F}_2}(\lambda;\mu)$, we conclude from (2.4) that

$$h_{\mathscr{F}_2}(\lambda;\mu) \leqq E^{\mu}\{h_{\mathscr{F}_2}(\lambda_\omega;\mu_\omega)\} + h_{\mathscr{F}_1}(\lambda;\mu). \tag{2.5}$$

Now we must show this inequality in the reverse direction. The inequality desired is trivial if $h_{\mathscr{F}_2}(\lambda;\mu) = \infty$; thus assume $h_{\mathscr{F}_2}(\lambda;\mu) < \infty$ which means by Theorem

2.1 that $\mu \ll \lambda$ on $\mathcal{F}_2$. Let $g(\omega)=d\mu/d\lambda|_{\mathcal{F}_2}$. As for the regular conditional probability distributions, we see that $\mu_\omega \ll \lambda_\omega$ on $\mathcal{F}_2$ for almost all ω (λ measure). Moreover,

$$g(\omega)=\frac{d\mu}{d\lambda}\bigg|_{\mathcal{F}_1}(\omega)\,\frac{d\mu_\omega}{d\lambda_\omega}\bigg|_{\mathcal{F}_2}(\omega),$$

and hence, from Theorem 2.1,

$$\begin{aligned} E^\mu\{\log g(\omega)\} &= E^\mu\left\{\log\frac{d\mu}{d\lambda}\bigg|_{\mathcal{F}_1}\right\}+E^\mu\left\{\log\frac{d\mu_\omega}{d\lambda_\omega}\bigg|_{\mathcal{F}_2}\right\},\\ &= E^\mu\left\{\log\frac{d\mu}{d\lambda}\bigg|_{\mathcal{F}_1}\right\}+E^\mu\left\{E^{\mu_\omega}\left\{\log\frac{d\mu_\omega}{d\lambda_\omega}\bigg|_{\mathcal{F}_2}\right\}\right\}\\ &= E^\mu\left\{\log\frac{d\mu}{d\lambda}\bigg|_{\mathcal{F}_1}\right\}+E^\mu\{h_{\mathcal{F}_2}(\lambda_\omega;\mu_\omega)\}\\ &= h_{\mathcal{F}_1}(\lambda;\mu)+E^\mu\{h_{\mathcal{F}_2}(\lambda_\omega;\mu_\omega)\}. \end{aligned} \tag{2.6}$$

Again using Theorem 2.1 we get from (2.6)

$$h_{\mathcal{F}_2}(\lambda;\mu)=h_{\mathcal{F}_1}(\lambda,\mu)+E^\mu\{h_{\mathcal{F}_2}(\lambda_\omega;\mu_\omega)\}.$$

3. Entropy of Processes

Let Ω be a space of functions $\omega(\cdot)$ on $-\infty<t<\infty$ with values in a Polish space X. We shall assume that either Ω consists of continuous functions with uniform convergence on bounded intervals or it consists of functions with discontinuities only of the first kind and with convergence induced by the Skorohod topology on bounded intervals. In either case, Ω itself is a Polish space. For each $\omega\in\Omega$, we denote by $\omega(t)$ the value of the function $\omega(\cdot)$ at time t. We denote by Ω_t^+ the corresponding space of functions on $[t,\infty)$ with values in X and we denote by $\mathcal{F}_t^s$ the σ-fields in Ω generated by $\omega(\sigma)$ for $s\leqq\sigma\leqq t$. The space Ω_t^+ can, of course, be naturally identified with the space Ω and the σ-field $\mathcal{F}_\infty^t$. We shall use the notation θ_t for the translation map on Ω, i.e., $(\theta_t\omega)(s)=\omega(t+s)$.

Let Q be a stationary process on Ω and let $\{P_{t,x}\}$ be a homogeneous Markov family of measures on Ω_t^+. We define $P_{t,\omega(t)}$ to be $P_{t,x}$ with the starting point $x=\omega(t)$. We denote by $\{Q_{t,\omega}\}$ the regular conditional probability distributions of Q given $\mathcal{F}_t^{-\infty}$. We now define the entropy of the stationary process Q with respect to the Markov process $P_{0,x}$ at time $t>0$ by

$$H(t,Q)=E^Q\{h_{\mathcal{F}_t^0}(P_{0,\omega(0)};Q_{0,\omega})\}. \tag{3.1}$$

Here, $h_{\mathcal{F}_t^0}(P_{0,\omega(0)};Q_{0,\omega})$ is the entropy introduced in the previous section. It is easy to check that $H(t,Q)$ is well defined for all $t>0$ with $0\leqq H(t,Q)\leqq\infty$.

THEOREM 3.1. *Either $H(t, Q) \equiv \infty$ for all $t > 0$ or there exists a constant $H(Q) < \infty$ such that*

$$H(t, Q) = tH(Q) \quad \text{for all} \quad t > 0. \tag{3.2}$$

Proof: Pick $0 < t_1 < t_2$ and consider $h_{\mathscr{F}^0_{t_2}}(P_{0,\omega(0)}; Q_{0,\omega})$. From Lemma 2.3,

$$\begin{aligned} h_{\mathscr{F}^0_{t_2}}(P_{0,\omega(0)}; Q_{0,\omega}) &= h_{\mathscr{F}^0_{t_1}}(P_{0,\omega(0)}; Q_{0,\omega}) \\ &\quad + E^{Q_{0,\omega}}\{h_{\mathscr{F}^0_{t_2}}((P_{0,\omega(0)})_{t,\bar\omega}; (Q_{0,\omega})_{t,\bar\omega})\}, \end{aligned} \tag{3.3}$$

where $(P_{0,\omega(0)})_{t_1,\bar\omega}$ means the r.c.p.d. of $P_{0,\omega(0)}$ with respect to the σ-field $\mathscr{F}^0_{t_1}$ and where $\bar\omega$ is the variable, i.e., $E^{Q_{0,\omega}}\{\ \}$ indicates integration with respect to $\bar\omega$ under $Q_{0,\omega}$ measure. Of course, $(Q_{0,\omega})_{t_1,\bar\omega}$ is the analogous r.c.p.d.

It will be convenient to express (3.3) in an alternative notation. Let $\bar\omega$ be a trajectory in Ω, $t \in (-\infty, \infty)$, and P be a measure on $\mathscr{F}^s_\infty$ with $s \leqq t$. Suppose $P\{\omega: \omega(t) = \bar\omega(t)\} = 1$. Then, we define $\delta_{\bar\omega} \otimes_t P$ to be a new measure on Ω such that

$$\begin{aligned} &(\delta_{\bar\omega} \otimes_t P)\{\omega(t_1) \in A_1, \omega(t_2) \in A_2, \cdots, \omega(t_n) \in A_n\} \\ &\quad = \chi_{A_1}(\bar\omega(t_1)) \cdots \chi_{A_k}(\bar\omega(t_k)) P\{\omega(t_{k+1}) \in A_{k+1}, \cdots, \omega(t_n) \in A_n\}, \end{aligned}$$

where $t_1 < t_2 < \cdots < t_k \leqq t \leqq t_{k+1} < t_{k+2} < \cdots < t_n$. Now, $Q_{0,\omega}\{\bar\omega: \bar\omega(0) = \omega(0)\} = 1$, and $(P_{0,\omega(0)})_{t_1,\bar\omega} = \delta_{\bar\omega} \otimes_{t_1} P_{t_1,\bar\omega(t_1)}$ on $\mathscr{F}^0_\infty$. Moreover, $(Q_{0,\omega})_{t_1,\bar\omega} = Q_{t_1,\bar\omega} = \delta_{\bar\omega} \otimes_{t_1} Q_{t_1,\bar\omega}$ on $\mathscr{F}^0_\infty$. Thus, we can rewrite (3.3) as

$$\begin{aligned} h_{\mathscr{F}^0_{t_2}}(Q_{0,\omega(0)}; Q_{0,\omega}) &= h_{\mathscr{F}^0_{t_1}}(P_{0,\omega(0)}; Q_{0,\omega}) \\ &\quad + E^{Q_{0,\omega}}\{h_{\mathscr{F}^0_{t_2}}(\delta_{\bar\omega} \otimes_{t_1} P_{t_1,\bar\omega(t_1)}; \delta_{\bar\omega} \otimes_{t_1} Q_{t_1,\bar\omega})\}. \end{aligned} \tag{3.4}$$

We now take expectations on both sides of (3.4) with respect to Q measure (ω is the variable of integration), and noting that

$$E^Q\{E^{Q_{0,\omega}}\{h_{\mathscr{F}^{t_1}_{t_2}}(P_{t_1,\bar\omega(t_1)}; Q_{t_1,\bar\omega})\} = E^Q\{h_{\mathscr{F}^{t_1}_{t_2}}(P_{t_1,\omega(t_1)}; Q_{t_1,\omega})\},$$

we obtain

$$\begin{aligned} E^Q\{h_{\mathscr{F}^0_{t_2}}(P_{0,\omega(0)}; Q_{0,\omega})\} &= E^Q\{h_{\mathscr{F}^0_{t_1}}(P_{0,\omega(0)}; Q_{0,\omega})\} \\ &\quad + E^Q\{h_{\mathscr{F}^{t_1}_{t_2}}(P_{t_1,\omega(t_1)}; Q_{t_1,\omega})\}. \end{aligned} \tag{3.5}$$

By definition (3.1), the left side of (3.5) is $H(t_2, Q)$ and the first term on the right of (3.5) is $H(t_1, Q)$. By the stationarity of the Q process and the homogeneity of the Markov process we see that the second term on the right of (3.5) is $H(t_2 - t_1, Q)$. Thus, $H(t, Q)$ is additive in t and from its definition it is clearly monotone in t. Hence, $H(t, Q)$ is linear in t unless it is identically infinite.

We shall call $H(Q)$ the entropy of the stationary process Q with respect to the Markov process $P_{0,x}$.

THEOREM 3.2. *Let $P_\omega = \delta_\omega \otimes_0 P_{0,\omega(0)}$. For all $t > 0$,*

$$H(t, Q) = \sup_{\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})} [E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}]. \tag{3.6}$$

Proof: Let $\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})$. Using (2.1), (3.1), and the fact that $Q_{0,\omega} = \delta_\omega \otimes_0 Q_{0,\omega}$, we obtain

$$\begin{aligned} E^Q\{\Phi\} &= E^Q\{E^{Q_{0,\omega}}\{\Phi\}\} \\ &\leqq E^Q\{h_{\mathscr{F}_t^{-\infty}}(P_\omega; Q_{0,\omega})\} + E^Q\{\log E^{P_\omega}\{e^\Phi\}\} \\ &= E^Q\{h_{\mathscr{F}_t^0}(P_{0,\omega(0)}; Q_{0,\omega})\} + E^Q\{\log E^{P_\omega}\{e^\Phi\}\} \\ &= H(t, Q) + E^Q\{\log E^{P_\omega}\{e^\Phi\}\}. \end{aligned}$$

Thus,

$$\sup_{\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})} [E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}] \leqq H(t, Q). \tag{3.7}$$

To show the inequality the other way, let

$$\sup_{\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})} [E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}] = l.$$

Then, by Jensen's inequality,

$$\sup_{\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})} [E^Q\{\Phi\} - \log E^Q\{E^{P_\omega}\{e^\Phi\}\}] \leqq l. \tag{3.8}$$

Let $\bar{Q}$ measure be defined by $\int P_\omega Q(d\omega)$. Then, from (3.8) we have, for every $\Phi \in \mathscr{B}(\mathscr{F}_t^{-\infty})$,

$$E^Q\{\Phi\} - \log E^{\bar{Q}}\{e^\Phi\} \leqq l,$$

which implies from (2.1) that

$$h_{\mathscr{F}_t^{-\infty}}(\bar{Q}; Q) \leqq l.$$

Using Lemma 2.3, we then get

$$h_{\mathscr{F}_t^{-\infty}}(\bar{Q}; Q) = h_{\mathscr{F}_0^{-\infty}}(\bar{Q}; Q) + E^Q\{h_{\mathscr{F}_t^{-\infty}}(\bar{Q}_\omega; Q_{0,\omega})\} \leqq l. \tag{3.9}$$

But, by the definition of P_ω, $\bar{Q} = Q$ on $\mathscr{F}_0^{-\infty}$, and therefore $h_{\mathscr{F}_0^{-\infty}}(\bar{Q}; Q) = 0$. Moreover, $\bar{Q}_\omega = P_\omega = \delta_\omega \otimes_0 P_{0,\omega(0)}$, and $Q_{0,\omega} = \delta_\omega \otimes_0 Q_{0,\omega}$ so that

$$E^Q\{h_{\mathscr{F}_t^0}(P_{0,\omega(0)}; Q_{0,\omega})\} = E^Q\{h_{\mathscr{F}_t^{-\infty}}(\bar{Q}_\omega; Q_{0,\omega})\}.$$

Hence, from (3.9) we obtain

$$H(t, Q) = E^Q\{h_{\mathscr{F}_t^0}(P_{0,\omega(0)}; Q_{0,\omega})\} \leqq l$$

which completes the proof.

THEOREM 3.3. *Under the hypothesis that the mapping $x \to P_{t,x}$ is weakly continuous, $H(Q)$ is lower semicontinuous and convex in Q.*

Proof: From Theorems 3.1 and 3.2,

$$H(Q) = H(1, Q) = \sup_{\Phi \in \mathscr{B}(\mathscr{F}_1^{-\infty})} [E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}]. \tag{3.10}$$

The supremum on the right of (3.10) is not changed if we restrict $\Phi \in \mathcal{B}(\mathcal{F}_1^{-\infty}) \cap C(\Omega)$. Thus, to show the lower semicontinuity of $H(Q)$, it suffices to show that, for each $\Phi \in \mathcal{B}(\mathcal{F}_1^{-\infty}) \cap C(\Omega)$, $E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}$ is continuous as a functional of Q. Clearly, $E^Q\{\Phi\}$ is a continuous functional of Q. Thus let $\psi(\omega) = E^{P_\omega}\{e^\Phi\}$ and it suffices to show that $\psi(\omega)$ is continuous almost everywhere with respect to Q measure for an arbitrary stationary Q. To show this it suffices to show that if $\omega_n \to \omega$ in Ω, then $P_{\omega_n} \Rightarrow P_\omega$. Assume, then that $\omega_n \to \omega$ and that ω has no jump at the origin, i.e., $\omega(0^+) = \omega(0^-)$. Then, $\omega_n(0) \to \omega(0)$ and hence $P_{\omega_n} \Rightarrow P_\omega$. It remains to observe that, for all stationary processes Q, $Q\{\omega : \omega$ has a jump at $0\} = 0$.

The convexity of $H(Q)$ follows from (3.10) when we note that $H(Q)$ is the supremum of a collection of linear functionals of Q.

Next (Theorem 3.5 below) we prove the surprising fact that $H(Q)$ is linear in Q, but first we need a lemma.

LEMMA 3.4. *Let $\mathcal{M}_\Omega$ be the space of probability measures on Ω. There exists a conditional probability $R_\omega : \Omega \to \mathcal{M}_\Omega$ such that $R_\omega = \delta_\omega \otimes_0 R_\omega$ and, for all stationary measures Q on Ω, $Q_{0,\omega} = R_\omega$ almost everywhere with respect to Q measure.*

Proof: As before, let $\mathcal{M}_S(\Omega)$ be the space of stationary measures on Ω and let $\mathcal{M}_E(\Omega) \subset \mathcal{M}_S(\Omega)$ be the subset of $\mathcal{M}_S(\Omega)$ consisting of the ergodic measures. $\mathcal{M}_E(\Omega)$ is the set of extremals of $\mathcal{M}_S(\Omega)$. It follows from an argument of Oxtoby [7] that there exists a subset $\Omega_0 \subset \Omega$ which is $\mathcal{F}_0^{-\infty}$ measurable and a $\mathcal{F}_0^{-\infty}$ measurable map $\pi_\omega : \Omega_0 \to \mathcal{M}_E(\Omega)$ with the following properties: $Q(\Omega_0) = 1$ for all $Q \in \mathcal{M}_S(\Omega)$ and $Q\{\omega : \pi_\omega = Q\} = 1$ for all $Q \in \mathcal{M}_E(\Omega)$.

Now, denote by $R(Q, \omega)$ the r.c.p.d. of Q given $\mathcal{F}_0^{-\infty}$. It is easily seen that this r.c.p.d. can always be selected so that it is jointly measurable in Q and ω. We define the desired R_ω by $R_\omega = R(\pi_\omega, \omega)$. Then, for all $Q \in \mathcal{M}_E(\Omega)$,

$$\int R_\omega Q(d\omega) = \int R(Q, \omega) Q(d\omega) = Q.$$

But, $R_\omega = R(\pi_\omega, \omega)$ is independent of Q and so we have a linear relation true for extremals and hence true for all $Q \in \mathcal{M}_S(\Omega)$.

THEOREM 3.5. *$H(Q)$ is linear in Q.*

Proof: From Theorem 3.1 and the preceding lemma,

$$\begin{aligned} H(Q) = H(1, Q) &= E^Q\{h_{\mathcal{F}_1^0}(P_{0,\omega(0)}; Q_{0,\omega})\} \\ &= E^Q\{h_{\mathcal{F}_1^0}(P_{0,\omega(0)}; R_\omega)\}, \end{aligned} \tag{3.11}$$

where R_ω, as defined in the preceding lemma, is independent of Q. Thus, the integrand on the right of (3.11) is independent of Q and so $H(Q)$ is linear in Q.

In Theorem 3.2 we proved that, for $P_\omega = \delta_\omega \otimes_0 P_{0,\omega(0)}$ and for all $t>0$,

$$H(t, Q) = \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^{-\infty})} [E^Q\{\Phi\} - E^Q\{\log E^{P_\omega}\{e^\Phi\}\}].$$

Now define

$$\begin{aligned} \bar{H}(t, Q) &= \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^0)} [E^Q\{\Phi\} - E^Q\{\log E^{P_{0,\omega(0)}}\{e^\Phi\}\}] \\ &= \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^0) \cap C(\Omega)} [E^Q\{\Phi\} - E^Q\{\log E^{P_{0,\omega(0)}}\{e^\Phi\}\}]. \end{aligned} \tag{3.12}$$

Clearly, $\bar{H}(t, Q) \leqq H(t, Q) = tH(Q)$.

THEOREM 3.6.

$$\lim_{T\to\infty} \frac{\bar{H}(t, Q)}{t} = H(Q). \tag{3.13}$$

Proof: Define $\bar{P}$ measure on $\mathcal{F}_\infty^0$ by $\bar{P} = \int P_{0,\omega(0)} Q(d\omega)$. By Jensen's inequality and (2.1),

$$\begin{aligned} \bar{H}(t, Q) &\geqq \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^0)} [E^Q\{\Phi\} - \log E^Q\{E^{P_{0,\omega(0)}}\{e^\Phi\}\}] \\ &= \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^0)} [E^Q\{\Phi\} - \log E^{\bar{P}}\{e^\Phi\}] \\ &= h_{\mathcal{F}_t^0}(\bar{P}; Q). \end{aligned}$$

Hence, to show (3.13), it suffices to show

$$\varliminf_{t\to\infty} \frac{1}{t} h_{\mathcal{F}_t^0}(\bar{P}; Q) \geqq H(Q). \tag{3.14}$$

From Lemma 2.3,

$$h_{\mathcal{F}_{t+1}^0}(\bar{P}; Q) - h_{\mathcal{F}_t^0}(\bar{P}; Q) = E^Q\{h_{\mathcal{F}_{t+1}^0}(\bar{P}_{t,\omega}^0; Q_{t,\omega}^0)\}, \tag{3.15}$$

where $\bar{P}_{t,\omega}^0$ and $Q_{t,\omega}^0$ are the r.c.p.d. of $\bar{P}$ and Q, respectively, given $\mathcal{F}_t^0$. But,

$$\begin{aligned} E^Q\{h_{\mathcal{F}_{t+1}^0}(\bar{P}_{t,\omega}^0; Q_{t,\omega}^0)\} &= E^Q\{h_{\mathcal{F}_{t+1}^0}(\delta_\omega \otimes_t \bar{P}_{t,\omega}^0; \delta_\omega \otimes_t Q_{t,\omega}^0)\} \\ &= E^Q\{h_{\mathcal{F}_{t+1}^t}(\bar{P}_{t,\omega}^0; Q_{t,\omega}^0)\} \\ &= E^Q\{h_{\mathcal{F}_{t+1}^t}(P_{t,\omega(t)}; Q_{t,\omega}^0)\} \\ &= E^Q\{h_{\mathcal{F}_1^0}(P_{0,\omega(0)}; Q_{0,\omega}^{-t})\}, \end{aligned} \tag{3.16}$$

using the homogeneity of the Markov family and the stationarity of the Q process. By the martingale convergence theorem, $Q_{0,\omega}^{-t} \Rightarrow Q_{0,\omega}$ as $t\to\infty$ almost everywhere with respect to Q measure. As noted earlier, $h(\lambda; \mu)$ is lower-semicontinuous in μ, and hence

$$\varliminf_{t\to\infty} h_{\mathcal{F}_1^0}(P_{0,\omega(0)}; Q_{0,\omega}^{-t}) \geqq h_{\mathcal{F}_1^0}(P_{0,\omega(0)}; Q_{0,\omega}). \tag{3.17}$$

Thus, by Fatou's lemma, (3.15), (3.16), and (3.17),

$$\varliminf_{t\to\infty} [h_{\mathscr{F}^0_{t+1}}(\bar{P}; Q) - h_{\mathscr{F}^0_t}(\bar{P}; Q)] \geqq E^Q\{h_{\mathscr{F}^0_1}(P_{0,\omega(0)}; Q_{0,\omega})\} \tag{3.18}$$

$$= H(1, Q) = H(Q).$$

Using the usual Caesaro argument, we obtain (3.14) and (3.18). This completes the proof of Theorem 3.6.

Let $C_S(\Omega)$ be the space of functions Φ on Ω which are bounded and measurable on Ω and such that, for all $Q \in \mathscr{M}_S(\Omega)$,

$$Q\{\text{discontinuities of } \Phi\} = 0.$$

For $\Phi \in C_S(\Omega)$, the linear functional of Q, $\int \Phi Q(d\omega)$, is then continuous in Q.

LEMMA 3.7. *Let* $Y_1 = \{\Phi: \Phi \in \mathscr{B}(\mathscr{F}^0_t), E^{P_{0,x}}\{e^{\Phi}\} \leqq 1$, *for all* $x\}$ *and let* $Y_2 = \{\Phi: \Phi \in \mathscr{B}(\mathscr{F}^0_t) \cap C_S(\Omega), E^{P_{0,x}}\{e^{\Phi}\} \leqq 1$, *for all* $x\}$. *Then, with* $\bar{H}(t, Q)$ *defined by* (3.12),

$$\bar{H}(t, Q) = \sup_{\Phi \in Y_1} E^Q\{\Phi\} = \sup_{\Phi \in Y_2} E^Q\{\Phi\}. \tag{3.19}$$

Proof: From (3.12),

$$\bar{H}(t, Q) = \sup_{\Phi \in \mathscr{B}(\mathscr{F}^0_t)} [E^Q\{\Phi\} - E^Q\{\log E^{P_{0,\omega(0)}}\{e^{\Phi}\}\}]$$

$$\geqq \sup_{\Phi \in Y_1} E^Q\{\Phi\} \geqq \sup_{\Phi \in Y_2} E^Q\{\Phi\}.$$

Conversely, suppose $\sup_{\Phi \in Y_2} E^Q\{\Phi\} = l$. Let $\psi \in \mathscr{B}(\mathscr{F}^0_t) \cap C(\Omega)$ and define $\bar{\psi} = \log E^{P_{0,x}}\{e^{\psi}\}$. Clearly, $\bar{\psi} \in C(R_1)$ and therefore $\bar{\psi}(\omega(0))$ is continuous as a function of ω on the set where ω is continuous at 0. This is a set of ω of Q measure 1 for every $Q \in \mathscr{M}_S(\Omega)$, i.e., $\bar{\psi}(\omega(0)) \in C_S(\Omega)$. Moreover,

$$E^{P_{0,x}}\{e^{\psi - \bar{\psi}(\omega(0))}\} = e^{-\bar{\psi}(x)} E^{P_{0,x}}\{e^{\psi}\}$$

$$= e^{-\bar{\psi}(x)} e^{\bar{\psi}(x)} = 1.$$

Thus, if we let $\Phi = \psi(\omega) - \bar{\psi}(\omega(0))$, then $\Phi \in \mathscr{B}(\mathscr{F}^0_t) \cap C_S(\Omega)$ and, as we just noted, $E^{P_{0,x}}\{e^{\Phi}\} = 1$. Therefore, Φ, as defined, is in Y_2 and so $E^Q\{\Phi\} \leqq l$. Hence,

$$\begin{aligned} E^Q\{\psi\} &= E^Q\{\Phi\} + E^Q\{\bar{\psi}(\omega(0))\} \\ &\leqq l + E^Q\{\bar{\psi}(\omega(0))\}. \end{aligned} \tag{3.20}$$

But,

$$\bar{H}(t, Q) = \sup_{\psi \in \mathscr{B}(\mathscr{F}^0_t) \cap C(\Omega)} [E^Q\{\psi\} - E^Q\{\log E^{P_{0,\omega(0)}}\{e^{\psi}\}\}]$$

$$= \sup_{\psi \in \mathscr{B}(\mathscr{F}^0_t) \cap C(\Omega)} [E^Q\{\psi\} - E^Q\{\bar{\psi}(\omega(0))\}].$$

Consequently, (3.20) implies $\bar{H}(t, Q) \leqq l$ and the proof is complete.

4. Upper Bounds

In this section we obtain upper bounds on $\Gamma_{t,x}(A)$ as $t\to\infty$ for appropriate sets A. The main results are Lemma 4.4 and Theorem 4.6. We need some preparatory lemmas.

LEMMA 4.1. *Let $\phi:\Omega\to R_1$ be F_T^0 measurable and such that $E^{P_{0,x}}\{e^{\phi(\omega)}\}\leqq 1$ for all x. Then,*

$$E^{P_{0,x}}\left\{\exp\left\{\frac{1}{T}\int_0^t \phi(\theta_s\omega)\,ds\right\}\right\}\leqq 1 \quad \text{for all} \quad t. \tag{4.1}$$

Proof: Define

$$\psi_s(\omega)=\sum_{\substack{k:k\geqq 0\\ s+kT\leqq t}}\phi(\theta_{s+kT}\omega)$$

so that we can rewrite the left side of (4.1) as $E^{P_{0,x}}\{\exp\{(1/T)\int_0^T\psi_s(\omega)\,ds\}\}$. From Jensen's inequality,

$$E^{P_{0,x}}\left\{\exp\left\{\frac{1}{T}\int_0^T\psi_s(\omega)\,ds\right\}\right\}\leqq E^{P_{0,x}}\left\{\frac{1}{T}\int_0^T\exp\{\psi_s(\omega)\}\,ds\right\}. \tag{4.2}$$

Since $\phi(\theta_{s+kT}\omega)$ is $\mathcal{F}_{s+(k+1)T}^{s+kT}$ measurable, we have from the hypothesis on ϕ and successive conditioning that

$$E^{P_{0,x}}\{\exp\{\psi_s(\omega)\}\}=E^{P_{0,x}}\left\{\exp\left\{\sum_k\phi(\theta_{s+kT}\omega)\right\}\right\}\leqq 1. \tag{4.3}$$

Inequalities (4.3) and (4.2) give us (4.1).

COROLLARY. *Let $\Phi\in\mathcal{B}(\mathcal{F}_T^0)$ and be such that $E^{P_{0,x}}\{e^{\Phi(\omega)}\}\leqq 1$ for all x. Then, for all t,*

$$E^{\Gamma_{t,x}}\left\{\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega)Q(d\omega)\right\}\right\}\leqq\exp\left\{2\sup_{\omega\in\Omega}|\phi(\omega)|\right\}. \tag{4.4}$$

Proof: By definition of $\Gamma_{t,x}$ measure,

$$E^{\Gamma_{t,x}}\left\{\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega)Q(d\omega)\right\}\right\}=E^{P_{0,x}}\left\{\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega')R_{t,\omega}(d\omega')\right\}\right\},$$

and since

$$\left|\int_0^t\Phi(\theta_s,\omega)\,ds-t\int_\Omega\Phi(\omega')R_{t,\omega}(d\omega')\right|\leqq 2T\sup_{\omega\in\Omega}|\Phi(\omega)|, \tag{4.5}$$

we have from Lemma 4.1, for all t,

$$E^{\Gamma_{t,x}}\left\{\exp\left\{\frac{t}{T}\int_\Omega\Phi(\omega)Q(d\omega)\right\}\right\}\leqq\exp\left\{2\sup_{\omega\in\Omega}|\Phi(\omega)|\right\}$$

In what follows we shall use the notation

$$J(A)=\overline{\lim_{t\to\infty}}\frac{1}{t}\log\sup_{x\in X}\Gamma_{t,x}(A)$$

for $A\subset\mathcal{M}_S(\Omega)$.

LEMMA 4.2. *Let E_T be the set of $\Phi\in\mathcal{B}(\mathcal{F}_T^0)\cap C_S(\Omega)$ such that, for all x, $E^{P_{0,x}}\{e^{\Phi(\omega)}\}\leqq 1$. Then, for any set $A\subset\mathcal{M}_S(\Omega)$,*

$$J(A)\leqq-\sup_{\substack{l,A_1,A_2,\cdots,A_l\\ A\subset\bigcup_{j=1}^l A_j}}\inf_{1\leqq j\leqq l}\sup_{T>0}\sup_{\Phi\in E_T}\inf_{Q\in A_j}\frac{1}{T}\int_\Omega\Phi(\omega)Q(d\omega). \tag{4.6}$$

Proof: From (4.4) we see that, for any set $A\subset\mathcal{M}_S(\Omega)$ and any $\Phi\in E_T$,

$$\Gamma_{t,x}(A)\leqq\exp\left\{2\sup_{\omega\in\Omega}|\Phi(\omega)|\right\}\cdot\exp\left\{-\frac{t}{T}\inf_{Q\in A}\int_\Omega\Phi(\omega)Q(d\omega)\right\}$$

Thus,

$$J(A)\leqq-\inf_{Q\in A}\frac{1}{T}\int_\Omega\Phi(\omega)Q(d\omega). \tag{4.7}$$

Since (4.7) holds for all $\Phi\in E_T$ and all $T>0$,

$$J(A)\leqq-\sup_{T>0}\sup_{\Phi\in E_T}\inf_{Q\in A}\frac{1}{T}\int_\Omega\Phi(\omega)Q(d\omega).$$

From the definition of $J(A)$ we see that $J(A\cup B)\leqq\max(J(A),J(B))$. Thus, if $A_1,A_2,\cdots,A_l$ are sets in $\mathcal{M}_S(\Omega)$ with $A\subset\bigcup_{j=1}^l A_j$, then $J(A)\leqq\max_{1\leqq j\leqq l}\{J(A_j)\}$. Finally, taking the supremum over all such finite coverings of A we obtain (4.6).

LEMMA 4.3. *Let A be compact in $\mathcal{M}_S(\Omega)$. For any $\varepsilon>0$, there exists an l and open sets $G_1,G_2,\cdots,G_l$ in $\mathcal{M}_S(\Omega)$ such that $A\subset\bigcup_{j=1}^l G_j$ and*

$$\inf_{1\leqq j\leqq l}\sup_{T>0}\sup_{\Phi\in E_T}\inf_{Q\in G_j}\frac{1}{T}\int_\Omega\Phi(\omega)Q(d\omega)\geqq\inf_{Q\in A}H(Q)-\varepsilon. \tag{4.8}$$

In particular, for any compact set A in $\mathcal{M}_S(\Omega)$,

$$\sup_{\substack{l;A_1,A_2,\cdots,A_l\\ A\subset\bigcup_{j=1}^l A_j}}\inf_{1\leqq j\leqq l}\sup_{T>0}\sup_{\Phi\in E_T}\inf_{\phi\in A_j}\frac{1}{T}\int_\Omega\Phi(\omega)Q(d\omega)\geqq\inf_{Q\in A}H(Q). \tag{4.9}$$

Proof: Let $\inf_{Q\in A}H(Q)=\eta$. From Theorem 3.6 and Lemma 3.7 it follows that, given $Q\in A$ and $\varepsilon>0$, there exists T_Q and $\Phi_Q\in E_{T_Q}$ such that

$$\frac{1}{T_Q}\int_\Omega\Phi_Q(\omega)Q(d\omega)\geqq\eta-\tfrac{1}{2}\varepsilon. \tag{4.10}$$

Since $\Phi_Q \in C_S(\Omega)$, the integral on the left of (4.10) is a continuous functional of Q which implies that there exists a neighborhood G_Q of Q in $\mathcal{M}_S(\Omega)$ such that

$$\frac{1}{T_Q}\int_\Omega \Phi_Q(\omega)Q(d\omega) \geqq \eta - \varepsilon \tag{4.11}$$

for all Q in G_Q. The neighborhoods $\{G_Q\}$ form an open covering of the compact set A and therefore there exists $G_{Q_1}, G_{Q_2}, \cdots, G_{Q_l}$ such that $A \subset \bigcup_{j=1}^{l} G_{Q_j}$. This and (4.11) yield (4.8) and hence (4.9).

LEMMA 4.4. *Let A be closed in $\mathcal{M}_S(\Omega)$ and such that the family of one-dimensional marginals of Q as Q varies over A forms a tight family of measures on X. Then*

$$J(A) \leqq -\inf_{Q\in A} H(Q). \tag{4.12}$$

Proof: Let $\mathcal{M}(X)$ be the space of probability measures on X. Let $A_M \subset \mathcal{M}(X)$ be the family of one-dimensional marginals of Q as Q varies over A. Since, by hypothesis, A_M is tight, given $\varepsilon_n \to 0$ there exist compact sets $K_n \subset X$ such that $\mu(K_n) \geqq 1-\varepsilon_n$ for all n and all $\mu \in A_M$. We assumed that $x \to P_{0,x}$ is weakly continuous, and therefore, given $\eta_n \to 0$ there exists a sequence of compact sets $C_n \subset D_X[0, 1]$ such that $P_{0,x}(C_n) \geqq 1-\eta_n$ for all $x \in K_n$. Thus, if we let $\nu > 0$, denote the complement of C_n by $\tilde{C}_n$, and take $x \in K_n$,

$$\begin{aligned} E^{P_{0,x}}\{\exp\{\nu\chi_{K_n}(\omega(0))\chi_{\tilde{C}_n}(\omega)\}\} &= E^{P_{0,x}}\{\exp\{\nu\chi_{\tilde{C}_n}(\omega)\}\} \\ &= e^\nu P_{0,x}\{\tilde{C}_n\} + P_{0,x}\{C_n\} \\ &= 1 + (e^\nu - 1)P_{0,x}\{\tilde{C}_n\} \\ &\leqq 1 + (e^\nu - 1)\eta_n . \end{aligned}$$

On the other hand, if $x \notin K_n$, $E^{P_{0,x}}\{\exp\{\nu\chi_{K_n}(\omega(0))\chi_{\tilde{C}_n}(\omega)\}\} = 1$. Hence, for all x and all $\nu > 0$,

$$E^{P_{0,x}}\{\exp\{\nu\chi_{K_n}(\omega(0))\chi_{\tilde{C}_n}(\omega)\}\} \leqq 1 + \eta_n(e^\nu - 1). \tag{4.13}$$

Let $\phi(\omega) = \nu\chi_{K_n}(\omega(0))\chi_{\tilde{C}_n}(\omega) - \log[1+\eta_n(e^\nu-1)]$ so that from (4.13) we observe that $E^{P_{0,x}}\{e^{\phi(\omega)}\} \leqq 1$, and thus, using Lemma 4.1 (with $T = 1$), we obtain

$$E^{P_{0,x}}\left\{\exp\left\{\nu\int_0^t \chi_{K_n}(\omega(0))\chi_{\tilde{C}_n}(\theta_s\omega)\,ds\right\}\right\} \leqq \exp\{t\log[1+\eta_n(e^\nu-1)]\}. \tag{4.14}$$

From (4.5),

$$\begin{aligned} \int_0^t \chi_{K_n}(\omega(s))\chi_{\tilde{C}_n}(\theta_s\omega)\,ds &= \int_0^t \chi_{\tilde{C}_n}(\theta_s\omega)\,ds - \int_0^t \chi_{\tilde{K}_n}(\omega(s))\chi_{\tilde{C}_n}(\theta_s\omega)\,ds \\ &\geqq \int_0^t \chi_{\tilde{C}_n}(\theta_s\omega)\,ds - \int_0^t \chi_{\tilde{K}_n}(\omega(s))\,ds \\ &\geqq tR_{t,\omega}(\tilde{C}_n) - 1 - tR_{t,\omega}\{\omega' : \omega'(0) \in \tilde{K}_n\}, \end{aligned}$$

which implies, in view of (4.14), and the definition of $\Gamma_{t,x}$ measure,

$$E^{\Gamma_{t,x}}\left\{\exp\left\{\nu t\left[\dot{Q}(\tilde{C}_n)-\frac{1}{t}-Q\{\omega:\omega(0)\in\tilde{K}_n\}\right]\right\}\right\} \tag{4.15}$$
$$\leqq \exp\{t\log[1+\eta_n(e^{\nu}-1)]\}.$$

Since $Q\in A$, $Q\{\omega:\omega(0)\in\tilde{K}_n\}\leqq\varepsilon_n$ and, therefore, from (4.15), for each n,

$$\Gamma_{t,x}\left\{A\cap\left\{Q:Q(\tilde{C}_n)\geqq\frac{1}{t}+2\varepsilon_n\right\}\right\} \tag{4.16}$$
$$\leqq\exp\{t\log[1+\eta_n(e^{\nu}-1)]-\varepsilon_n\nu t\}.$$

Inequality (4.16) holds for any $\nu>0$, sequence $\varepsilon_n\to 0$, and sequence $\eta_n\to 0$. In particular, let us choose, for $\lambda>0$, $\nu=\lambda n^2$, $\varepsilon_n=1/n$, and $\eta_n=\exp\{-\lambda n^2\}$. Then,

$$t\log[1+\eta_n(e^{\nu}-1)]=t\log[1+\exp\{-\lambda n^2\}(\exp\{\lambda n^2\}-1)],$$

and $\varepsilon_n\nu t=(1/n)\cdot\lambda n^2 t=\lambda nt$. Thus, from (4.16), for each n,

$$\Gamma_{t,x}\left\{A\cap\left\{Q:Q(\tilde{C}_n)\geqq\frac{1}{t}+\frac{2}{n}\right\}\right\}$$
$$\leqq e^{t\log 2-\lambda nt}.$$

This implies

$$\Gamma_{t,x}\left\{A\cap\bigcup_{n=1}^{\infty}\left\{Q:Q(\tilde{C}_n)\geqq\frac{1}{t}+\frac{2}{n}\right\}\right\}\leqq e^{t\log 2}\sum_{n=1}^{\infty}e^{-\lambda nt}$$
$$=e^{t\log 2}\left(\frac{e^{-\lambda t}}{1-e^{-\lambda t}}\right).$$

If we let $A_t=\{Q: Q(\tilde{C}_n)\leqq 1/t+2/n \text{ for all } n\}$, then we have just shown

$$\Gamma_{t,x}\{A\cap\tilde{A}_t\}\leqq e^{t\log 2}\left(\frac{e^{-\lambda t}}{1-e^{-\lambda t}}\right). \tag{4.17}$$

We should note that the set A_t depends on λ since the sets C_n depend on the choice of η_n, and η_n was selected in terms of λ. From (4.17) we get

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\sup_{x\in X}\Gamma_{t,x}(A\cap\tilde{A}_t)\leqq\log 2-\lambda. \tag{4.18}$$

Let $A_\infty=\bigcap_{t>0}A_t=\{Q\in\mathcal{M}_S(\Omega): Q(\tilde{C}_n)\leqq 2/n, \text{ for all } n\}$. The restriction of Q to $\mathcal{F}_1^0$ constitutes a tight family of measures on $D_X[0,1]$. But, for stationary processes, tightness when restricted to some interval implies overall tightness and hence A_∞ is a compact set in $\mathcal{M}_S(\Omega)$. Since A is closed, $A\cap A_\infty$ is compact in $\mathcal{M}_S(\Omega)$. Thus, from Lemmas 4.2 and 4.3, $J(A\cap A_\infty)\leqq-\inf_{Q\in A\cap A_\infty}H(Q)\leqq$

$-\inf_{Q\in A} H(Q)$ which implies that for any $\varepsilon > 0$ there exists an open set $G_\varepsilon \supset A \cap A_\infty$ such that

$$J(G_\varepsilon) \leqq -\inf_{Q\in A} H(Q) + \varepsilon. \tag{4.19}$$

In Lemma 4.5 to follow, we show that, given any open set $G \subset \mathcal{M}_S(\Omega)$ such that $G \supset A \cap A_\infty$, there exists a t_0 such that $t \geqq t_0$ implies $A \cap A_t \subset G$. Using this, (4.18), and (4.19), we have, for any $\varepsilon > 0$, $\lambda > 0$, and $t \geqq t_0$,

$$\begin{aligned}\Gamma_{t,x}(A) &\leqq \Gamma_{t,x}(A \cap A_t) + \Gamma_{t,x}(A \cap \tilde{A}_t)\\ &\leqq \Gamma_{t,x}(G_\varepsilon) + \Gamma_{t,x}(A \cap \tilde{A}_t)\end{aligned}$$

which means

$$\begin{aligned}J(A) &\leqq \max\,(J(G_\varepsilon), \log 2 - \lambda)\\ &\leqq \max\left(-\inf_{Q\in A} H(Q) + \varepsilon, \log 2 - \lambda\right).\end{aligned} \tag{4.20}$$

Since the left side of (4.20) is independent of ε and λ we now let $\varepsilon \to 0$ and $\lambda \to \infty$ obtaining the desired (4.12).

LEMMA 4.5. *Let $A \subset \mathcal{M}_S(\Omega)$ be closed, $A_t = \{Q : Q(\tilde{C}_n) < 2/n + 1/t \text{ for all } n\}$, and $A_\infty = \bigcap_{t>0} A_t$. Let G be a neighborhood in $\mathcal{M}_S(\Omega)$ such that $G \supset A \cap A_\infty$. Then, there exists a t_0 such that $t \geqq t_0$ implies $A \cap A_t \subset G$.*

Proof: Let $\{t_k\} \to \infty$ and let $Q_k \in A \cap A_{t_k}$. Assume $Q_k \in \tilde{G}$ for every k. We want to show that this leads to a contradiction. Since $Q_k \in A_{t_k}$, we have $Q_k(\tilde{C}_n) < 2/n + 1/t_k$ for all n. Thus $\overline{\lim}_{k\to\infty} Q_k(\tilde{C}_n) \leqq 2/n$ for each n. This last implies that the family of stationary measures $\{Q_k\}$ is tight on $[0, 1]$. As noted earlier, this means the family $\{Q_k\}$ is tight in $\mathcal{M}_S(\Omega)$. Thus, if $\bar{Q}$ is any limit point, we must have $\bar{Q} \in A \cap A_\infty$ since $A \cap A_\infty$ is closed, and also $\bar{Q} \in \tilde{G}$ since $\tilde{G}$ is closed. Hence, we have contradicted the assumption, $A \cap A_\infty \subset G$.

We should note that Lemma 4.4 gives us the desired upper estimate for closed sets A in $\mathcal{M}_S(\Omega)$, i.e., $J(A) \leqq -\inf_{Q\in A} H(Q)$ but only under the additional hypothesis that the family A_M of one-dimensional marginals of Q as Q varies over A forms a tight family of measures in X. Now, of course, if X is itself compact, then this last hypothesis on A_M is automatically satisfied. However, if X is not compact, then we can replace the hypothesis of tightness on A_M by imposing additional hypotheses on the Markov process $P_{0,x}$. We still obtain the upper estimate cited above for closed sets but we must give up a bit on the uniformity in x. The estimates obtained above involving $J(A)$ were (by definition of $J(A)$) uniform for all $x \in X$. Now, under the additional hypotheses to be imposed on $P_{0,x}$ (when X is not compact) we will get the same upper bounds but they will hold uniformly only for x in compact subsets of X.

THEOREM 4.6. *Assume the Markov process $P_{0,x}$ satisfies hypotheses* (i)–(v). *Let A be a closed set in $\mathcal{M}_S(\Omega)$. Then, for any compact set K in X,*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \sup_{x\in K} \Gamma_{t,x}(A) \leqq -\inf_{Q\in A} H(Q). \tag{4.21}$$

Proof: By hypothesis there exists a sequence $\{u_n(x)\}$ of functions in the domain of L, the infinitesimal generator of the Markov process $P_{0,x}$, with properties (i)–(v) (see the introduction). Consider, then, the equation for $u(x, t)$:

$$u_t = Lu + V_n(x)u,$$
$$u(x, 0) = u_n(x).$$

By the Feynman-Kac formula the solution is given by

$$u(x, t) = E^{P_{0,x}}\left\{\exp\left\{\int_0^t V_n(\omega(s))\,ds\right\} u_n(\omega(t))\right\},$$

but, from hypothesis (iii), $-(Lu_n/u_n)(x) = V_n(x)$ so that the solution $u(x, t) \equiv u_n(x)$, and thus

$$E^{P_{s,x}}\left\{\exp\left\{\int_s^t V_n(\omega(s))\,ds\right\} u_n(\omega(t))\right\} = u_n(x). \tag{4.22}$$

From hypothesis (ii), for any compact set K there exists a constant C_K such that $\sup_{x\in K} \sup_n u_n(x) \leqq C_K$. Also, from hypothesis (i), there exists a constant c such that $u_n(x) \geqq c > 0$ for all x and n. Thus, from (4.22) we conclude that, for any compact set K,

$$\sup_{x\in K} \sup_n E^{P_{0,x}}\left\{\exp\left\{\int_0^t V_n(\omega(s))\,ds\right\}\right\} \leqq \frac{\sup_{x\in K} \sup_n u_n(x)}{\inf_{x\in X} u_n(x)} \leqq \frac{C_K}{c}.$$

Using hypothesis (iv) we get from Fatou's lemma that, for all t,

$$\sup_{x\in K} E^{P_{0,x}}\left\{\exp\left\{\int_0^t V(\omega(s))\,ds\right\}\right\} \leqq \frac{C_K}{c} = M. \tag{4.23}$$

By the definition of $\Gamma_{t,x}$ measure we have

$$E^{\Gamma_{t,x}}\{e^{tF(Q)}\} = E^{P_{0,x}}\{\exp\{tF(R_{t,\omega})\}\},$$

hence, in particular, if

$$F(R_{t,\omega}) = \int_\Omega V(\omega'(0)) R_{t,\omega}(d\omega') = \frac{1}{t}\int_0^t V(\omega(s))\,ds,$$

then $F(Q) = \int_X V(y)\mu(dy)$, where μ is the one-dimensional marginal of Q. Thus, (4.23) becomes

$$\sup_{x\in K} E^{\Gamma_{t,x}}\left\{\exp\left\{t\int_X V(y)\mu(dy)\right\}\right\}\leqq M. \tag{4.24}$$

Let $K_l=\{x\in X\colon\ V(x)\leqq l\}$. Using hypothesis (iii) and the constant C which appears there, we get from (4.24)

$$\sup_{x\in K} E^{\Gamma_{t,x}}\{\exp\{t[l\mu(\tilde{K}_l)-C]\}\}\leqq M.$$

Thus, for all $x\in K$,

$$\Gamma_{t,x}\{Q\colon \mu(\tilde{K}_l)\geqq\delta_l\}\leqq M\exp\{t(C-l\delta_l)\}. \tag{4.25}$$

In particular, take $\lambda>0$ and choose $l_n=Cn+\lambda n^2$, $\delta_{l_n}=1/n$. From (4.25) we get, for all $x\in K$,

$$\Gamma_{t,x}\left\{Q\colon \mu(\tilde{K}_{l_n})\geqq\frac{1}{n}\right\}\leqq Me^{-\lambda nt},$$

which implies that, for all $x\in K$,

$$\Gamma_{t,x}\left\{\bigcup_{n=1}^{\infty}\left\{Q\colon \mu(\tilde{K}_{l_n})\geqq\frac{1}{n}\right\}\right\}\leqq M\frac{e^{-\lambda t}}{1-e^{-\lambda t}}. \tag{4.26}$$

If we let

$$A_\lambda=\bigcap_{n=1}^{\infty}\left\{Q\colon \mu(\tilde{K}_{l_n})<\frac{1}{n}\right\},$$

then (4.26) says that

$$\Gamma_{t,x}(\tilde{A}_\lambda)\leqq M\frac{e^{-\lambda t}}{1-e^{-\lambda t}}. \tag{4.27}$$

Now, let A be a closed set in $\mathcal{M}_S(\Omega)$. From the definition of A_λ we see that the marginals of Q as Q varies over A_λ form a tight family. Since $A\cap A_\lambda$ is closed, we have from Lemma 4.4 that, for each $\lambda>0$,

$$J(A\cap A_\lambda)\leqq-\inf_{Q\in A\cap A_\lambda}H(Q)\leqq-\inf_{Q\in A}H(Q). \tag{4.28}$$

Since $A\subset(A\cap A_\lambda)\cup\tilde{A}_\lambda$, we conclude from (4.27) and (4.28) that, for every $\lambda>0$,

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\sup_{x\in K}\Gamma_{t,x}(A)\leqq\max\left(-\inf_{Q\in A}H(Q),-\lambda\right).$$

If we let $\lambda\to\infty$ in this last inequality, we get (4.21).

5. Lower Bounds

We need additional hypotheses on the Markov process $P_{0,x}$ in order to obtain the desired lower bounds. Let $p(t, x, dy)$ be the transition probability associated with $P_{0,x}$. We shall assume that there exists a density function for $p(1, x, dy)$ with respect to a reference measure α on X, i.e., there exists, for all $x \in X$, a function $p(1, x, y) > 0$ for almost all y (α measure) such that for all $x \in X$:

I: $p(1, x, dy) = p(1, x, y)\alpha(dy)$,

and

II: $p(1, x, \cdot)$ as a mapping from $X \to L_1(\alpha)$ is continuous.

We begin with three lemmas. The main result of this section is Theorem 5.5.

LEMMA 5.1. *Let $Q \in \mathcal{M}_S(\Omega)$ be such that $H(Q) < \infty$ and let μ be the marginal distribution of Q. Then, $\mu \ll \alpha$.*

Proof: Let $A \subset X$ be such that $\alpha(A) = 0$. We want to show that if $Q \in \mathcal{M}_S(\Omega)$ such that $H(Q) < \infty$, then $\mu(A) = 0$, where μ is the marginal of Q. From hypothesis I, for all $x \in X$,

$$P_{0,x}(\omega: \omega(1) \in A) = \int_A p(1, x, y)\alpha(dy) = 0. \tag{5.1}$$

Since $H(Q) < \infty$ and $H(1, Q) = H(Q)$, we know from (3.1) and Theorem 2.1 that $Q_{0,\omega}$ is absolutely continuous with respect to $P_{0,\omega(0)}$ for almost all ω (Q measure). Thus, from (5.1) we conclude that for almost all ω (Q measure) $Q_{0,\omega}\{\omega': \omega'(1) \in A\} = 0$. Hence, $E^Q\{Q_{0,\omega}\{\omega': \omega'(1) \in A\}\} = 0$, i.e., $\mu(A) = 0$.

LEMMA 5.2. *The reference measure α on X is absolutely continuous with respect to $p(1, x, \cdot)$ uniformly for x in compact subsets of X. In particular, if $H(Q) < \infty$, and μ is the marginal of Q, then $\mu \ll p(1, x, \cdot)$ uniformly for x in compact sets.*

Proof: The second statement of the lemma follows from the first and Lemma 5.1; thus we want to prove $\alpha \ll p(1, x, \cdot)$ uniformly for x in compact sets. Let $A \subset X$ such that $\alpha(A) = \delta > 0$. By hypothesis *I*, $p(1, x, A) \geqq \varepsilon(x, \delta)$. By hypothesis II, $p(1, x, \cdot)$ as a mapping from $X \to L_1(\alpha)$ is continuous and therefore $p(1, x', A) \geqq \varepsilon(x, \delta)/2$ for all x' in a neighborhood of x. The result now follows by the usual compactness argument.

LEMMA 5.3. *Let $Q \in \mathcal{M}_S(\Omega)$ be such that $H(Q) < \infty$ and let μ be the marginal of Q. Let $0 \leqq \phi_n(y) \leqq 1$ for $n = 1, 2, \cdots$, where $y \in X$. Assume*

$$\int_X \phi_n(y)\mu(dy) \geqq c_1 > 0 \quad \text{for} \quad n = 1, 2, \cdots.$$

Then, for any compact set $K \subset X$, there exists $c_2 > 0$ such that, for all $x \in K$, and all $n = 1, 2, 3, \cdots$,

$$\int_X \phi_n(y) p(1, x, dy) \geqq c_2.$$

Proof: Assume the contrary, i.e., for some sequence $\{x_n\} \in K$,

$$\lim_{n \to \infty} \int_X \phi_n(y) p(1, x_n, dy) = 0.$$

This implies $p(1, x_n, \{y \in X : \phi_n(y) \geqq \varepsilon\}) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. From Lemma 5.2 we conclude that $\mu\{y \in X: \phi_n(y) \geqq \varepsilon\} \to 0$ as $n \to \infty$ for every $\varepsilon > 0$. Since $\phi_n(y) \leqq 1$, the bounded convergence theorem then implies $\int_X \phi_n(y) \mu(dy) \to 0$ as $n \to \infty$ which is a contradiction.

THEOREM 5.4. *Let $Q \in \mathcal{M}_S(\Omega)$ be ergodic and such that $H(Q) < \infty$. Let K_1 be a compact set in X such that $\alpha(K_1) > 0$, and let K_2 be any compact set in X. For any neighborhood N of Q in $\mathcal{M}_S(\Omega)$, we have uniformly, for $x \in K_2$,*

$$\varliminf_{t \to \infty} \frac{1}{t} \log P_{0,x}\{\omega : R_{t,\omega} \in N, \omega(t) \in K_1\} \geqq -H(Q). \tag{5.2}$$

Proof: As before, let $Q_{0,\omega}$ be the r.c.p.d. of Q given $\mathcal{F}_0^{-\infty}$. Let

$$\psi(\omega, t) = \log \left| \frac{dQ_{0,\omega}}{dP_{0,\omega(0)}} \right|_{\mathcal{F}_t^0},$$

so that we have

$$\psi(\omega, t+s) = \psi(\omega, t) + \psi(\theta_t \omega, s)$$

and also, from (3.1) and Theorem 2.1,

$$E^Q\{\psi(\omega, t)\} = tH(Q).$$

Hence, by the ergodic theorem, for almost all ω (Q measure)

$$\lim_{t \to \infty} \frac{\psi(\omega, t)}{t} = H(Q). \tag{5.3}$$

Stated in another way, (5.3) says that

$$\lim_{t \to \infty} \frac{\psi(\omega', t)}{t} = H(Q) \tag{5.4}$$

almost everywhere ω'($Q_{0,\omega}$ measure) for almost all ω (Q measure). Again, from the ergodic theorem, for almost all ω (Q measure), and any neighborhood N of Q,

$$\lim_{t \to \infty} Q_{0,\omega}\{\omega' : R_{t,\omega'} \in N\} = 1. \tag{5.5}$$

With μ denoting the marginal of Q, let K_3 be a compact set in X such that $\mu(K_3) \geqq \frac{1}{2}$. Let $D_t = \{\omega\colon R_{t,\omega} \in N,\ \omega(t) \in K_3\}$ and consider $P_{0,\omega(0)}\{D_t\}$. Although $Q_{0,\omega}$ is absolutely continuous with respect to $P_{0,\omega(0)}$ the reverse is not necessarily true; however, for $\varepsilon > 0$,

$$\begin{aligned} P_{0,\omega(0)}\{D_t\} &\geqq \int_{D_t} \frac{dP_{0,\omega(0)}}{dQ_{0,\omega}}\bigg|_{\mathscr{F}_t^0} dQ_{0,\omega} \\ &= \int_{D_t} e^{-\psi(\omega',t)} Q_{0,\omega}(d\omega') \\ &\geqq e^{-t(H(Q)+\varepsilon)} Q_{0,\omega}\left\{D_t\left\{\omega'\colon \frac{1}{t}\psi(\omega',t) \leqq H(Q)+\varepsilon\right\}\right\}. \end{aligned}$$

If we let $F_t = D_t \cap \{\omega'\colon (1/t)\psi(\omega',t) \leqq H(Q)+\varepsilon\}$, and $\phi(t,x) = [P_{0,x}\{D_t\} \exp\{t(H(Q)+\varepsilon)\}] \wedge 1$, then we have just shown that

$$\phi(t,\omega(0)) \geqq Q_{0,\omega}\{F_t\}. \tag{5.6}$$

Taking expectations on both sides of (5.6) with respect to Q measure we get

$$\int_X \phi(t,x)\mu(dx) \geqq E^Q\{Q_{0,\omega}\{F_t\}\} = Q\{F_t\}. \tag{5.7}$$

Moreover, it follows from (5.4) and (5.5) that

$$\lim_{t\to\infty} Q\{F_t\} = \mu(K_3)$$

and so, from (5.7) and the choice of K_3, we obtain

$$\varliminf_{t\to\infty} \int_X \phi(t,x)\mu(dx) \geqq \tfrac{1}{2}. \tag{5.8}$$

Given the neighborhood N of Q in $\mathscr{M}_S(\Omega)$, we can pick another neighborhood N_1 such that $\bar{N}_1 \subset N$ and we pick $T < \infty$ and $\varepsilon > 0$ such that, for any $Q' \in N_1$ and any Q'' satisfying $\|Q'-Q''\|_{\mathscr{F}_T^0} < \varepsilon$, it follows that $Q'' \in N$. Here the norm is the variation norm. Now $\|R_{t,\omega} - R_{t-2,\theta_1\omega}\|_{\mathscr{F}_t^0} \leqq (T+2)/t$ so that by taking t large enough this norm can be made less than ε. Therefore, for the neighborhood N_1 just picked and for t large enough,

$$\begin{aligned} &P_{0,x}\{R_{t,\omega} \in N,\ \omega(t) \in K_1\} \\ &\qquad \geqq P_{0,x}\{R_{t-2,\theta_1\omega} \in N_1,\ \omega(t) \in K_1\} \\ &\qquad = \int_X \int_X p(1,x,dy) P_{0,y}\{R_{t-2,\omega} \in N_1,\ \omega(t-2) \in dz\} p(1,z,K_1) \\ &\qquad \geqq \left[\int_X p(1,x,dy) P_{0,y}\{R_{t-2,\omega} \in N_1,\ \omega(t-2) \in K_3\right] \cdot W, \end{aligned} \tag{5.9}$$

where $W = \inf_{z\in K_3} p(1, z, K_1)$. Using the notation introduced earlier in this proof we can rewrite (5.9) as

$$e^{t(H(Q)+\varepsilon)}P_{0,x}\{R_{t,\omega}\in N, \omega(t)\in K_1\}\geqq W\int_X \phi(t-2, y)p(1, x, dy). \tag{5.10}$$

From (5.8), (5.10), and Lemma 5.3 we have uniformly, for $x\in K_2$,

$$\varliminf_{t\to\infty} \frac{1}{t}\log P_{0,x}\{R_{t,\omega}\in N, \omega(t)\in K_1\}\geqq -H(Q). \tag{5.11}$$

In obtaining (5.11) we also used the fact that since $\alpha(K_1)>0$ and K_3 is compact we have $W>0$.

THEOREM 5.5. *Let $Q\in \mathcal{M}_S(\Omega)$ be such that $H(Q)<\infty$ and let N be any neighborhood of Q; then, uniformly for x in compact subsets of X,*

$$\varliminf_{t\to\infty} \frac{1}{t}\log \Gamma_{t,x}(N)\geqq -H(Q). \tag{5.12}$$

Thus, if G is open in $\mathcal{M}_S(\Omega)$, we have, uniformly for x in compact subsets of X,

$$\varliminf_{t\to\infty} \frac{1}{t}\log \Gamma_{t,x}(G)\geqq -\inf_{Q\in G} H(Q). \tag{5.13}$$

Proof: We can assume without loss of generality that $Q=\sum_{j=1}^{l}\pi_j Q_j$ where each Q_j is an ergodic member of $\mathcal{M}_S(\Omega)$ and the π_j are non-negative constants whose sum is 1. Since $H(Q)$ is linear by Theorem 3.5 we have $H(Q)=\sum_{j=1}^{l}\pi_j H(Q_j)$. Divide the interval $[0, t]$ into subintervals of length $\pi_j t$ and let t_j, $j=1, 2, \cdots, l$, be the right-hand endpoints of these subintervals. Let $K\subset X$ be compact. For each of the ergodic stationary processes Q_j, $j=1, 2, \cdots l$, let N_j be a neighborhood of Q_j such that if $Q_j'\in N_j$ for $j=1, 2, \cdots, l$ and $\|Q''-\sum_{j=1}^{l}\pi_j Q_j'\|_{\mathcal{F}_T^0}<\varepsilon$ for some choice of T and ε, then $Q''\in N$. Then,

$$\begin{aligned}\Gamma_{t,x}(N) &= P_{0,x}\{\omega: R_{t,\omega}\in N\}\\ &\geqq P_{0,x}\{\omega: R_{t_j-t_{j-1}}, \theta_{t_{j-1}}\omega\in N_j, j=1, 2, \cdots, l\}\\ &\geqq P_{0,x}\{\omega: R_{t_j-t_{j-1}}, \theta_{t_{j-1}}\omega\in N_j, \omega(t_j)\in K, j=1, 2, \cdots, l\}.\end{aligned}$$

Hence, if K_0 is the compact set in X for which we want to show uniformity in the estimate, we get from the preceding inequality

$$\begin{aligned}\inf_{x\in K_0}\Gamma_{t,x}(N) &\geqq \inf_{x\in K_0} P_{0,x}\{\omega: R_{t_1,\omega}\in N_1, \omega(t_1)\in K\}\\ &\cdot\prod_{j=1}^{l-1}\inf_{x\in K} P_{0,x}\{\omega: R_{t_j-t_{j-1}}, \omega\in N_j, \omega(t_j-t_{j-1})\in K\}.\end{aligned} \tag{5.14}$$

Now, if we apply Theorem 5.4 to (5.14) using the fact that the Q_i are ergodic and the previously noted linearity of $H(Q)$ we obtain (5.12).

6. The Contraction Principle

In this section we show the connection between the entropy function $H(Q)$ and the I-function introduced by the authors in their earlier papers. The relationship is clear from the following theorem. For the definition of the I-function see [2-III].

THEOREM 6.1. *For $Q \in \mathcal{M}_S(\Omega)$ let $q(Q)$ denote its one-dimensional marginal. Then,*

$$\inf_{Q:q(Q)=\mu} H(Q) = I(\mu). \tag{6.1}$$

Proof: Let $Q \in \mathcal{M}_S(\Omega)$ such that $H(Q) = l$ and $q(Q) = \mu$. Let $\Phi \in \mathcal{B}(\mathcal{F}_t^0)$ and be such that

$$E^{P_{0,x}}\{e^{\Phi}\} \leqq 1 \quad \text{for all} \quad x. \tag{6.2}$$

We earlier defined (in (3.12))

$$\bar{H}(t, Q) = \sup_{\Phi \in \mathcal{B}(\mathcal{F}_t^0)} [E^Q\{\Phi\} - E^Q\{\log E^{P_{0,\omega(0)}}\{e^{\Phi}\}\}],$$

and noted that $\bar{H}(t, Q) \leqq tH(Q)$. Thus, for the Φ selected,

$$E^Q\{\Phi\} \leqq lt. \tag{6.3}$$

In particular, let $\mathcal{U}$ be the space of functions $u \in \mathcal{B}(X)$ for each of which there exist constants c and C such that, for all x, $0 < c \leqq u \leqq C < \infty$ and, for $h > 0$ and $u \in \mathcal{U}$, take

$$\Phi(\omega) = \log \frac{u(\omega(h))}{(T_h u)(\omega(0))},$$

where T_h is the semigroup associated with the Markov process $P_{0,x}$. Now, this particular Φ is an element of $\mathcal{B}(\mathcal{F}_h^0)$ and direct calculation shows that Φ satisfies condition (6.2). Thus, from (6.3) we conclude

$$E^Q\left\{\log \frac{u(\omega(h))}{(T_h u)(\omega(0))}\right\} \leqq lh,$$

i.e.,

$$\begin{aligned} &E^Q\{\log u(\omega(h))\} - E^Q\{\log (T_h u)(\omega(0))\} \\ &= \int_X \log u(x)\mu(dx) - \int_X \log (T_h u)(x)\mu(dx) \\ &= \int_X \log \frac{u(x)}{(T_h u)(x)} \mu(dx) \leqq lh. \end{aligned} \tag{6.4}$$

Since (6.4) holds for every $u \in \mathcal{U}$ we get

$$\sup_{u\in\mathcal{U}} \int_X \log \frac{u(x)}{(T_h u)(x)} \mu(dx) \leqq lh. \tag{6.5}$$

In the notation of [2-III], (6.5) states that $(1/h)I_h(\mu) \leqq l$. But in [2-III] we proved $\lim_{h\to 0}(1/h)I_h(\mu) = I(\mu)$, and therefore from (6.5) we conclude that $I(\mu) \leqq l$. Since this was true for any Q such that $q(Q)=\mu$ we have

$$I(\mu) \leqq \inf_{Q:q(Q)=\mu} H(Q). \tag{6.6}$$

We want now to show the inequality in the other direction. Let μ be a probability measure on X and suppose $I(\mu) = l < \infty$. With $I_h(\mu)$ defined as above in (6.5), we know from [2-III] that, for all $h>0$, $I_h(\mu) \leqq hI(\mu) = hl$. In [2-III] we proved that there exists a bivariate distribution $\rho(dx, dy)$ on $X\times X$ such that both marginals of ρ are equal to μ, and if $\rho_x(dy)$ denotes the r.c.p.d. of the second component given the first, then for all $h>0$

$$\int_X h(p(h, x, \cdot); \rho_x(\cdot))\mu(dx) = I_h(\mu) \leqq hl. \tag{6.7}$$

In (6.7), $h(\alpha; \beta)$ is, of course, the entropy of β with respect to α, as introduced in Section 2, and $p(t, x, \cdot)$ is the transition function for the Markov process $P_{0,x}$.

Let n be a positive integer, let $h = 1/n$ and consider the grid $\{jh\}$, $j = 0, \pm 1, \pm 2, \cdots$. Define a Markov chain on this grid with stationary marginal μ and transition probability $\rho_x(dy)$. Given a sample of this Markov chain, let us interpolate between the grid points by random trajectories whose distribution is $P^h_{x,y}$, where $P^h_{x,\omega(h)}$ is the r.c.p.d. of $P_{0,x}$ given $\mathcal{F}^h_h$ and where, in the interpolation, x and y are the end points, i.e., values of the Markov chain at adjacent grid points. Let $Q^{(h)}$ be the measure induced by this interpolation procedure from the Markov chain on the grid. We note that $P^h_{x,y}$ is defined for each x for almost all y with respect to $p(h, x, \cdot)$ measure, but $\rho_x(\cdot)$ is absolutely continuous with respect to $p(h, x, \cdot)$ for almost all x (μ measure) and hence $P^h_{x,y}$ is defined almost everywhere with respect to $\rho(dx, dy)$ measure. Thus the interpolation procedure is well defined.

Let P_μ be the Markov process on $\mathcal{F}^0_\infty$ with initial distribution μ, i.e., $P_\mu = \int P_{0,x}\mu(dx)$. Consider $h_{\mathcal{F}^0_1}(P_\mu; Q^{(h)})$. Let $\mathcal{F}^{(h)}$ be the σ-field generated by $\omega(jh)$, $0 \leqq j \leqq 1/h = n$. Then, by Lemma 2.3,

$$h_{\mathcal{F}^0_1}(P_\mu; Q^{(h)}) = h_{\mathcal{F}^{(h)}}(P_\mu; Q^{(h)}) + E^{Q^{(h)}}\{h_{\mathcal{F}^0_1}(P_{\mu,\omega}; Q^{(h)}_\omega)\}, \tag{6.8}$$

where $P_{\mu,\omega}$ and $Q^{(h)}_\omega$ are, respectively, the r.c.p.d. of P_μ and $Q^{(h)}$ given $\mathcal{F}^{(h)}$. Notice that, by construction, $P_{\mu,\omega} = Q^{(h)}_\omega$ on $\mathcal{F}^0_1$ for almost all ω with respect to $Q^{(h)}$ measure, and therefore $E^{Q^{(h)}}\{h_{\mathcal{F}^0_1}(P_{\mu,\omega}; Q^{(h)}_\omega)\} = 0$. Hence, from (6.8) we get

$$h_{\mathcal{F}^0_1}(P_\mu; Q^{(h)}) = h_{\mathcal{F}^{(h)}}(P_\mu; Q^{(h)}). \tag{6.9}$$

In Lemma 6.2 to follow we show, using (6.7), that $h_{\mathscr{F}^{(h)}}(P_\mu; Q^{(h)}) \leqq nhl = l$. Hence, from (6.9), we conclude that $h_{\mathscr{F}_1^0}(P_\mu;\ Q^{(h)}) \leqq l$. This last inequality and Lemma 2.2 imply that the family of measures $\{Q^{(h)}\}$ is tight. Thus, let Q be any weak limit of $Q^{(h)}$ as $h \to 0$ $(n \to \infty)$ and we see that Q is stationary with marginal μ and, moreover, $H(Q) \leqq l$. Hence, $\inf_{Q: q(Q)=\mu} H(Q) \leqq l = I(\mu)$ and the proof is complete except for the lemma.

LEMMA 6.2. *Let $\pi(x, dy)$ be any Markov chain transition probability with state space X and let μ be a probability measure on X. Let $\rho(dx, dy)$ be a probability measure on $X \times X$ with both marginals equal to μ, conditional measure $\rho_x(dy)$, and let*

$$\tau = \int_X h(\pi(x, \cdot); \rho_x(\cdot))\mu(dx).$$

Let P_μ be the π-chain with initial distribution μ and let R_μ be the stationary ρ-chain with initial distribution μ. Let $\mathscr{F}_n$ be the σ-field generated by $x_0, x_1, \cdots, x_n$, the first n steps of the chain. Then

$$h_{\mathscr{F}_n}(P_\mu; R_\mu) = n\tau. \tag{6.10}$$

Proof: If $\tau = \infty$ the result is obvious, so assume $\tau < \infty$. This means $\rho_x(dy) \ll \pi(x, dy)$ for almost all x (μ measure). Let

$$\theta(x, y) = \frac{d\rho_x(\cdot)}{d\pi(x, \cdot)},$$

and we see that

$$\left.\frac{dR_\mu}{dP_\mu}\right|_{\mathscr{F}_n} = \theta(x_0, x_1)\theta(x_1, x_2) \cdots \theta(x_{n-1}, x_n).$$

Now,

$$h_{\mathscr{F}_n}(P_\mu; R_\mu) = E^{R_\mu}\left\{\log \left.\frac{dR_\mu}{dP_\mu}\right|_{\mathscr{F}_n}\right\}$$

$$= \sum_{i=1}^n E^{R_\mu}\{\log \theta(x_{i-1}, x_i)\}.$$

But R_μ is stationary, and hence

$$\sum_{i=1}^n E^{R_\mu}\{\log \theta(x_{i-1}, x_i)\} = nE^{R_\mu}\{\log \theta(x_0, x_1)\}$$

$$= n\tau.$$

We finish this section with two theorems which, although not used directly in this paper, are used in applying this theory to the polaron problem in [6]. We include them here since the contraction principle is used in the proof.

THEOREM 6.3. *Let $\mathscr{A}$ be a family of stationary processes satisfying*
(i) *the family $\mathscr{A}_M$ of one-dimensional marginals is tight,*
(ii) $H(Q) \leqq l < \infty$ *for all* $Q \in \mathscr{A}$.
Then, $\mathscr{A}$ is tight in $\mathscr{M}_S(\Omega)$.

Proof: Let $Q \in \mathscr{A} \subset \mathscr{M}_S(\Omega)$ with marginal μ and let $\Phi \in \mathscr{B}(\mathscr{F}_T^0)$. Then,

$$E^Q\{\Phi\} \leqq TH(Q) + \int \log E^{P_{0,x}}\{e^{\Phi}\}\mu(dx). \tag{6.11}$$

Let $K \subset X$ be compact and let $\varepsilon > 0$ be given. Since we have assumed everywhere that $x \to P_{0,x}$ is weakly continuous, there exists a set $C \subset \Omega$ which is compact in $D_X[0, 1]$ such that $P_{0,x}(\tilde{C}) < \varepsilon$ for all $x \in K$. As a special case of (6.11) choose, for $\lambda > 0$,

$$\Phi(\omega) = \lambda \int_0^{T-1} \chi_K(\omega(s))\chi_{\tilde{C}}(\theta_s\omega)\, ds.$$

Now, this Φ is an element of $\mathscr{B}(\mathscr{F}_T^0)$, and from (6.11)

$$E^Q\{\Phi\} \leqq Tl + \int \log E^{P_{0,x}}\left\{\exp\left\{\lambda \int_0^{T-1} \chi_K(\omega(s))\chi_{\tilde{C}}(\theta_s\omega)\, ds\right\}\right\}\mu(dx) \tag{6.12}$$

since $H(Q) \leqq l$. On the other hand, for this Φ,

$$\begin{aligned} E^Q\{\Phi\} &= \lambda \int_0^{T-1} E^Q\{\chi_K(\omega(s))\chi_{\tilde{C}}(\theta_s\omega)\}\, ds \\ &\geqq \lambda(T-1)[Q(\tilde{C}) - \mu(\tilde{K})]. \end{aligned} \tag{6.13}$$

From (6.12) and (6.13) we get

$$\begin{aligned} Q(\tilde{C}) &\leqq \mu(\tilde{K}) + \frac{Tl}{\lambda(T-1)} \\ &\quad + \frac{1}{\lambda(T-1)} \int \log E^{P_{0,x}}\left\{\exp\left\{\lambda \int_0^{T-1} \chi_K(\omega(s))\chi_{\tilde{C}}(\theta_s\omega)\, ds\right\}\right\}\mu(dx). \end{aligned} \tag{6.14}$$

In particular, take $T = 2$ and using the same estimates as in (4.13) we get

$$Q(\tilde{C}) \leqq \mu(\tilde{K}) + \frac{2l}{\lambda} + \frac{1}{\lambda}\log[1 + \varepsilon(e^{\lambda} - 1)]. \tag{6.15}$$

Since the one-dimensional marginals $\mathscr{A}_M$ form a tight family, one can clearly choose K, ε, λ to make the right side of (6.15) as small as one pleases for all $Q \in \mathscr{A}$ simultaneously, showing $\mathscr{A}$ to be tight in $\mathscr{M}_S(\Omega)$.

THEOREM 6.4. *Assume the Markov process $P_{0,x}$ satisfies hypotheses* (i)–(v) (*in the introduction*). *Let $\mathscr{A}$ be a family in $\mathscr{M}_S(\Omega)$ such that $H(Q) \leqq l < \infty$ for all $Q \in \mathscr{A}$. Then, $\mathscr{A}$ is tight in $\mathscr{M}_S(\Omega)$.*

Proof: If $\mu \in \mathscr{A}_M$, the family of one-dimensional marginals in $\mathscr{A}$, then, by the contraction principle,

$$I(\mu)=\inf_{Q:q(Q)=\mu} H(Q).$$

But since $\mu \in \mathscr{A}_M$, there is a Q with $H(Q) \leqq l$ and marginal μ so that $I(\mu) \leqq l$. This is true for all $\mu \in \mathscr{A}_M$. We proved in [2-III] that if, under hypotheses (i)–(v), for all $\mu \in \mathscr{A}_M$, $I(\mu) \leqq l$, then $\mathscr{A}_M$ is tight. The result now follows from Theorem 6.3.

7. Special Properties of the Ornstein-Uhlenbeck Process

In the case when X is not compact, to get appropriate upper bounds required imposing hypotheses (i)–(v) on the Markov process $P_{0,x}$, as we saw in Section 4. We show now that the O–U process in d-dimensions satisfies (i)–(v). Also in this section we prove (Theorem 7.1) an entropy identity for the O–U process which we use in the polaron problem in [6].

For the O–U process in d-dimensions, the infinitesimal generator is

$$L=\tfrac{1}{2}\Delta - kx \cdot \nabla, \tag{7.1}$$

where $k>0$ is an arbitrary positive constant.

To show (i)–(v) are satisfied, define $\theta(x)=x$ for $|x| \leqq 1$, bounded on R_1, θ' and θ'' uniformly bounded on R_1, $\theta' \geqq 0$, and θ chosen to be odd. Define

$$u_n(x)=\prod_{i=1}^{d} \cosh(n\theta(x_i/n)).$$

Clearly, $u_n(x) \geqq 1$ for all x and all n so that hypothesis (i) is satisfied. As $n \to \infty$, $u_n(x) \to \prod_{i=1}^{d} \cosh x_i$ uniformly on compact sets which means hypothesis (ii) is satisfied.

Using (7.1), we find, in $d=1$,

$$\begin{aligned} -\frac{Lu_n(x)}{u_n} &= V_n(x) \\ &= -\left\{\frac{1}{2}\left[\theta'\left(\frac{x}{n}\right)\right]^2 + \frac{1}{2n}\theta''\left(\frac{x}{n}\right)\tanh\left(n\theta\left(\frac{x}{n}\right)\right)\right. \\ &\qquad \left. - kx\theta'\left(\frac{x}{n}\right)\tanh\left(n\theta\left(\frac{x}{n}\right)\right)\right\} \end{aligned}$$

and hence, showing that there is a constant c such that $V_n(x) \geqq -c$ for all n and all x is equivalent to showing that the expression inside the curly braces is bounded. But this follows from the fact that θ' and θ'' are bounded and $x\tanh(n\theta(x/n)) \geqq 0$ for all x. A similar argument holds in d-dimensions. So hypothesis (iii) holds.

If we let $f(x)=\cosh(n\theta(x/n))$ and $\psi_n(x)=(\frac{1}{2}f''-kxf')/f$, then in d-dimensions, $V_n(x)=-\sum_{i=1}^d \psi_n(x_i)$. As $n\to\infty$,

$$V_n(x)\to-\sum_{i=1}^{d}(\tfrac{1}{2}-kx_i\tanh x_i).$$

Thus, if we define

$$V(x)=\sum_{i=1}^{d}(kx_i\tanh x_i-\tfrac{1}{2}),$$

we see that $V_n(x)\to V(x)$ as $n\to\infty$ for all x. Hence, hypothesis (iv) is satisfied. Hypothesis (v) is satisfied because there is a constant c_1 such that, for all x,

$$V(x)\geqq\sum_{i=1}^{d}k|x_i|-c_1.$$

THEOREM 7.1. *Let $H_k(Q)$ be the entropy of a d-dimensional stationary process Q with respect to the d-dimensional* O–U *process with constant $k>0$. Let $H_0(Q)$ be the entropy of Q with respect to d-dimensional Brownian motion. Then,*

$$H_k(Q)=H_0(Q)+\tfrac{1}{2}k^2\int\|x\|^2\mu(dx)-\tfrac{1}{2}dk, \tag{7.2}$$

where μ is the marginal of Q.

Proof: By definition,

$$H_k(Q)=E^Q\{h_{\mathscr{F}_1^0}(P^k_{0,\omega(0)};Q_{0,\omega})\},$$

where $P^k_{0,x}$ is the O–U process with parameter k. Brownian motion is $P^0_{0,x}$ and we now use the fact that $P^k_{0,x}$ and $P^0_{0,x}$ are mutually absolutely continuous. Indeed, for any pair $\alpha\approx\beta$,

$$\begin{aligned}h_\Sigma(\alpha;\lambda)&=E^\lambda\left\{\log\frac{d\lambda}{d\alpha}\Big|_\Sigma\right\}\\&=E^\lambda\left\{\log\frac{d\lambda}{d\beta}\Big|_\Sigma+\log\frac{d\beta}{d\alpha}\Big|_\Sigma\right\}\\&=h_\Sigma(\beta,\lambda)+E^\lambda\left\{\log\frac{d\beta}{d\alpha}\Big|_\Sigma\right\}.\end{aligned}$$

Apply this with $\lambda=Q_{0,\omega}$, $\alpha=P^k_{0,\omega(0)}$, $\beta=P^0_{0,\omega(0)}$ and we get $(\Sigma=\mathscr{F}_1^0)$

$$H_k(Q)=H_0(Q)+E^Q\left\{E^{Q_{0,\omega}}\left\{\log\frac{dP^0_{0,\omega(0)}}{dP^k_{0,\omega(0)}}\Big|_{\mathscr{F}_1^0}\right\}\right\}. \tag{7.3}$$

From the Cameron-Martin formula,

$$\log\frac{dP^0_{0,\omega(0)}}{dP^k_{0,\omega(0)}}\Big|_{\mathscr{F}_1^0}=\tfrac{1}{2}k\|\omega(1)\|^2+\tfrac{1}{2}k^2\int_0^1\|\omega(s)\|^2\,ds-\tfrac{1}{2}kd-\tfrac{1}{2}k\|\omega(0)\|^2$$

and since Q is stationary we have

$$E^{Q}\left\{E^{Q_{0,\omega}}\left\{\log \frac{dP^{0}_{0,\omega(0)}}{dP^{k}_{0,\omega(0)}}\bigg|_{\mathcal{F}^{0}_{1}}\right\}\right\} = E^{Q}\left\{\log \frac{dP^{0}_{0,\omega(0)}}{dP^{k}_{0,\omega(0)}}\bigg|_{\mathcal{F}^{0}_{1}}\right\}$$

$$= E^{Q}\left\{\tfrac{1}{2}k^{2}\int_{0}^{1}\|\omega(s)\|^{2}\,ds\right\} - \tfrac{1}{2}kd$$

$$= \tfrac{1}{2}k^{2}\int \|x\|^{2}\mu(dx) - \tfrac{1}{2}kd$$

which gives us (7.2).

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Bibliography

[1] Donsker, M. D., and Varadhan, S. R. S., *Asymptotic Evaluation of Certain Wiener Integrals for Large Time*, Functional Integration and its Applications, Proceedings of the International Conference held at the Cumberland Lodge, Windsor Great Park, London, in April 1974. A. M. Arthurs, Ed., Clarendon, Oxford, 1975.

[2] Donsker, M. D., and Varadhan, S. R. S., *Asymptotic evaluation of certain Markov process expectations for large time, I*, Comm. Pure Appl. Math. 28, 1975, pp. 279–301; III, Comm. Pure Appl. Math. 29, 1976, pp. 389–461.

[3] Varadhan, S. R. S., *Asymptotic probabilities and differential equations*, Comm. Pure Appl. Math. 19, 1966, pp. 261–286.

[4] Feynman, R. P., *Statistical Mechanics, a Set of Lectures*, Benjamin, New York, 1972.

[5] Pekar, G. S., *Research on Electron Theory of Crystals*, Translation Series No. AEC-tr. 5575, Physics AEC, Division of Technical Information, Washington, DC, 1963.

[6] Donsker, M. D., and Varadhan, S. R. S., *Asymptotics for the polaron*, to appear.

[7] Oxtoby, J. C., *Ergodic sets*, Bull. Amer. Math. Soc. 58, 1952, pp. 116–136.

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