From Fine Analysis to Instance Optimality

Jérémy Barbay

Departmento de Ciencias de la Computacion
Universidad de Chile

Outline

1. The Convex Hull Paradox
   - $O(n \lg n)$
   - $O(nh)$
   - Worst Case Complexity?
2. Fine analysis of the convex hull
   - $O(n \lg h)$ in 2D
   - $O(n \lg h)$ in 3D
   - $O(nH(C))$, instance optimal
3. Similar Paradoxes

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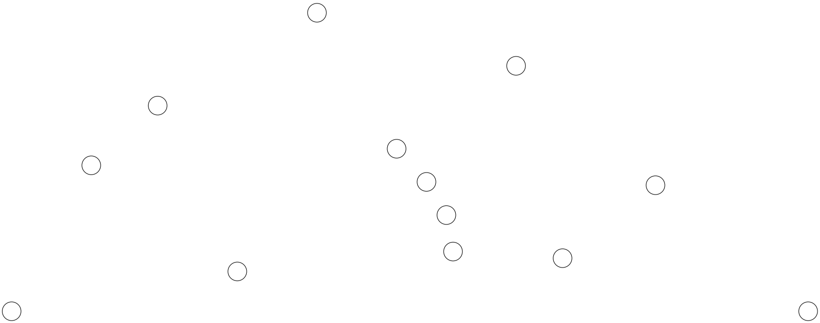
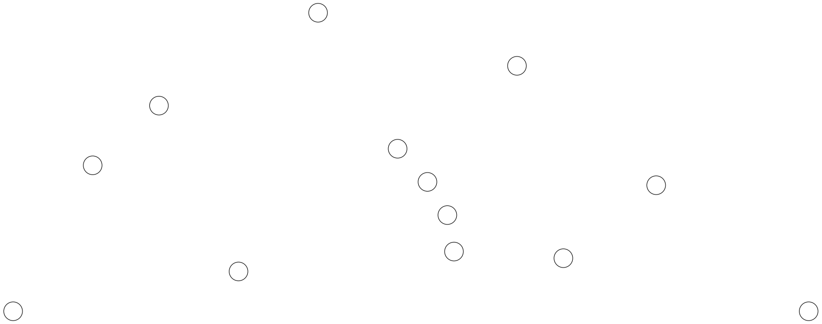
The Planar Convex Hull



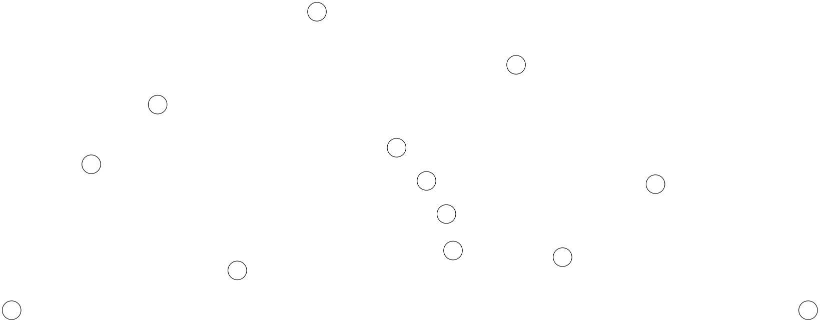
- Can one of you define it?
- What is the best complexity known for it?

2d Convex Hull in $O(n \lg n)$



1. Sort the points by x-coordinates;
2. Scan them, backtracking if necessary.

2d Convex Hull in $O(nh)$



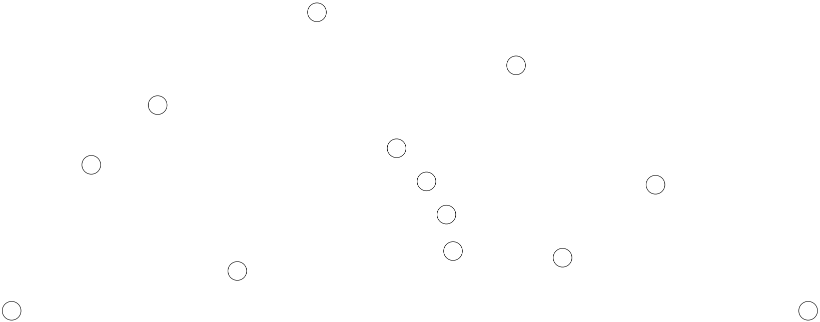
1. Find the left-most point
2. Compute the $n - 1$ slopes with the other points
3. Choose the highest slope
4. Iterate

Worst case Complexity of 2d Convex Hull

- Question ill-defined: the worst case over what?
  - all instances of fixed size n?
  - all instances of fixed input size n and output size h?
- For each we have distinct lower bounds:
  - $\Omega(n \lg n)$, which is tight; and
  - $\Omega(n \lg h)$, which is not tight!
- So what is the complexity of 2d convex hull?

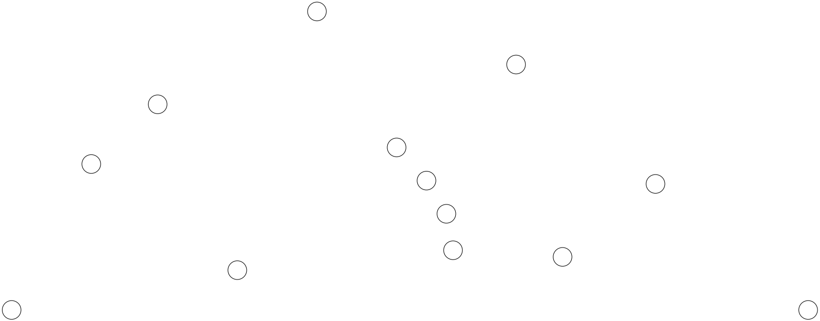
Outline

2d Convex Hull in $O(n \lg h)$ in 2D



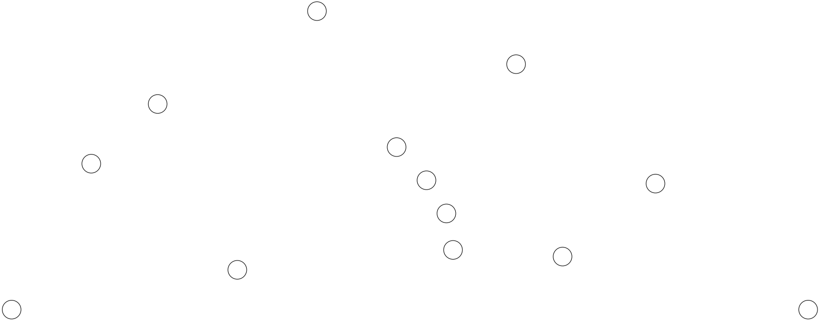
1. Compute the point m of median x-coordinate;
2. Partition the points by $m.x$;
3. Compute the highest edge (a, b) intersecting the line $x = m.x$;
4. Recurse on each side;

2d Convex Hull in $O(n \lg h)$ in 3D



1. Start with a small guess for h;
2. Group the instances in n/h x-sorted groups of size h;
3. Simulate the $O(nh)$ algorithms on the groups;
4. If it did not suffice, merge the group two by two and iterate.

Convex Hull in $O(n(1 + H(C)))$



- Algorithm: a variant of [Kirkpatrick, Seidel]
  1. Compute the points leftmost l and rightmost r;
  2. Compute the point m of median x-coordinate;
  3. Compute the highest edge (a, b) intersecting the line $x = m.x$;
  4. Remove all points contained in the polygon (l, a, b, r);
  5. Recurse on each side;

Instance Optimality: definitions

Definition (Instance Optimality)

An algorithm is instance-optimal if its cost is at most a constant factor from the cost of any other algorithm A' running on the same input, for *every* input instance.

Unfortunately, for many problems, this requirement is too stringent.

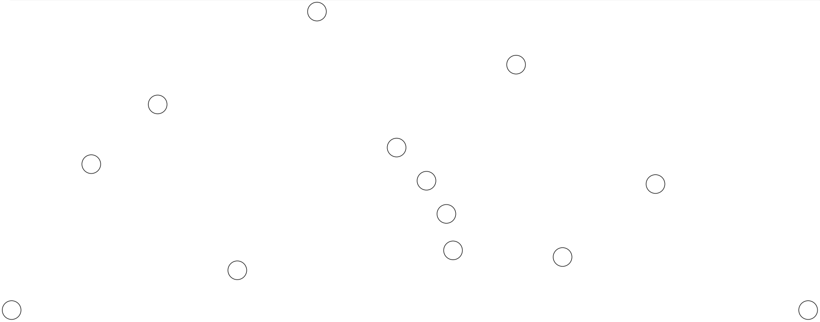
Definition (Input Order Oblivious Instance Optimality)

For a set S of n elements in $\mathcal{D}$, let $T_A(S)$ denote the maximum running time of A on input σ over all $n!$ possible permutations σ of S. Let $\mathrm{OPT}(S)$ denote the minimum of $T_{A'}(S)$ over all correct algorithms $A' \in \mathcal{A}$. If $A \in \mathcal{A}$ is a correct algorithm such that $T_A(S) \leq O(1) \cdot \mathrm{OPT}(S)$ for every set S, then we say A is instance-optimal in the order-oblivious setting.

Certificate and Instance Optimal Proof

Definition (Certificate)

A *Certificate* for an instance I and a solution S is the description of a sequence of steps to check the validity of S for I.



Example

For the convex hull, a list of triangles and the points they cover.

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Optimal Prefix Free Codes [In Progress]

- $O(n \lg n)$ classical algorithm.
- $O(n)$ algorithm when frequencies are sorted.
- $O(n)$ algorithm when frequencies are all within a factor of 2.
- $O(n)$ algorithm when frequencies are all distinct by factor of 2.
- Adaptive Results for k distinct code lengths:
  - Belal and Elmasry claim $O(nk)$ in STACS 2006.
  - Belal and Elmasry claim $O(n4^k)$ in ARXIV 2012.
- A lower bound of $\Omega(n \lg k)$ in the worst case over instances resulting in k distinct code lengths.
- Conjectures:
  - $O(n \lg k)$ adaptive algorithm?
  - $O(nH(n_1, \dots, n_k))$ instance optimal algorithm?
  - $O(n)$ algorithm in word-RAM?

Optimal Minimax Trees [Open]

- Tree minimizing the max weight+height of a leaf.
- $O(n \lg n)$ classical algorithm [Golumbic];
- $O(n)$ algorithm when weights partially sorted by fractional part [Drmota, Szpankowski];
- $O(nd \lg \lg n)$ where d is the number of distinct values $\lceil w_i \rceil$ [Kirkpatrick and Klawe]
- $O(n)$ algorithm in word-RAM [Gawrichowski,Gagie]!

Optimal Alphabetic Binary Search Tree [Open]

- $O(n \lg n)$ classical Hu-Tucker algorithm;
- $o(n \lg n)$ algorithms in many particular cases;
- $O(n)$ algorithm when frequencies "can be sorted in linear time";
- A lower bound of $\Omega(n \lg k)$ in the worst case over instances resulting in k distinct code lengths.

Summary

- $O(nk)$ and $O(n \lg n)$ suggests $O(n \lg k)$
- and (Input Order Oblivious) Instance Optimality.

- Outlook
  - Input Order Adaptive Instance Optimality.
  - Full Instance Optimality (Kolmogorov's complexity?)
  - 1-instance optimality!