CAYLEY POLYNOMIALS

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Dedicated to Bruce Allison on the occasion of his 60th birthday

ABSTRACT. We consider a polynomial version of the Cayley numbers. Namely, we define the ring of Cayley polynomials in terms of generators and relations in the category of alternative algebras. The ring turns out to be an octonion algebra over an ordinary polynomial ring. Also, a localization (a ring of quotients) of the ring of Cayley polynomials gives another description of an octonion torus. Finally, we find a subalgebra of a prime nondegenerate alternative algebra so that the subalgebra is an octonion algebra over the center.

INTRODUCTION

Nonassociative analogues of Laurent polynomials naturally appeared in the classification of extended affine Lie algebras and Lie tori. These Lie algebras are a natural generalization of affine Kac-Moody Lie algebras (see [AABGP], [N], [Y2]). As the affine Kac-Moody Lie algebras are coordinatized by the ring of Laurent polynomials in one variable, extended affine Lie algebras or Lie tori are coordinatized by nonassociative analogues of Laurent polynomials in several variables. Those Lie algebras have types classified by finite irreducible root systems, and the coordinate algebras depend on the types. In particular, such Lie algebras of type A_2 are coordinatized by alternative algebras, and an alternative analogue of Laurent polynomials (which is not associative) was found in [BGKN]. The coordinate algebra is called an *octonion torus*. It turns out that the coordinate algebras of extended affine Lie algebras or Lie tori of type A_2, C_3 or F_4, which are not associative, are exactly octonion tori (see also [AG], [BY], [Y1], [Y3]).

An octonion torus (an octonion n-torus) is defined by a Cayley-Dickson process over a ring of Laurent polynomials. More precisely, it is obtained by the Cayley-Dickson process three times over $F[z_1^{\pm 1}, \ldots , z_n^{\pm 1}]$ with $n \geq 3$, where F is a field of characteristic $\neq 2$, taking

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the structure constants z_1, z_2 and z_3, i.e., in the standard notation for the Cayley-Dickson process (see §1),

$$(F[z_1^{\pm 1}, \ldots, z_n^{\pm 1}], z_1, z_2, z_3).$$

To study the algebra structure, it is enough to consider the case $n = 3$, the octonion 3-torus $(F[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}], z_1, z_2, z_3)$, which is also called the *Cayley torus*. Our goal is to find a simple presentation of the Cayley torus, or essentially, a simple presentation of its subalgebra

$$D = (F[z_1, z_2, z_3], z_1, z_2, z_3).$$

The algebra D also appears as a certain subalgebra of a free alternative algebra generated by more than three elements, which was discovered by Dorofeev (see Remark 3.2).

We define the algebra $F_C[t_1, t_2, t_3]$ over F by the following relations

$$t_2 t_1 = -t_1 t_2, \quad t_3 t_1 = -t_1 t_3, \quad t_3 t_2 = -t_2 t_3 \quad \text{and} \quad (t_1 t_2) t_3 = -t_1 (t_2 t_3)$$

in the category of alternative algebras, and call it the ring of *Cayley polynomials* or a *universal octonion algebra* (since it covers all the octonion algebras over F). Then we show that $F_C[t_1, t_2, t_3]$ is isomorphic to D. In particular, $F_C[t_1, t_2, t_3]$ is an octonion algebra over the center $F[t_1^2, t_2^2, t_3^2]$ (ordinary commutative associative polynomials in three variables t_1^2, t_2^2 and t_3^2). Also, the Cayley torus can be viewed as the ring of quotients of $F_C[t_1, t_2, t_3]$ by the monomials of the center $F[t_1^2, t_2^2, t_3^2]$. As corollaries, we obtain a simple presentation of the Cayley torus and also a presentation of any octonion algebra over F. Moreover, the base field F can be generalized to a ring Φ of scalars containing $1/2$, and so we will set up the notions above over Φ.

Finally, we will discuss about *Cayley-Dickson rings*. Such a ring embeds into an octonion algebra over a field. However, the ring itself is not necessarily an octonion algebra in general. We note that our ring $F_C[t_1, t_2, t_3]$ or an octonion torus is a Cayley-Dickson ring and also an octonion algebra. We will see in Proposition 4.2 that there exists a subring B of a Cayley-Dickson ring R (or of a prime nondegenerate alternative algebra R over Φ) so that B is an octonion algebra over the center of R, and the central closure $\overline{B}$ coincides with the central closure $\overline{R}$.

Throughout the paper let Φ be a unital commutative associative ring containing $1/2$. Also, all algebras are assumed to be unital.

§1 Cayley-Dickson Process

We recall the Cayley-Dickson Process over a ring Φ of scalars (see [M] for detail). For an algebra B over Φ, we assume that B is *faithful*, i.e., for all $\alpha \in \Phi$, $\alpha 1 = 0 \implies \alpha = 0$. Let $*$ be a *scalar involution* of B over Φ, i.e., an anti-automorphism of period 2 with $bb^* \in \Phi 1$. Let $\mu \in \Phi$ be a *cancellable scalar*, i.e., $\mu b = 0$ for some $b \in B \implies b = 0$. The Cayley-Dickson algebra (or process) over Φ with structure constant μ constructed from $B = (B, *)$ is a new

algebra $B \oplus B$ with product $(a, b)(c, d) = (ac + \mu db^*, a^*d + cb)$ for $a, b, c, d \in B$. Letting $v = (0, 1)$ we can write this algebra as $B + vB$ with multiplication

(1.1) $$(a + vb)(c + vd) = (ac + \mu db^*) + v(a^*d + cb).$$

We call v the *basic generator*. Note that $v^2 = \mu$. The algebra $B + vB$ also has the new involution $*$, defined by $(a + vb)^* = a^* - vb$, which is scalar. So one can continue the process, and we write, for example, (B, μ, ν) instead of $((B, \mu), \nu))$. Note that B is faithful $\Rightarrow (B, \mu)$ is faithful, and ν is cancellable for $B \Leftrightarrow \nu$ is cancellable for (B, μ).

Let μ_1, μ_2, μ_3 be any cancellable scalars of Φ. The Cayley-Dickson process twice starting from Φ with trivial involution, say (Φ, μ_1, μ_2), is called a *quaternion algebra*, which is a 4-dimensional free Φ-module and an associative but not commutative algebra, and three times, say $(\Phi, \mu_1, \mu_2, \mu_3)$, is called an *octonion algebra*, which is an 8-dimensional free Φ-module and an alternative but not associative algebra. Note that quaternion algebras and octonion algebras are central, and if Φ is a field, they are simple.

Lemma 1.2. *Let v_1, v_2 and v_3 be the basic generators in each step of an octonion algebra $(\Phi, \mu_1, \mu_2, \mu_3)$ so that $v_1^2 = \mu_1$, $v_2^2 = \mu_2$ and $v_3^2 = \mu_3$. Then $v_2 v_1 = -v_1 v_2$, $v_3 v_1 = -v_1 v_3$, $v_3 v_2 = -v_2 v_3$ and $(v_1 v_2) v_3 = -v_1 (v_2 v_3)$.*

Proof. One can easily check these identities from (1.1). $\square$

We will consider quaternion algebras and octonion algebras over various rings of scalars, not necessarily Φ in the following sections.

§2 HAMILTON POLYNOMIALS

The associative algebra over Φ with generators t_1 and t_2 and the relation $t_1 t_2 = -t_2 t_1$ is called the ring of *Hamilton polynomials* or a *universal quaternion algebra*, denoted $\Phi_H[t_1, t_2]$. Note that the center of $\Phi_H[t_1, t_2]$ is equal to $\Phi[t_1^2, t_2^2]$ (the ordinary commutative associative polynomials over Φ in two variables t_1^2 and t_2^2), and $\Phi_H[t_1, t_2]$ is a quaternion algebra over $\Phi[t_1^2, t_2^2]$, i.e., $(\Phi[t_1^2, t_2^2], t_1^2, t_2^2)$ using the notation in §1. (Consider the base ring as $\Phi[t_1^2, t_2^2]$ instead of Φ. Then t_1^2 and t_2^2 are cancellabe elements of $\Phi[t_1^2, t_2^2]$.) Also, it is clear that any quaternion algebra over Φ is a homomorphic image of $\Phi_H[t_1, t_2]$.

Note that the multiplicative subset $S := \{t_1^r t_2^s\}_{r,s \in 2\mathbb{N}}$ of the center $\Phi[t_1^2, t_2^2]$ does not contain zero divisors of $\Phi_H[t_1, t_2]$, and so one can construct the ring of quotients $S^{-1}\Phi_H[t_1, t_2]$ (see e.g. [SSSZ, p.185]). Then $S^{-1}\Phi_H[t_1, t_2]$ is still a quaternion algebra, i.e.,

$$S^{-1}\Phi_H[t_1, t_2] = (\Phi[t_1^{\pm 2}, t_2^{\pm 2}], t_1^2, t_2^2).$$

Give the degrees $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ for t_1, t_2, t_1^{-1} and t_2^{-1}, respectively. (Note that t_1 and t_2 are invertible in $S^{-1}\Phi_H[t_1, t_2]$ and $t_1^{-1} = t_1^{-2}t_1$ and $t_2^{-1} = t_2^{-2}t_2$.) Then $S^{-1}\Phi_H[t_1, t_2]$ becomes a $\mathbb{Z}^2$-graded algebra, called the *quaternion 2-torus* or the *Hamilton torus*. Note that $\Phi_H[t_1, t_2]$ embeds into $S^{-1}\Phi_H[t_1, t_2]$.

If A is an associative algebra over Φ generated by invertible elements a and b, and they satisfy $ab = -ba$, then A is a homomorphic image of $S^{-1}\Phi_H[t_1, t_2]$ via $t_1 \mapsto a$ and $t_2 \mapsto b$,

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using the universal property of the ring of quotients. Also, the associative algebra L over Φ with generators $t_1^{\pm 1}$ and $t_2^{\pm 1}$ and relations $t_1 t_1^{-1} = t_2 t_2^{-1} = 1$ has a natural $\mathbb{Z}^2$-grading as above. So there is a natural graded homomorphism from $S^{-1}\Phi_H[t_1, t_2]$ onto $L/(t_1 t_2 + t_2 t_1)$. On the other hand, since $S^{-1}\Phi_H[t_1, t_2]$ has the relations defining $L/(t_1 t_2 + t_2 t_1)$, there is a natural graded homomorphism from $L/(t_1 t_2 + t_2 t_1)$ onto $S^{-1}\Phi_H[t_1, t_2]$. Hence they are graded isomorphisms. Thus the Hamilton torus $S^{-1}\Phi_H[t_1, t_2]$ has a presentation in the category of associative algebra; generators $t_1^{\pm 1}$ and $t_2^{\pm 1}$ with relations $t_1^{-1} t_1 = t_2 t_2^{-1} = 1$ and $t_1 t_2 = -t_2 t_1$. Because of the presentation, it is reasonable to write

$$S^{-1}\Phi_H[t_1, t_2] = \Phi_H[t_1^{\pm 1}, t_2^{\pm 1}].$$

Also, a *quaternion n-torus* $(n \geq 2)$ is defined as

$$\Phi_H[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] := \Phi_H[t_1^{\pm 1}, t_2^{\pm 1}] \otimes_\Phi \Phi[t_3^{\pm 1}, \ldots, t_n^{\pm 1}],$$

where $\Phi[t_3^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the ordinary Laurent polynomial algebra over Φ in $(n-2)$-variables.

The following proposition is well-known in ring theory. One can prove it in the same way as in Theorem 3.6 for octonion algebras.

Proposition 2.1. *Any quaternion algebra over Φ, say (Φ, μ_1, μ_2) for cancellable scalars μ_1, μ_2 of Φ, is isomorphic to $\Phi_H[t_1, t_2]/(t_1^2 - \mu_1, t_2^2 - \mu_2)$. Hence (Φ, μ_1, μ_2) has a presentation in the category of associative algebras; generators t_1 and t_2 with relations $t_1^2 = \mu_1$, $t_2^2 = \mu_2$ and $t_1 t_2 = -t_2 t_1$.*

In particular, if Φ is a field, A is an associative algebra over Φ generated by a_1 and a_2, and they satisfy $a_1 a_2 = -a_2 a_1$, $a_1^2 = \mu_1$ and $a_2^2 = \mu_2$, then A is isomorphic to (Φ, μ_1, μ_2).

§3 Cayley Polynomials

We will use the commutator $[a, b] = ab - ba$ and the associator $(a, b, c) = (ab)c - a(bc)$ in the subsequent claims. Alternative algebras are defined by two idendtiites: $(a, a, b) = 0 = (b, a, a)$. We have the alternative law $(a, b, c) = -(b, a, c) = (b, c, a)$, etc., and the flexible law $(a, b, a) = 0$, and so we can omit the parentheses for $(ab)a = a(ba)$. We will use the middle Moufang identity $(ab)(ca) = a(bc)a$ in Proposition 3.1. Recall that the center of an alternative algebra A is defined as $\{z \in A \mid [z, a] = (z, a, b) = 0 \text{ for all } a, b \in A\}$.

The alternative algebra over Φ with generators t_1, t_2, t_3 and the *Cayley relations*

(C) $t_2 t_1 = -t_1 t_2, \quad t_3 t_1 = -t_1 t_3, \quad t_3 t_2 = -t_2 t_3 \quad \text{and} \quad (t_1 t_2) t_3 = -t_1 (t_2 t_3)$

is called the ring of *Cayley polynomials* or a *universal octonion algebra*, denoted $\Phi_C[t_1, t_2, t_3]$. Note that any octonion algebra over Φ is a homomorphic image of $\Phi_C[t_1, t_2, t_3]$ by Lemma 1.2.

Let Z be the center of $\Phi_C[t_1, t_2, t_3]$. Our main goal is to show that $\Phi_C[t_1, t_2, t_3]$ is an octonion algebra over Z.

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Claim 1. $\Phi_C[t_1, t_2, t_3]$ *has the identities* $(t_i t_j)t_k = -t_i(t_j t_k)$ *(anti-associativity) and* $(t_i t_j)t_k = -t_k(t_i t_j)$ *(anti-commutativity) for any distinct* $i, j, k \in \{1, 2, 3\}$.

Proof. By the anti-commutativity in (C), it suffices to show three identities for the anti-commutativity $(t_i t_j)t_k = -t_k(t_i t_j)$, say $k = 1, 2, 3$. However, we need to prove five identities for the anti-associativity $(t_i t_j)t_k = -t_i(t_j t_k)$.

By the alternative law, we have $(t_1, t_2, t_3) = -(t_2, t_1, t_3)$. So

$$
\begin{aligned}
(t_2 t_1)t_3 + t_2(t_1 t_3) &= (t_2 t_1)t_3 - (t_2, t_1, t_3) + (t_2 t_1)t_3 \\
&= (t_1, t_2, t_3) - 2(t_1 t_2)t_3 \quad \text{(since } t_2 t_1 = -t_1 t_2) \\
&= 0 \quad \text{(since } (t_1 t_2)t_3 = -t_1(t_2 t_3)).
\end{aligned}
$$

Hence,

(a1) $$(t_2 t_1)t_3 = -t_2(t_1 t_3) \quad \text{(anti-associativity)}.$$

Also, from $(t_1, t_2, t_3) = -(t_1, t_3, t_2)$, we have $(t_1 t_2)t_3 - t_1(t_2 t_3) = -(t_1 t_3)t_2 + t_1(t_3 t_2)$. Since $t_3 t_2 = -t_2 t_3$, we get

(1) $$(t_1 t_2)t_3 = -(t_1 t_3)t_2.$$

For the rest of argument, we will use the identities in (C) without mentioning. By (1), we have $(t_1 t_3)t_2 + t_1(t_3 t_2) = -(t_1 t_2)t_3 + t_1(t_3 t_2) = 0$. Hence,

(a2) $$(t_1 t_3)t_2 = -t_1(t_3 t_2) \quad \text{(anti-associativity)}.$$

By (a1) and (1), we get

(2) $$t_2(t_1 t_3) = -(t_1 t_3)t_2 \quad \text{(anti-commutativity for } k = 2).$$

Also, $(t_3 t_1)t_2 + t_3(t_1 t_2) = -(t_1, t_2, t_3) + 2(t_3 t_1)t_2 = -(t_1, t_2, t_3) - 2(t_1 t_3)t_2 = -(t_1, t_2, t_3) + 2(t_1 t_2)t_3 = 0$ by (1). Hence,

(a3) $$(t_3 t_1)t_2 = -t_3(t_1 t_2) \quad \text{(anti-associativity)}.$$

By (1) and (a3), we get

(3) $$t_3(t_1 t_2) = -(t_1 t_2)t_3 \quad \text{(anti-commutativity for } k = 3).$$

Now, $(t_3 t_2)t_1 + t_3(t_2 t_1) = (t_3, t_2, t_1) + 2t_3(t_2 t_1) = -(t_1, t_2, t_3) - 2t_3(t_1 t_2) = -(t_1, t_2, t_3) + 2(t_1 t_2)t_3 = 0$ by (3). Hence,

(a4) $$(t_3 t_2)t_1 = -t_3(t_2 t_1) \quad \text{(anti-associativity)}.$$

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By (a4), the main involution on the free alternative algebra $\Phi\langle t_1, t_2, t_3\rangle$, i.e., the involution determined by $t_1 \mapsto t_1$, $t_2 \mapsto t_2$ and $t_3 \mapsto t_3$, preserves the relations (C). So applying for the induced involution to (a2), we get

(a5) $$(t_2 t_3)t_1 = -t_2(t_3 t_1) \quad \text{(anti-associativity)}.$$

Finally,

$$\begin{aligned}
(t_3 t_2)t_1 + t_1(t_3 t_2) &= -t_3(t_2 t_1) + t_1(t_3 t_2) \quad \text{by (a4)} \\
&= t_3(t_1 t_2) - t_1(t_2 t_3) \\
&= -(t_1 t_2)t_3 - t_1(t_2 t_3) \quad \text{by (3)} \\
&= 0.
\end{aligned}$$

Hence,
$$(t_3 t_2)t_1 = -t_1(t_3 t_2) \quad \text{(anti-commutativity for } k = 1). \quad \square$$

Claim 2. $t_1^2, t_2^2, t_3^2 \in Z$.

Proof. We have $[t_1^2, t_i] = (t_1^2, t_i, t_j) = 0$ for all $i, j \in \{1, 2, 3\}$. Indeed, except for the cases $(i, j) = (2, 3)$ and $(3, 2)$ in the second identity, this follows from (C) and Artin's Theorem, that is, any subalgebra generated by 2 elements is associative. For the cases $(i, j) = (2, 3)$ and $(3, 2)$, one can use the identity $(a^2, b, c) = (a, ab + ba, c)$ [SSSZ, (17), p.36] for any alternative algebra. Hence, $(t_1^2, t_2, t_3) = (t_1^2, t_3, t_2) = 0$ by (C). Thus, t_1^2 is central for a generating set of $\Phi_C[t_1, t_2, t_3]$, and by the theorem of Bruck and Kleinfeld [SSSZ, Lemma 16, p.289], we obtain $t_1^2 \in Z$. By the symmetry of our relations (C) with Claim 1, we also obtain $t_2^2, t_3^2 \in Z$. $\square$

Claim 3. $(t_1 t_2)^2 = -t_1^2 t_2^2$, $(t_1 t_3)^2 = -t_1^2 t_3^2$, $(t_2 t_3)^2 = -t_2^2 t_3^2$ and $(t_1(t_2 t_3))^2 = t_1^2 t_2^2 t_3^2$. In particular, they are all in the center Z.

Proof. Again, by Artin's Theorem, we have $(uv)^2 = uvuv$ (no parentheses are needed). So, for example, $(t_1 t_2)^2 = -t_1^2 t_2^2$, or for the last one, $(t_1(t_2 t_3))^2 = t_1(t_2 t_3)t_1(t_2 t_3) = -(t_2 t_3)t_1^2(t_2 t_3)$ (by Claim 1) $= -t_1^2(t_2 t_3)^2 = t_1^2 t_2^2 t_3^2$. $\square$

We can give a natural $\mathbb{N}^3$-grading to $\Phi_C[t_1, t_2, t_3]$, defining $\deg t_1 = (1, 0, 0)$, $\deg t_2 = (0, 1, 0)$ and $\deg t_3 = (0, 0, 1)$. This is possible because $\Phi_C[t_1, t_2, t_3]$ is defined by the homogeneous relations (C).

Proposition 3.1. *Let $t = t_{i_1} \cdots t_{i_k}$ be an element of degree (ℓ, m, n) in $\Phi_C[t_1, t_2, t_3]$, omitting various parentheses, where $t_{i_j} = t_1, t_2$ or t_3, so that the total degree of t is $\ell + m + n = k$. Then $t = \pm(t_1^\ell t_2^m)t_3^n$.*

Proof. It is clear for $k = 1, 2$ or 3 by Claim 1. Suppose that the total degree $k > 3$. Then there exists at least one of the following parts in t: (i) $a(bc)$, (ii) $(ab)c$ or (iii) $(ab)(cd)$ for some $a, b, c, d \in \{t_1, t_2, t_3\}$. For (i), if two of a, b, c are the same, then $a(bc) = \pm t_p^2 t_q$ for some $p, q \in \{1, 2, 3\}$ by Claim 1 and 2. Hence, $t = \pm t' t_p^2$ for some t' which is a product of

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t_{i_j}'s with total degree $(k-2)$ by Claim 2, and by induction, t' has the desired form, and so does t by Claim 2. Otherwise, $a(bc) = \pm(t_1 t_2)t_3$. Consider a next part s so that t has a part $s((t_1 t_2)t_3)$ or $((t_1 t_2)t_3)s$. (s can be one of t_1, t_2, t_3.) By induction and Claim 1 and 2, $s = \pm t_i z_1$, $\pm t_i t_j z_2$ $(i \neq j)$ or $\pm(t_1 t_2)t_3 z_3$, where z_1, z_2, z_3 are products of even power of t_{i_j}'s and so $z_1, z_2, z_3 \in Z$. If $z_i \neq 1$, then one can use induction again for the part after taking off z_i from t. Hence we can assume that $s = t_i$, $t_i t_j$ $(i \neq j)$ or $\pm(t_1 t_2)t_3$. Then $s((t_1 t_2)t_3)$ or $((t_1 t_2)t_3)s$ has, correspondingly, the factor t_i^2, $(t_i t_j)^2 = -t_i^2 t_j^2$ or $((t_1 t_2)t_3)^2 = t_1^2 t_2^2 t_3^2$ (by Claim 1 and 3), and so $t = \pm t' t_i^2$, $t = \pm t' t_i^2 t_j^2$ or $t = \pm t' t_1^2 t_2^2 t_3^2$ for some t' which is a product of t_{i_j}'s with total degree $k-2$, $k-4$ or $k-6$. Hence, by induction, t' has the desired form, and so does t by Claim 2. The case (ii) returns to the case (i) by Claim 1. For (iii), two of a, b, c, d should be the same, and so by Claim 1, 2 and the middle Moufang identity, $(ab)(cd) = \pm t_p^2(t_q t_r)$ for some $p, q, r \in \{1, 2, 3\}$. Hence, by the same argument in the first case of (i), t has the desired form. $\square$

Now, let $\Phi[z_1, z_2, z_3]$ be the ordinary polynomial algebra over Φ in three variables, and let

$$D = (\Phi[z_1, z_2, z_3], z_1, z_2, z_3)$$

be the octonion algebra, i.e., the Cayley-Dickson process over $\Phi[z_1, z_2, z_3]$ three times with structure constants z_1, z_2 and z_3 starting with trivial involution. Let v_1, v_2 and v_3 be the basic generators in each step of D so that $v_1^2 = z_1$, $v_2^2 = z_2$ and $v_3^2 = z_3$. Then D has a natural $\mathbb{N}^3$-grading, defining $\deg v_1 = (1, 0, 0)$, $\deg v_2 = (0, 1, 0)$ and $\deg v_3 = (0, 0, 1)$. It is easily seen that every homogeneous space is a 1-dimensional free Φ-module.

Remark 3.2. The algebra D appears as a subalgebra of a free alternative algebra, discovered by Dorofeev (see [SSSZ, Theorem 13, p.296]). More precisely, let $\mathcal{F}$ be the free alternative algebra over Φ generated by distinct elements a, b and c. Let $u = [a, b]$, $v = (a, b, c)$ and $w = (u, v, a)$. Then the subalgebra of $\mathcal{F}$ generated by u, v and w is isomorphic to D via $u \mapsto v_1$, $v \mapsto v_2$ and $w \mapsto v_3$.

We now prove our main theorem.

Theorem 3.3. $\Phi_C[t_1, t_2, t_3]$ is graded isomorphic to D. In particular, $Z = \Phi[t_1^2, t_2^2, t_3^2]$, which is the ordinary polynomial algebra over Φ in three variables t_1^2, t_2^2, t_3^2, and $\Phi_C[t_1, t_2, t_3]$ is an octonion algebra (Z, t_1^2, t_2^2, t_3^2).

Proof. By Lemma 1.2, there exists the epimorphism from $\Phi_C[t_1, t_2, t_3]$ onto D defined by $t_1 \mapsto v_1$, $t_2 \mapsto v_2$ and $t_3 \mapsto v_3$. So it is enough to show that every homogeneous space for the natural $\mathbb{N}^3$-grading of $\Phi_C[t_1, t_2, t_3]$ is generated by one element. But this follows from Proposition 3.1. $\square$

We note that the multiplicative subset

$$S := \{z_1^p z_2^q z_3^r\}_{p, q, r \in \mathbb{N}}$$

of the center $\Phi[z_1, z_2, z_3]$ of D does not contain zero divisors of the octonion algebra D (which is 8-dimensional over the center), and so the ring of quotients $S^{-1}D$ is also 8-dimensional

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over the center $\Phi[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$ with generators v_1, v_2 and v_3, and the multiplication table respect to the generators is the same as the multiplication table on D. Hence it is the octonion algebra

$$S^{-1}D = (\Phi[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}], z_1, z_2, z_3).$$

Note that v_1, v_2 and v_3 are invertible in $S^{-1}D$, and $v_1^{-1} = z_1^{-1}v_1$, $v_2^{-1} = z_2^{-1}v_2$ and $v_3^{-1} = z_3^{-1}v_3$. Thus defining $\deg v_1^{-1} = (-1, 0, 0)$, $\deg v_2^{-1} = (0, -1, 0)$ and $\deg v_3^{-1} = (0, 0, -1)$, $S^{-1}D$ has a $\mathbb{Z}^3$-grading, and the $\mathbb{Z}^3$-graded algebra $S^{-1}D$ is called the *octonion 3-torus* or the *Cayley torus*. Note that D embeds into $S^{-1}D$. Clearly, every homogeneous space of the Cayley torus is a 1-dimensional free Φ-module.

Corollary 3.4. *Let $T = \{t_1^p t_2^q t_3^r\}_{p,q,r \in 2\mathbb{N}}$ be the subset of $\Phi_C[t_1, t_2, t_3]$.*
(1) T is a multiplicative subset of the center of $\Phi_C[t_1, t_2, t_3]$, which does not contain zero divisors of $\Phi_C[t_1, t_2, t_3]$, and $T^{-1}\Phi_C[t_1, t_2, t_3]$ can be identified with the Cayley torus $S^{-1}D$ via $t_1 \mapsto v_1$, $t_2 \mapsto v_2$ and $t_3 \mapsto v_3$.
(2) If A is an alternative algebra over Φ generated by a_1, a_2 and a_3, and they satisfy the Cayley relations $a_1 a_2 = -a_2 a_1$, $a_1 a_3 = -a_3 a_1$, $a_2 a_3 = -a_3 a_2$ and $(a_1 a_2)a_3 = -a_1(a_2 a_3)$, then A is a homomorphic image of $\Phi_C[t_1, t_2, t_3]$ via $t_1 \mapsto a_1$, $t_2 \mapsto a_2$ and $t_3 \mapsto a_3$, and a_1^2, a_2^2 and a_3^2 are central in A.
If, moreover, a_1, a_2 and a_3 are invertible, then A is a homomorphic image of the Cayley torus $T^{-1}\Phi_C[t_1, t_2, t_3]$ via the same map.
(3) The Cayley torus has a presentation in the category of alternative algebras; generators $t_1^{\pm 1}$, $t_2^{\pm 1}$ and $t_3^{\pm 1}$ with relations $t_i t_i^{-1} = 1$ for $i = 1, 2, 3$ and the Cayley relations (C).

Proof. (1) is now clear by Theorem 3.3. For (2), let $\varphi : \Phi_C[t_1, t_2, t_3] \longrightarrow A$ be the epimorphism defined by $t_1 \mapsto a_1$, $t_2 \mapsto a_2$ and $t_3 \mapsto a_3$. Then the elements of $\varphi(T)$ are central in A by Theorem 3.3. For the second statement, since $\varphi(T)$ are invertible, φ extends to $T^{-1}\Phi_C[t_1, t_2, t_3]$ by the universal property of the ring of quotients.

For (3), let Q be the alternative algebra having the presentation in the assertion. Define $\deg t_1^{\pm 1} = (\pm 1, 0, 0)$, $\deg t_2^{\pm 1} = (0, \pm 1, 0)$ and $\deg t_3^{\pm 1} = (0, 0 \pm 1)$. Let Q^α be the space generated by the monomials of degree $\alpha \in \mathbb{Z}^3$. Then $Q = \sum_{\alpha \in \mathbb{Z}^3} Q^\alpha$. Since the Cayley torus $T^{-1}\Phi_C[t_1, t_2, t_3]$ has the relations in the assertion, there is a natural homomorphism from Q onto the Cayley torus so that Q^α is mapped onto the homogeneous space of degree α in the Cayley torus. Hence $Q = \oplus_{\alpha \in \mathbb{Z}^3} Q^\alpha$ (the sum becomes direct). On the other hand, by (2), there is a natural graded homomorphism from $T^{-1}\Phi_C[t_1, t_2, t_3]$ onto Q. Hence they are graded isomorphisms. $\square$

We note that Part (3) of Corollary 3.4 was obtained independently by Bruce Allison (unpublished). Because of the presentation of the Cayley torus, it is reasonable to write

$$T^{-1}\Phi_C[t_1, t_2, t_3] = \Phi_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}].$$

Also, an *octonion n-torus* $(n \geq 3)$ $(\Phi[z_1^{\pm 1}, \ldots, z_n^{\pm 1}], z_1, z_2, z_3)$ can be written as

$$\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] := \Phi_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \otimes_\Phi \Phi[t_4^{\pm 1}, \ldots, t_n^{\pm 1}]$$

where $\Phi[t_4^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the Laurent polynomial algebra over Φ in $(n-3)$-variables. Letting $P := \Phi[t_4^{\pm 1}, \ldots, t_n^{\pm 1}]$, the octonion torus $\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ can be considered as the Cayley torus over P, i.e., $P_C[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$.

One can start with the alternative algebra $\Phi_C[t_1, \ldots, t_n]$ over Φ with generators $t_1, \ldots, t_n$ $(n \geq 3)$ and the Cayley relations (C), and the central relations

(Z) $[t_i, t_k] = (t_i, t_j, t_k) = 0$ for $i < j < k$ with $i, j = 1, \ldots, n$ and $k = 4, \ldots, n$.

Then by the same argument as above, we obtain the following:

Theorem 3.5. *$\Phi_C[t_1, \ldots, t_n]$ is graded isomorphic to $(\Phi[z_1, \ldots, z_n], z_1, z_2, z_3)$. In particular, the center $Z = \Phi[t_1^2, t_2^2, t_3^2, t_4, \ldots, t_n]$, and $\Phi_C[t_1, \ldots, t_n]$ is an octonion algebra (Z, t_1^2, t_2^2, t_3^2).*

Moreover, generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$ with the Cayley relations (C), the central relations (Z), and the invertible relations $t_i t_i^{-1} = 1$ for $i = 1, \ldots, n$ give a presentation of an octonion torus $\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ in the category of alternative algebras.

An octonion torus (under the name of the alternative torus) was found in [BGKN] on the classification of extended affine Lie algebras (under the name of quasi-simple Lie algebras). The generators and relations of an octonion torus will be useful for determining generators and relations for certain extended affine Lie algebras.

The following theorem gives a presentation of an octonion algebra over Φ.

Theorem 3.6. *Any octonion algebra over Φ, say $(\Phi, \mu_1, \mu_2, \mu_3)$ for cancellable scalars μ_1, μ_2, μ_3 of Φ, is isomorphic to $\Phi_C[t_1, t_2, t_3]/(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$. Hence $(\Phi, \mu_1, \mu_2, \mu_3)$ has a presentation in the category of alternative algebras; generators t_1, t_2 and t_3 with Cayley relations (C) and $t_1^2 = \mu_1$, $t_2^2 = \mu_2$ and $t_3^2 = \mu_3$.*

In particular, if Φ is a field, A is an alternative algebra over Φ generated by a_1, a_2 and a_3, and they satisfy $a_1^2 = \mu_1$, $a_2^2 = \mu_2$, $a_3^2 = \mu_3$, and the Cayley relations $a_1 a_2 = -a_2 a_1$, $a_1 a_3 = -a_3 a_1$, $a_2 a_3 = -a_3 a_2$ and $(a_1 a_2)a_3 = -a_1(a_2 a_3)$, then A is isomorphic to $(\Phi, \mu_1, \mu_2, \mu_3)$.

Proof. Let $B := \Phi_C[t_1, t_2, t_3]/(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$. Let v_1, v_2 and v_3 be the basic generators of $(\Phi, \mu_1, \mu_2, \mu_3)$ so that $v_1^2 = \mu_1$, $v_2^2 = \mu_2$ and $v_3^2 = \mu_3$. Let $\varphi : \Phi_C[t_1, t_2, t_3] \longrightarrow (\Phi, \mu_1, \mu_2, \mu_3)$ be the epimorphism defined by $t_1 \mapsto v_1$, $t_2 \mapsto v_2$ and $t_3 \mapsto v_3$. Since the ideal $(t_1^2 - \mu_1, t_2^2 - \mu_2, t_3^2 - \mu_3)$ is contained in the kernel of φ, φ descends to an epimorphism $\overline{\varphi} : B \longrightarrow (\Phi, \mu_1, \mu_2, \mu_3)$. By Theorem 3.3, $\Phi_C[t_1, t_2, t_3]$ is an 8-dimensional free $\Phi[t_1^2, t_2^2, t_3^2]$-module with basis $\{1, t_1, t_2, t_3, t_1 t_2, t_1 t_3, t_2 t_3, (t_1 t_2)t_3\}$. So $\{1, \overline{t}_1, \overline{t}_2, \overline{t}_3, \overline{t_1 t_2}, \overline{t_1 t_3}, \overline{t_2 t_3}, \overline{(t_1 t_2)t_3}\}$ generates B over Φ, where $^-$ is the canonical map from $\Phi_C[t_1, t_2, t_3]$ onto B. Since $\overline{\varphi}(\overline{t}_i) = v_i$ $(i = 1, 2, 3)$ and $\{1, v_1, v_2, v_3, v_1 v_2, v_1 v_3, v_2 v_3, (v_1 v_2)v_3\}$ is linearly independent over Φ, $\overline{\varphi}$ is injective and we get $B \cong (\Phi, \mu_1, \mu_2, \mu_3)$.

For the second statement, one gets $A \cong B$ since A is a homomorphic image of the simple algebra B and $A \neq 0$. $\square$

Let R be an alternative ring with a nonzero center Z which does not contain zero divisors of R (e.g. R is prime). Then $Z^* = Z \setminus \{0\}$ is a multiplicative subset of Z, and one can construct the ring of quotients $(Z^*)^{-1}R$, which is called the *central closure* of R, denoted $\overline{R}$. We note that R embeds into $\overline{R}$, $\overline{Z} = (Z^*)^{-1}Z$ is a field of fractions of Z, $\overline{R}$ is a central $\overline{Z}$-algebra, and $\overline{R} \cong \overline{Z} \otimes_Z R$. Moreover, R is called a *Cayley-Dickson ring* if the central closure $\overline{R}$ is an octonion algebra over $\overline{Z}$ (see [SSSZ, p.193]). For example, if Φ is a domain, our ring of Cayley polynomials $\Phi_C[t_1, t_2, t_3]$ or an octonion torus $\Phi_C[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is a Cayley-Dickson ring so that the central closure is $(\overline{Z}, t_1^2, t_2^2, t_3^2)$, where $\overline{Z} = \overline{\Phi}(t_1^2, t_2^2, t_3^2)$ or $\overline{\Phi}(t_1^2, t_2^2, t_3^2, t_4, \ldots, t_n)$ (rational function fields over the field of fractions $\overline{\Phi}$ in 3 or n variables), respectively. However, a Cayley-Dickson ring is not necessarily an octonion algebra over the center (see Example 4.3).

Lemma 4.1. *Let A be an alternative algebra over Φ with center Z which does not contain zero divisors of A. Assume that A is generated by a_1, a_2 and a_3, and they satisfy the Cayley relations $a_1 a_2 = -a_2 a_1$, $a_1 a_3 = -a_3 a_1$, $a_2 a_3 = -a_3 a_2$ and $(a_1 a_2)a_3 = -a_1(a_2 a_3)$, then A is an octonion algebra over Z, isomorphic to (Z, a_1^2, a_2^2, a_3^2).*

Proof. Since A embeds into $\overline{A}$, a_1, a_2 and a_3 also satisfy the Cayley relations in $\overline{A}$. Also, $a_1^2, a_2^2, a_3^2 \in Z$ by Corollary 3.4 (2). Hence, by Theorem 3.6, $\overline{A}$ is an octonion algebra over the field $\overline{Z}$, i.e., $\overline{A} = (\overline{Z}, a_1^2, a_2^2, a_3^2)$. In particular, A is an 8-dimensional free Z-module with basis $\{1, a_1, a_2, a_3, a_1 a_2, a_1 a_3, a_2 a_3, (a_1 a_2)a_3\}$, which is also a basis of $\overline{A}$. Moreover, A and $\overline{A}$ has the same multiplication table relative to the basis, and so $A = (Z, a_1^2, a_2^2, a_3^2)$. $\square$

Let us state a celebrating theorem in alternative theory [SSSZ, Theorem 9, p.194]:

Slater's Theorem. *Any prime nondegenerate alternative algebra that is not associative is a Cayley-Dickson ring.*

(It is also true that every Cayley-Dickson ring is a prime nondegenerate ring [SSSZ, Proposition 3, p.193].) Using the theorem, we have the following:

Proposition 4.2. *Let R be a prime nondegenerate alternative algebra that is not associative over Φ, and $Z = Z(R)$ its center. Then there exist a subalgebra A of R so that $Z(A) \subset Z$ and A is an octonion algebra over the center $Z(A)$. Moreover, the subalgebra $B := ZA$ is an octonion algebra over the center Z and the central closures of B and R coincide, i.e., $\overline{B} = \overline{R}$.*
   Also, $\overline{R}$ is a base field extension of $\overline{A}$, namely, $\overline{R} \cong \overline{Z} \otimes_K \overline{A}$, where $K = \overline{Z(A)}$.

Proof. By Slater's Theorem, $\overline{R}$ is an octonion algebra over the field $\overline{Z}$. Let v_1, v_2 and v_3 be the basic generators and so they satisfy the Cayley relations. Note that $v_1 = z_1^{-1}r_1$, $v_2 = z_2^{-1}r_2$ and $v_3 = z_3^{-1}r_3$ for some $z_1, z_2, z_3 \in Z^*$ and $r_1, r_2, r_3 \in R$. So $a_1 := z_1 z_2 z_3 v_1$, $a_2 := z_1 z_2 z_3 v_2$ and $a_3 := z_1 z_2 z_3 v_3$ also satisfy the Cayley relations and they are in R. Let A be the subalgebra of R generated by a_1, a_2 and a_3. Note that if $z \in Z(A)$, then z is, in particular, central for a generating set $\{a_1, a_2, a_3\}$ of A, and so is for a generating set $\{v_1, v_2, v_3\}$ of $\overline{R}$. Hence, by the theorem of Bruck and Kleinfeld [SSSZ, Lemma 16, p.289],

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z is central for $\overline{R}$, and so $z \in Z$. Thus, $Z(A)$ does not contain zero divisors of A, and hence by Lemma 4.1, $A = (Z(A), a_1^2, a_2^2, a_3^2)$.

Now, we have $Z \subset Z(B) \subset Z$, and so $Z = Z(B)$. Thus by Lemma 4.1 again, $B = (Z, a_1^2, a_2^2, a_3^2)$. Finally, for any $r \in R$, there exists some $z \in Z^*$ such that $zr \in B$. In fact, $r = f(v_1, v_2, v_3) = f(z_1^{-1}z_2^{-1}z_3^{-1}a_1, z_1^{-1}z_2^{-1}z_3^{-1}a_2, z_1^{-1}z_2^{-1}z_3^{-1}a_3)$ for some polynomial f over $\overline{Z}$. So there exists $z \in Z^*$ such that $zf(z_1^{-1}z_2^{-1}z_3^{-1}a_1, z_1^{-1}z_2^{-1}z_3^{-1}a_2, z_1^{-1}z_2^{-1}z_3^{-1}a_3) = g(a_1, a_2, a_3)$ for some polynomial g over Z. Hence $zr = g(a_1, a_2, a_3) \in B$. Thus $r = z^{-1}b$ for some $b \in B$, and so $\overline{R} \subset \overline{B}$. Since the other inclusion is clear, we obtain $\overline{R} = \overline{B}$.

For the last statement, let φ be a K-linear map from $\overline{Z} \otimes_K \overline{A}$ to $\overline{R}$ defined by $\varphi(u_i \otimes w_j) = u_i w_j$ for a basis $\{u_i\}$ of $\overline{Z}$ over K and a basis $\{w_i\}$ of $\overline{A}$ over K. Then φ is a homomorphism, and $\overline{Z}$-linear. Since $\overline{Z} \otimes_K \overline{A}$ and $\overline{R}$ are both 8-dimensional over $\overline{Z}$, it is enough to show that φ is onto. For $v^{-1}r \in \overline{R}$ ($v \in Z^*$ and $r \in R$), there exists $z \in Z^*$ and $b \in B$ such that $zr = b$ by the above, and so $zr = \sum_k z_k a_k$ for some $z_k \in Z$ and $a_k \in A$. Hence $v^{-1}r = v^{-1}z^{-1}\sum_k z_k a_k = \varphi(\sum_k v^{-1}z^{-1}z_k \otimes a_k)$, and so φ is onto. $\square$

Let us finally give an example of a prime nondegenerate algebra which is not an octonion algebra over the center.

Example 4.3. For simplicity, let F be a field of characteristic $\neq 2$, and let $F[z]$ be the ordinary polynomial algebra over F. Let $F[z]_C[t_1, t_2, t_3]$ be the ring of Cayley polynomials over $F[z]$. Let R be the F-subalgebra of $F[z]_C[t_1, t_2, t_3]$ generated by t_1, t_2, t_3 and zt_1. Then the center $Z = Z(R) = F[t_1^2, t_2^2, t_3^2, z^2t_1^2, zt_1^2]$, and R is a 12-dimensional free Z-module with basis

$$\{1, t_1, t_2, t_3, t_1t_2, t_1t_3, t_2t_3, (t_1t_2)t_3, zt_1, zt_1t_2, zt_1t_3, zt_1(t_2t_3)\}.$$

Hence R is not an octonion algebra over Z. But the central closure $\overline{R}$ is an octonion algebra over $\overline{Z} = F(z, t_1^2, t_2^2, t_3^2)$, i.e., $\overline{R} = (\overline{Z}, t_1^2, t_2^2, t_3^2)$ (by Theorem 3.6), and so R is a Cayley-Dickson ring. One can take a subalgebra A of R in Proposition 4.2 as $A = F_C[t_1, t_2, t_3]$.

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