1 Parameterisation of a Race Track

According to Mathias, an easy way to describe the motion of a car on a race track is to use the so called curvilinear coordinate system. The first step to use curvilinear coordinate system is to find a parameterisation of the centre line of the track by arc length of it.

Here, $\gamma_c(s)$ maps the arc length s to the Cartesian coordinate of the track

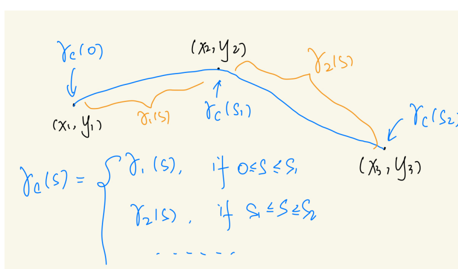
$$\gamma_{\mathrm{c}}(s) = \begin{pmatrix} x_{\mathrm{r}}(s) \\ y_{\mathrm{r}}(s) \end{pmatrix}$$

.

However, it is almost impossible to find an elegant mathematical representation of the centre line, and also, real-world censor often provides discrete data, for example, point clouds. Therefore, one way to solve the problem is to use a set of ordered points to describe the centre line and interpolate between the points.

The formulation of this problem is:

- **Given:** A set of ordered points (x_i, y_i), $i = 1, \ldots, n$
- **Output:** A smooth curve that goes through every point. In particular, the curve needs to be:
  - C^2 smooth at each (x_i, y_i)



Solution: Cubic Spline

A 3rd-order polynomial, which has 4 free parameters, can be used to interpolate between two points such that It satisfies 4 equations for each piece:

$$\begin{cases} \gamma_i(s_i) = \gamma_{i+1}(s_i) = (x_{i+1}, y_{i+1}) \\ \gamma_i(s_{i-1}) = \gamma_{i-1}(s_{i-1}) = (x_i, y_i) \\ \gamma_i'(s_i) = \gamma_{i+1}'(s_i) \\ \gamma_i''(s_i) = \gamma_{i+1}''(s_i) \end{cases}$$

Then

$$\gamma(s) = \begin{cases} \gamma_1(s), & s_0 \le s \le s_1 \\ \gamma_2(s), & s_1 \le s \le s_2 \\ \vdots \end{cases}$$

is C^2 smooth.

2 Optimize a reference path

According to Mathias, to simplify the problem, we can find an optimal path using MPC by setting the vehicle's velocity to be 1. The objective is then to maximize the vehicle's progress relative to the reference curve and to minimize the total curvature of the path. The curvature of the path is considered as a control variable and it is related to vehicle's tire model.

2.1 Problem formulation

Using (s, r) to represent the vehicle's position where s is the projection on the centre line and r is the lateral distance relative to it. Let $\kappa(t)$ be the curvature of the path at time t, T be the predictive time scale, t_i be the initial time, $\psi(t)$ be the path angle at time t, ψ_{ref} be the centre line's angle at the vehicle's position at time t.

Side notes – coordinate transform:

$$\kappa_{\text{ref}}(s) = \frac{x'_{\text{ref}}(s)y''_{\text{ref}}(s) - y'_{\text{ref}}(s)x''_{\text{ref}}(s)}{(x'_{\text{ref}}(s)^2 + y'_{\text{ref}}(s)^2)^{3/2}}$$

$$x(t) = x_{\text{ref}}(S(t)) - r(t) \cdot \sin(\psi_{\text{ref}}(S(t)))$$
$$y(t) = y_{\text{ref}}(S(t)) + r(t) \cdot \cos(\psi_{\text{ref}}(S(t)))$$

According to Mathias, one possible formulation of this problem is the following,

Minimize the objective function:

$$-a_1 s(t_i + T) + a_2 \int_{t_i}^{t_i+T} \kappa(t)^2 \, dt$$

subject to:

- System dynamics:

$$\dot{s}(t) = \frac{\cos\big(\psi(t) - \psi_{\text{ref}}(t)\big)}{1 - r(t)\kappa_{\text{ref}}(s(t))},$$
$$\dot{r}(t) = \sin\big(\psi(t) - \psi_{\text{ref}}(t)\big),$$
$$\dot{\psi}(t) = \kappa(t),$$
$$\dot{\psi}_{\text{ref}}(t) = \kappa_{\text{ref}}(s(t)) \cdot \dot{s}(t)$$

- Initial conditions:

$$\big(s(t_i),\, r(t_i),\, \psi(t_i),\, \psi_{\text{ref}}(t_i)\big) = \big(s_i,\, r_i,\, \psi_i,\, \psi_{\text{ref},i}\big)$$

- State constraints:

$$r(t) \in [r_{\min},\, r_{\max}]$$

- Control constraints:

$$\kappa(t) \in [\kappa_{\min},\, \kappa_{\max}]$$

2.2 Problem discretization

For simplicity, $s(t_i)$ is written as s_i and likewise. h is the stepsize for discretization time between t_i and $t_i + T$. The number of discretization is $N - i + 1$. The trapezoidal rule for numerical integration and the Euler method for numerical ODE are used.

Objective:

$$-a_1 S_N + a_2 \sum_{n=1}^{N} \frac{h}{2}(\kappa_n + \kappa_{n-1})$$

Subject to:

$$\begin{cases} S_{j+1} = S_j + h \cdot \frac{\cos(\psi_j - \psi_{\text{ref},j})}{1 - r_j \kappa_{\text{ref}}(s_j)} \\ r_{j+1} = r_j + h \cdot \sin(\psi_j - \psi_{\text{ref},j}) \\ \psi_{j+1} = \psi_j + h\kappa_j \\ \psi_{\text{ref},j+1} = \psi_{\text{ref},j} + h \cdot \kappa_{\text{ref}}(s_j) \cdot \frac{\cos(\psi_j - \psi_{\text{ref},j})}{1 - r_j \kappa_{\text{ref}}(s_j)} \end{cases} \qquad \text{for } j = i, \ldots, N-1$$

Initial conditions:

$$(s_i, r_i, \psi_i, \psi_{ref,i})$$

State constraints:

$$r_j \in [r_{\min}, r_{\max}] \quad \text{for } j = i, \ldots, N$$

Control constraints:

$$\kappa_j \in [\kappa_{\min}, \kappa_{\max}] \quad \text{for } j = i, \ldots, N$$

2.3 Solution

TODO