Parameterisation of a Race Track

According to Mathias, an easy way to describe the motion of a car on a race track is to use the so called curvilinear coordinate system. The first step to use curvilinear coordinate system is to find a parameterisation of the centre line of the track by arc length of it.

Here, $\gamma_c(s)$ maps the arc length s to the Cartesian coordinate of the track

$$\gamma_{\mathrm{c}}(s) = \begin{pmatrix} x_{\mathrm{r}}(s) \\ y_{\mathrm{r}}(s) \end{pmatrix}$$

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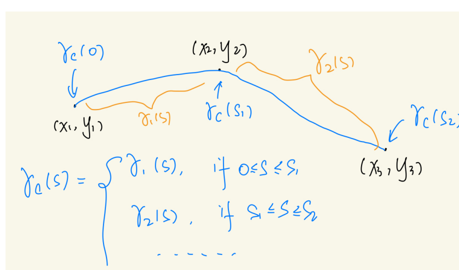
However, it is almost impossible to find an elegant mathematical representation of the centre line, and also, real-world censor often provides discrete data, for example, point clouds. Therefore, one way to solve the problem is to use a set of ordered points to describe the centre line and interpolate between the points.

The formulation of this problem is:

- **Given:** A set of ordered points (x_i, y_i), $i = 1, \ldots, n$
- **Output:** A smooth curve that goes through every point. In particular, the curve needs to be:
  - C^2 smooth at each (x_i, y_i)



Solution: Cubic Spline

A 3rd-order polynomial, which has 4 free parameters, can be used to interpolate between two points such that It satisfies 4 equations for each piece:

$$\begin{cases} \gamma_i(s_i) = \gamma_{i+1}(s_i) = (x_{i+1}, y_{i+1}) \\ \gamma_i(s_{i-1}) = \gamma_{i-1}(s_{i-1}) = (x_i, y_i) \\ \gamma_i'(s_i) = \gamma_{i+1}'(s_i) \\ \gamma_i''(s_i) = \gamma_{i+1}''(s_i) \end{cases}$$

Then

$$\gamma(s) = \begin{cases} \gamma_1(s), & s_0 \le s \le s_1 \\ \gamma_2(s), & s_1 \le s \le s_2 \\ \vdots \end{cases}$$

is C^2 smooth.